Power laws, fractals, and the structure and dynamics of vascular networks

Van Savage
UCLA Medical School, Dept. of Biomathematics

SFI CSSS, 2009

Outline

I. General review of power laws

   1. Identifying power laws
   2. Some Examples
   3. Self Similarity
   4. Universality classes and critical points

II. Dimensional Analysis

III. Introduction to scaling in biology (aka Allometry)

   1. Examples of processes that vary systematically with body size and temperature over a large range.
   2. Theory for these patterns

IV.Conclusions

I. Review of power laws

1. Types of Power Laws

$$y = Ax^b$$

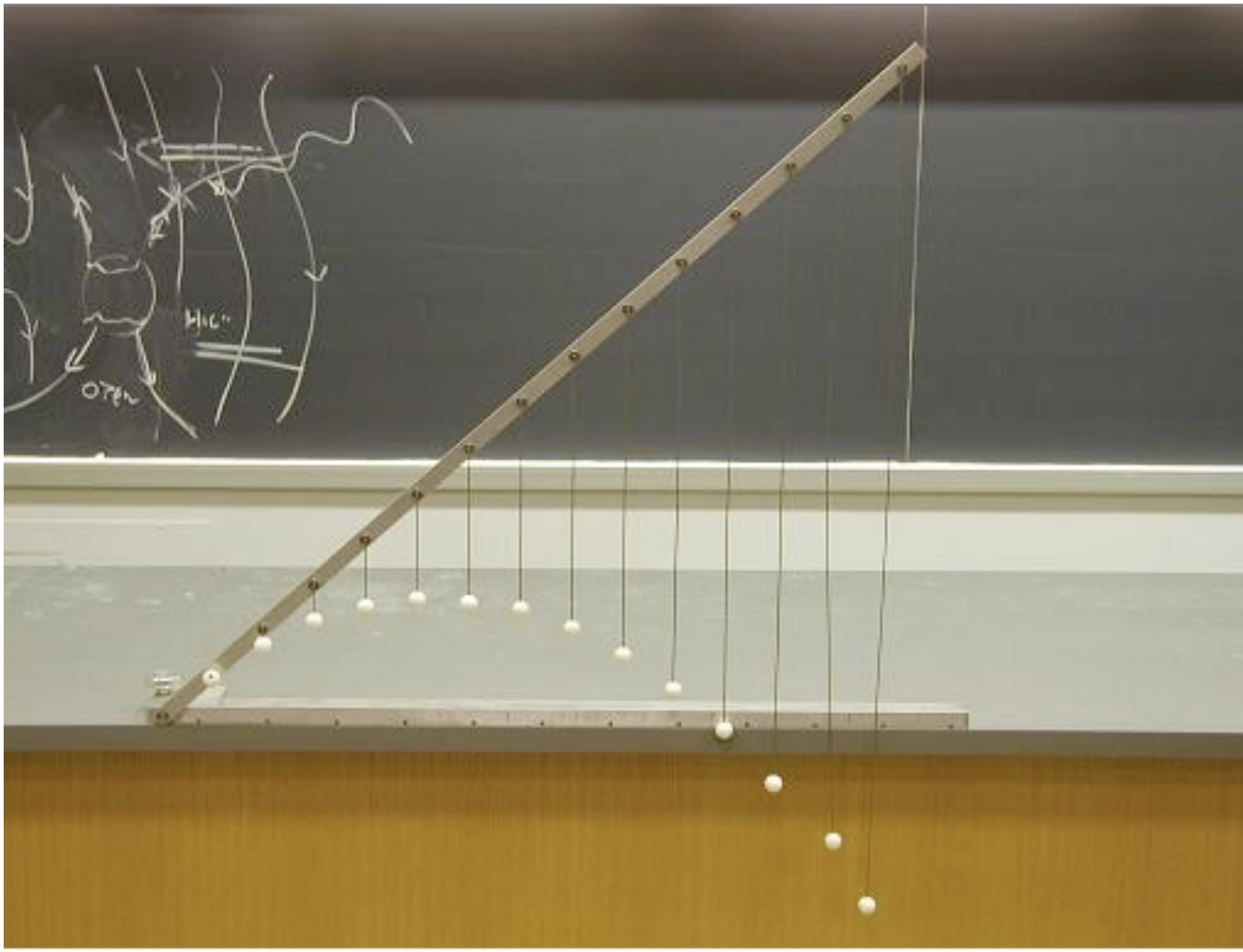
1. Physical laws--planetary orbits, parabolic motion of thrown objects, classical forces, etc. (these are really idealized notions and do not exist in real world)

2. Scaling relations--relate two fundamental parameters in a system like lifespan to body mass in biology (Physical Laws are special/strong case)

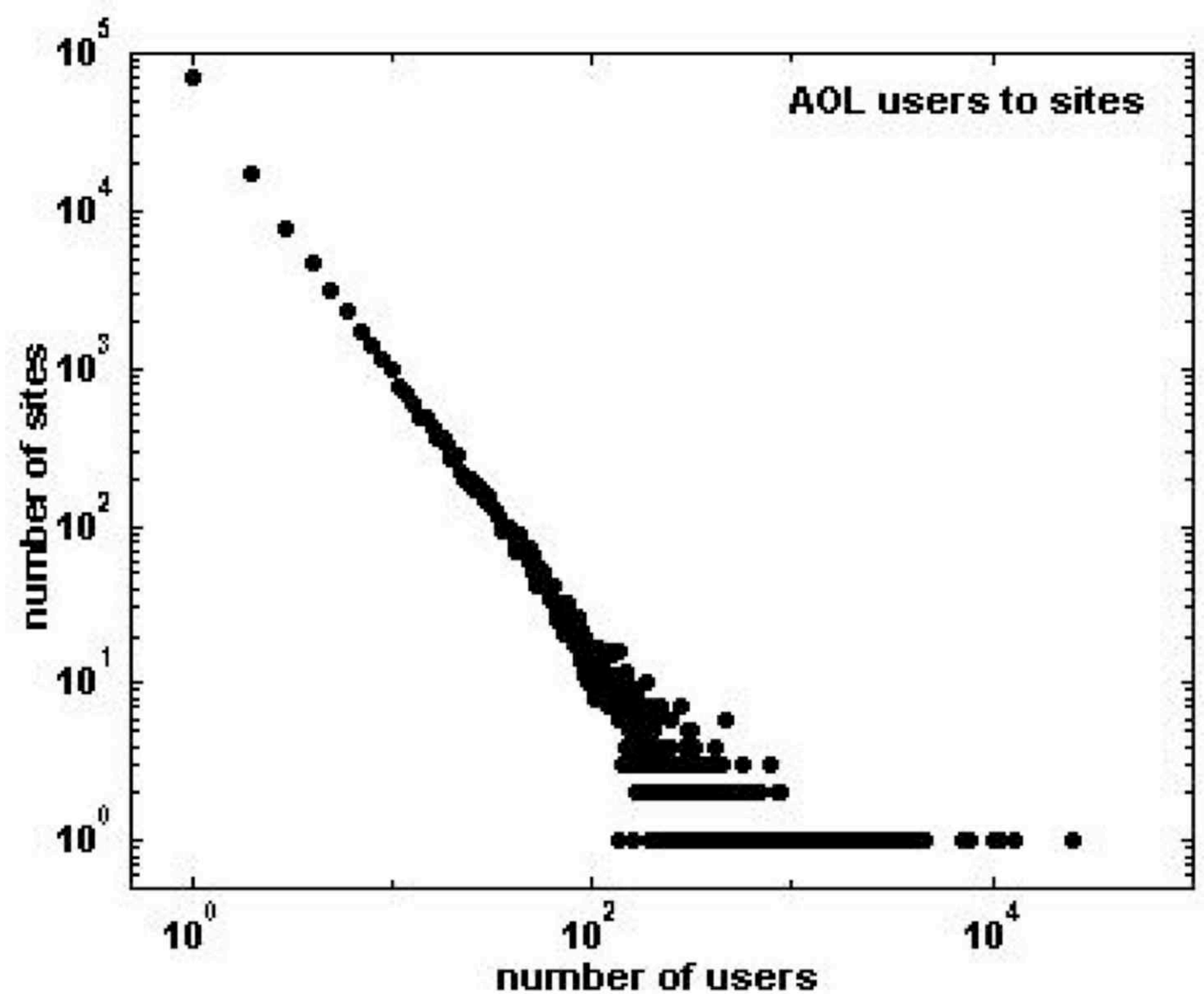
3. Statistical distributions

1. Examples of Power Laws

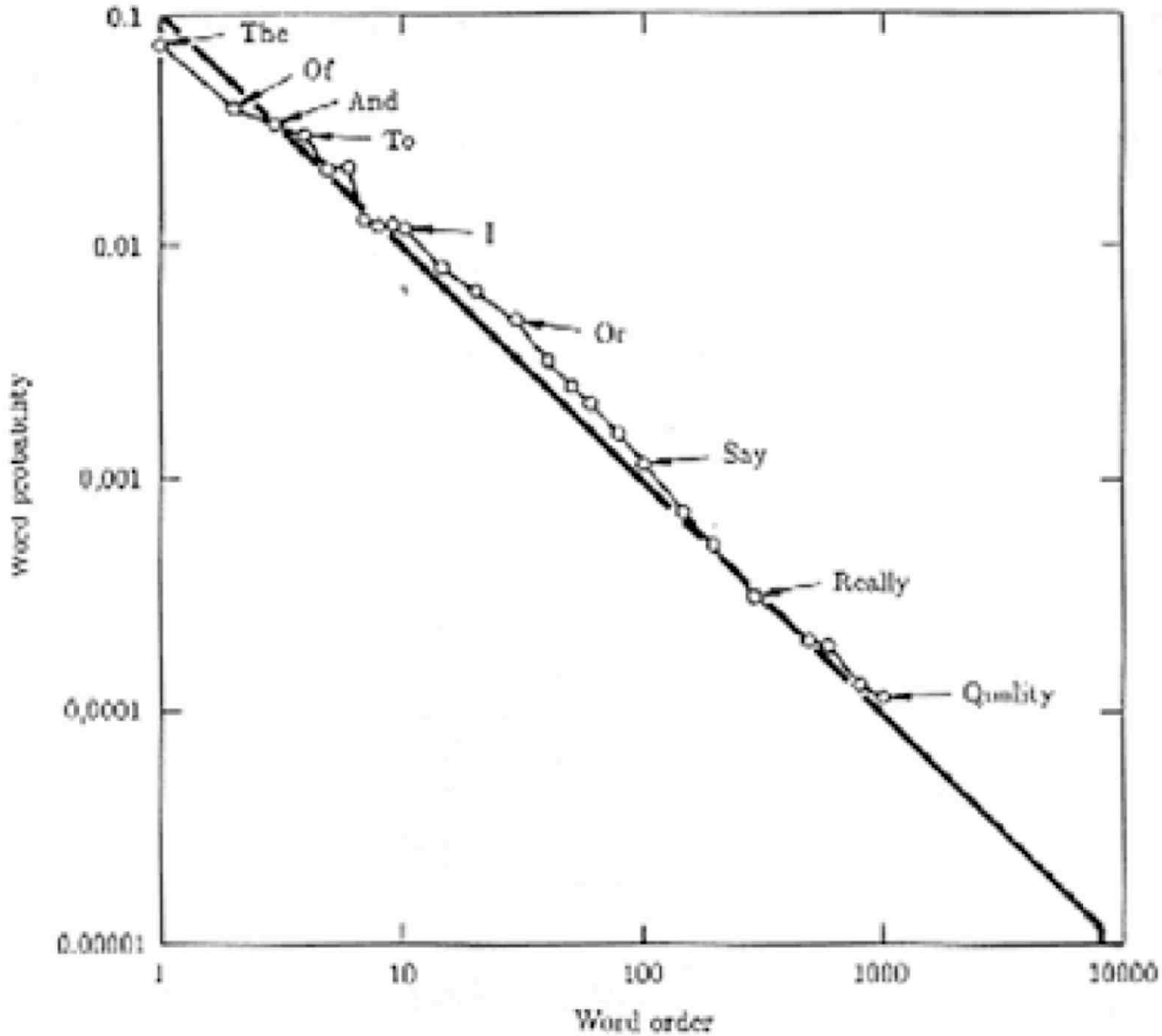
Parabolic Motion



Web Sites

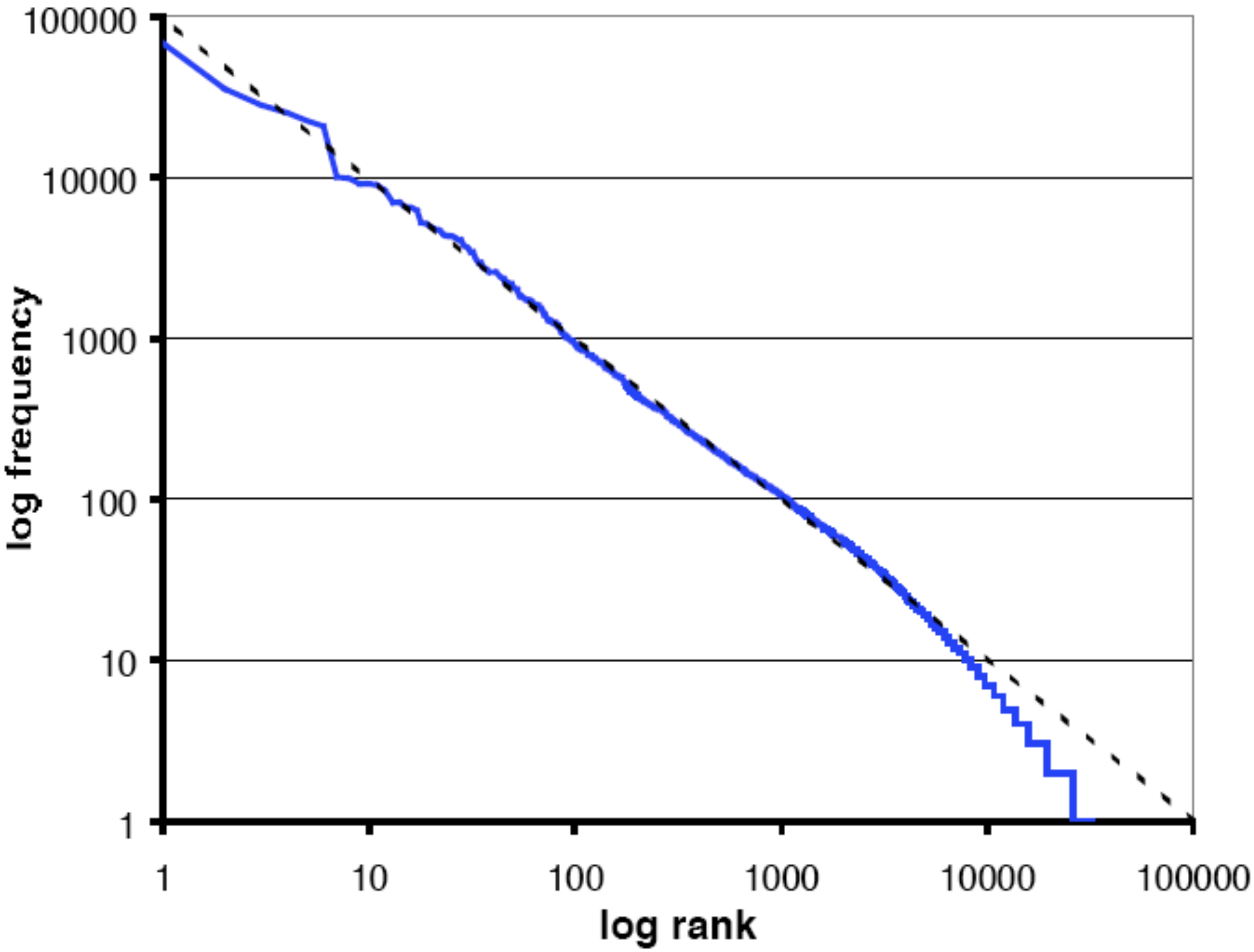

AOL users to sites
number of sites
number of users
10^5
10^4
10^3
10^2
10^1
10^0
10^0
10^2
10^4

