NONEXTENSIVE STATISTICAL MECHANICS
AND THERMODYNAMICS

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Rio de Janeiro, Brazil
Santa Fe Institute, New Mexico, USA
Complexity Science Hub Vienna, Austria

Santa Fe, June 2017
World champion of complexity!

(Photo taken in London by Luciano Pietronero)
It is the natural (or artificial or social) system itself which, through its geometrical-dynamical properties, mandates the specific informational tool --- entropy --- to be meaningfully used for the study of its thermostatistical and thermodynamical properties.

**TRIADIC CANTOR SET:**

\[
\begin{align*}
10 \text{ cm} & \\
\hline
& \hline
& \hline
& \hline
\end{align*}
\]

\[
d_F = \frac{\ln 2}{\ln 3} = 0.6309...
\]

Hence the interesting measure is

\[
(10 \text{ cm})^{0.6309...} \approx 4.275 \text{ cm}^{0.6309}
\]
The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.
### Entropic Forms

- Concave
- Extensive
- Lesche-stable
- Finite entropy production per unit time
- Pesin-like identity (with largest entropy production)
- Composable (unique trace form; Enciso-Tempesta)
- Topsoe-factorizable (unique)
- Amari-Ohara-Matsuzoe conformally invariant geometry (unique)
- Biro-Barnafoldi-Van thermostat universal independence (unique)
- Nonadditive (if $q \neq 1$)

### Entropic Functionals

#### BG Entropy ($q = 1$)

- Equiprobability
- $p_i = \frac{1}{W}$ (for all $i$)
- $\forall p_i \ (0 \leq p_i \leq 1)$
- $\left( \sum_{i=1}^{W} p_i = 1 \right)$
- $k \ln W$
- $-k \sum_{i=1}^{W} p_i \ln p_i$

#### Entropy $S_q$ ($q$ real)

- $k \frac{W^{1-q} - 1}{1-q}$
- $1 - \sum_{i=1}^{W} p_i^q$
- $k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

Possible generalization of Boltzmann-Gibbs statistical mechanics

**DEFINITIONS**: $q$–logarithm:  
\[ \ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \quad (x > 0; \ ln_1 x = \ln x) \]

$q$–exponential:  
\[ e_q^x \equiv [1 + (1-q) x]^{\frac{1}{1-q}} \quad (e_1^x = e^x) \]

**Hence, the entropies can be rewritten**:

<table>
<thead>
<tr>
<th></th>
<th>equal probabilities</th>
<th>generic probabilities</th>
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<tbody>
<tr>
<td><strong>BG entropy</strong></td>
<td>$k \ln W$</td>
<td>$k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$</td>
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<tr>
<td>$(q = 1)$</td>
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<tr>
<td><strong>entropy $S_q$</strong></td>
<td>$k \ln_q W$</td>
<td>$k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$</td>
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<tr>
<td>$(q \in R)$</td>
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Full bibliography (regularly updated):

http://tsallis.cat.cbpf.br/biblio.htm

6198 articles by 12672 scientists (96 countries)

[26 June 2017]
<table>
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<th>Country</th>
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**12672 SCIENTISTS**  **96 COUNTRIES**

[Updated 26 June 2017]
BOOKS AND SPECIAL ISSUES ON NONEXTENSIVE STATISTICAL MECHANICS

1999

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2002

2004

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2006

2006

2007

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2009

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2011

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2012

2013

2016

~2017

\[ S_q = k \ln_q W \equiv k \frac{W^{1-q} - 1}{1 - q} \]
$e_q^x = \left[ 1 + (1 - q)x \right]^{\frac{1}{1-q}}$
\[ e_q^x = \left[ 1 + (1 - q)x \right]^{\frac{1}{1-q}} \]
\[
\frac{dy}{dx} = -a_q y^q \quad \text{with } y(0) = 1
\]

\[
\Rightarrow y = e^{-a_q x} = \frac{1}{\left[1 + (q-1)a_q x\right]^{q-1}}
\]
TYPICAL SIMPLE SYSTEMS:

- Short-range space-time correlations
- Markovian processes (*short memory*), Additive noise
- Strong chaos (positive maximal Lyapunov exponent), **Ergodic**, Riemannian geometry
- Short-range many-body interactions, weakly quantum-entangled subsystems
- Linear and homogeneous Fokker-Planck equations, Gaussians
  \[ W(N) \propto \mu^N \quad (\mu > 1) \]
- Boltzmann-Gibbs entropy (additive)
- Exponential dependences (Boltzmann-Gibbs weight, ...)

