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Stability of Equilibria in Games with Procedurally Rational Players[1]

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One approach to the modeling of bounded rationality in games is based on the dynamics of evolution and learning. An alternative static and equilibrium-based approach has been developed recently by Osborne and Rubinstein. This paper formalizes Osborne and Rubinstein's dynamic interpretation of their equilibrium concept, uses the criterion of dynamic stability as an equilibrium refinement, and shows that stable equilibria can involve the playing of strictly dominated actions while dominant strategy equilibria can be unstable. These effects cannot occur under standard evolutionary game dynamics. Sufficient conditions for the instability of equilibria are provided for symmetric and asymmetric games. *Journal of Economic Literature* Classification Number: C72.

1. INTRODUCTION

Dissatisfaction with the stringent rationality requirements of the standard equilibrium approach to strategic behavior has fueled the growth of a large literature on the dynamics of evolution and learning in games (Weibull, 1995; Vega-Redondo, 1996; Fudenberg and Levine, 1997; Samuelson, 1997; Young, 1998). The focus of this literature has been on questions such as whether trajectories converge to Nash equilibria, whether dominated actions are eliminated along convergent or nonconvergent trajectories, and whether stability can provide an effective equilibrium selection criterion. The unifying theme in this otherwise diverse body of work is that individu-

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als have only a limited understanding of the strategic environment in which they interact, and adjust their behavior in accordance with some adaptive process. This process is typically assumed to be payoff monotonic: successful actions are more likely to be adopted with greater frequency than are less successful ones.

An entirely different approach to the modeling of bounded rationality has been developed recently by Osborne and Rubinstein (1998). The approach is static and equilibrium-based, but relies on less stringent assumptions regarding the knowledge and understanding of players than does the standard theory of Nash equilibrium. Their equilibrium concept, $S(1)$ equilibrium, is based on an explicit process of reasoning on the part of players and therefore corresponds to procedural rather than substantive rationality in the sense of Simon (1978). Although $S(1)$ equilibrium is a static concept, Osborne and Rubinstein interpret it as a steady state of a dynamic process of sampling. This paper formalizes their informal description of the dynamic process and thereby facilitates a comparison of this approach with the explicitly dynamic approach of evolutionary game theory. It turns out that the two approaches give rise to radically different static and dynamic predictions. For instance, dynamically stable $S(1)$ equilibria can involve the playing of strictly dominated actions, and equilibria in which strictly *dominant* actions are played with probability 1 can be unstable. The criterion of dynamic stability also yields a refinement of $S(1)$ equilibrium that is both intuitively appealing and effective in application. For instance, although it is the case that all strict Nash equilibria are also $S(1)$ equilibria, some strict Nash equilibria may be unstable with respect to the dynamics. This provides a simple basis for selection among strict Nash equilibria in certain coordination games which, unlike the commonly used criterion of stochastic stability, relies neither on the ultra-long run nor on the presence of rare mutations.

The paper is organized as follows. Osborne and Rubinstein's informal dynamic interpretation of $S(1)$ equilibrium is formalized in Section 2. The use of dynamic stability as a method of selection among alternative equilibria is developed in Section 3. It is shown in Section 4 that strictly dominated actions may be played with positive probability in stable $S(1)$ equilibria, and that equilibria in which only dominant actions are played can be unstable. This can occur in symmetric games only if the number of players plus and the number of actions is at least five. Sufficient conditions for the instability of strict Nash equilibria are provided in Section 5 for symmetric games. These conditions are easy to verify and are satisfied in many commonly studied games. The case of asymmetric games with multiple player populations is examined in Section 6, and sufficient conditions for the instability of strict Nash equilibria are provided also for this case. Section 7 concludes.

2. EQUILIBRIUM AND DYNAMICS

Consider first the case of symmetric n-player games. As in Osborne and Rubinstein, let $A = \{a_1, \ldots, a_m\}$ represent the finite set of actions available to each player, and let $u(a_i, b)$ denote the payoff to a player of choosing the action $a_i \in A$ when the remaining $n-1$ players choose the action profile $b \in A^{n-1}$. This payoff function represents each player's ordinal preferences over the set of outcomes. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a probability distribution on A, and let $v(a_i, \alpha)$, $i = 1, \ldots, m$, be the random variables yielding $u(a_i, b)$ with probability $\Pr(b)$ for each $b \in A^{n-1}$. Here $\Pr(b)$ is the probability that the action profile chosen by the remaining $n-1$ players is b, assuming that each action is chosen independently subject to the distribution α on A. Finally, let $w(a_i, \alpha)$ be the probability that, when each random variable $v(x, \alpha)$ is drawn independently and exactly once, the action a_i yields the best outcome. In the case of realizations in which a_i is not unique in yielding the best outcome, the probability is weighted by the reciprocal of the total number of tied alternatives. An $S(1)$ equilibrium is defined as a probability distribution α on the set of actions A with the property that

$$w(a_i, \alpha) = \alpha_i \qquad \text{for every action } a_i \in A.$$

The concept of $S(1)$ equilibrium is based on the idea that players sample each action exactly once and select the action which yields the highest payoff. In the case of ties, one of the tied alternatives is picked at random, with each alternative having the same probability of being chosen. The probability with which a given action is chosen under this procedure will depend, of course, on the (mixed) strategies chosen by the remaining players during each of the sampling periods. An $S(1)$ equilibrium is a mixed strategy α with the following property: if all other players adopt this strategy throughout the sampling procedure, then the probability that action a_i is best under the sampling procedure is precisely α_i.

One interpretation of $S(1)$ equilibria that is advanced by Osborne and Rubinstein is that it is the steady state of a dynamic process involving a large population of individuals who are randomly matched to play the game. Each member of the population adopts the same action throughout his/her stay in the population, and the population composition changes as a result of new entrants and departures. When entering, a player samples each action once and selects that which yields the best outcome according the procedure described above. In this case, an $S(1)$ equilibrium is a distribution of actions in the incumbent population which induces the same distribution of actions in the flow of entrants.

