Quantum statistical complexity
Sharpening Occam's razor with quantum mechanics

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Outline

1. How to define complexity
2. How to measure complexity
3. Using quantum mechanics to model a complex system
4. Connecting complexity to physical theories
1. How to define complexity
General definition

“This leads us to the following tentative definition of complexity: A complex system is an ensemble of many similar elements which are interacting in a disordered way, resulting in robust organisation and memory.”

From “What is a complex system”, J Ladyman, J Lambert, K Wiesner (2010), http://www.maths.bristol.ac.uk/~enxkw/Publications.html
2. How to measure complexity
Data-driven definition

Since statistical complexity is a measure applied to data, we offer the quantitative definition of a complex system:

“A system is complex if it can generate data series with high statistical complexity.”

From “What is a complex system”, J Ladyman, J Lambert, K Wiesner (2010), http://www.maths.bristol.ac.uk/~enxkw/Publications.html
Information-theoretic measures of complexity

Take an infinite sequence of random variables $X_{-\infty}^\infty$, drawn according to a probability distribution $P(X_{-\infty}^\infty)$ (not necessarily i.i.d) (stationary process).


$$EMC = \sum_{N=0}^{\infty} (h^N - h)$$


$$E = \lim_{N \to \infty} I(X_{-N}^{-1}; X_0^N)$$

The two measures are equivalent:

$$EMC = E$$
Grouping strings into equivalence classes

Identify finite set of equivalence classes, obeying

\[ \eta(x^n) = \{ y^m : Pr(z^l|y^m) = Pr(z^l|x^n) \forall l \} \]

\( S_0 = \{3141592, 5926535, \ldots \} \)

\( S_1 = \{4159265, \ldots \} \)

\[ \ldots \]
Statistical complexity

This set of equivalence classes is a “sufficient statistic” for the process, called “causal states” (Shalizi, Crutchfield. J. Stat. Phys. (2001) 104, 817-879).

The Statistical Complexity is the Shannon entropy over the stationary distribution of effective (causal) states:

$$C_\mu = H(S)$$

The **Statistical Complexity** is the minimum amount of information needed to optimally predict a process.
Minimal and optimal - Constructing an automaton

The resulting automaton / hidden Markov model is called **ε-machine**. It is the unique, minimal, optimal predictor (Shalizi, Crutchfield. J. Stat. Phys. (2001) 104, 817-879)
Applications of statistical complexity

“Occam’s razor”

“Plurality is not to be posited without necessity.”
3. Using quantum-mechanics to model a complex system
Room for improvement

“Given a stochastic process $P (X^\infty_{-\infty})$ with excess entropy $E$ and statistical complexity $C_\mu$. Let its corresponding $\epsilon$-machine have transition probabilities $T_{i,j}^{(r)}$. Then $C_\mu > E$ iff there exists a non-zero probability that two different causal states, $S_j$ and $S_k$ will both make a transition to a coinciding causal state $S_l$ upon emission of a coinciding output $r \in \Sigma$, i.e.

$$T_{j,l}^{(r)}, T_{k,l}^{(r)} \neq 0$$

“

Quantum mechanical state

Classical bit vs quantum bit (qubit)

$$|\psi\rangle = c_1 |0\rangle + c_2 |1\rangle$$

[cqed.org]
Quantum ‘causal state’*

Alphabet \( \mathcal{A} \), transition probabilities \( T_{jk} \), causal states \( S_j \)

Form a new, ‘quantum’ causal states:

\[
|S_j\rangle = \sum_{k=1}^{n} \sum_{r \in \mathcal{A}} \sqrt{T_{jk}^{(r)}} |r\rangle |k\rangle
\]

* The term causal state is strictly speaking not accurate since minimality has not been proven.

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Quantum statistical complexity

Define ‘quantum statistical complexity’ $C_q$ as

$$C_q = -Tr \rho \log \rho$$

where $\rho$ is the density matrix over quantum ‘causal states’.
Quantum statistical complexity

“Consider any stochastic process $P(X_{\infty})$ with excess entropy $E$, and that the optimal classical system that generates such statistics has entropy $C_{\mu} > E$. Then we may construct a quantum system that exhibits identical statistics, with internal entropy $C_q < C_{\mu}$.”
Example: Biased flip

\[ |S_A\rangle = \sqrt{p}|1\rangle|B\rangle + \sqrt{1-p}|0\rangle|A\rangle \]

\[ |S_B\rangle = \sqrt{p}|0\rangle|A\rangle + \sqrt{1-p}|1\rangle|B\rangle \]
Quantum beats classical

Following Occam we would favour the quantum model over the classical model.
4. Connecting complexity to physical theories
Thermodynamic limit

Wiesner et al. arxiv:0905.2918.
Conclusion

**Complexity**: A system is complex if it can generate data series with high statistical complexity.

**Occam’s quantum razor**: We may construct a quantum system that exhibits identical statistics, with complexity $C_q < C_\mu$.

**Physical theories**: Complexity measures can be imbedded in existing physical theories.