Word usage



0.1
0.01
0,001
0,0001
0.00001
Word probability
1
10
100
1000
10000
Word order
The
Of
And
To
I
Or
Say
Really
Quality

Word Usage



Unigrams		
Freq	Token	Meaning
620619	的	Of
308326	国	State
219543	一	One
209497	中	Centre / Middle
176905	在	In / At
159861	和	And
143359	人	Human
139713	了	Perfective marker
133696	会	Get together/Meeting/Association
128805	年	Year

2. Identifying Power Laws

$$y = ax^b$$

$$\ln y = b \ln x + \ln a$$

Linear plot: slope=b and intercept=ln a

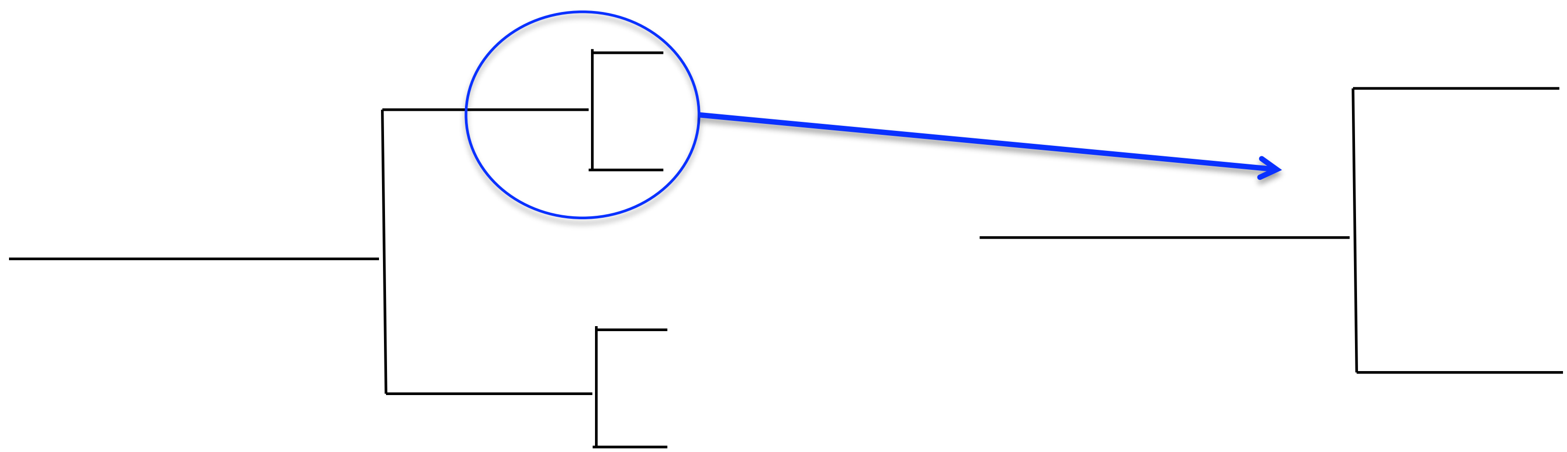
- Need big range on *x*- and *y*-axes to determine power laws because this minimizes effects of noise and errors
- Can give good measure of *b,* the exponent

Identifying Power Laws

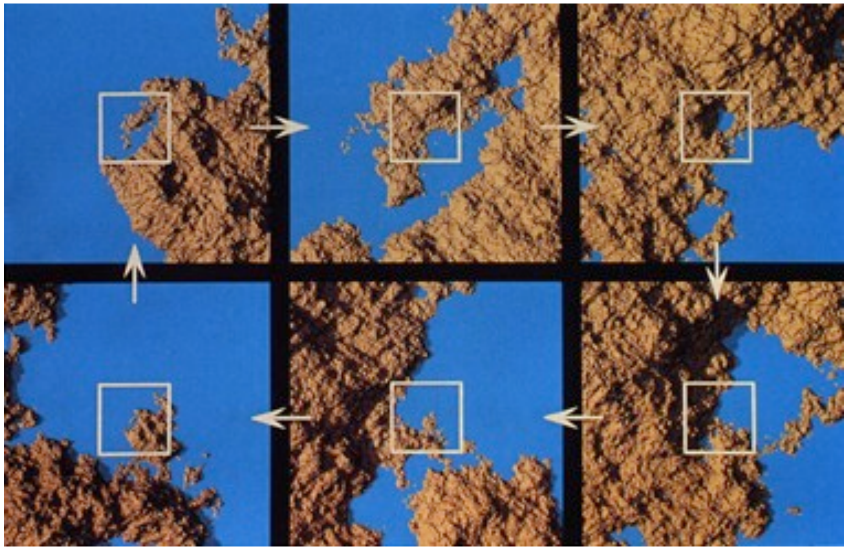
- Maximum likelihood methods are good for identifying power laws if used in correct way
- Whether to curve fit in linear or logarithmic space depends on distribution of errors because regressions make assumptions about these: homoscedascity->variance in *y* is independent of value of *x*
- Body size (for population, variance in body size or heart rate varies linearly with *x*)--logarithmic space

3. Self Similarity and Fractals

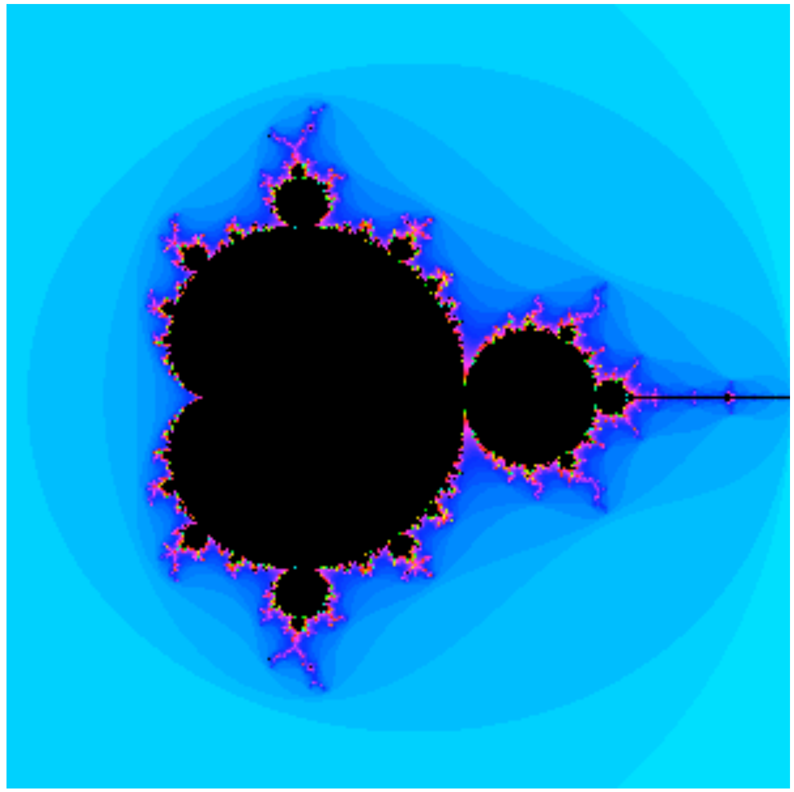
Imagine taking a picture of smaller piece and magnifying it, and then it looks like original part.



Other examples of self similarity

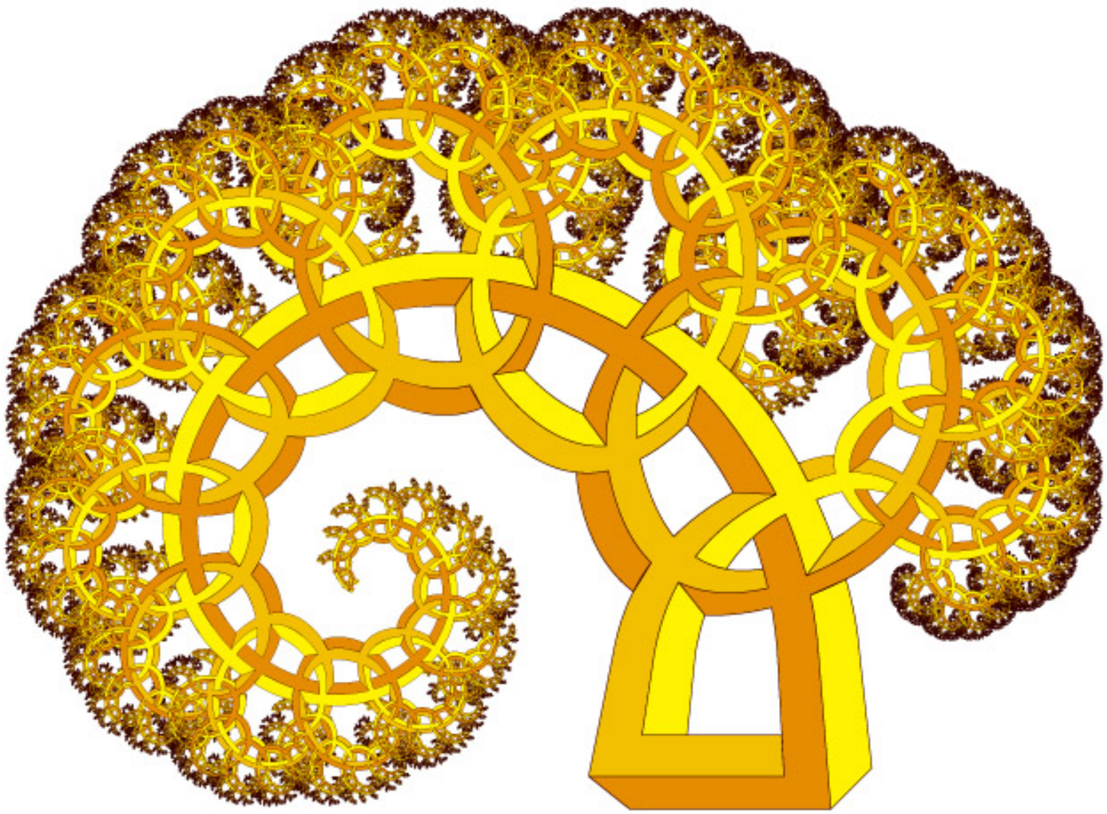


Generate complex fractals by iterating a simple pattern over and over



1. Branching process that is self similar and repeated at every scale
2. Changes our concepts of measuring distances by ruler and thus our concept of area, volumes, and dimensions
3. Watch “Hunting the Hidden Dimension” on NOVA

Can generate tree-like structures



Power Laws ⇔ Self Similarity

Equation form for self similarity: $$f(\lambda x) = \lambda^k f(x)$$

->

$$f(x) = ax^k$$

$$f(\lambda x) = a(\lambda x)^k = \lambda^k a x^k = \lambda^k f(x)$$

<-

Differentiate with respect to λ

Chain Rule

$$k\lambda^{k-1} f(x) = \frac{df(\lambda x)}{d\lambda} = \frac{d(\lambda x)}{d\lambda}\frac{df(\lambda x)}{d(\lambda x)} = x\frac{df(\lambda x)}{d(\lambda x)}$$