This dynamic process may be formalized as follows. Let $\alpha(t)$ represent the distribution of actions in the population at time t. That is, the proportion of the population choosing action $a_i \in A$ is given by $\alpha_i(t)$. Let $\dot{\alpha}_i(t)$ represent the rate of change of this proportion. Under the dynamics of sampling described above, the representation in the population of an action that is "best" with a higher (lower) probability than it is currently being played should increase (decrease). This suggests the dynamics

$$\dot{\alpha}_i(t) = F(\alpha(t)),$$

where F satisfies

$$w(a_i, \alpha(t)) > \alpha_i(t) \quad \Leftrightarrow \quad \dot{\alpha}_i(t) > 0,$$
$$\alpha(t_0) \in \Omega^m \quad \Rightarrow \quad \alpha(t) \in \Omega^m \text{ for all } t > t_0,$$

and $\Omega^m = \{x \in \mathbf{R}^m \mid x_i \geq 0 \text{ and } \sum_{i=1}^m x_i = 1\}$ is the set of probability distributions on A. We assume that F is Lipschitz continuous (to ensure that trajectories are well defined from all initial conditions) and refer to any system of differential equations $\dot{\alpha}_i(t) = F(\alpha(t))$ which satisfies the above conditions as a *sampling dynamic*. As an example, consider the following:

$$\dot{\alpha}_i(t) = w(a_i, \alpha(t)) - \alpha_i(t). \tag{1}$$

It is a straightforward matter to verify that (1) satisfies each of the above conditions for a sampling dynamic. Furthermore, this particular specification may be derived from first principles as follows.[2] Suppose that time is divided into small discrete periods of length h and that the proportion of individuals who exit the population is $1 - e^{-\lambda h}$ in any given period. New entrants arrive at the same rate and select actions on the basis of the sampling procedure described above, choosing action a_i with probability $w(a_i, \alpha(t))$, where $\alpha(t)$ is the population composition at the beginning of the period. Then

$$\alpha_i(t+h) = e^{-\lambda h}\alpha_i(t) + (1 - e^{-\lambda h})w(a_i, \alpha(t)),$$

so that

$$\dot{\alpha}_i(t) = \lim_{h \to 0}\left(\frac{1 - e^{-\lambda h}}{h}\right)(w(a_i, \alpha(t)) - \alpha_i(t)) = \lambda\left(w(a_i, \alpha(t)) - \alpha_i(t)\right).$$

Setting $\lambda = 1$ by choice of time units yields (1). This simple specification will be used for illustrative purposes in the numerical examples which follow, and, for convenience, will also be used in the statement and proof of all formal results. It is easy to verify, however, that all results will continue to hold without modification for arbitrary sampling dynamics.

Clearly, a distribution α is a rest point of a sampling dynamic if and only if it is an $S(1)$ equilibrium. Not all rest points of the dynamics will

[2] I am grateful to an anonymous referee for suggesting a derivation along these lines.

be stable, however. In examining the question of stability, the following standard definitions (Hirsch and Smale, 1974) will be used below.

DEFINITION. A rest point α is *stable* if, for every neighborhood $U \subset \mathbf{R}^m$ containing α, there is a neighborhood $V \subset U$ such that if $\alpha(t_0) \in V \cap \Omega^m$, then $\alpha(t) \in U \cap \Omega^m$ for all $t > t_0$.

In other words, α is stable if, for every neighborhood of α, it is possible to find an open set of initial conditions from which trajectories never leave this neighborhood. A rest point α is *unstable* if it is not stable. A sufficient condition for instability is that one or more of the eigenvalues of the Jacobean, when evaluated at the rest point, has positive real part.

DEFINITION. A rest point α is *asymptotically stable* if it is stable and if there is some neighborhood $U \subset \mathbf{R}^m$ such that all trajectories initially in $U \cap \Omega^m$ converge to α.

A sufficient condition for asymptotic stability (and hence stability) is that all eigenvalues of the Jacobean, when evaluated at the rest point, have negative real part. In the next section, it is shown that stability (and asymptotic stability) can provide a method of selection among $S(1)$ equilibria.

3. EQUILIBRIUM SELECTION

Although the set of Nash equilibria of a game will not generally coincide with the set of $S(1)$ equilibria, every strict symmetric Nash equilibrium corresponds to an $S(1)$ equilibrium. Specifically, if $(\alpha^*, \ldots, \alpha^*)$ is a strict Nash equilibrium, then α^* is an $S(1)$ equilibrium. Hence coordination games have multiple $S(1)$ equilibria. The following example shows that stability with respect to the dynamics (1) can provide an equilibrium refinement that eliminates some equilibria in such games.

EXAMPLE 1 (coordination). Consider the symmetric game with the payoff matrix (payoffs correspond to the row player; by symmetry the column player's payoff is given by the transpose)

	a_1	a_2
a_1	1	x
a_2	y	0

where $y < 1$ and $x < 0$. For any strategy α^* which places probability 1 on either action, (α^*, α^*) is a strict Nash equilibrium and hence α^* is an $S(1)$ equilibrium. If $x < y$, then all probability distributions on the action space are $S(1)$ equilibria (see Example 2 in Osborne and Rubinstein.) In this case, all distributions are also stable $S(1)$ equilibria, although none is asymptotically stable. Suppose, however, that $x > y$. Then the probability that the first action is best under the sampling procedure is

$$w(a_1, \alpha) = \alpha_1 + \alpha_1(1 - \alpha_1).$$

Hence we have

$$\dot{\alpha}_1 = \alpha_1(1 - \alpha_1) \geq 0,$$

with strict inequality holding whenever $\alpha_1 \in (0, 1)$. The equilibrium $\alpha^* = (1, 0)$ is therefore the only stable equilibrium, and all trajectories except the one originating at the unstable equilibrium converge to it. Note that this is also the unique stochastically stable equilibrium, although this equivalence need not hold more generally.

The above example is interesting because strict Nash equilibria are always stable with respect to any deterministic payoff monotonic evolutionary selection dynamics, such as the replicator dynamics (Weibull, 1995). Evolutionary approaches to the equilibrium selection problem in coordination games have therefore focused on the criterion of stochastic stability (Kandori *et al.*, 1993; Young, 1993). In models which combine payoff monotonic selection dynamics with rare mutations, stochastically stable states are states which occur with positive probability as the mutation rate approaches zero. The notion of stochastic stability is appropriate only for the ultra-long run, since movements across basins of attraction of the deterministic dynamics can become very infrequent as the mutation rate is decreased (Ellison, 1993). In contrast, the use of the dynamics (1) to select among strict Nash equilibria provides a refinement that depends neither on mutations nor on the ultra-long run. In Section 5, easily verifiable sufficient conditions are provided which can be applied directly to the equilibrium selection problem in coordination games.