TYPICAL COMPLEX SYSTEMS:

- Long-range space-time correlations
- Non-Markovian processes (*long memory*), Additive and multiplicative noises
- Weak chaos (zero maximal Lyapunov exponent), **Nonergodic**, Multifractal geometry
- Long-range many-body interactions, strongly quantum-entangled subsystems
- Nonlinear and/or inhomogeneous Fokker-Planck equations, \( q \)-Gaussian
  \[ e.g., \quad W(N) \propto N^\rho \quad (\rho > 0) \]
- Entropy \( S_q \) (nonadditive)
- \( q \)-exponential dependences (asymptotic power-laws)

An entropy is additive if, for any two probabilistically independent systems $A$ and $B$,

$$S(A + B) = S(A) + S(B)$$

Therefore, since

$$\frac{S_q(A + B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1 - q) \frac{S_q(A)}{k} \frac{S_q(B)}{k},$$

$S_{BG}$ and $S^R_{q\text{Renyi}} (\forall q)$ are additive, and $S_q (\forall q \neq 1)$ is nonadditive.

**EXTENSIVITY:**

Consider a system $\Sigma \equiv A_1 + A_2 + ... + A_N$ made of $N$ (not necessarily independent) identical elements or subsystems $A_1$ and $A_2$, ..., $A_N$.

An entropy is extensive if

$$0 < \lim_{N \rightarrow \infty} \frac{S(N)}{N} < \infty, \ i.e., \ S(N) \propto N \ (N \rightarrow \infty)$$
EXTENSIVITY OF THE ENTROPY \((N \to \infty)\)

\(W \equiv\) total number of possibilities with nonzero probability, assumed to be equally probable

If \(W(N) \sim \mu^N\) \((\mu > 1)\)

\[\Rightarrow S_{BG}(N) = k_B \ln W(N) \propto N \quad \text{OK!}\]

If \(W(N) \sim N^\rho\) \((\rho > 0)\)

\[\Rightarrow S_q(N) = k_B \ln_q W(N) \propto [W(N)]^{1-q} \propto N^{\rho(1-q)}\]

\[\Rightarrow S_{q=1-1/\rho}(N) \propto N \quad \text{OK!}\]

If \(W(N) \sim \nu^{N^n}\) \((\nu > 1; \ 0 < \gamma < 1)\)

\[\Rightarrow S_{\delta}(N) = k_B \left[ \ln W(N) \right]^\delta \propto N^{\gamma \delta}\]

\[\Rightarrow S_{\delta=1/\gamma}(N) \propto N \quad \text{OK!}\]

IMPORTANT:

\[\mu^N \gg \nu^{N^n} \gg N^\rho \quad \text{if} \quad N \gg 1\]

All happy families are alike; each unhappy family is unhappy in its own way.

Leo Tolstoy (Anna Karenina, 1875-1877)
<table>
<thead>
<tr>
<th>SYSTEMS</th>
<th>ENTROPY $S_{BG}$ (ADDITIVE)</th>
<th>ENTROPY $S_q$ (NONADDITIVE)</th>
<th>ENTROPY $S_\delta$ (NONADDITIVE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W(N)$</td>
<td>EXTENSIVE</td>
<td>NONEXTENSIVE</td>
<td>NONEXTENSIVE</td>
</tr>
<tr>
<td>(equiprobable)</td>
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<tr>
<td>$e.g., \mu^N$ $(\mu &gt; 1)$</td>
<td>EXTENSIVE</td>
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<td>$e.g., N^\rho$ $(\rho &gt; 0)$</td>
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<td>EXTENSIVE $(q = 1 - 1/\rho)$</td>
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<td>$e.g., v^{N^\gamma}$ $(\nu &gt; 1; 0 &lt; \gamma &lt; 1)$</td>
<td>NONEXTENSIVE</td>
<td>NONEXTENSIVE</td>
<td>EXTENSIVE $(\delta = 1/\gamma)$</td>
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</table>
King Thutmose I
18th Dynasty
circa 1500 BC
A theory is the more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability. Therefore the deep impression that classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that, within the framework of applicability of its basic concepts, it will never be overturned.