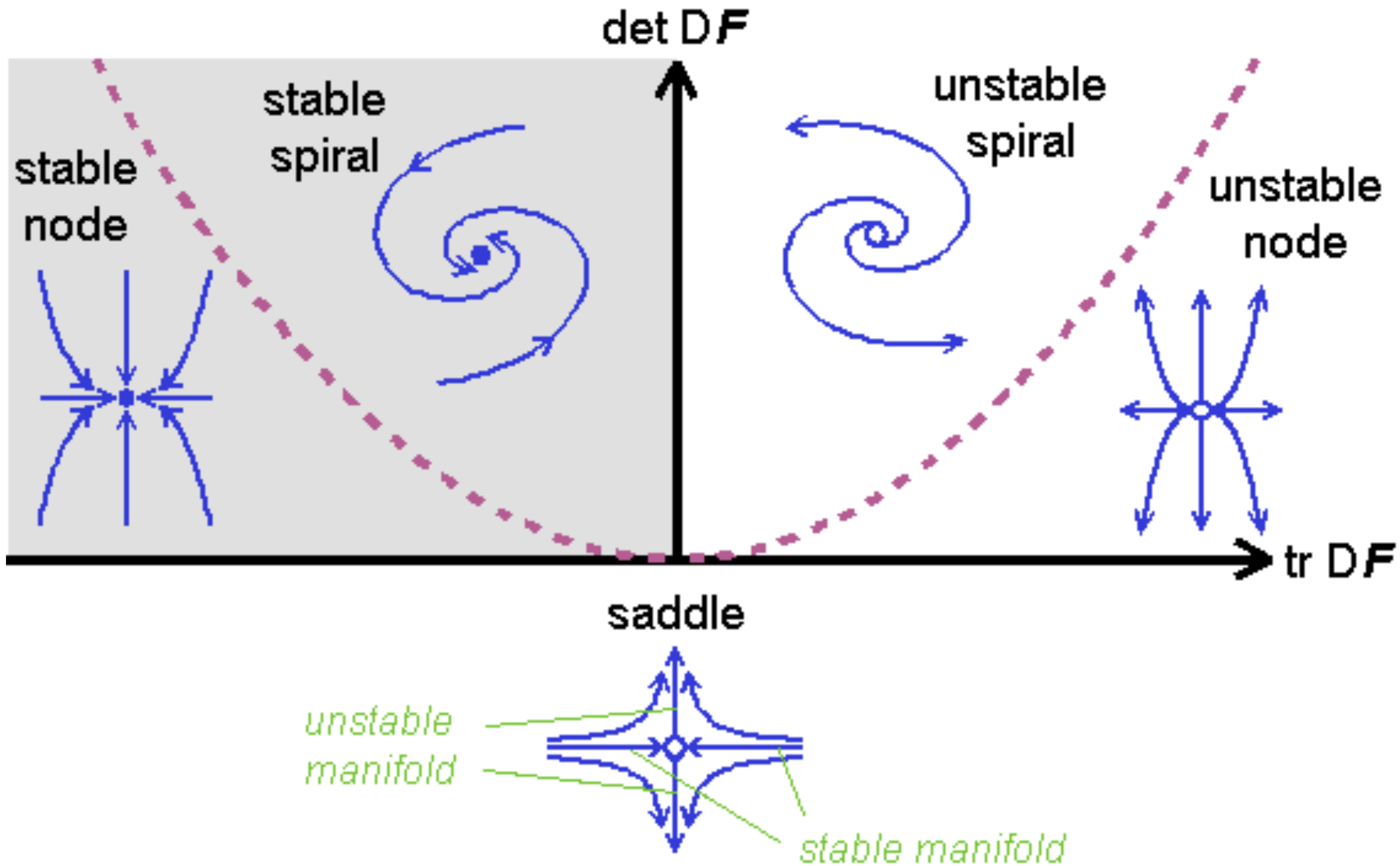
Free to choose λ=1

$$x\frac{df(x)}{dx} = kf(x) \Rightarrow \frac{df}{f} = k\frac{dx}{x}$$

$$\Rightarrow f(x) = ax^k$$

4. Fixed Points and Universality

Dynamical systems flow relative to fixed points



What is functional form as fixed point is approached? This describes region and dynamics that are relevant for many scientific questions.

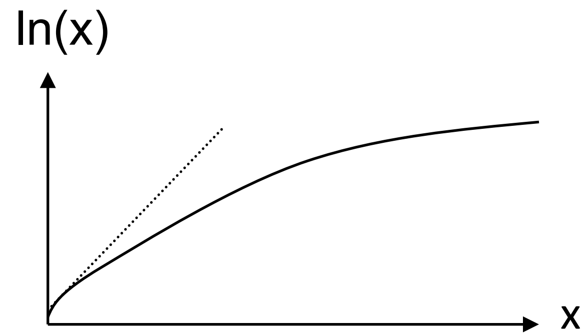
Non-power-law functions often behave as power laws near critical points

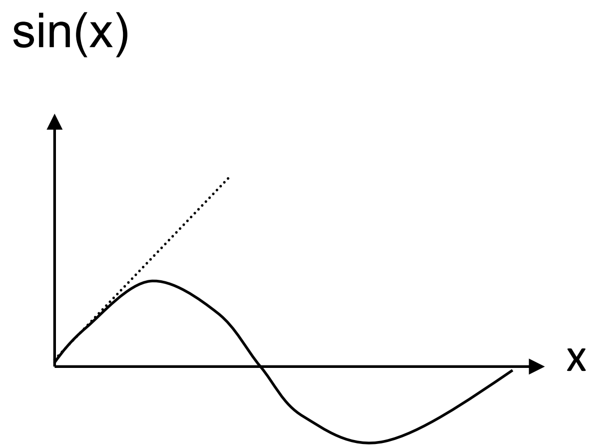
- Other functions commonly occur in nature: e^x, sin(x), cosh(x), $J_\nu(x)$, Ai(x)
- These functions can generally be expressed in Taylor or power series near critical points (phase transitions, etc.). When x is close to x*, difference is small, and first term dominates.

$$f(x) = \sum_{k=p}^{N} \frac{(x-x^*)^k}{k!} f^{(k)}(x-x^*) \sim C(x-x^*)^p$$

p-exponent of leading-order term

Any functions with the same first term in their series expansion behave the same near critical points, which is of great physical interest, even if they behave very differently elsewhere. Source of universality classes.





Near x=0, both of these functions scale like x!

III. Dimensional Analysis and Power Laws

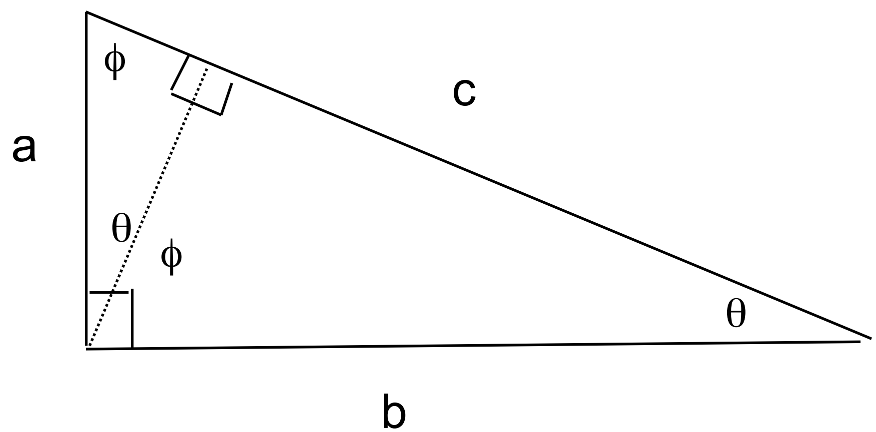
Dimensional Analysis

- Often used in physics
- For reasons given thus far, many processes should scale as a power law.
- Given some quantity, f, that we want to determine, we need to intuit what other variables on which it must depend, $\{x_1, x_2, \ldots, x_n\}$.
- Assume f depends on each of these variables as a power law.
- Use consistency of units to obtain set of equations that uniquely determine exponents.

$$f(x_1, x_2, \ldots, x_n) = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$$

Example 1: Pythagorean Theorem

- Hypotenuse, c, and smallest angle, θ, uniquely determine right triangles.
- Area=f(c, θ), DA implies Area=c^2g(θ).

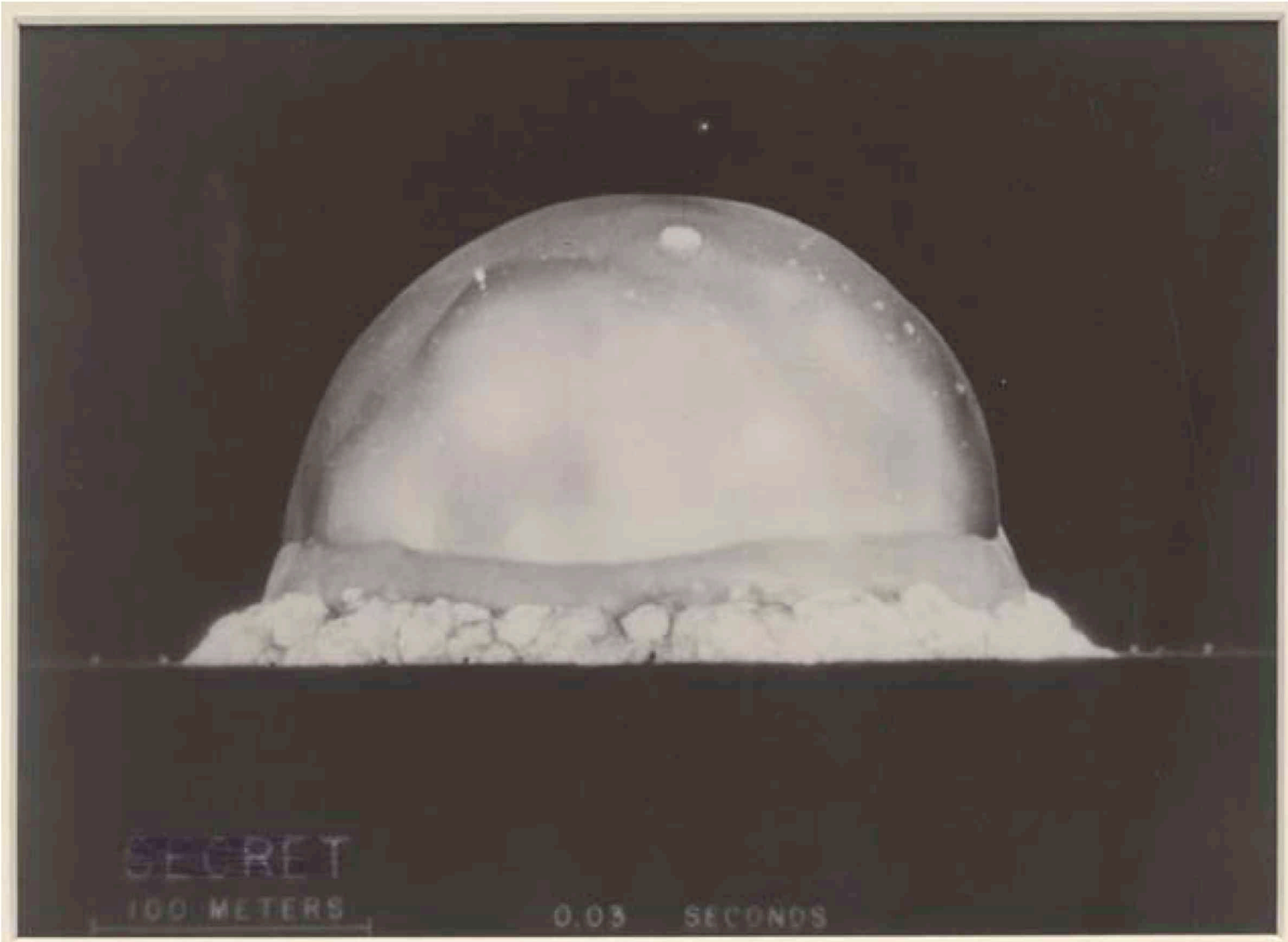


Area of whole triangle=sum of area of smaller triangles

$$a^2 g(\theta) + b^2 g(\theta) = c^2 g(\theta)$$
$$\Rightarrow a^2 + b^2 = c^2$$

Example: Nuclear Blast

- US government wanted to keep energy yield of nuclear blasts a secret.
- Pictures of nuclear blast were released in Life magazine with time stamp
- Using DA, G. I. Taylor determined energy of blast and government was upset because they thought there had been a leak of information


SECRET
100 METERS
0.03 SECONDS

- Radius, R, of blast depends on time since explosion, t, energy of explosion, E, and density of medium, ρ, that explosion expands into
- [R]=m, [t]=s, [E]=kg*m²/s², ρ=kg/m³
- $R = t^p E^q \rho^k$

$$1 = 2q - 3k \quad \text{m}$$

$$0 = p - 2q \quad \text{s}$$

$$0 = q + k \quad \text{kg}$$

q=1/5, k=-1/5, p=2/5

$$R = (E/\rho)^{1/5} t^{2/5} \Rightarrow E = \frac{R^5 \rho}{t^2}$$

unknown constant coefficient can be determined from y-intercept of regression of log-log plot of time series

Pitfalls of Dimensional Analysis

- Miss constant factors
- Miss dimensionless ratios
- But, can get far with a good bit of ignorance!!!