4. STRICTLY DOMINATED STRATEGIES

One of the more striking results in Osborne and Rubinstein is the finding that strategies that are strictly dominated by pure strategies can receive positive probability in $S(1)$ equilibria. It is interesting to raise the question, therefore, of whether this can occur at *stable* $S(1)$ equilibria. It turns out that not only can $S(1)$ equilibria which place positive probability on strictly dominated strategies be dynamically stable, they can be uniquely so. In other words, the $S(1)$ equilibrium which corresponds to the strict Nash equilibrium in which the *dominant* strategy is played with probability 1 can be unstable. The "voluntary exchange" example given by Osborne and Rubinstein itself turns out to have this property.

EXAMPLE 2 (voluntary exchange). Consider the symmetric game with the following payoff matrix:

	a_1	a_2	a_3
a_1	2	5	8
a_2	1	4	7
a_3	0	3	6

The first action strictly dominates the other two in this game and therefore the unique Nash equilibrium is (α^*, α^*) where $\alpha^* = (1, 0, 0)$. The conditions for an $S(1)$ equilibrium are as follows:

$$\alpha_1 = \alpha_1^3 + \alpha_2(1-\alpha_3)^2 + \alpha_3,$$
$$\alpha_2 = \alpha_1\alpha_2(1-\alpha_3) + \alpha_3(1-\alpha_3),$$
$$\alpha_3 = \alpha_1^2\alpha_2 + \alpha_3(1-\alpha_3)^2.$$

Exactly two probability distributions satisfy these conditions:

$$\alpha = (1, 0, 0)$$
$$\alpha = (0.52, 0.28, 0.20).$$

Each of the two strictly dominated strategies are played with positive probability in the latter of these equilibria, while the former corresponds to the unique strict Nash equilibrium of the game. In order to determine which, if any, of these equilibria are stable, consider the dynamics (1) applied to this game. Substitution for α_3 yields the following two-dimensional system:

$$\dot{\alpha}_1 = \alpha_1^3 + \alpha_2(\alpha_1+\alpha_2)^2 + (1-\alpha_1-\alpha_2) - \alpha_1,$$
$$\dot{\alpha}_2 = \alpha_1\alpha_2(\alpha_1+\alpha_2) + (1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2) - \alpha_2.$$

The Jacobean of the system is given by

$$\begin{pmatrix} 3\alpha_1^2 + 2\alpha_1\alpha_2 + 2\alpha_2^2 - 2 & \alpha_1^2 + 4\alpha_1\alpha_2 + 3\alpha_2^2 - 1 \\ 2\alpha_1\alpha_2 + \alpha_2^2 - 2\alpha_1 - 2\alpha_2 + 1 & \alpha_1^2 + 2\alpha_1\alpha_2 - 2\alpha_1 - 2\alpha_2 \end{pmatrix}.$$

Consider first the equilibrium $\alpha = (1, 0, 0)$. Here the Jacobean is

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

with eigenvalues 1 and -1. This equilibrium is therefore unstable. Consider next the equilibrium $\alpha = (0.52, 0.28, 0.20)$, in which strictly dominated strategies are played with positive probability. Here the Jacobean is

$$\begin{pmatrix} -0.75 & 0.07 \\ -0.23 & -1.04 \end{pmatrix},$$

with eigenvalues −0.84 and −0.95. This equilibrium is therefore locally asymptotically stable, and trajectories which are initially sufficiently close to it converge it.

The above example shows that regardless of whether trajectories converge to an equilibrium, strictly dominated strategies will continue to be played indefinitely along all paths. Simulation results suggest, moreover, that the stable interior $S(1)$ equilibrium in this example attracts all trajectories except for that which originates at the unstable equilibrium. Figure 1 depicts the dynamics of $\alpha(t)$ starting from the initial condition $\alpha(0) = (0.98, 0.01, 0.01)$, which is very close to the dominant strategy equilibrium. Convergence to the interior equilibrium is rapid and monotonic.

To get some intuition for why the dominant strategy equilibrium is unstable, observe that when the population composition is close (but not equal) to the state $\alpha = (1, 0, 0)$, an entrant who samples the three available actions is very likely to draw an opponent playing a_1 on each occasion, and will therefore also adopt a_1. However, there is some probability that when a_1 is sampled an opponent playing a_1 is drawn, but an opponent playing something other than a_1 is drawn when either a_2 or a_3 is sampled. This

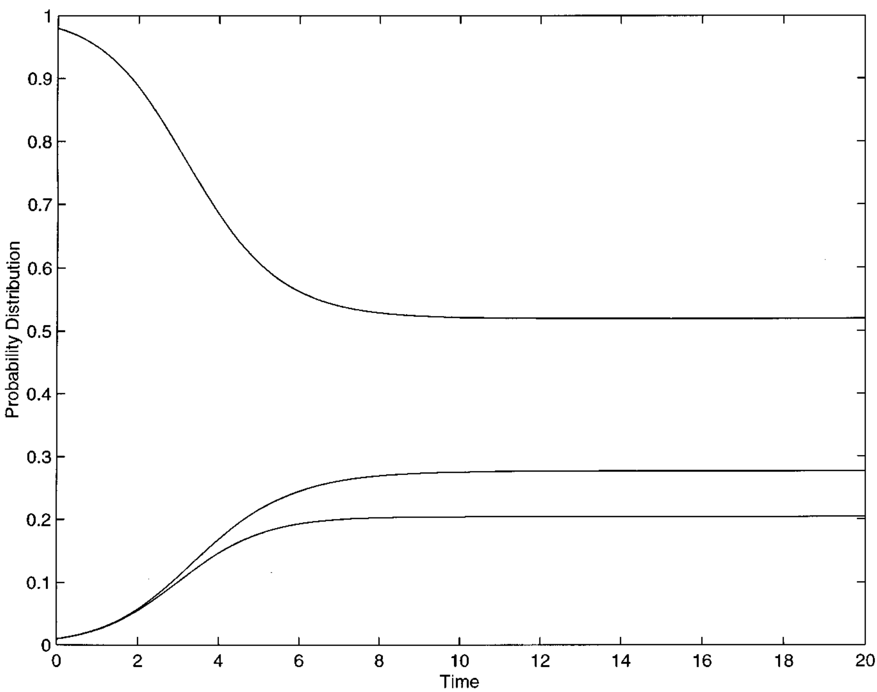

FIG. 1. Convergence to the interior $S(1)$ equilibrium.