Albert Einstein (1949)
THERMODYNAMICS:

\[ G(V, T, p, \mu, H, \ldots) = U(V, T, p, \mu, H, \ldots) - TS(V, T, p, \mu, H, \ldots) \]
\[ + pV - \mu N(V, T, p, \mu, H, \ldots) - HM(V, T, p, \mu, H, \ldots) - \ldots \]

where \[ S, V, N, M, \ldots \] scale like \( V \equiv L^d \) (extensivity)
\[ T, p, \mu, H, \ldots \] scale like \( L^\theta \)
\[ G, U, \ldots \] scale like \( L^\varepsilon \)

hence \( \varepsilon = \theta + d \)

Dividing by \( L^{\theta+d} \) we obtain

\[ g\left(\frac{T}{L^\theta}, \frac{p}{L^\theta}, \frac{\mu}{L^\theta}, \frac{H}{L^\theta}, \ldots\right) = u\left(\frac{T}{L^\theta}, \frac{p}{L^\theta}, \frac{\mu}{L^\theta}, \frac{H}{L^\theta}, \ldots\right) - \frac{T}{L^\theta} s\left(\frac{T}{L^\theta}, \frac{p}{L^\theta}, \frac{\mu}{L^\theta}, \frac{H}{L^\theta}, \ldots\right) \]
\[ + \frac{p}{L^\theta} - \frac{\mu}{L^\theta} n\left(\frac{T}{L^\theta}, \frac{p}{L^\theta}, \frac{\mu}{L^\theta}, \frac{H}{L^\theta}, \ldots\right) - \frac{H}{L^\theta} m\left(\frac{T}{L^\theta}, \frac{p}{L^\theta}, \frac{\mu}{L^\theta}, \frac{H}{L^\theta}, \ldots\right) - \ldots \]

where all variables are intensive.
- Classical short-range-interacting many-body Hamiltonian systems
  (i.e., $\alpha > d$) $\Rightarrow \theta = 0 \Rightarrow \varepsilon = d$ (textbooks)
- Classical long-range-interacting many-body Hamiltonian systems
  (i.e., $0 \leq \alpha < d$) $\Rightarrow \theta = d - \alpha \Rightarrow \varepsilon = 2d - \alpha$ (widely verified in the literature)
- Schwarzschild (3+1)-dimensional black hole $\Rightarrow M_{bh} \propto L \Rightarrow \varepsilon = 1 \Rightarrow \theta = 1 - d$
- Banados-Teitelboim-Zanelli (2+1)-dimensional black hole $\Rightarrow \varepsilon = 2 \Rightarrow \theta = 2 - d$
The relation between $S$ and $W$ given in Eq. (1) is the only reasonable given the proposition that the entropy of a system consisting of subsystems is equal to the sum of entropies of the subsystems.

(Free translation by Tobias Micklitz)
Boltzmann-Gibbs entropy is sufficient but not necessary for the likelihood factorization required by Einstein

Constantino Tsallis\textsuperscript{1,2} and Hans J. Haubold\textsuperscript{3}

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2 Santa Fe Institute - 1399 Hyde Park Road, Santa Fe, NM 87501, USA
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published online 21 May 2015
Einstein 1910 (reversal of Boltzmann formula):