Summary

- Self-similarity and fractals $\Leftrightarrow$ Power Laws
- Behavior near critical point $\Rightarrow$ Power Laws
- But, Power Laws $\not\Rightarrow$ near critical points
- Dimensional Analysis assumes power law form and this is partially justified by necessity of matching units

The essence of mathematics is not to make simple things complicated, but to make complicated things simple.--S. Gudder

A little philosophy of science

- Many major, “universal” patterns in different fields are power laws
- Can often explain these without knowledge of all the details of the system
- Art of science is knowing system well enough to have intuition about which details are important

Explaining existence of single power law is not enough

- Much better to predict value of exponent and not just that it is a power law
- To really believe a theory we need multiple pieces of evidence (possibly multiple power laws) and need to be able to predict many of these.
- Understanding dynamics and some further details allows one to predict deviations from power law, and that is a very strong test and leads to very precise results

II. Power laws in biology

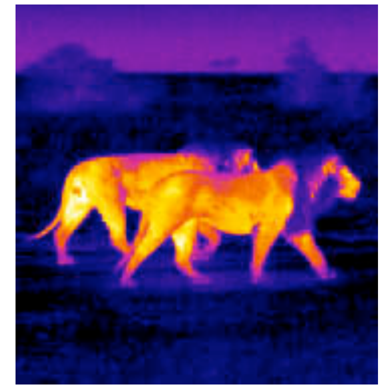
Metabolic rate--“Fire of life”--power for maintenance, growth, and reproduction

heat loss:

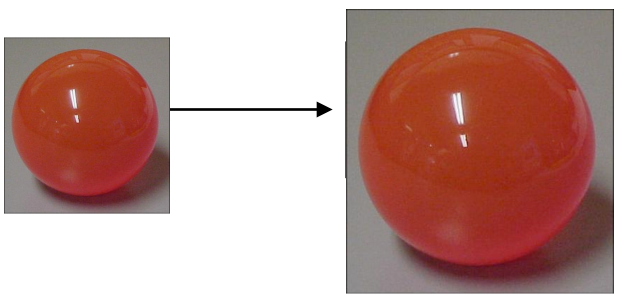
metabolic rate=heat loss rate $\propto$ surface area

↑

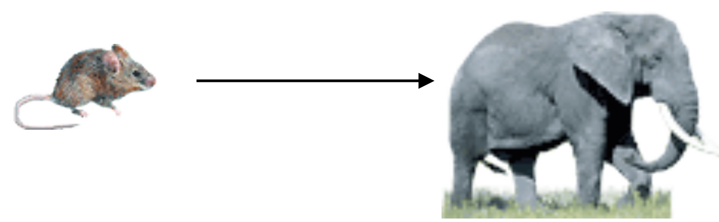
Stefan-Boltzmann law



isometry--shape stays same



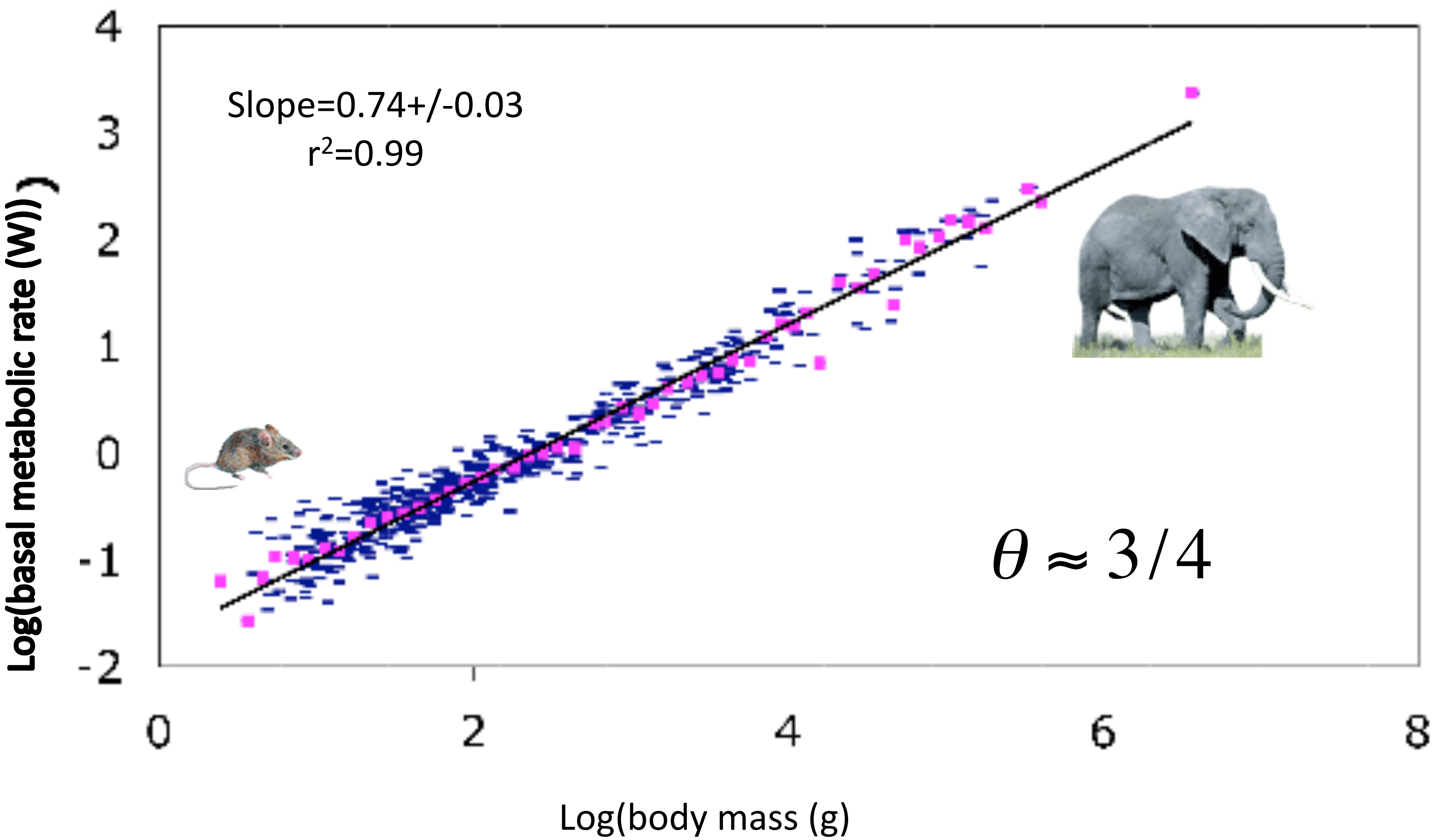
allometry--shape changes



$\Rightarrow$metabolic rate=heat loss rate $\propto$ surface area $\propto$ (volume)$^{2/3}$

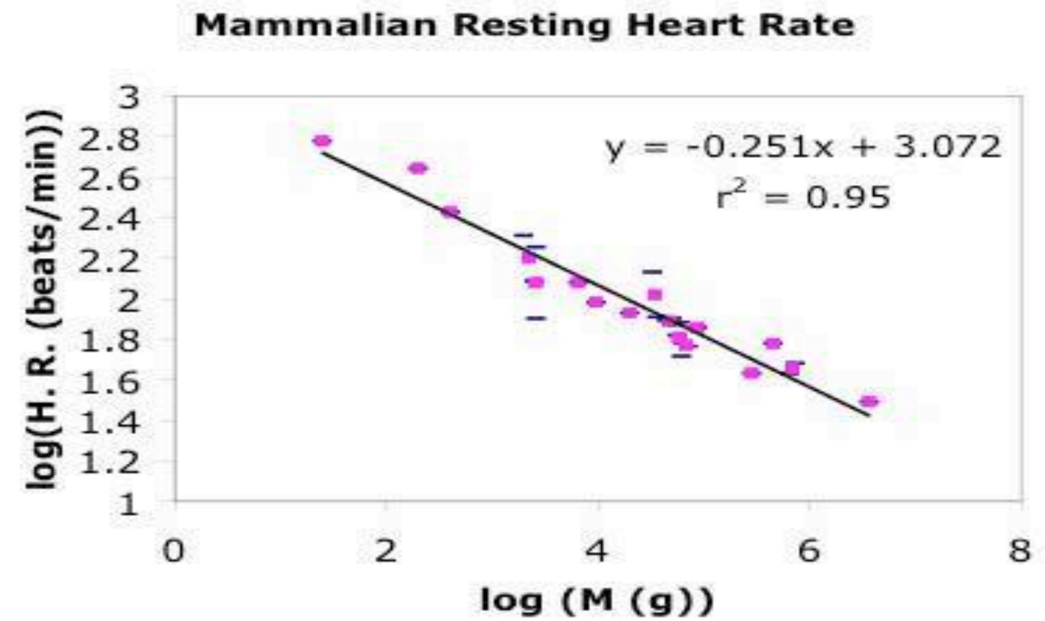
Animals are not isometric, so not a good null hypothesis.

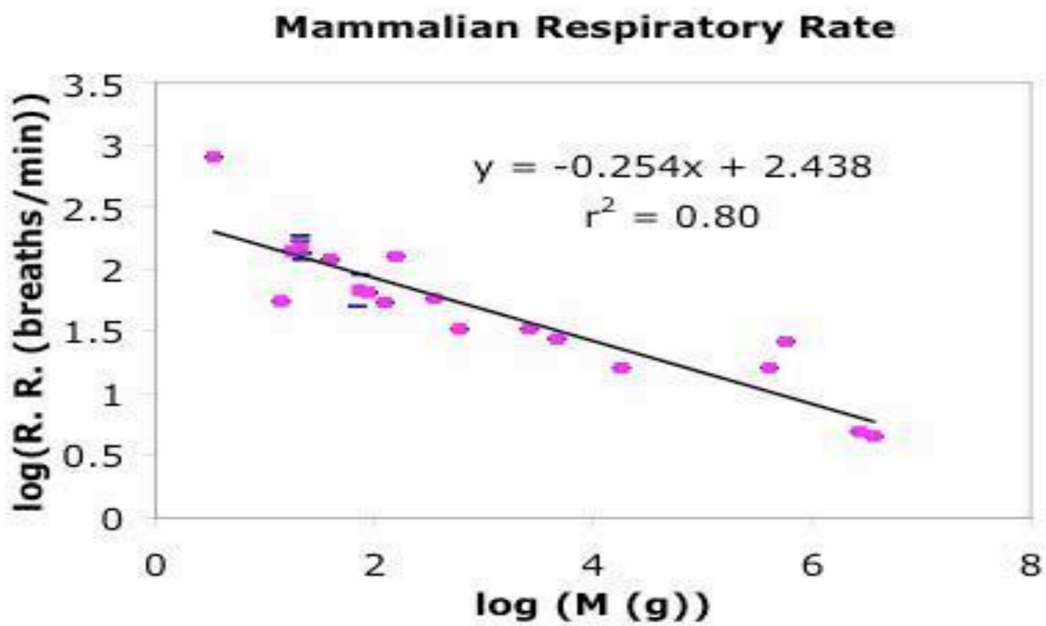
Mass dependence of metabolic rate



Savage et al., *Func. Eco.*, 2004

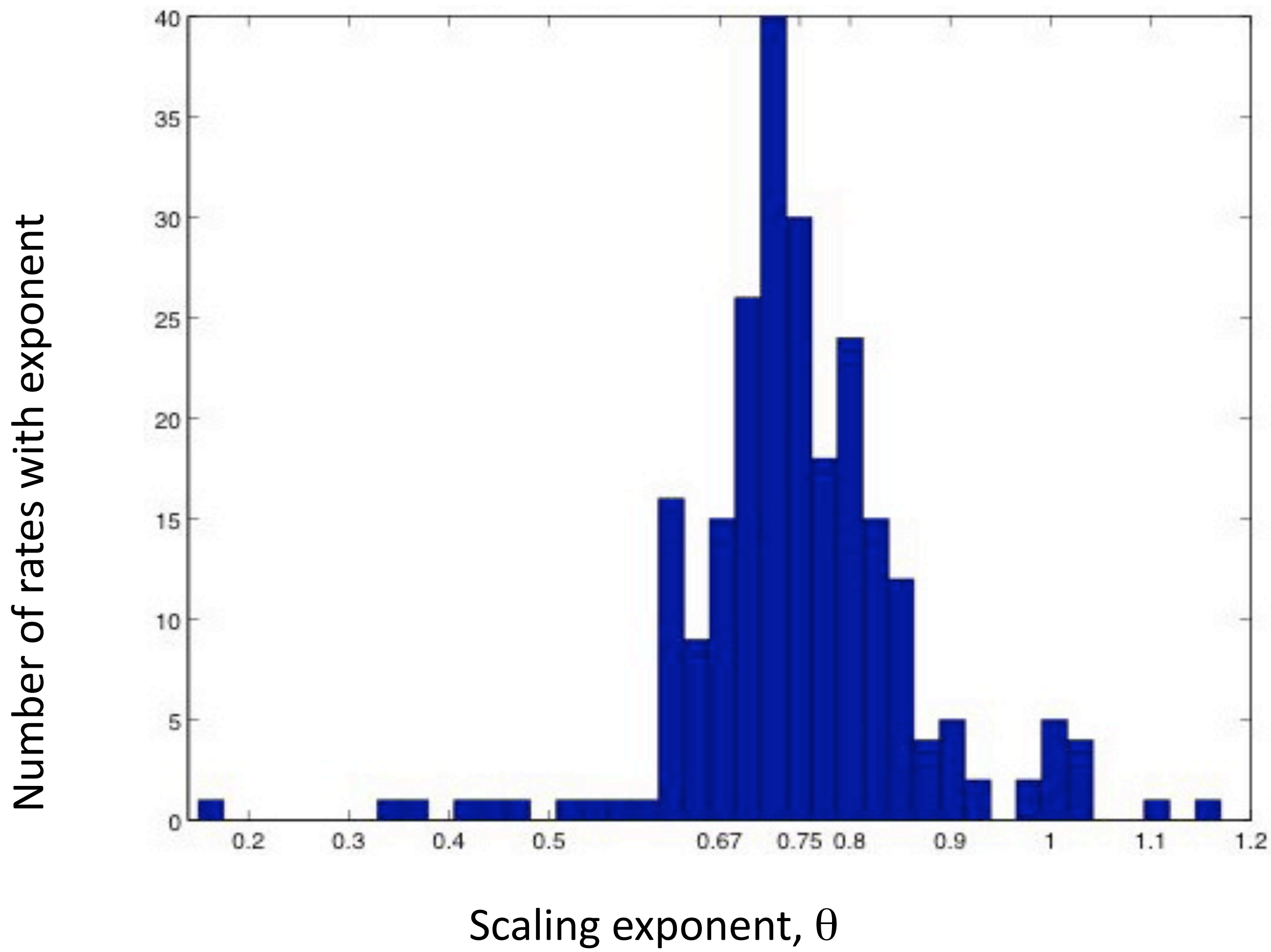
Typical Mass-Specific Rates





Savage, et al.,
Func. Eco., 2004

3/4-power scaling



Rates at the cellular, individual, and population level for many different taxa scale like this. Many times and lengths also scale.