probability will be very small, but for the state $\alpha = (1, 0, 0)$ to be unstable, it is sufficient that it be larger than the share of the incumbent population which is *not* playing a_1. In the example, this is indeed the case. This reasoning shows that the instability of dominant strategy equilibria cannot occur unless it is possible for a player to obtain higher payoffs at some other action profile. An implication of this is that the efficient strict Nash equilibrium in symmetric common interest games cannot be unstable. The link between efficiency and instability is subtle, however, and is explored formally in the section to follow.

The finding that strictly dominated strategies can be played in $S(1)$ equilibria of two-person games does not hold when each player has only two actions (Osborne and Rubinstein, 1998). The following example shows that such strategies can indeed be played in equilibria of games having two actions, provided that the number of players is at least three.

EXAMPLE 3 (three-player prisoners' dilemma). Consider the symmetric three-person game with two actions and the following payoff matrix

	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
a_1	1	3	5
a_2	0	2	4

The action a_1 strictly dominates a_2 and each player plays a_1 with probability 1 in the unique Nash equilibrium. Hence $\alpha = (1, 0)$ is an $S(1)$ equilibrium. There is a second $S(1)$ equilibrium, however, corresponding to the mixed strategy $(x, 1 - x)$, where

$$x = \frac{1}{6} \frac{\left(\sqrt[3]{\left(20 + 4\sqrt{29}\right)}\right)^2 - 4 + 2\sqrt[3]{\left(20 + 4\sqrt{29}\right)}}{\sqrt[3]{\left(20 + 4\sqrt{29}\right)}} \approx 0.718.$$

Of the two $S(1)$ equilibria, only the latter is stable. To see this, consider a mixed strategy $\alpha = (\alpha_1, \alpha_2)$. Then

$$\begin{aligned} \dot{\alpha}_1 &= w(a_1, \alpha) - \alpha_1 \\ &= \alpha_1^4 + 2\alpha_1\alpha_2\left(2\alpha_1\alpha_2 + \alpha_1^2\right) + \alpha_2^2 - \alpha_1 \\ &= \alpha_1^4 + 2\alpha_1(1 - \alpha_1)\left(2\alpha_1(1 - \alpha_1) + \alpha_1^2\right) + (1 - \alpha_1)^2 - \alpha_1. \end{aligned}$$

It may be verified that $\dot{\alpha}_1 > 0$ for all $\alpha_1 \in [0, x)$ and $\dot{\alpha}_1 < 0$ for all $\alpha_1 \in (x, 1)$ where x is as defined above. Hence the equilibrium $\alpha = (1, 0)$ is unstable and all trajectories except that which originates at this unstable equilibrium converge to the equilibrium in which strictly dominated strategies are played with positive probability.

The finding that strictly dominated strategies that are dominated by pure strategies can be played with positive probability at asymptotically stable $S(1)$ equilibria may be contrasted with the fact that in the standard theory of evolutionary games, payoff monotonic dynamics eliminate all such strategies, and even weak payoff positive dynamics can never converge to a state in which strictly dominated strategies are played (Samuelson and Zhang, 1992; Weibull, 1995). Strictly dominated strategies can survive under payoff monotonic dynamics only along nonconvergent trajectories, and this too only if they are not dominated by a pure strategy.[3]

While the analysis in this section concerns the possibility of dominated actions being played at stable $S(1)$ equilibria, it is possible to construct examples in which the same holds for stable $S(K)$ equilibria. The concept of $S(K)$ equilibrium generalizes that of $S(1)$ equilibrium to allow for the possibility that the decision-maker samples each action $K \geq 1$ times before determining which one to adopt. Each action in this case is associated with a distribution of consequences. Determining the action which has the best distribution of consequences is not possible if payoffs have a strictly ordinal interpretation. If payoffs correspond to von Neumann–Morgenstern utilities, then comparisons among distributions of consequences are possible and $S(K)$ equilibrium may be defined by a straightforward generalization of the definition of $S(1)$ equilibrium. As is the case with $S(1)$ equilibria, dominant strategy $S(K)$ equilibria may be unstable, and $S(K)$ equilibria in which dominated actions are played can be stable. It can be shown, for instance, that both of these effects occur for $S(2)$ equilibria in Example 3, provided that the payoff in the middle of the second row is replaced by $2 + \eta$ for any $\eta \in (0, 1)$.

5. INSTABILITY OF EQUILIBRIA

In two of the examples of the previous section, dominant strategy equilibria were shown to be unstable. In this section, simple and easily verifiable sufficient conditions for the instability of arbitrary strict symmetric Nash equilibria (including dominant strategy equilibria) are given.

Consider a symmetric game with $n \geq 2$ players, $m \geq 2$ actions, and action space $A = \{a_1, a_2, \ldots, a_m\}$.

[3]Strategies that are strictly dominated by a *mixed* strategy can survive along nonconvergent paths under payoff monotonic selection dynamics, for instance, in a continuous-time version of Dekel and Scotchmer's (1992) example; see Björnerstedt (1993) or Weibull (1995) for details. It is also possible for strategies that are strictly dominated by a pure strategy to survive along nonconvergent paths under weak payoff positive selection dynamics; see Sethi (1998) for an example.

