Appendix from E. H. van Nes and M. Scheffer, "Slow Recovery from Perturbations as a Generic Indicator of a Nearby Catastrophic Shift"
(Am. Nat., vol. 169, no. 6, p. 738)

Technical Explanation of Slowing Down of Recovery Rates

Thresholds for Regime Shifts in Mathematical Terms

The threshold point where a regime shift takes place is called a catastrophic bifurcation in mathematics (see also "Glossary"). In models, there are two common ways of undergoing a regime shift (Strogatz 1994): by crossing a fold bifurcation or a transcritical bifurcation (fig. A1). A fold bifurcation (also called saddle node) is a bifurcation where at the critical parameter value two equilibria appear: a stable equilibrium and an unstable saddle point. When approaching this bifurcation from the left in figure A1, ecological resilience is decreasing until the critical threshold is reached. The other catastrophic bifurcation is the transcritical bifurcation. At the critical threshold, two equilibria cross and exchange stability (Strogatz 1994). In most ecological models, this implies the extinction of a species. Both kinds of thresholds are common in ecological models. The first three models analyzed here have a fold bifurcation; the other three models have a transcritical bifurcation.

Why Does Recovery after Perturbations Slow Down?

Wissel (1984) proved mathematically that for all continuous differential equations, the recovery rate will slow to zero close to a threshold point (bifurcation). This can be understood as follows: crossing a bifurcation, the stability of the equilibrium changes. Therefore, at the bifurcation point, the equilibrium is neither stable nor unstable (fig. 5*b*). This implies that trajectories in the neighborhood of the equilibrium are neither attracted nor repelled from the equilibrium and recovery rates for small disturbances are zero. If we change the control parameter in the direction away from the regime shift, the equilibrium becomes stable, and thus recovery from small disturbances will have a speed greater than zero. By continuity, there must be a zone between these points where the recovery rate decreases gradually to zero.

When Can We Expect Critical Slowing Down?

An important point to note is that a decrease of recovery rates does not necessarily indicate a switch to an alternative state (i.e., a catastrophic bifurcation). It also occurs with other local bifurcations. In all situations where the stability of one of equilibria of the model changes, we can expect critical slowing down for the reasons explained above. Examples of such thresholds are Hopf bifurcations (where the model becomes cyclic) and pitchfork and noncatastrophic transcritical bifurcations (Strogatz 1994). Critical slowing down can even happen if a threshold is approached that merely implies an increased sensitivity of the system (fig. 5*c*).

Glossary

Table A1
Glossary

Term	Definition
Attractor	Behavior of the model if simulation time goes to infinity
Bifurcation	A critical threshold in parameters where the nature of the model behavior (attractor or equilibria) changes
Critical slowing down	Slowing down of recovery rates on perturbations close to a local bifurcation at which the stability of an equilibrium changes
Eigenvalues of equilibria	Parameters of the linearized model close to an equilibrium (can be complex numbers), determined by linearizing deviations from an equilibrium (see various textbooks); it is a common way to assess the stability properties of equilibria, and the dominant (i.e., maximum) real eigenvalue is also an approximation of the recovery rate to equilibrium
Equilibrium	A point where there is no change in state variables; equilibria can be unstable, meaning that trajectories move away from the equilibrium
Global bifurcation	This bifurcation cannot be detected from a change in the properties (eigenvalues) of the equilibria; examples are the homoclinic, heteroclinic, and infinite-period bifurcations
Local bifurcation	This threshold can be detected from changes in eigenvalues of the equilibria; examples are fold, transcritical, and Hopf bifurcation

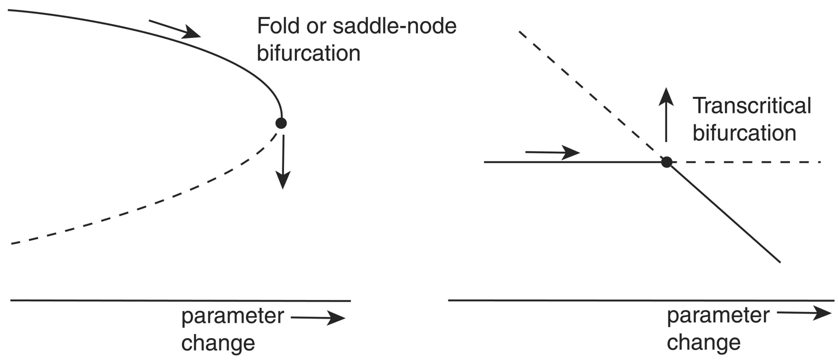

Figure A1: Two main kinds of bifurcations that may lead to a regime shift. The fold or saddle node bifurcation implies that two new equilibria appear. The transcritical bifurcation is not necessarily catastrophic. It implies a crossing of two equilibria (Strogatz 1994). The arrows indicate from which direction it should be approached to induce a catastrophic shift.