Savage et al., *Func. Eco.*, 2004

Power laws in vascular networks

The pig: a case study

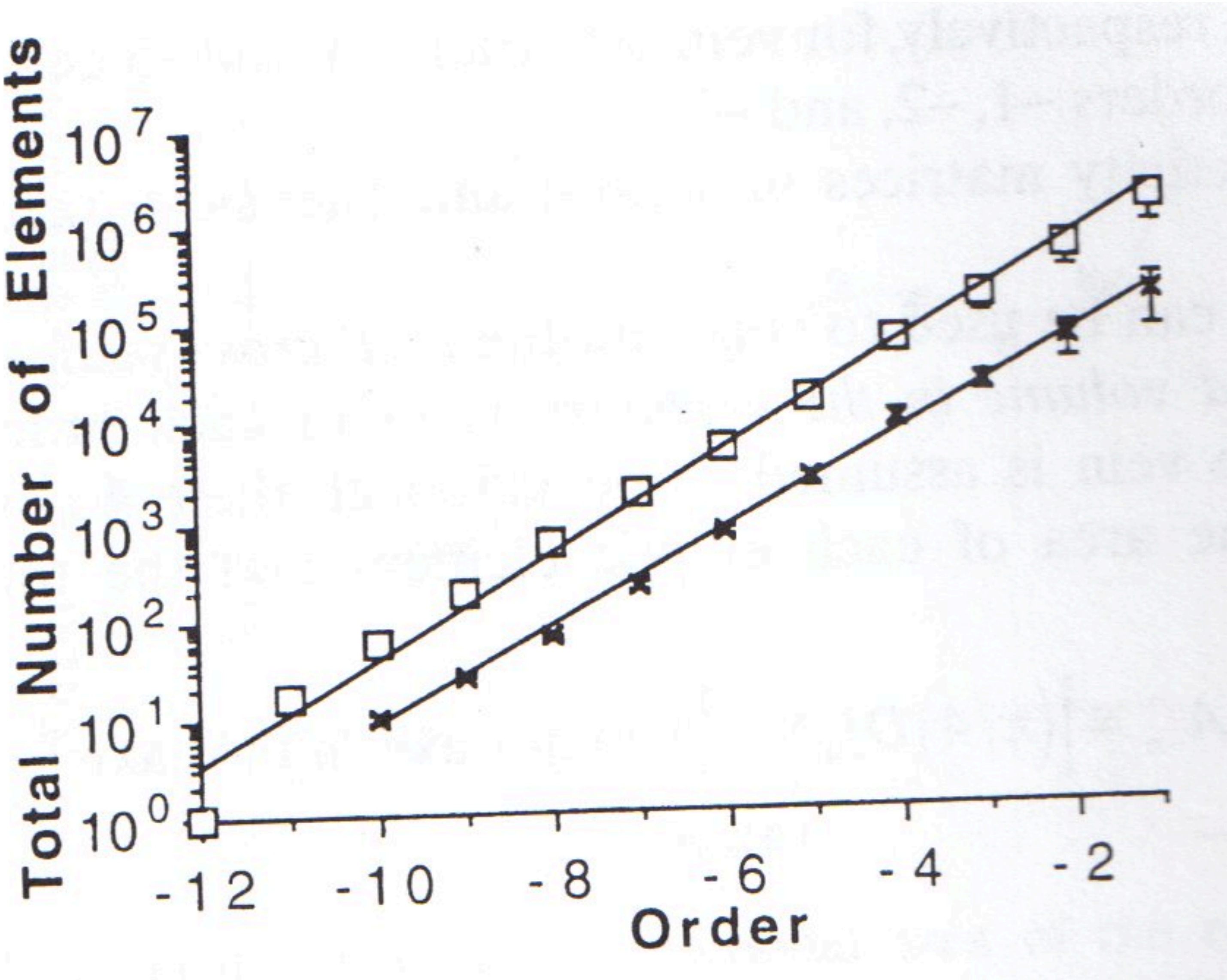
Properties of pig vascular systems: Capillary numbers

	Right coronary artery	Left anterior descending coronary artery
Arterial capillaries	1,187,194 ± 515,552	1,273,281 ± 813,674
Arterial capillaries	Whole heart	3,018,384 ± 1,697,777
Venous capillaries	Whole heart	5,085,977 ± 2,085,250

Properties of pig vascular systems: Number of units

Order	RCA Number of vessel elements	LAD Number of vessel elements	LCX Number of vessel elements	and LCCA Number of vessel elements
11	1	1		2
10	10	7	1	18
9	35	37 ± 2	10	83 ± 2
8	114 ± 1	113 ± 9	51	283 ± 11
7	403 ± 5	348 ± 32	144 ± 4	909 ± 44
6	1,458 ± 44	1,385 ± 162	638 ± 51	3,524 ± 247
5	7,354 ± 649	6,386 ± 11,052	2,148 ± 312	16,093 ± 2,117
4	20,074 ± 3,739	17,985 ± 5,676	7,554 ± 2,338	46,194 ± 12,089
3	51,915 ± 13,644	44,456 ± 19,672	17,820 ± 8,001	115,638 ± 42,301
2	138,050 ± 46,070	140,293 ± 72,949	56,915 ± 29,829	339,873 ± 152,326
1	393,294 ± 158,657	368,554 ± 221,134	149,386 ± 90,276	923,339 ± 480,169

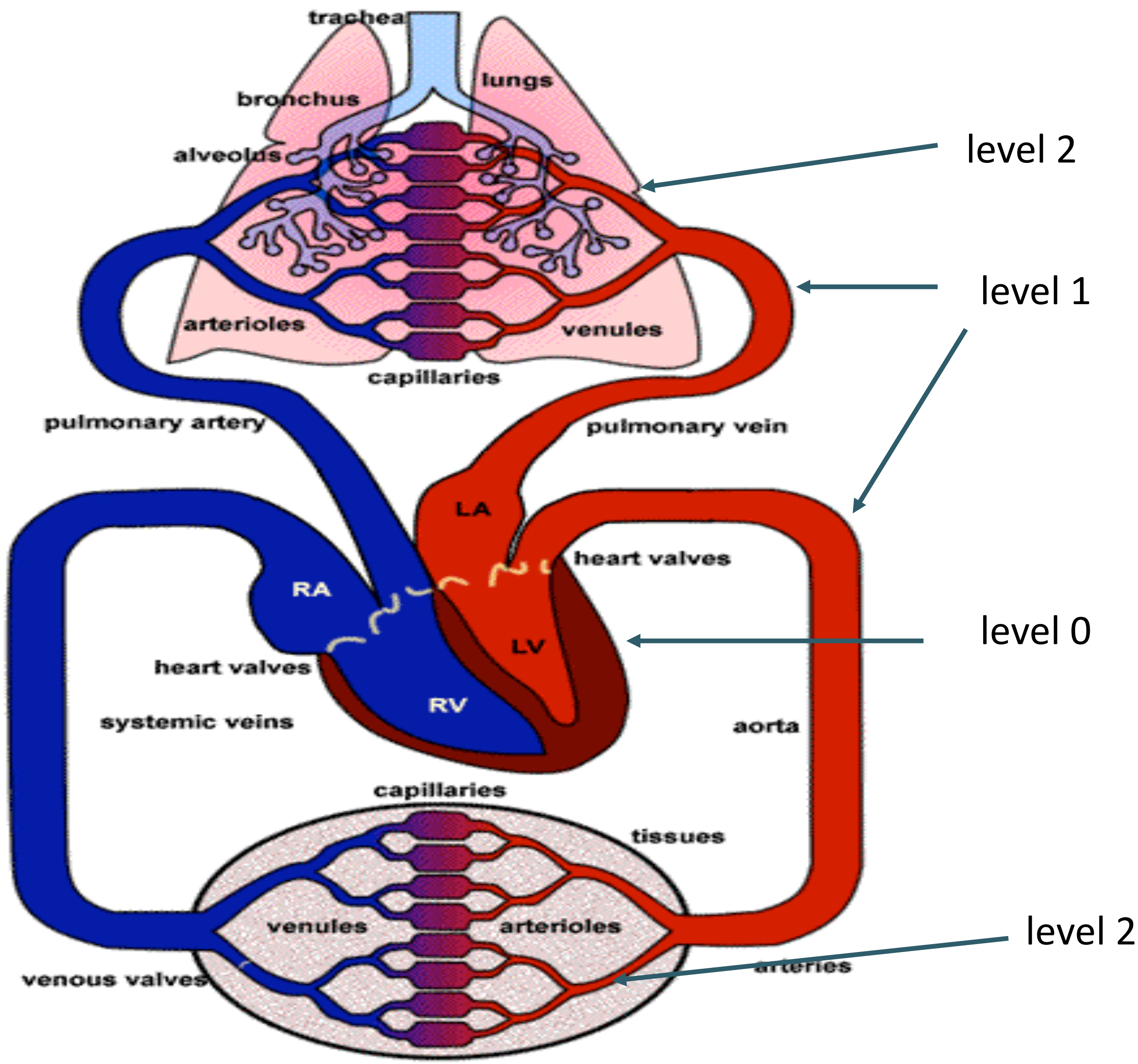
Scaling for number of elements in pig



Focus on hierarchical, branching network

branching
ratio

$$n = 2$$



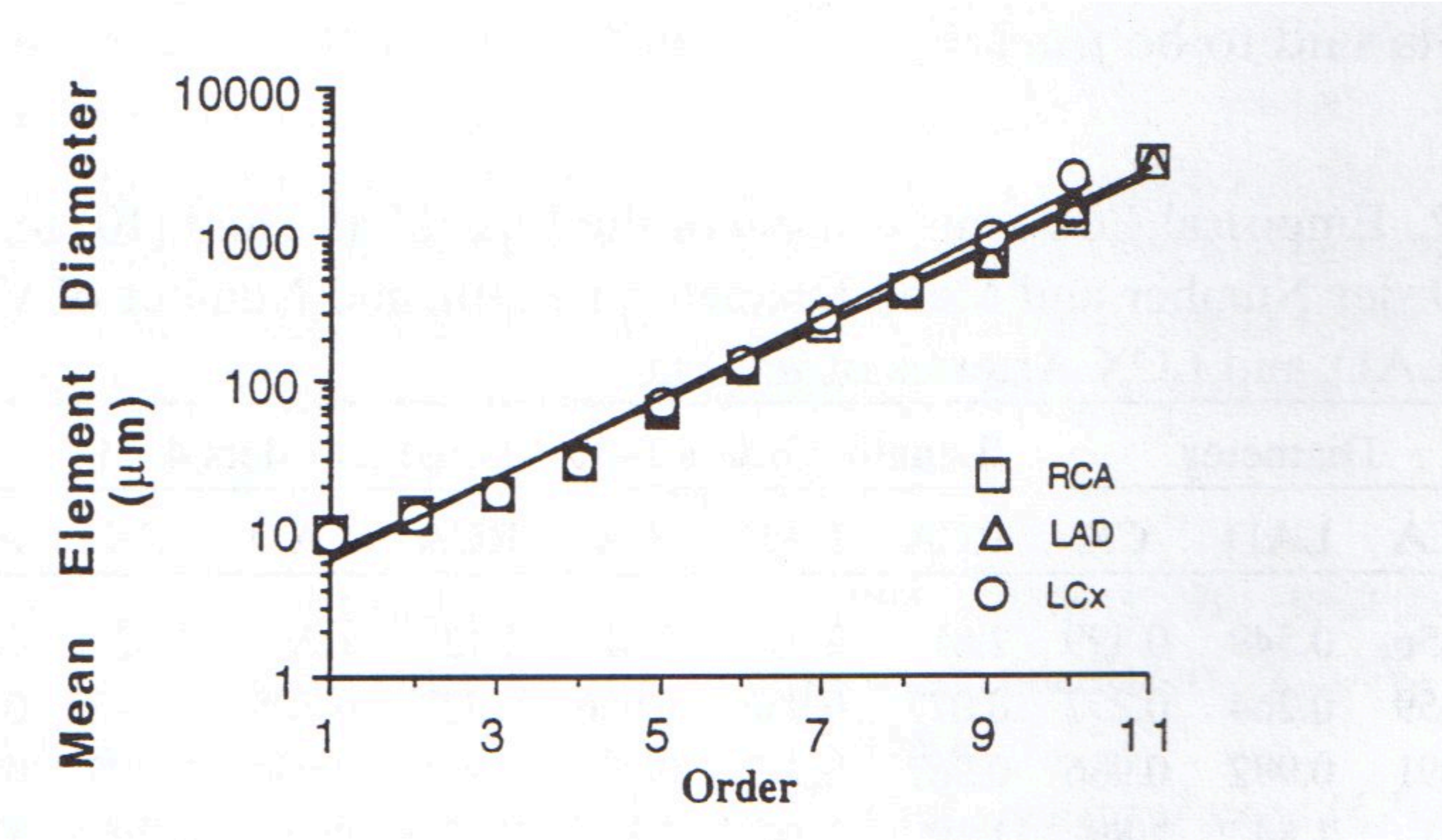
Properties of pig vascular systems: Geometry of vessels

TABLE 7.2:1. Diameters and Lengths (Mean ± SD) of Vessel Elements of Each Order in the LAD, LCX, and RCA Arteries of the Pig