For any two independent systems $A$ and $B$, the likelihood function should satisfy

$$\Omega(A + B) = \Omega(A) \Omega(B) \quad \text{(Einstein principle)}$$

$q = 1$:

$$S_{BG} = k_B \ln W \quad \text{hence} \quad \Omega(\{ p_i \}) \propto e^{S_{BG}(\{ p_i \})/k_B}$$

$$\Omega(A + B) \propto e^{S_{BG}(A+B)/k_B} = e^{S_{BG}(A)/k_B + S_{BG}(B)/k_B} = e^{S_{BG}(A)/k_B} e^{S_{BG}(B)/k_B} \propto \Omega(A) \Omega(B) \quad \text{OK!}$$

$\forall q$:

$$S_q = k_B \ln_q W \quad \text{hence} \quad \Omega(\{ p_i \}) \propto e^{S_q(\{ p_i \})/k_B}$$

$$\Omega(A + B) \propto e^{S_q(A+B)/k_B} = e^{S_q(A)/k_B + S_q(B)/k_B + (1-q)[S_q(A)/k_B][S_q(B)/k_B]} = e^{S_q(A)/k_B} e^{S_q(B)/k_B} \propto \Omega(A) \Omega(B) \quad \text{OK \ \forall q!}$$
COMPOSITION OF VELOCITIES OF INERTIAL SYSTEMS (d=1)

\[ v_{13} = v_{12} + v_{23} \quad \text{(Galileo)} \]

\[ v_{13} = \frac{v_{12} + v_{23}}{1 + \frac{v_{12}}{c} \frac{v_{13}}{c}} \quad \text{(Einstein)} \]

**Newton mechanics:**
It satisfies Galilean additivity but violates Lorentz invariance (hence mechanics can not be unified with Maxwell electromagnetism)

**Einstein mechanics (Special relativity):**
It satisfies Lorentz invariance (hence mechanics is unified with Maxwell electromagnetism) but violates Galilean additivity

**Question:** which is physically more fundamental, the additive composition of velocities or the unification of mechanics and electromagnetism?
Special relativity recovers Newtonian/Galilean mechanics as particular case:

\[
v_{13} = \frac{v_{12} + v_{23}}{1 + \frac{v_{12}}{c} \frac{v_{13}}{c}} \sim v_{12} + v_{23}
\]

if \( 1/c \to 0 \) or \( \forall 1/c \neq 0 \) with \( v/c \to 0 \)

\( q \)-statistics recovers Boltzmann-Gibbs statistics as particular case:

\[
e_q^{-\beta E} \equiv \frac{1}{\left[1 + (q-1)\beta E\right]^{q-1}} \sim e^{-\beta E} = e^{-E/kT}
\]

if \( (q-1) \to 0 \) or \( \forall (q-1) \neq 0 \) with \( \beta E = \frac{E}{kT} \to 0 \)
\( q \)-GAUSSIANS:

\[
p_q(x) \propto e_q^{-(x/\sigma)^2} \equiv \frac{1}{\left[1 + (q-1)(x/\sigma)^2\right]^{\frac{1}{q-1}}} \quad (q < 3)
\]

On a $q$-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

Generalization of symmetric $\alpha$-stable Lévy distributions for $q>1$

Sabir Umarov,$^{1,a)}$ Constantino Tsallis,$^{2,3,b)}$ Murray Gell-Mann,$^{3,c)}$ and Stanly Steinberg$^{4,d)}$

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$^2$Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
$^3$Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA
$^4$Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, USA

(Received 10 November 2009; accepted 4 January 2010; published online 3 March 2010)

See also:
H.J. Hilhorst, JSTAT P10023 (2010)
M. Jauregui, C. T. and E.M.F. Curado, JSTAT P10016 (2011)
A. Plastino and M.C. Rocca, Physica A 392, 3952 (2013)
S. Umarov and C. T., J Phys A 49, 415204 (2016)
**CENTRAL LIMIT THEOREM**

\( N^{1/[\alpha(2-q)]} \)-scaled attractor \( F(x) \) when summing \( N \to \infty \) \( q \)-independent identical random variables

with symmetric distribution \( f(x) \) with \( \sigma_Q = \int dx \ x^2 |f(x)|^Q / \int dx \ |f(x)|^Q \left( Q \equiv 2q-1, \ q_1 = \frac{1+q}{3-q} \right) \)