DEFINITION. An action profile $(a_q, a_q, \ldots, a_q)$ in a symmetric n-player game is *inferior* if, for every $i \neq q$, there exists $j(i) \neq q$ such that

$$u(a_{j(i)}, a_i, a_q, \ldots, a_q) > u(a_q, a_q, \ldots, a_q).$$

It is *twice inferior* if, for every action $i \neq q$, there exist $j(i) \neq q$ and $k(i) \neq q$ such that $j \neq k$ and

$$u(a_{j(i)}, a_i, a_q, \ldots a_q) \geq u(a_{k(i)}, a_i, a_q, \ldots, a_q) > u(a_q, a_q, \ldots, a_q).$$

This definition states the following. A symmetric action profile $(a_q, a_q, \ldots, a_q)$ is inferior if, when $n-2$ of the other $n-1$ players take the action a_q while the remaining player selects $a_i \neq a_q$, there exists at least one response a_j ($j \neq q$) by player 1 which yields an outcome that is strictly preferred by player 1 to the outcome at $(a_q, a_q, \ldots, a_q)$. A symmetric action profile is twice inferior if, when $n-2$ of the other $n-1$ players take the action a_q while the remaining player selects $a_i \neq a_q$, there exist at least *two* distinct responses a_j and a_k ($j, k \neq q$) by player 1 which yield an outcome that is preferred by player 1 to the outcome at $(a_q, a_q, \ldots, a_q)$. Clearly, no action profile can be twice inferior in games having only two actions. We shall say that a symmetric strict Nash equilibrium $(\alpha^*, \ldots, \alpha^*)$ is inferior if $\alpha_q^* = 1$ and $(a_q, \ldots, a_q)$ is inferior. In other words, a strict Nash equilibrium is inferior if the action profile that is played with probability 1 in the equilibrium is inferior. In Example 3, the dominant strategy equilibrium is inferior and in Example 2, the dominant strategy equilibrium is twice inferior.

The following result provides sufficient conditions for instability of equilibria in games having at least three players.

THEOREM 1. *If $(\alpha^*, \ldots, \alpha^*)$ is an inferior strict Nash equilibrium of a symmetric game with three or more players, then α^* is unstable under the sampling dynamics* (1).

Proof. Let $(\alpha^*, \ldots, \alpha^*)$ be a symmetric strict Nash equilibrium and assume without loss of generality that $\alpha_1^* = 1$. Note that α^* is an $S(1)$ equilibrium and hence a rest point of the dynamics (1). Consider a mixed strategy $\alpha = (1-\varepsilon, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m) \in \Omega^m$, where $\varepsilon \in (0, 1)$. Since $\alpha \in \Omega^m$, we have $\varepsilon_i \geq 0$ for all $i \in \{2, \ldots, m\}$, and $\sum_2^m \varepsilon_i = \varepsilon$. Consider the following event: when a_1 is sampled, the outcome is $(a_1, a_1, \ldots, a_1)$, and when $a_{j(2)}$ is sampled, the outcome is $(a_{j(2)}, x)$, where $x \in A^{n-1}$ contains some permutation of the actions $\{a_2, a_1, \ldots, a_1\}$. The probability of this event is

$$(1-\varepsilon)^{n-1}(n-1)(1-\varepsilon)^{n-2}\varepsilon_2,$$

and if it occurs, a_1 will not yield the best outcome. Next consider the following event: when a_1 is sampled, the outcome is $(a_1, a_1, \ldots, a_1)$; when

$a_{j(2)}$ is sampled, the outcome is $(a_{j(2)}, x)$, where $x \in A^{n-1}$ does *not* contain a permutation of the actions $\{a_2, a_1, \ldots, a_1\}$; when $a_{j(3)}$ is sampled, the outcome is $(a_{j(3)}, y)$, where $y \in A^{n-1}$ contains some permutation of the actions $\{a_3, a_1, \ldots, a_1\}$. The probability of this event (regardless of whether or not $a_{j(2)} = a_{j(3)}$) is at least

$$(1-\varepsilon)^{n-1}(1-(n-1)(1-\varepsilon)^{n-2}\varepsilon_2)(n-1)(1-\varepsilon)^{n-2}\varepsilon_3,$$

and if it occurs, a_1 will not yield the best outcome. Note that the two events described are mutually exclusive. Reasoning in this manner, we obtain the following bound for the probability that a_1 will not yield the best outcome under the sampling procedure:

$$\begin{aligned} 1 - w(a_1, \alpha) \geq {} & (1-\varepsilon)^{n-1}(n-1)(1-\varepsilon)^{n-2}\varepsilon_2 \\ & + (1-\varepsilon)^{n-1}(1-(n-1)(1-\varepsilon)^{n-2}\varepsilon_2)(n-1)(1-\varepsilon)^{n-2}\varepsilon_3 \\ & + \cdots + (1-\varepsilon)^{n-1}\left(\prod_{i=2}^{m}(1-(n-1)(1-\varepsilon)^{n-2}\varepsilon_{i-1})\right) \\ & (n-1)(1-\varepsilon)^{n-2}\varepsilon_m. \end{aligned}$$

Let $o(\varepsilon^2)$ represent terms that are second order or higher in ε and/or ε_i $(\varepsilon^2, \varepsilon\varepsilon_i, \varepsilon_i\varepsilon_j, \ldots)$. Then the above inequality may be written as

$$w(a_1, \alpha) \leq 1 - (n-1)\sum_{i=2}^{m}\varepsilon_i + o(\varepsilon^2) \leq 1 - 2\varepsilon + o(\varepsilon^2), \tag{2}$$

since $n \geq 3$. Note that there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon < \bar{\varepsilon}$, $o(\varepsilon^2) < \varepsilon$. Hence, for all $\varepsilon < \bar{\varepsilon}$, we have

$$w(a_1, \alpha) < 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon = \alpha_1. \tag{3}$$

Let $N_{\bar{\varepsilon}}$ be defined as follows:

$$N_{\bar{\varepsilon}} = \{x \in \Omega^m \mid x = (1-\varepsilon, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m) \text{ with } \varepsilon < \bar{\varepsilon}\}.$$

Then from (1) and (3) we have $\dot{\alpha}_1 < 0$ for all $\alpha \in N_{\bar{\varepsilon}} \backslash \alpha^*$. Hence all trajectories initially in $N_{\bar{\varepsilon}} \backslash \alpha^*$ eventually leave $N_{\bar{\varepsilon}}$, and α^* is unstable. ∎

Many experimental public goods games in which zero contributions are a dominant strategy belong to the class to which Theorem 1 applies (Ledyard, 1995). Consider, for instance, the following example.