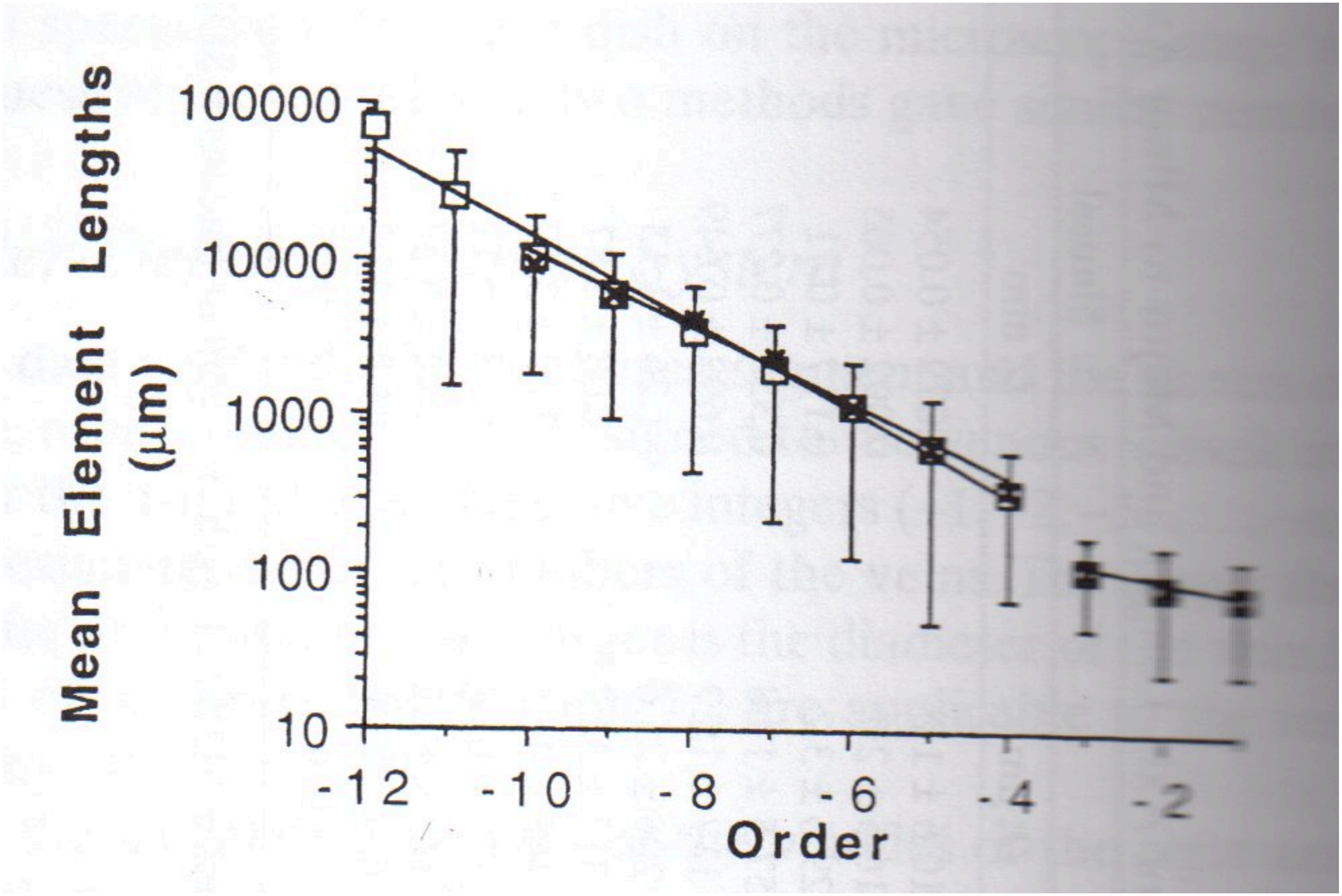
	LAD		LCX		RCA	
Order	Diameter (μm)	Length (mm)	Diameter (μm)	Length (mm)	Diameter (μm)	Length (mm)
1	9.0 ± 0.73	0.115 ± 0.066	9.0 ± 0.073	0.115 ± 0.066	9.3 ± 0.84	0.125 ± 0.084
2	12.3 ± 1.3	0.136 ± 0.088	12.3 ± 1.3	0.136 ± 0.088	12.8 ± 1.4	0.141 ± 0.103
3	17.7 ± 2.2	0.149 ± 0.104	17.7 ± 2.2	0.149 ± 0.104	17.7 ± 2.1	0.178 ± 0.105
4	30.5 ± 6.0	0.353 ± 0.154	27.5 ± 6.1	0.405 ± 0.170	28.6 ± 5.4	0.253 ± 0.174
5	66.2 ± 13.6	0.502 ± 0.349	73.2 ± 14.2	0.908 ± 0.763	63.1 ± 11.3	0.545 ± 0.415
6	139 ± 24.1	1.31 ± 0.914	139 ± 26.2	1.83 ± 1.34	132 ± 22.2	1.64 ± 1.13
7	308 ± 56.6	3.54 ± 2.11	279 ± 38.4	4.22 ± 2.26	256 ± 30.1	3.13 ± 2.11
8	462 ± 40.9	4.99 ± 3.02	462 ± 56.1	6.98 ± 3.92	428 ± 47.5	5.99 ± 3.53
9	714 ± 81.8	9.03 ± 6.13	961 ± 193	21.0 ± 15.6	706 ± 75.2	9.06 ± 5.56
10	1,573 ± 361	20.3 ± 17.9	2,549	47.5	1,302 ± 239	16.1 ± 13.3
11	3,171	45.9			3,218	78.1

LAD = left anterior descending artery; LCX = circumflex artery; RCA = right coronary artery. Data from Kassab et al. (1993a).

Scaling for vessel radii in pig



Scaling for vessel lengths in pig



Approximate power laws exist for structure of pig vascular system

III. Base metabolic scaling model

Theories are approximations that hope to impart deeper understanding

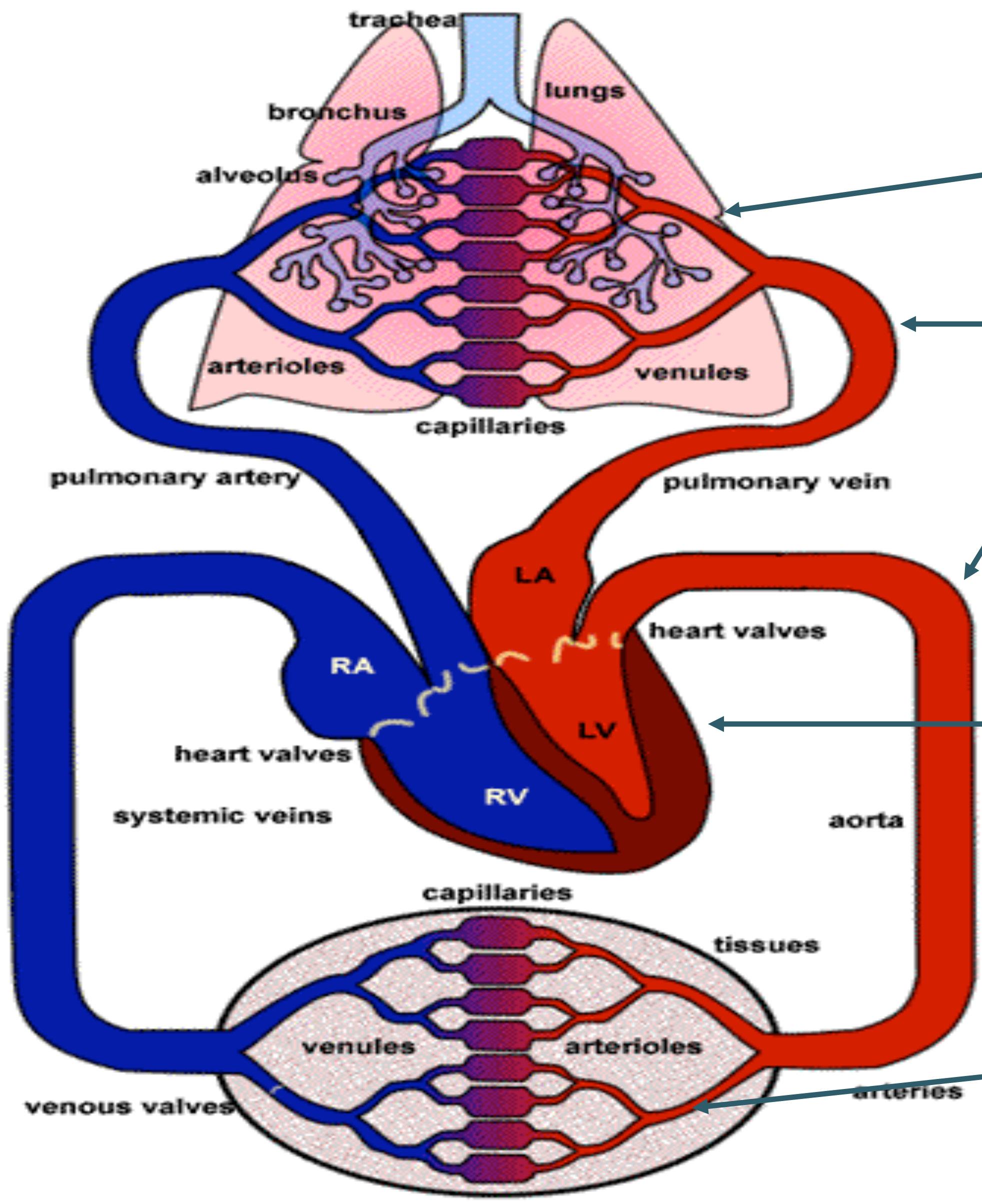
We all know that art [theory] is not truth. Art [theory] is a lie that makes us realize truth, at least the truth that is given us to understand. The artist [theorist] must know the manner whereby to convince others of the truthfulness of his lies.

--Pablo Picasso

Focus on hierarchical, branching network

branching
ratio

$$n = 2$$



level 2

level 1

level 0

level 2

Model has three assumptions

i. Minimization of energy to deliver resources

constraint on vessel radii

ii. Space filling to feed all cells

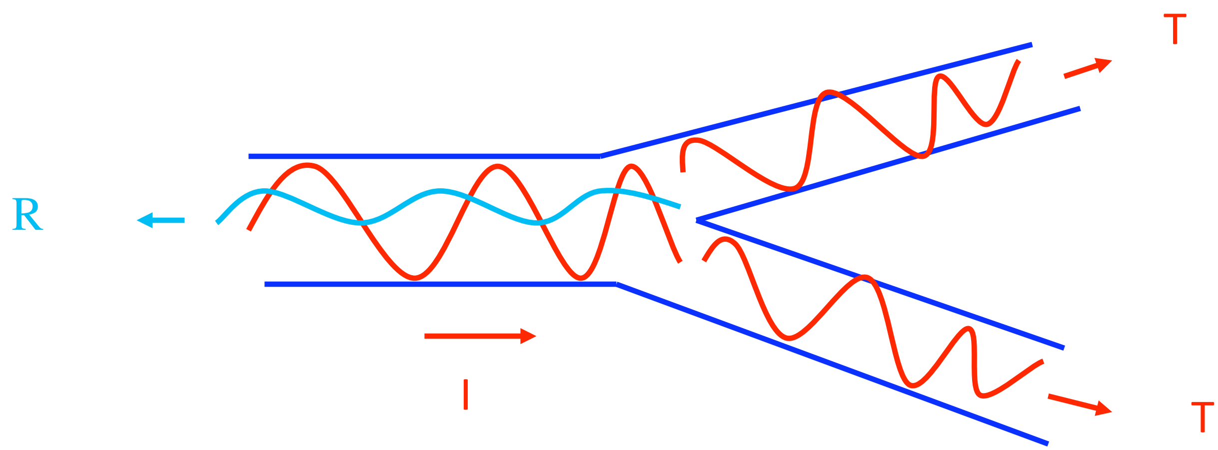
constraint on vessel lengths

iii. Capillaries are invariant in size

sets overall scale for network

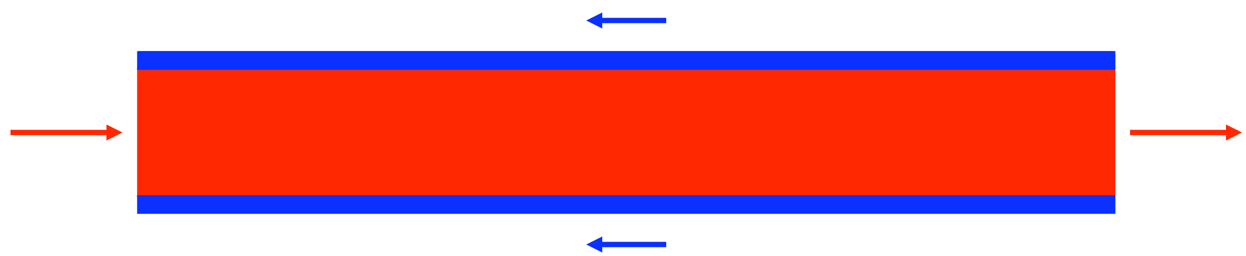
i. Energy minimization

Reflection at junctions (important for larger vessels, pulsatile flow)