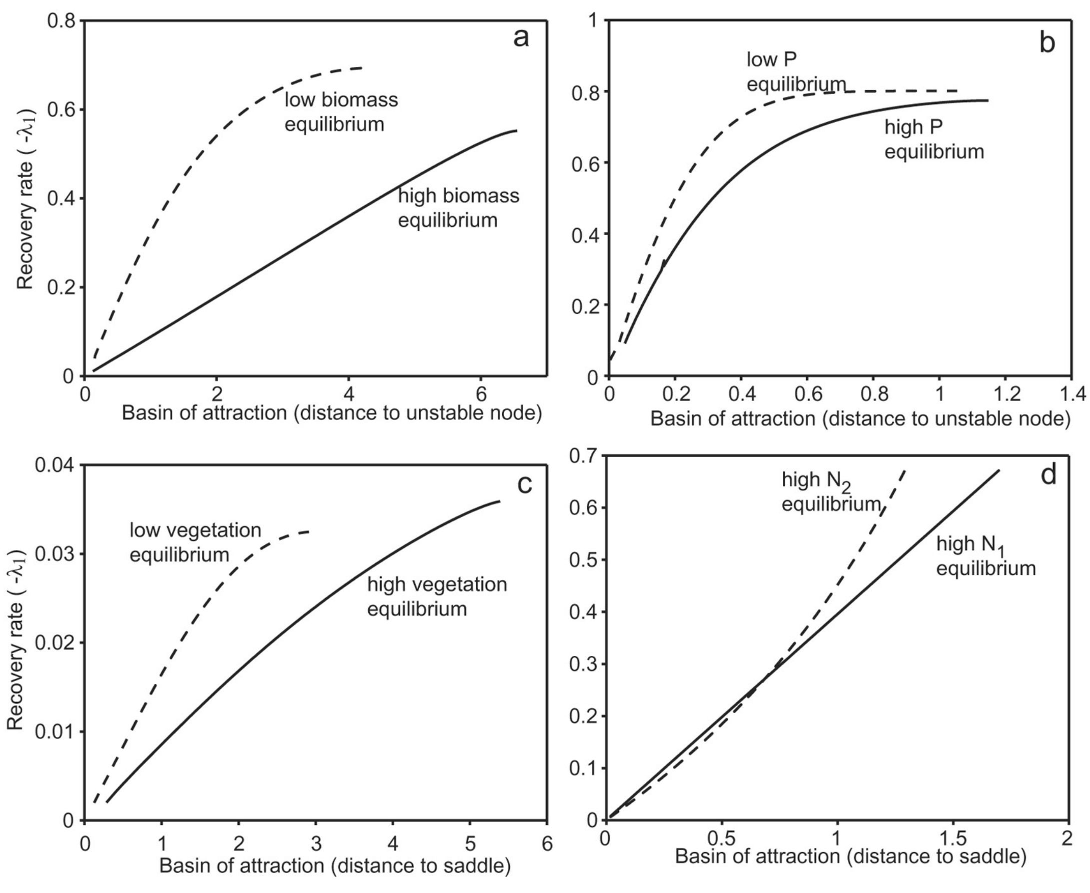

Figure A2: Relations between economic resilience (size of the basin of attraction) and the recovery rate from disturbances (determined as the real part of the dominant eigenvalue) of the four simple models: *a*, logistic growth with grazing (May 1977); *b*, nutrient recycling (Carpenter et al. 1999); *c*, macrophytes in shallow lakes (Scheffer 1998), and *d*, two species Lotka-Volterra. The size of the basin of attraction was determined as the distance to the unstable equilibrium (or saddle) that delimits both equilibria. For parameters, see table 1.