EXAMPLE 4 (private provision of public goods). Each of $n \geq 3$ individuals has an endowment $e = (m-1)x$, all or part of which may be contributed to the provision of a public good in finite increments x. The action space $A = \{a_1, \ldots, a_m\}$, where a_i represents a contribution of $(i-1)x$ units. Let

a_j represent the contribution of player j. The total contribution is $\sum_{j=1}^{n} a_j$ and the payoff to player j is $\pi_j = e - a_j + \beta \sum_{j=1}^{n} a_j$, where $1/n < \beta < 1$. Clearly a_1 is a strictly dominant action. The unique Nash equilibrium is $(\alpha^*, \ldots, \alpha^*)$, where $\alpha_1^* = 1$. Since this equilibrium is strict and symmetric, α^* is an $S(1)$ equilibrium. If $2\beta > 1$ the conditions for Theorem 1 are satisfied and this equilibrium is therefore unstable.

Example 4 implies that some positive contributions will be observed at any stable $S(1)$ equilibrium in this class of games, which accords with much of the experimental evidence.[4]

Theorem 1 states that inferiority is sufficient for instability of equilibria in symmetric games with three or more players. The following simple example shows that inferiority is *not* sufficient for instability in two-player games.

EXAMPLE 5 (prisoners' dilemma). Consider the symmetric game represented by the following payoff matrix

	a_1	a_2
a_1	1	3
a_2	0	2

The action a_1 is strictly dominant and hence $\alpha_1^* = 1$ at the unique Nash equilibrium (α^*, α^*); α^* is also the unique $S(1)$ equilibrium in this game. The dynamics (1) applied to this game are as follows:

$$\dot{\alpha}_1 = w(a_1, \alpha) - \alpha_1 = \alpha_1^2 + (1 - \alpha_1) - \alpha_1 = (1 - \alpha_1)^2.$$

Hence $\dot{\alpha}_1 > 0$ for all $\alpha_1 \in [0, 1)$ and, despite the fact that the unique Nash equilibrium is inferior, α^* is globally stable under the dynamics (1).

Example 5 illustrates that inferiority is not sufficient for instability when the number of players $n = 2$. Twice inferiority, however, is sufficient.

THEOREM 2. *If* (α^*, α^*) *is a twice inferior strict Nash equilibrium of a symmetric two-player game, then* α^* *is unstable under the sampling dynamics* (1)

Proof. Let (α^*, α^*) be a symmetric strict Nash equilibrium and assume without loss of generality that $\alpha_1^* = 1$. Note that α^* is an $S(1)$ equilibrium and hence a rest point of the dynamics (1). Consider a mixed strategy $\alpha = (1 - \varepsilon, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m) \in \Omega^m$, where $\varepsilon \in (0, 1)$. Since $\alpha \in \Omega^m$, we have $\varepsilon_i \geq 0$ for all $i \in \{2, \ldots, m\}$, and $\sum_2^m \varepsilon_i = \varepsilon$. Now consider the following event: when a_1 is sampled, the outcome is (a_1, a_1); when $a_{j(2)}$ is sampled,

[4]There are, of course, alternative interpretations of the experimental evidence. For instance, the experimental data are also consistent with the hypothesis that some individuals have preferences that are altruistic (or interdependent in other, more complex, ways).

the outcome is $(a_{j(2)}, a_2)$. The probability of this event is $(1-\varepsilon)\varepsilon_2$, and if it occurs, a_1 will not yield the best outcome. Next consider the following event: when a_1 is sampled, the outcome is (a_1, a_1); when $a_{j(2)}$ is sampled, the outcome is *not* $(a_{j(2)}, a_2)$; when $a_{k(2)}$ is sampled, the outcome is $(a_{k(2)}, a_2)$. The probability of this event is $(1-\varepsilon)(1-\varepsilon_2)\varepsilon_2$, and if it occurs, a_1 will not yield the best outcome. Note that the two events described are mutually exclusive. Hence the probability that action a_2 will yield a better outcome than action a_1 under sampling is at least

$$(1-\varepsilon)\varepsilon_2 + (1-\varepsilon)(1-\varepsilon_2)\varepsilon_2 = (1-\varepsilon)\varepsilon_2(2-\varepsilon_2).$$

Reasoning in this manner, we obtain the following bound for the probability that a_1 will not yield the best outcome under the sampling procedure:

$$1 - w(a_1, \alpha) \geq (1-\varepsilon)\varepsilon_2(2-\varepsilon_2) + \cdots + (1-\varepsilon)\left(\prod_{i=3}^{m}(1-\varepsilon_{i-1})^2\right)\varepsilon_m(2-\varepsilon_m).$$

Let $o(\varepsilon^2)$ represent terms that are second order or higher in ε and/or ε_i $(\varepsilon^2, \varepsilon\varepsilon_i, \varepsilon_i\varepsilon_j, \ldots)$. Then the above inequality may be written as

$$w(a_1, \alpha) \leq 1 - 2\sum_{i=2}^{m}\varepsilon_i + o(\varepsilon^2) = 1 - 2\varepsilon + o(\varepsilon^2).$$

This is the same as inequality (2) in the proof of Theorem 1 above, and the argument used thereafter applies unchanged to the present case. ■

A profile (a_q, a_q) can be twice inferior even when a_q is a strictly dominant action, as Example 2 illustrates. The instability of the equilibrium which places probability 1 on the strictly dominant action implies that strictly dominated actions must be played with positive probability indefinitely along all trajectories, regardless of whether or not such trajectories converge to an $S(1)$ equilibrium. The following example illustrates a further application of Theorem 2.

EXAMPLE 6 (three-action coordination game). Consider the symmetric game represented by the following payoff matrix

	a_1	a_2	a_3
a_1	2	6	6
a_2	0	7	3
a_3	1	4	8

This game has three strict Nash equilibria, each of which corresponds to an $S(1)$ equilibrium. There are no other $S(1)$ equilibria. Of the three equilibria, the one at (α^*, α^*) with $\alpha_1^* = 1$ is twice inferior and hence α^* is unstable by direct application of Theorem 2.