<table>
<thead>
<tr>
<th>( q = 1 ) [independent]</th>
<th>( q \neq 1 ) (i.e., ( Q \equiv 2q-1 \neq 1 )) [globally correlated]</th>
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<tbody>
<tr>
<td>( \sigma_Q &lt; \infty ) ((\alpha = 2))</td>
<td>( F(x) = G_q(x) \equiv G_{(3q-1)/(1+q_1)}(x) ), with same ( \sigma_Q ) of ( f(x) )</td>
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<td>( \sigma_Q \to \infty ) ((0 &lt; \alpha &lt; 2))</td>
<td>( G_q(x) \sim \begin{cases} G(x) &amp; \text{if }</td>
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<td>( L_\alpha(x) \sim \begin{cases} G(x) &amp; \text{if }</td>
<td>x</td>
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(Intermediate regime)

(Distant regime)

\[ \text{S. Umarov, C. T., M. Gell-Mann and S. Steinberg, Milan J Math 76, 307 (2008)} \]

\[ \text{S. Umarov, C. T., M. Gell-Mann and S. Steinberg, J Math Phys 51, 033502 (2010)} \]
The standard map: From Boltzmann-Gibbs statistics to Tsallis statistics

Ugur Timakli, Ernesto P. Borges

As well known, Boltzmann-Gibbs statistics is the correct way of thermostatically approaching ergodic systems. On the other hand, nontrivial ergodicity breakdown and strong correlations typically drag the system into out-of-equilibrium states where Boltzmann-Gibbs statistics fails. For a wide class of such systems, it has been shown in recent years that the correct approach is to use Tsallis statistics instead. Here we show how the dynamics of the paradigmatic conservative (area-preserving) standard map exhibits, in an exceptionally clear manner, the crossing from one statistics to the other. Our results unambiguously illustrate the domains of validity of both Boltzmann-Gibbs and Tsallis statistical distributions. Since various important physical systems from particle confinement in magnetic traps to autoionization of molecular Rydberg states, through particle dynamics in accelerators and comet dynamics, can be reduced to the standard map, our results are expected to enlighten and enable an improved interpretation of diverse experimental and observational results.
STANDARD MAP  (Chirikov 1969)

\[
p_{i+1} = p_i - K \sin x_i \pmod{2\pi}
\]
\[
x_{i+1} = x_i + p_{i+1} \pmod{2\pi}
\]
\[(i = 0, 1, 2, ...)\]

(area-preserving)

Particle confinement in magnetic traps, particle dynamics in accelerators, comet dynamics, ionization of Rydberg atoms, electron magneto-transport
Tirnakli and Borges
Nature / Scientific Reports 6, 23644 (2016)
The standard map: From Boltzmann-Gibbs statistics to Tsallis statistics

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Universal Internucleotide Statistics in Full Genomes: A Footprint of the DNA Structure and Packaging?

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Figure 1. The procedure of the assessment of the four internucleotide interval sequences from the DNA primary sequence.
\[ A = \frac{1}{1 + (q-1) \beta (l/L)^{q-1}} \]
A novel automatic microcalcification detection technique using Tsallis entropy & a type II fuzzy index

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ABSTRACT

This article investigates a novel automatic microcalcification detection method using a type II fuzzy index. The thresholding is performed using the Tsallis entropy characterized by another parameter ‘q’, which depends on the non-extensiveness of a mammogram. In previous studies, ‘q’ was calculated using the histogram distribution, which can lead to erroneous results when pectoral muscles are included. In this study, we have used a type II fuzzy index to find the optimal value of ‘q’. The proposed approach has been tested on several mammograms. The results suggest that the proposed Tsallis entropy approach outperforms the two-dimensional non-fuzzy approach and the conventional Shannon entropy partition approach. Moreover, our thresholding technique is completely automatic, unlike the methods of previous related works. Without Tsallis entropy enhancement, detection of microcalcifications is meager; 80.21% Tps (true positives) with 8.1 Fps (false positives), whereas upon introduction of the Tsallis entropy, the results surge to 96.55% Tps with 0.4 Fps.
Brain tissue segmentation using q-entropy in multiple sclerosis magnetic resonance images