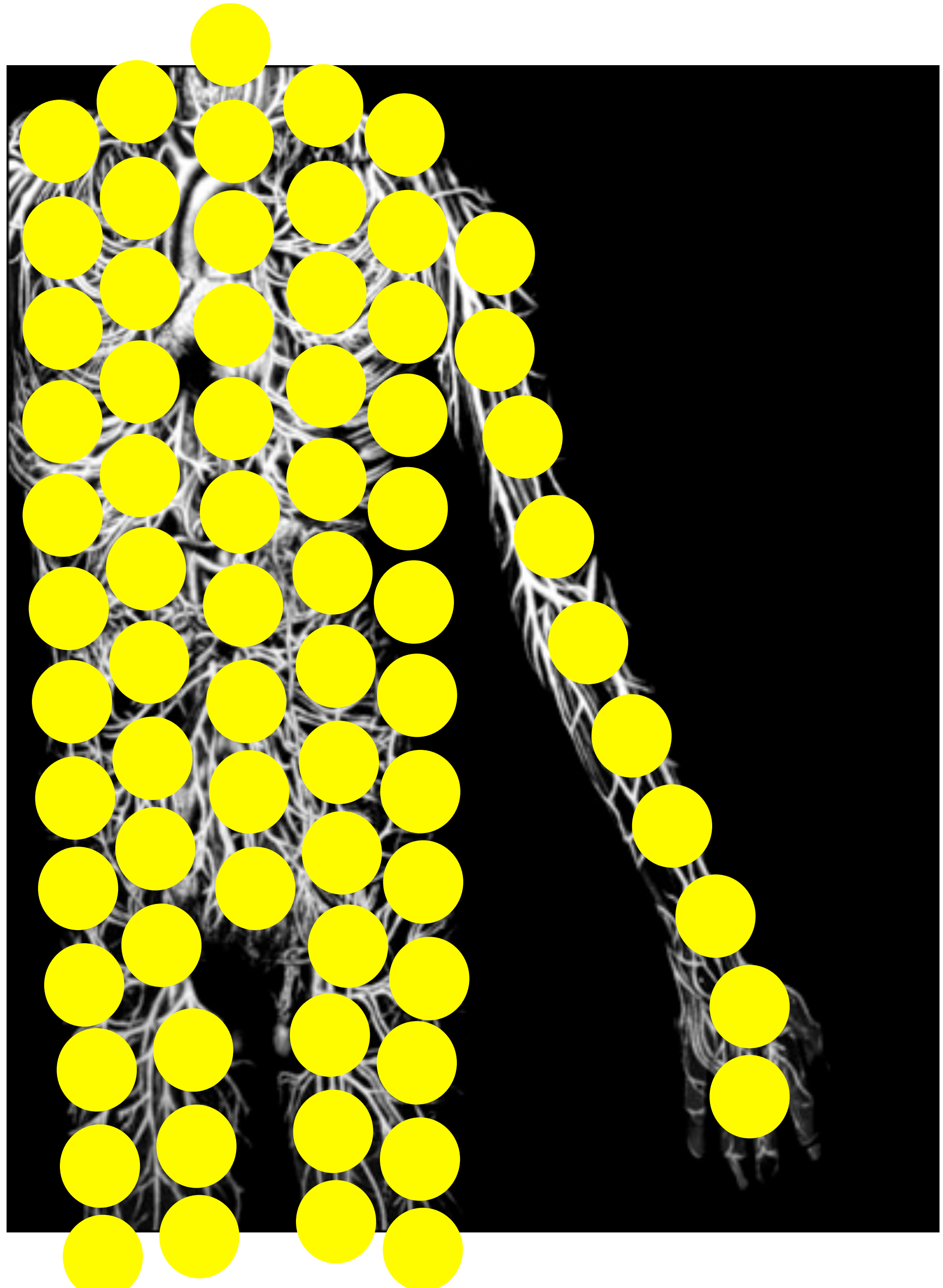


$$\beta = \frac{r_{k+1}}{r_k} = n^{-1/2}$$ ← area preserving

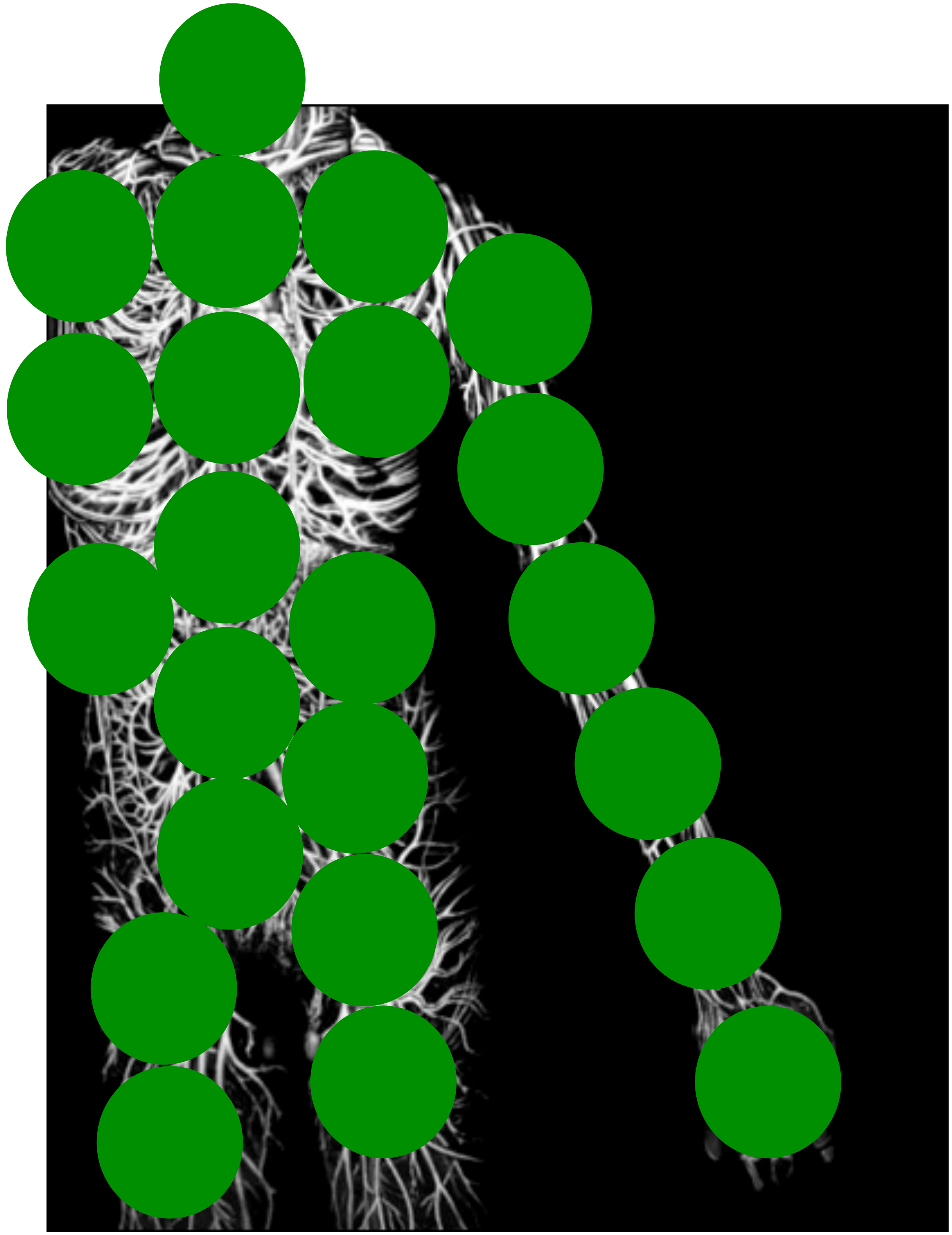
Dissipation (important for small vessels, Poiseuille flow)



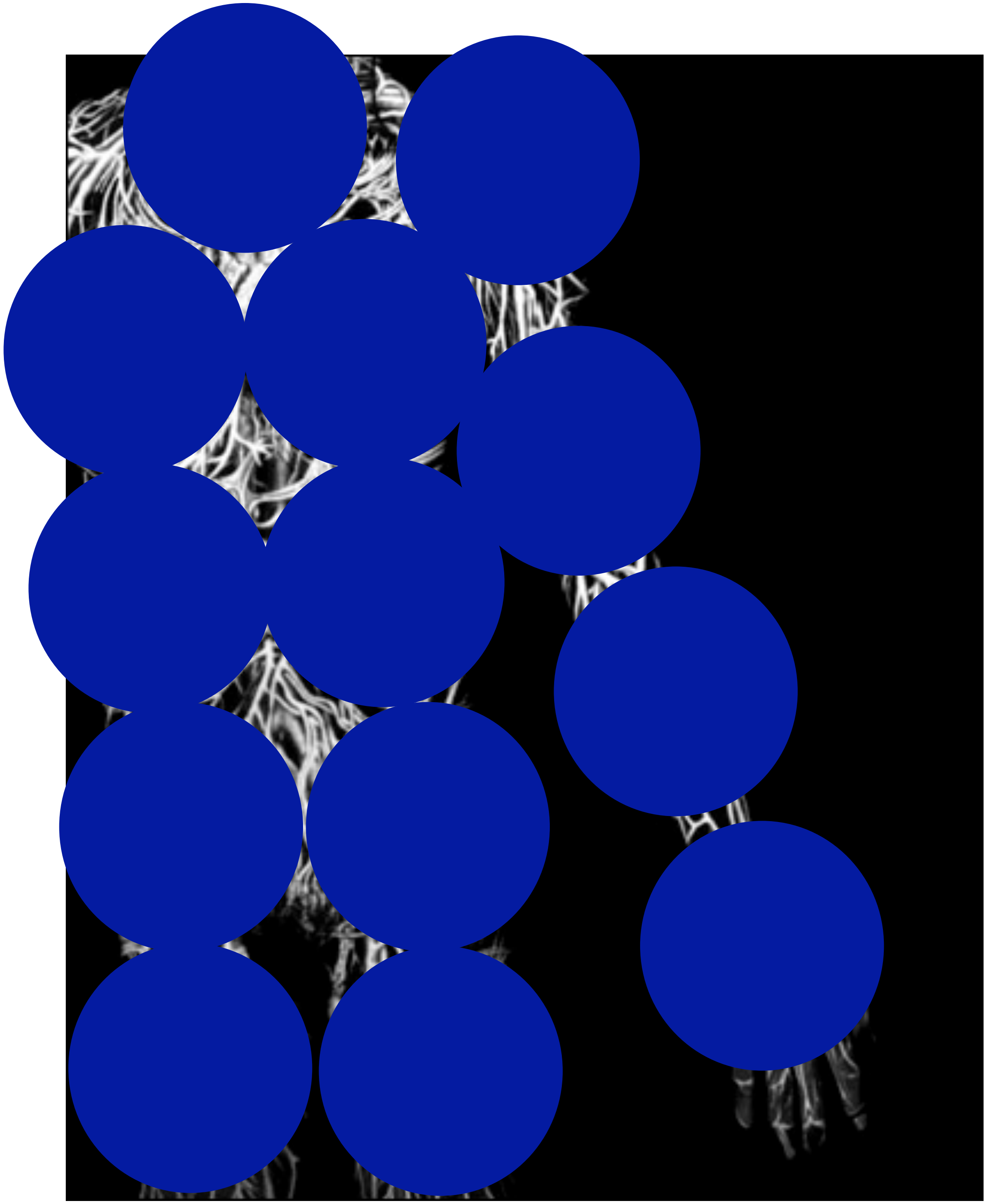
$$\beta = \frac{r_{k+1}}{r_k} = n^{-1/3}$$ ← area increasing



