Quantum statistical complexity

Sharpening Occam's razor with quantum mechanics

Karoline Wiesner
School of Mathematics
Centre for Complexity Sciences
University of Bristol

Collaborators

Mile Gu, Elisabeth Rieper, Vlatko Vedral*

(Physics, Singapore University, *Oxford University)

James Ladyman, James Lambert

(Philosophy, Bristol University)

Outline

1. How to define complexity
2. How to measure complexity
3. Using quantum mechanics to model a complex system
4. Connecting complexity to physical theories

1. How to define complexity

General definition

"This leads us to the following tentative definition of complexity: A complex system is an ensemble of many similar elements which are interacting in a disordered way, resulting in robust organisation and memory."

From "What is a complex system", J Ladyman, J Lambert, K Wiesner (2010), http://www.maths.bristol.ac.uk/~enxkw/Publications.html

2. How to measure complexity

Data-driven definition

Since statistical complexity is a measure applied to data, we offer the quantitative definition of a complex system:

"A system is complex if it can generate data series with high statistical complexity."

From "What is a complex system", J Ladyman, J Lambert, K Wiesner (2010), http://www.maths.bristol.ac.uk/~enxkw/Publications.html

Information-theoretic measures of complexity

Take an infinite sequence of random variables $X_{-\infty}^{\infty}$, drawn according to a probability distribution P($X_{-\infty}^{\infty}$) (not necessarily i.i.d) (stationary process).

Effective measure complexity (Grassberger, Int. J. Theor. Phys. 9 (1986) 907-938):

$$EMC = \sum_{N=0}^{\infty} (h^N - h)$$

Excess entropy (Crutchfield, Feldman, Chaos 13 (2003) 25-54):

$$\mathbf{E} = \lim_{N \to \infty} I(X_{-N}^{-1}; X_0^N)$$

The two measures are equivalent:

$$EMC = \mathbf{E}$$

Grouping strings into equivalence classes

Identify finite set of equivalence classes, obeying

$$\eta(x^n) = \{y^m \;:\; Pr(z^l|y^m) = Pr(z^l|x^n) \; \forall \; l\}$$

S_0 = {3141592, 5926535, ...}

S_1 = {4159265, ...}

...

Statistical complexity

This set of equivalence classes is a “sufficient statistic” for the process, called “causal states” (Shalizi, Crutchfield. J. Stat. Phys. (2001) 104, 817-879).