Example 6 shows that selection among strict Nash equilibria in certain games on the basis of the dynamics (1) can be very easy to implement by means of the sufficient conditions identified in Theorems 1 and 2. As discussed above, this invites a comparison with the criterion of stochastic stability, which is currently the standard basis for selection among strict Nash equilibria. For reasons discussed above, stochastic stability is most suitable as a selection criterion applied to the very long run. In contrast, the dynamics (1) are deterministic and can converge rapidly, providing a selection criterion that applies to the short run. Stochastic stability is, however, a more powerful criterion which can distinguish among equilibria that the sampling dynamics treat as identical (as in Example 1, for the case $x < y$). Which of the two methods is more appropriate in any given context will therefore depend on the time horizon over which selection is expected to occur.

6. MULTIPLE POPULATIONS

The analysis to this point has been based on the assumption that there is a single population from which all players are drawn. While this is a suitable assumption for symmetric games, it is not adequate for the analysis of asymmetric games since the action space and the potential payoff consequences of a given action generally differ across different player positions. In this case it is natural to assume that there exists a distinct population for each player position. Symmetric games too can be analyzed on the basis of multiple player populations (one for each player position). The dynamic properties of the multiple population case differ from those of the single population case and results that hold in one case need not carry over to the other even for symmetric games.

Consider, for simplicity, the case of a two-player asymmetric game with action spaces $A = \{a_1, \ldots, a_{m_1}\}$ and $B = \{b_1, \ldots, b_{m_2}\}$, respectively. Let $u_k(a_i, b_j)$ denote the payoff to player k when player 1 chooses the action $a_i \in A$ while player 2 chooses $b_j \in B$. These payoff functions represent each player's ordinal preferences over the set of outcomes. Let α be a probability distribution on A and β a probability distribution on B. Let $v_1(a_i, \beta)$ be the m_1 random variables yielding $u_1(a_i, b_j)$ with probability β_j for each $b_j \in B$. Similarly, let $v_2(b_j, \alpha)$ be the m_2 random variables yielding $u_2(a_i, b_j)$ with probability α_i for each $a_i \in A$. Let $w_1(a_i, \beta)$ be the probability that, when each random variable $v_1(x, \beta)$ is drawn independently and exactly once, the action a_i yields the best outcome. As before, in the case of realizations in which a_i is not unique in yielding the best outcome, the probability is weighted by the reciprocal of the total number of tied alternatives. Finally, let $w_2(b_j, \alpha)$ be the probability that, when each random variable $v_2(y, \alpha)$ is drawn independently and exactly once, the action b_j yields the best outcome, again with the same tie-breaking convention that the probability is weighted by the reciprocal of the total number of tied alternatives.

An $S(1)$ equilibrium in this case is defined as a pair of probability distributions (α, β) on the sets of actions A and, B respectively, with the property that

$$w_1(a_i, \beta) = \alpha_i \qquad \text{for every action } a_i \in A,$$
$$w_2(b_j, \alpha) = \beta_j \qquad \text{for every action } b_j \in B.$$

The dynamics (1) can easily be generalized to cover this case as follows:

$$\dot{\alpha}_i(t) = w(a_i, \beta(t)) - \alpha_i(t), \tag{4}$$
$$\dot{\beta}_i(t) = w(b_i, \alpha(t)) - \beta_i(t). \tag{5}$$

The above dynamics are defined for the state space $\mathbf{R}^{m_1+m_2}$. It is easily verified that under the dynamics (4) and (5), if $(\alpha(t_0), \beta(t_0)) \in \Omega^{m_1} \times \Omega^{m_2}$, then $(\alpha(t), \beta(t)) \in \Omega^{m_1} \times \Omega^{m_2}$ for all $t > t_0$. It is also easily seen that a state (α, β) is a rest point of these dynamics if an only if it is an $S(1)$ equilibrium. Furthermore, all of the above definitions and statements generalize in a straightforward manner to the case of three or more populations.

For symmetric games, if α is an $S(1)$ equilibrium in the single population case, then (α, α) must be an $S(1)$ equilibrium in the multiple population case. There may, however, be additional equilibria in the latter case, and the stability properties of equilibria which occur in both cases need not be identical, as the following example illustrates.

Example 7 (Hawk–Dove). Consider the game represented by the following payoff matrix

	b_1	b_2
a_1	$(0, 0)$	$(3, 1)$
a_2	$(1, 3)$	$(2, 2)$

In the single population case there is a unique $S(1)$ equilibrium $\alpha = (0.5, 0.5)$ which is globally asymptotically stable. To see this, observe that the dynamics (1) yield

$$\dot{\alpha}_1 = w(a_1, \alpha) - \alpha_1 = 1 - 2\alpha_1.$$

In the multiple population case, on the other hand, the dynamics are as follows:

$$\dot{\alpha}_1 = w(a_1, \beta) - \alpha_1 = 1 - \beta_1 - \alpha_1$$
$$\dot{\beta}_1 = w(b_1, \alpha) - \beta_1 = 1 - \alpha_1 - \beta_1.$$

In this case any pair of distributions (α, β) is an $S(1)$ equilibrium provided that $\alpha_1 + \beta_1 = 1$. The single population $S(1)$ equilibrium remains an equilibrium in the multiple population case, but it is no longer asymptotically stable.

While the stability of an equilibrium need not be maintained as one moves from the single population to the multiple population case, the following result shows that *instability* is maintained.

THEOREM 3. *If α^* is unstable under the single population dynamics* (1), *then (α^*, α^*) is unstable under the multiple population dynamics* (4) *and* (5).