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Abstract

The loss of brain volume has been used as a marker of tissue destruction and can be used as an index of the progression of neurodegenerative diseases, such as multiple sclerosis. In the present study, we tested a new method for tissue segmentation based on pixel intensity threshold using generalized Tsallis entropy to determine a statistical segmentation parameter for each single class of brain tissue. We compared the performance of this method using a range of different q parameters and found a different optimal q parameter for white matter, gray matter, and cerebrospinal fluid. Our results support the conclusion that the differences in structural correlations and scale invariant similarities present in each tissue class can be accessed by generalized Tsallis entropy, obtaining the intensity limits for these tissue class separations. In order to test this method, we used it for analysis of brain magnetic resonance images of 43 patients and 10 healthy controls matched for gender and age. The values found for the entropic q index were 0.2 for cerebrospinal fluid, 0.1 for white matter and 1.5 for gray matter. With this algorithm, we could detect an annual loss of 0.98% for the patients, in agreement with literature data. Thus, we can conclude that the entropy of Tsallis adds advantages to the process of automatic target segmentation of tissue classes, which had not been demonstrated previously.
The ideal q values for the segmentation of the classes are: CSF = 0.2, WM = 0.1, GM = 1.5, which have not been shown previously. These characteristics allow its application to clinical routine.

Figure 3. Maximum entropy segmentation example. A, Original image; B, image with the segmentation masks. Blue indicates cerebrospinal fluid, white indicates the gray matter, and red indicates the white matter.

Figure 6. Segmentation using Shannon and Tsallis entropies.
In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).
CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

\[ V(r) \sim -\frac{A}{r^\alpha} \quad (r \to \infty) \quad (A > 0, \quad \alpha \geq 0) \]

integrable if \( \alpha / d > 1 \) (short-ranged)

non-integrable if \( 0 \leq \alpha / d \leq 1 \) (long-ranged)
Influence of the interaction range on the thermostatistics of a classical many-body system

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\[ \alpha = 0.9 \]

yields

\[ q = 1.58 \]

\[ \alpha = 2 \]

yields

\[ q = 1 \]
ONE-BODY ENERGY DISTRIBUTION

\[ P(E_i) = P(\mu) e^{-\beta (E_i - \mu)} \]

- \( d = 1 \) \( q = 1.31 \) | \( \beta = 29 \)
- \( d = 2 \) \( \mu = 0.73 \) | \( P(\mu) = 2.8 \)
- \( d = 3 \)

\( U = 0.69 \)
\( \alpha/d = 0.80 \)
\( N = 262144 \)

\( \Delta t = [8,15] \cdot 10^5 \)
Fermi-Pasta-Ulam model with long-range interactions: Dynamics and thermostatistics

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\[
\mathcal{H} = \frac{1}{2} \sum_{n=1}^{N} p_n^2 + \frac{1}{2} \sum_{n=0}^{N} (x_{n+1} - x_n)^2 + \frac{b}{4\tilde{N}} \sum_{n=0}^{N} \sum_{m=n+1}^{N+1} \frac{(x_n - x_m)^4}{|n-m|^\alpha} (b > 0; \alpha \geq 0)
\]

\[ d = 1; \alpha = 2; a = 1; b = 10; u = 9; N = 8000 \]

yields \( q = 1 \)
$d = 1; \alpha = 0.9; a = 1; b = 10; u = 9; N = 8000$

yields $q = 1.22$
Experimental Validation of a Nonextensive Scaling Law in Confined Granular Media

The velocity distribution of sheared granular media shows unexpected similarities with turbulent fluid flows.