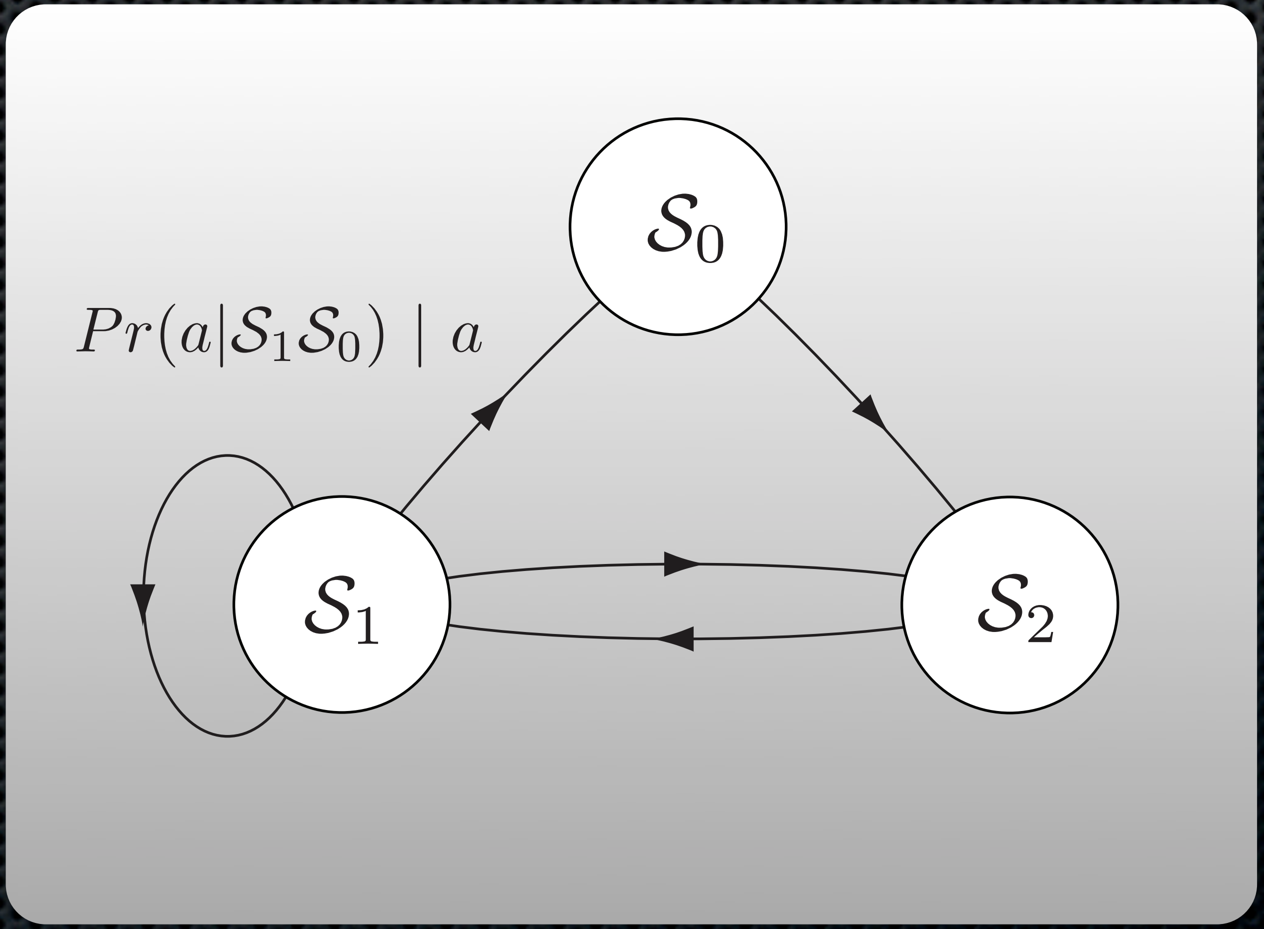
The Statistical Complexity is the Shannon entropy over the stationary distribution of effective (causal) states:

$$C_\mu = H(\mathcal{S})$$

The **Statistical Complexity** is the minimum amount of information needed to optimally predict a process.

Minimal and optimal - Constructing an automaton

The resulting automaton / hidden Markov model is called **ϵ-machine**. It is the unique, minimal, optimal predictor (Shalizi, Crutchfield. J. Stat. Phys. (2001) 104, 817-879)



Applications of statistical complexity

- Dynamical systems (Crutchfield, J.P. & Young, K. Inferring statistical complexity. Phys. Rev. Lett. 63, 105(1989).)
- Spin systems (Crutchfield, J.P. & Feldman, D.P. Statistical complexity of simple one-dimensional spin systems . Phys. Rev. E 55, R1239(1997).)
- Crystal growth (Varn, D.P., Canright, G.S. & Crutchfield, J.P. Discovering planar disorder in close-packed structures from x-ray diffraction: Beyond the fault model. Phys. Rev. B 66, 174110(2002).)
- Molecular dynamics (Li, C., Yang, H. & Komatsuzaki, T. Multiscale complex network of protein conformational fluctuations in single-molecule time series. Proceedings of the National Academy of Sciences 105, 536-541 (2008). D. Kelly et al. Inferring hidden Markov models from noisy time sequences. arXiv:1011.2969)
- Atmospheric turbulence (A. J. Palmer, C. W. Fairall, and W. A. Brewer, Complexity in the Atmosphere , IEEE Transactions on Geoscience and Remote Sensing 38, July 4 (2000).)
- Population dynamics (Crutchfield, J.P. & Görnerup, O. Objects that make objects: the population dynamics of structural complexity. Journal of the Royal Society Interface 22, 345-349(2006). Görnerup, O. & Crutchfield, J.P. Hierarchical Self-Organization in the Finitary Process Soup. Artificial Life 14, 245-254(2008).)
- Self-organisation (Shalizi, C.R., Shalizi, K.L. & Haslinger, R. Quantifying Self-Organization with Optimal Predictors. Phys. Rev. Lett. 93, 118701(2004).)
- Neural spike sequences (Tino, P. & Koteles, M. Extracting finite-state representations from recurrent neural networks trained on chaotic symbolic sequences. Neural Networks, IEEE Transactions on 10, 284-302(1999).)