Proof. Suppose that $\alpha^* \in \Omega^m$ is unstable under the dynamics (1). Then, by definition, there exists a neighborhood $U \subset \mathbf{R}^m$ of α^* such that all trajectories which originate at any $\alpha(t_0) \in U \cap \Omega^m$, $\alpha(t_0) \neq \alpha^*$, must eventually leave U. Now consider the multiple population dynamics (4) and (5) in the neighborhood of the equilibrium (α^*, α^*). Since the game is symmetric, any trajectory $(\alpha(t), \beta(t))$ which satisfies initial conditions $\alpha(t_0) = \beta(t_0)$ must satisfy $\alpha(t) = \beta(t)$ for all $t > t_0$. Moreover, the time path of $\alpha(t)$ will be identical under (1) to the time paths of $\alpha(t)$ and $\beta(t)$ under (4) and (5), provided that initial conditions $\alpha(t_0)$ in the former case are identical to the initial conditions $\alpha(t_0)$ and $\beta(t_0)$ in the latter case. Hence, all trajectories from initial conditions $(\alpha(t_0), \beta(t_0))$ satisfying $\alpha(t_0) = \beta(t_0) \in U \cap \Omega^m$ will eventually leave $U \times U$ under (4) and (5). Since every neighborhood of (α^*, α^*) contains some points (α, β) such that $\alpha = \beta$, there can be no neighborhood W of (α^*, α^*) such that trajectories initially in W remain in $U \times U$. Hence (α^*, α^*) is unstable. ■

Although Theorem 3 is stated and proved for the two population case, the generalization to multiple populations is straightforward. It implies, in particular, that dominant strategy equilibria which are unstable under the single population dynamics (as in Examples 2 and 3) remain unstable under the multiple population dynamics. In such cases, strictly dominated strategies will be played with positive probability along all trajectories, regardless of whether the trajectories converge.

Turning, finally, to the case of genuinely asymmetric games with multiple player populations, the following definition extends the notion of inferiority to the case in which the action spaces of the two players need not be the same.

DEFINITION. An action profile (a_q, b_r) in a two-player game is *inferior for player* 1 if, for every $i \neq r$ there exists $j(i) \neq q$ such that

$$u(a_{j(i)}, b_i) > u(a_q, b_r).$$

It is *twice inferior for player* 1 if, for every action $i \neq r$ there exist $j(i) \neq q$ and $k(i) \neq q$ such that $j \neq k$ and

$$u(a_{j(i)}, b_i) \geq u(a_{k(i)}, b_i) > u(a_q, b_r).$$

(Twice) inferiority for player 2 is defined analogously. An action profile (a_q, b_r) is (twice) inferior if it is (twice) inferior for both players.

This definition is consistent with that given for symmetric games above (for symmetric games the two definitions are identical.) We say that a strict Nash equilibrium (α^*, β^*) is (twice) inferior if the action profile that is played with probability 1 in this equilibrium is itself (twice) inferior. The following result provides sufficient conditions for the instability of equilibria in asymmetric games with multiple player populations.

THEOREM 4. *In any two-player game, all twice inferior strict Nash equilibria are unstable under the dynamics* (4) *and* (5).

Proof. Let (α^*, β^*) be a strict Nash equilibrium and assume without loss of generality that $\alpha_1^* = \beta_1^* = 1$. Note that (α^*, β^*) is a (multiple population) $S(1)$ equilibrium and hence a rest point of (4) and (5). Consider the pair of mixed strategies $\alpha = (1-\delta, \delta) \in \Omega^2$ and $\beta = (1-\varepsilon, \varepsilon) \in \Omega^2$, where $\varepsilon, \delta \in (0, 1)$. Using the same reasoning as in the proof of Theorem 2, the following inequalities may be obtained:

$$w(a_1, \beta) \leq 1 - 2\varepsilon + o(\varepsilon^2),$$
$$w(b_1, \alpha) \leq 1 - 2\delta + o(\delta^2).$$

Hence there exist $\bar{\delta}, \bar{\varepsilon} > 0$ such that for all $\delta < \bar{\delta}$ and $\varepsilon < \bar{\varepsilon}$, the following hold:

$$w(a_1, \beta) < 1 - \varepsilon \tag{6}$$
$$w(b_1, \alpha) < 1 - \delta. \tag{7}$$

Let $\eta = \min\{\bar{\delta}, \bar{\varepsilon}\}$ and let N_η be defined as follows:

$$N_\eta = \{(\alpha, \beta) \in \Omega^2 \times \Omega^2 \mid (\alpha, \beta) = ((1-\delta, \delta), (1-\varepsilon, \varepsilon)) \text{ with } \delta, \varepsilon < \eta\}.$$

Then from (4), (5) and (6), (7), the following hold for all $(\alpha, \beta) \in N_\eta$:

$$\dot{\alpha}_1 = w(a_1, \beta) - \alpha_1 < (1-\varepsilon) - (1-\delta) = \delta - \varepsilon,$$
$$\dot{\beta}_1 = w(b_1, \alpha) - \beta_1 < (1-\delta) - (1-\varepsilon) = \varepsilon - \delta.$$

Hence $\dot{\alpha}_1 + \dot{\beta}_1 < 0$ for all $(\alpha, \beta) \in N_\eta \backslash (\alpha^*, \beta^*)$, and all trajectories initially in $N_\eta \backslash (\alpha^*, \beta^*)$ eventually leave N_η. Therefore (α^*, β^*) is unstable. ■

Theorem 4 provides sufficient conditions for the instability of strict Nash equilibria in asymmetric two-player games. As in the case of symmetric games, these conditions can be satisfied at equilibria in which strictly dominant actions are played with probability 1. Hence the basic conclusion arising from the analysis of symmetric games remains unchanged: strictly dominated strategies may be played with positive probability along all trajectories even in asymmetric games with multiple player populations. As might be conjectured, the following asymmetric game analogue of Theorem 1 also holds: in games with three or more players, all inferior strict Nash equilibria are unstable under the multiple population dynamics. The proof of this claim follows the same logic as those of Theorems 1 and 4, but requires considerable additional notation, and is therefore omitted.

7. CONCLUSIONS

This paper has explored the dynamic implications of the sampling procedure that underlies Osborne and Rubinstein's equilibrium concept for games with procedurally rational players. Dynamic stability can serve as a criterion for selection among multiple $S(1)$ equilibria. Furthermore, since there is a correspondence between strict Nash equilibria and $S(1)$ equilibria, this criterion can also be used to address the standard (Nash) equilibrium selection problem. More significantly, both symmetric and asymmetric dominant strategy equilibria can be unstable, and stable equilibria can involve the playing of actions that are strictly dominated. In these two respects, the sampling dynamics yield predictions that differ starkly from those based on the standard theory of evolutionary games. This occurs because the dynamics do not generally satisfy the condition of payoff monotonicity that underlies most work in evolutionary game theory. Which of the two approaches is more suitable in particular applications will depend, naturally, on which of the dynamic specifications more accurately describes individual learning and adjustment in the environment being studied.

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