Gaël Combe, Vincent Richefeu, Marta Stasiak, and Allbens P.F. Atman

Combe, Richefeu, Stasiak and Atman
PRL 115, 238301 (2015)
FIG. 4. Verification of the Tsallis-Bukman scaling law for different regimes of diffusion. (top) Evolution of the measured diffusion exponent $\alpha$ as a function of $1/\sqrt{\Delta \gamma}$ the dashed line is a direct application of the scaling law from the fit of the values shown in Fig. 3, $\alpha(1/\sqrt{\Delta \gamma}) = 2/[3 - q(1/\sqrt{\Delta \gamma})]$. (Inset) a typical diffusion curve showing the mean square displacement fluctuations, $\langle x^2 \rangle$, in function of the shear strain, $\gamma$; it allows the assessment of the diffusion exponent, $\alpha$, for each strain window tested. In the case shown, it corresponds to the smallest strain window, the rightmost point in the curve at the main panel. Note that for a constant strain rate, $\gamma$ is proportional to time. (Bottom) Measure of the deviation of the data relative to the scaling law prediction, as a function of $1/\sqrt{\Delta \gamma}$, showing an agreement on the order of $\pm 2\%$.

$$\alpha = \frac{2}{3 - q}$$

Combe, Richefeu, Stasiak and Atman
PRL 115, 238301 (2015)

CT and DJ Bukman, PRE 54 (1996) R2197
LHC (Large Hadron Collider)

CMS, ALICE, ATLAS and LHCb detectors

~ 4000 scientists/engineers from ~ 200 institutions of ~ 50 countries
Tsallis fits to $p_T$ spectra and multiple hard scattering in $pp$ collisions at the LHC

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(Received 12 May 2013; published 5 June 2013)

Phenomenological Tsallis fits to the CMS, ATLAS, and ALICE transverse momentum spectra of hadrons for $pp$ collisions at LHC were recently found to extend over a large range of the transverse momentum. We investigate whether the few degrees of freedom in the Tsallis parametrization may arise from the relativistic parton-parton hard-scattering and related processes. The effects of the multiple hard-scattering and parton showering processes on the power law are discussed. We find empirically that whereas the transverse spectra of both hadrons and jets exhibit power-law behavior of $1/p_T^n$ at high $p_T$, the power indices $n$ for hadrons are systematically greater than those for jets, for which $n \sim 4-5$. 
\( q = 1 + \frac{1}{n} \)

\[ \langle E d^3 N / dp^3 \rangle_n \]

The rhs fit for \( pp \) collisions:

\[ \frac{dN}{dyd\vec{p}_T} = \frac{A \alpha_s^2 (1-x_{a0})^g (1-x_{b0})^g}{[1+m_r^2/m_{T0}^2]^{n/2}} \]

- \( \sqrt{s} = 7 \text{ TeV}, n = 5.48, m_{T0} = 1.14 \text{ GeV} \)
- \( \sqrt{s} = 0.9 \text{ TeV}, n = 5.55, m_{T0} = 0.896 \text{ GeV} \)
SIMPLE APPROACH: TWO-DIMENSIONAL SINGLE RELATIVISTIC FREE PARTICLE

C.Y. Wong, G. Wilk, L.J.L. Cirto and C. T.,

EPJ Web of Conferences 90, 04002 (2015), and PRD 91, 114027 (2015)

\[
\frac{1}{2\pi p_T \eta \rho p_T} \bigg|_{\eta = 0} = A e^{-\frac{E_T}{T}}
\]

\[ [A] = \text{GeV}^{-2} c^3 \]

\[ [T] = \text{GeV} \]

\[ E_T = \sqrt{m^2 c^4 + p_T^2 c^2} \]
Einstein (1905)

\[ E = (m^2 c^4 + p^2 c^4)^{1/2} \]

\[ m_p \approx 938.3 \text{ MeV}/c^2 \]
\[ m_e \approx 0.511 \text{ MeV}/c^2 \]

\[ E_p = 10^8 \text{ TeV (EECR)} \]

\[ E = mc^2 + p^2/2m \]

\[ E_e = 7 \text{ TeV} \]

\[ E_p = 7 \text{ TeV} \]
Tout le monde savait que c’ était impossible.
Il y avait un qui ne le savait pas.
Alors il est allé et il l’a fait.

Jean Cocteau (Marcel Pagnol, Winston Churchill, Mark Twain ...)
Si l’action n’a quelque splendeur de liberté, elle n’a point de grâce ni d’honneur.

Montaigne