“Occam’s razor”

“Plurality is not to be posited without necessity.”

3. Using quantum-mechanics to model a complex system

Room for improvement

“Given a stochastic process P ($X_{-\infty}^{\infty}$) with excess entropy E and statistical complexity C_μ. Let its corresponding **ε**-machine have transition probabilities $T_{i,j}^{(r)}$. Then C_μ > E iff there exists a non-zero probability that two different causal states, S_j and S_k will both make a transition to a coinciding causal state S_l upon emission of a coinciding output r ∊ **Σ**, i.e.

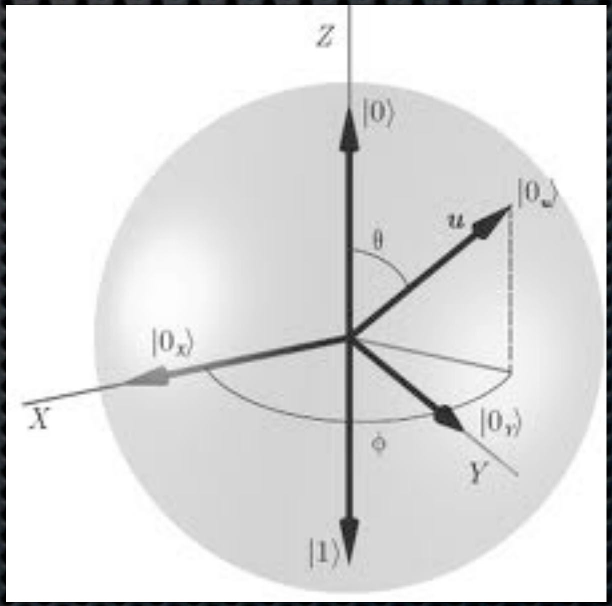
$$T_{j,l}^{(r)}, T_{k,l}^{(r)} \neq 0$$ “

(Wiesner et al. arxiv:0905.2918. Gu, Wiesner et al. 2010)

Quantum mechanical state

Classical bit vs quantum bit (qubit)

$$|\psi\rangle = c_1|0\rangle + c_2|1\rangle$$



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Quantum ‘causal state’*

Alphabet $\mathcal{A}$, transition probabilities T_{jk}, causal states S_j

Form a new, ‘quantum’ causal states:

$$|S_j\rangle = \sum_{k=1}^{n} \sum_{r \in \mathcal{A}} \sqrt{T_{jk}^{(r)}}\, |r\rangle |k\rangle$$

basis set from causal states

basis set from alphabet

square roots of transition probabilities

* The term causal state is strictly speaking not accurate since minimality has not been proven.

Quantum statistical complexity

Define ‘quantum statistical complexity’ C_q as

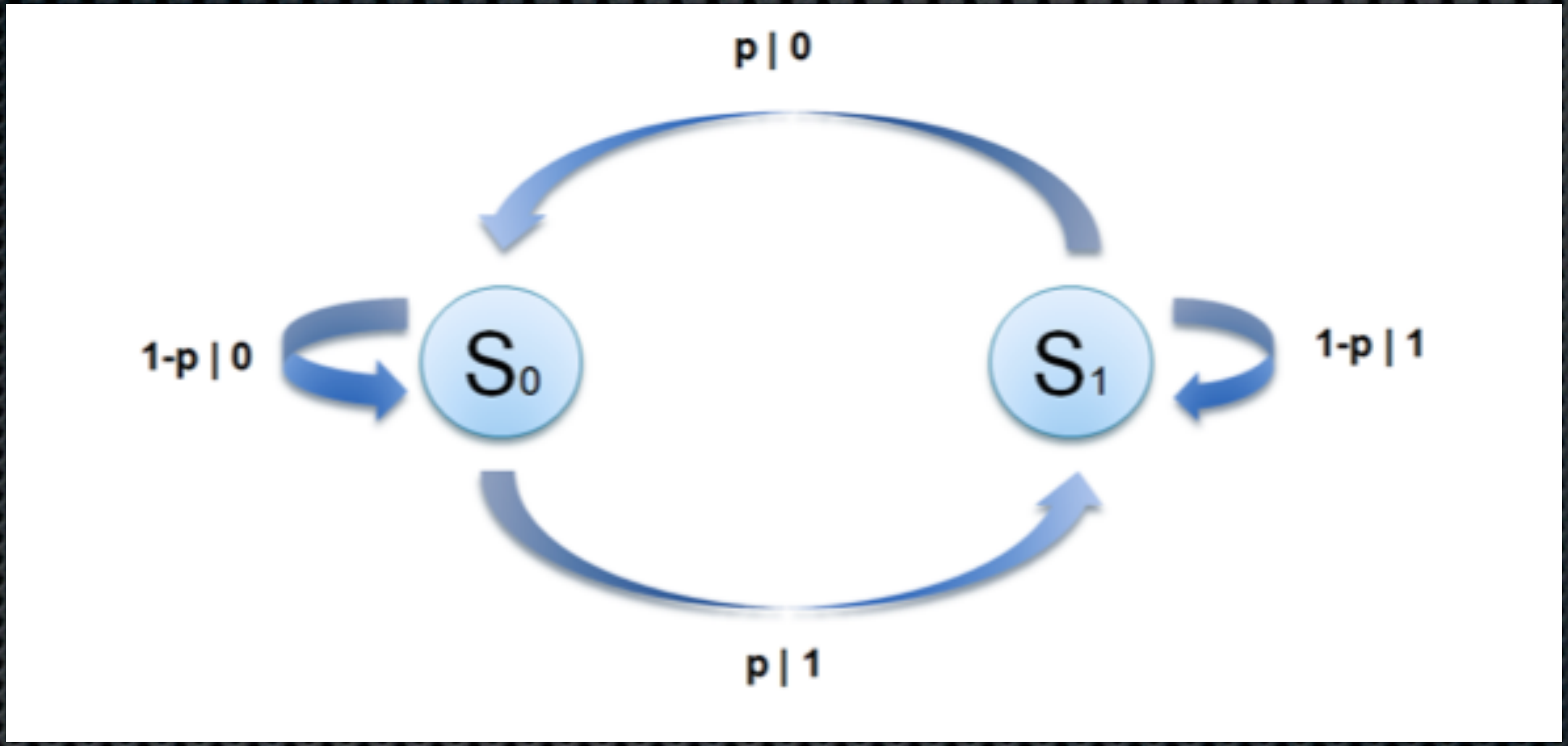
$$C_q = -Tr\,\rho \log \rho$$

where ρ is the density matrix over quantum ‘causal states’.

Quantum statistical complexity

"Consider any stochastic process P($X_{-\infty}^{\infty}$) with excess entropy E, and that the optimal classical system that generates such statistics has entropy C_μ >E. Then we may construct a quantum system that exhibits identical statistics, with internal entropy $C_q < C_\mu$."

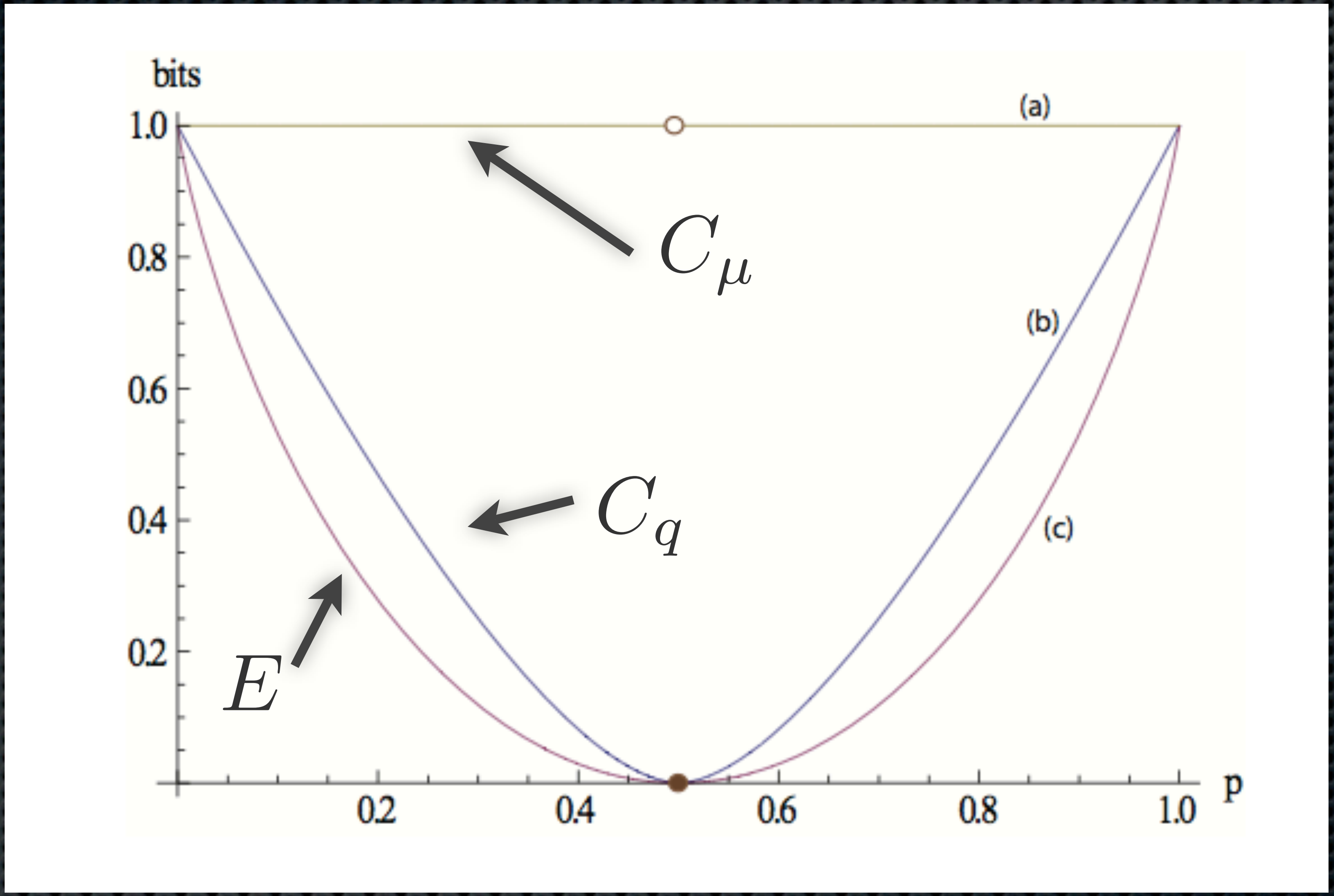
Example: Biased flip



$$|S_A\rangle = \sqrt{p}|1\rangle|B\rangle + \sqrt{(1-p)}|0\rangle|A\rangle$$

$$|S_B\rangle = \sqrt{p}|0\rangle|A\rangle + \sqrt{(1-p)}|1\rangle|B\rangle$$

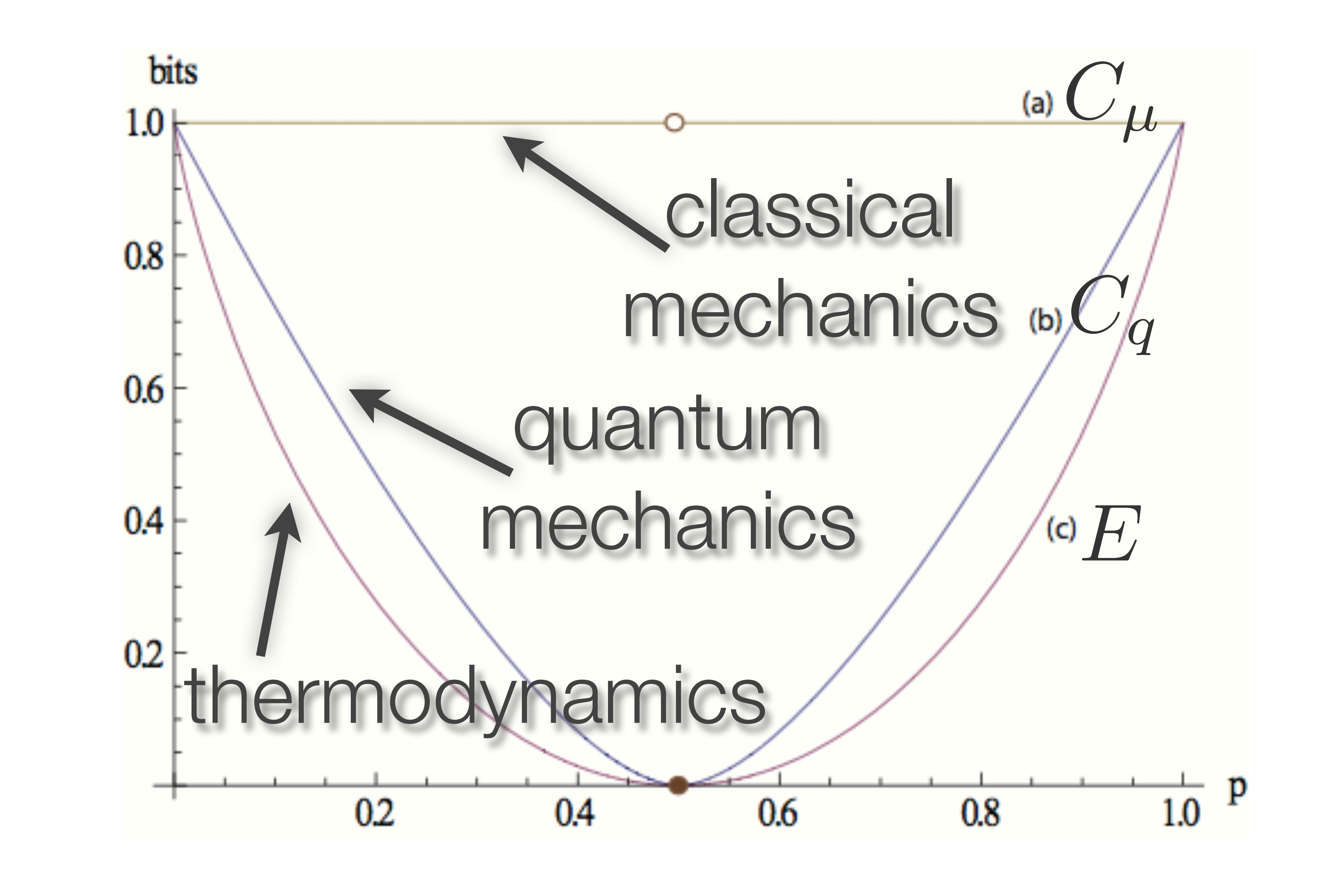
Quantum beats classical



Following Occam we would favour the quantum model over the classical model.

4. Connecting complexity to physical theories

Thermodynamic limit



Wiesner et al. arxiv:0905.2918.

Conclusion

Complexity: A system is complex if it can generate data series with high statistical complexity.

Occam's quantum razor: We may construct a quantum system that exhibits identical statistics, with complexity $C_q < C_\mu$.

Physical theories: Complexity measures can be imbedded in existing physical theories.