Space filling

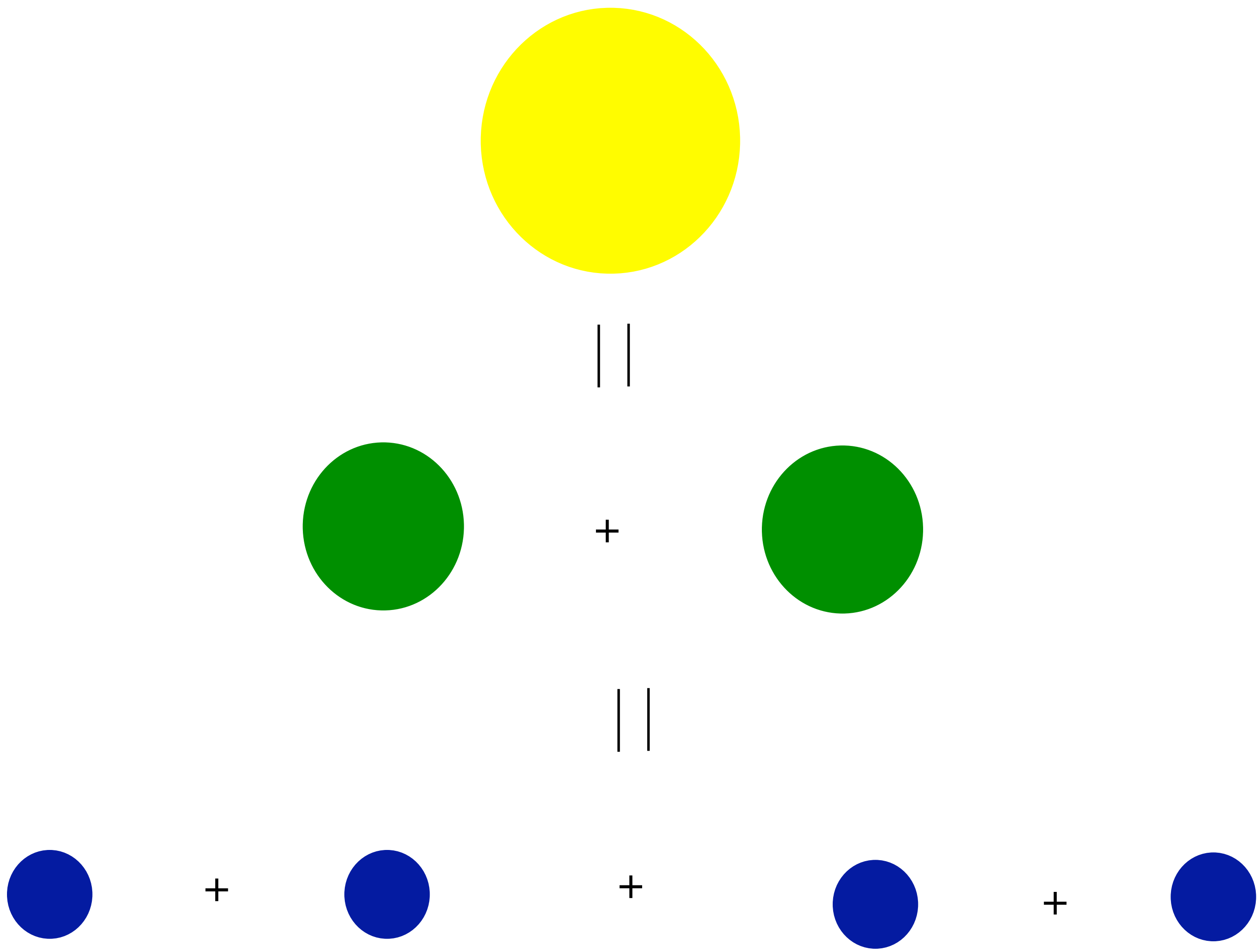


Space filling



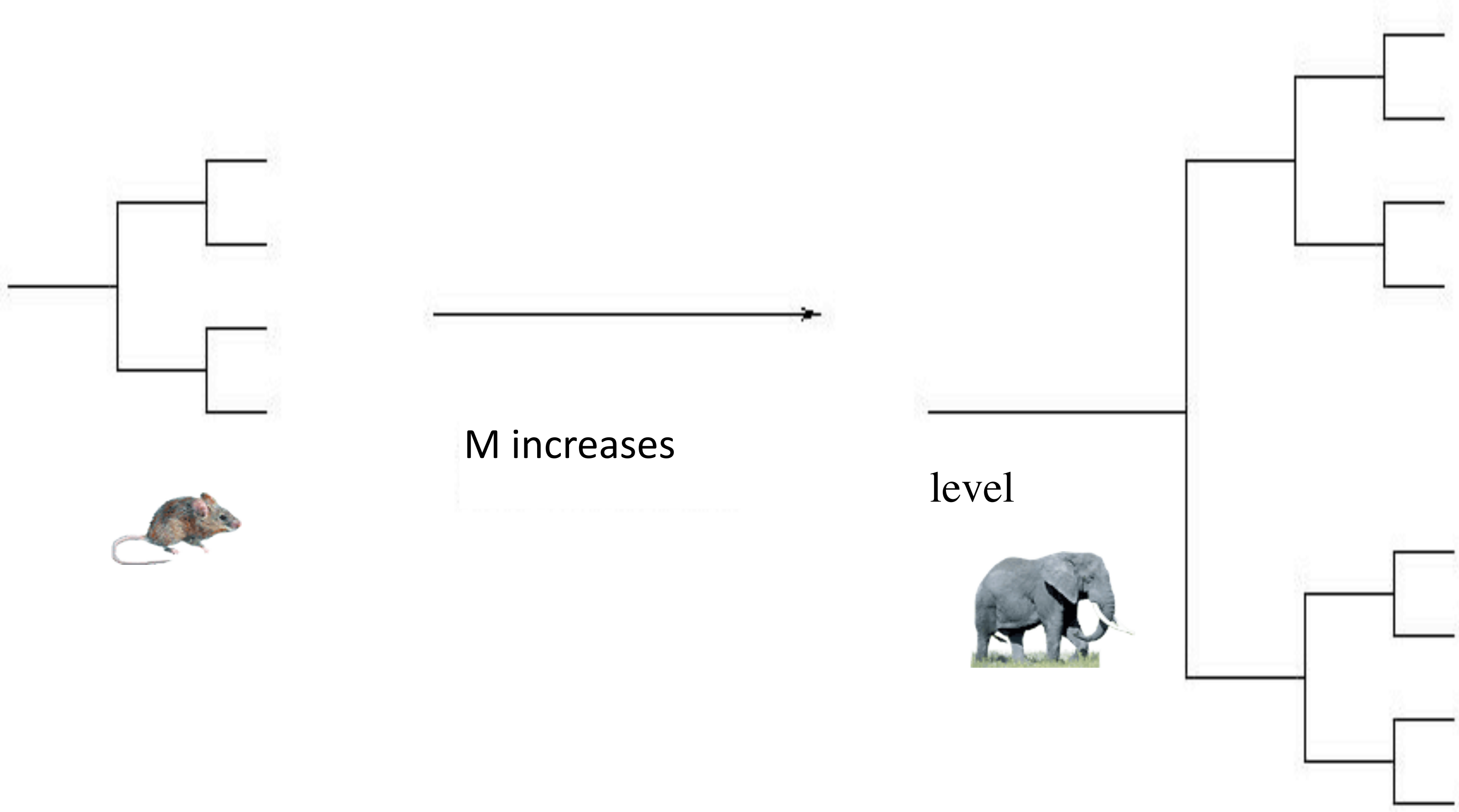
Space filling

Space Filling



$$N_k C l_k^3 = N_{k+1} C l_{k+1}^3 \longrightarrow \gamma = \frac{l_{k+1}}{l_k} = \left(\frac{N_k}{N_{k+1}}\right)^{1/3} = n^{-1/3}$$

iii. Terminal units are invariant



Body size changes network size

Metabolic rate and body mass

number of capillaries

capillary volume

blood volume

body mass

$$M \propto V_{blood} = \sum_{levels} N_k \pi r_k^2 l_k \propto N_{cap}^{4/3} V_{cap} \propto B^{4/3}$$

met rate

leading order

follows from energy min. and space filling

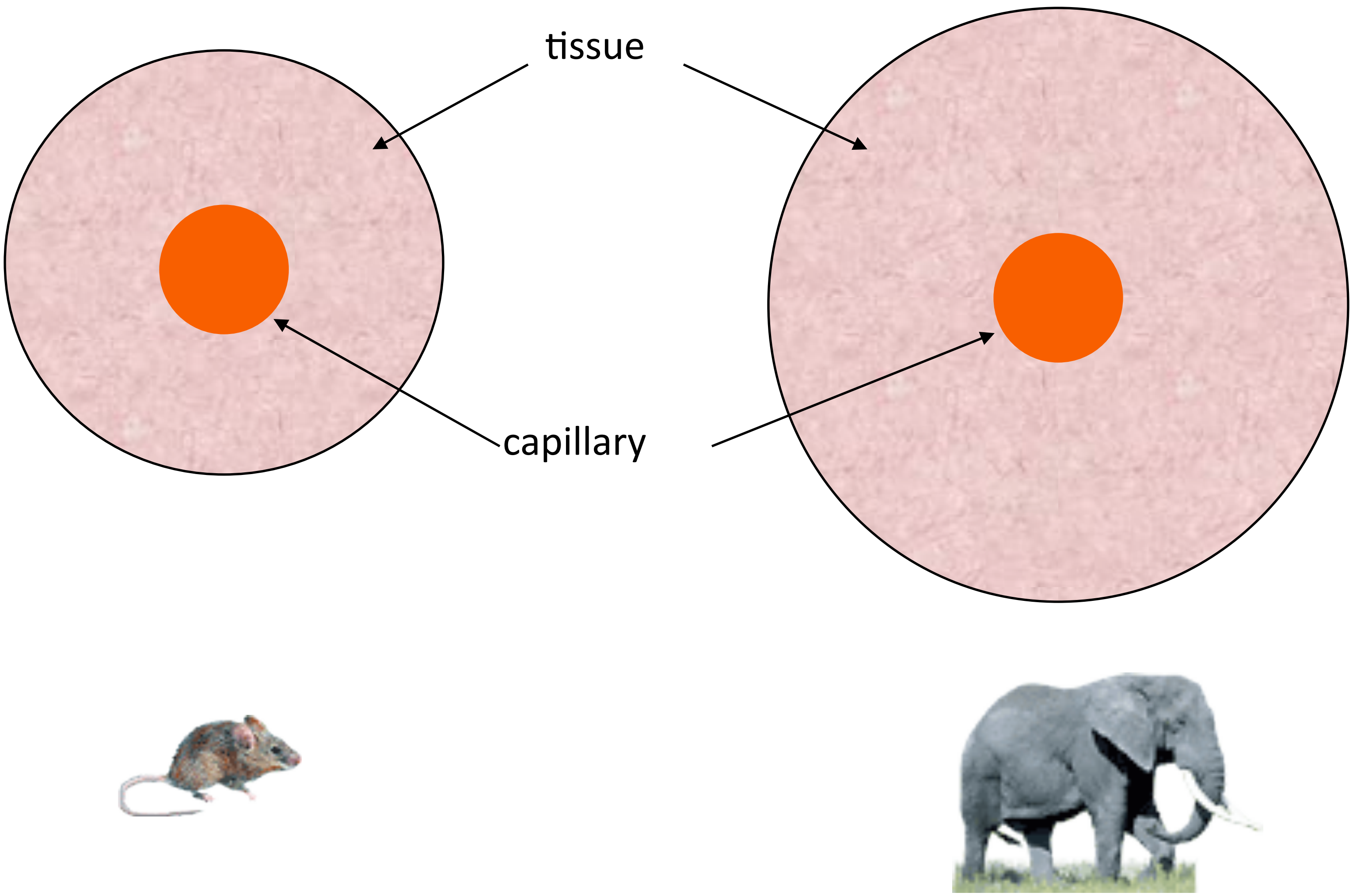
use scale factors to relate each level to terminal units

invariance of terminal units $B=N_{cap}B_{cap}$

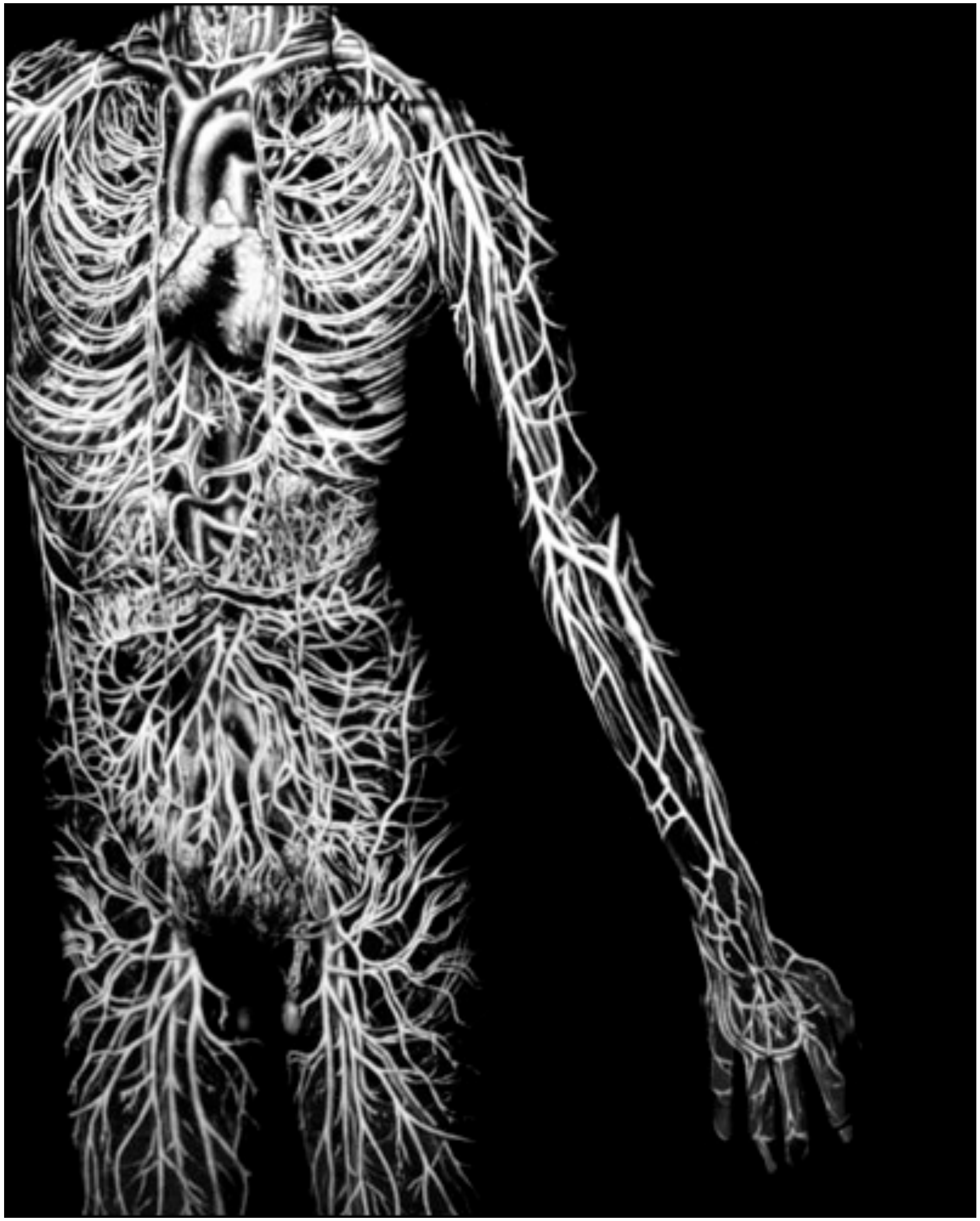
$$B \propto M^{3/4}$$

West et al., *Science,* 1997

Capillary density changes
Each capillary feeds more cells in larger organisms

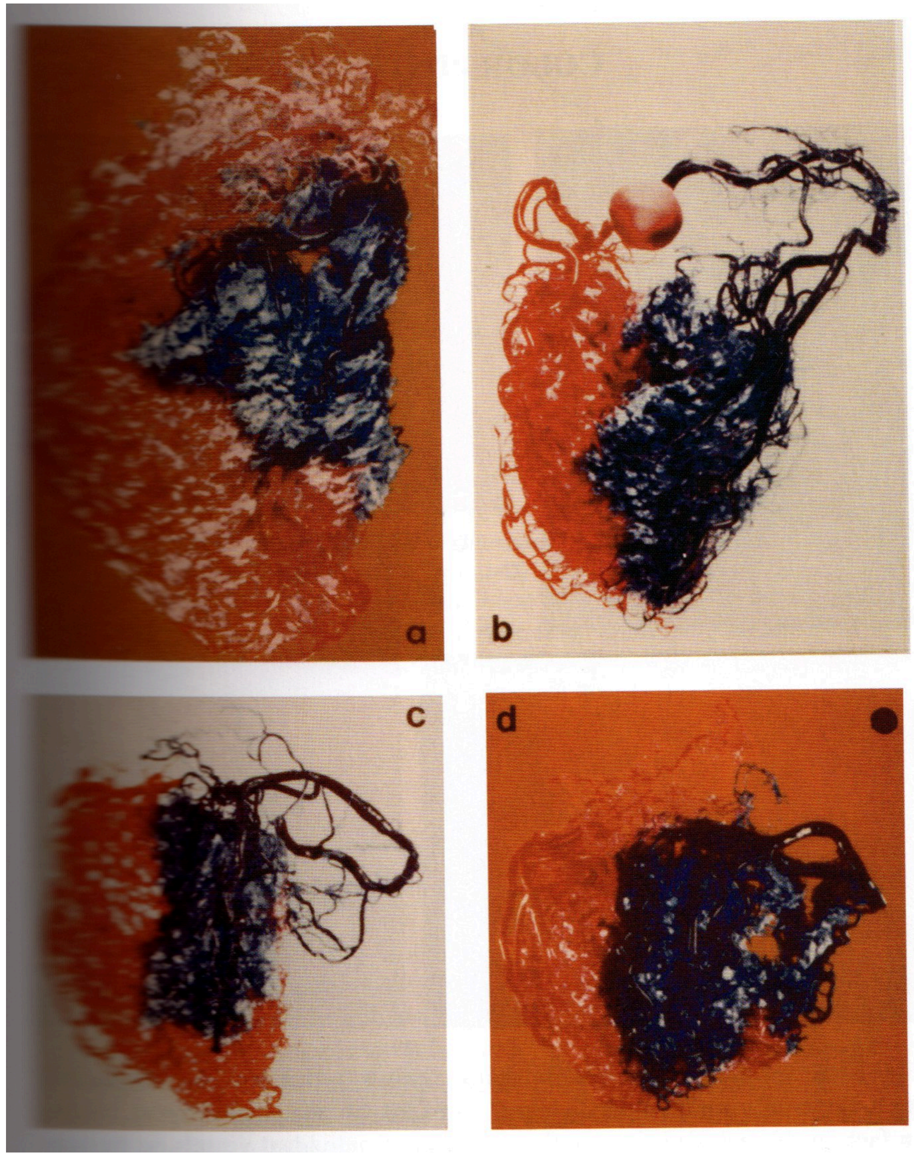


What do we mean by vascular networks? What do they look like and what are there properties?

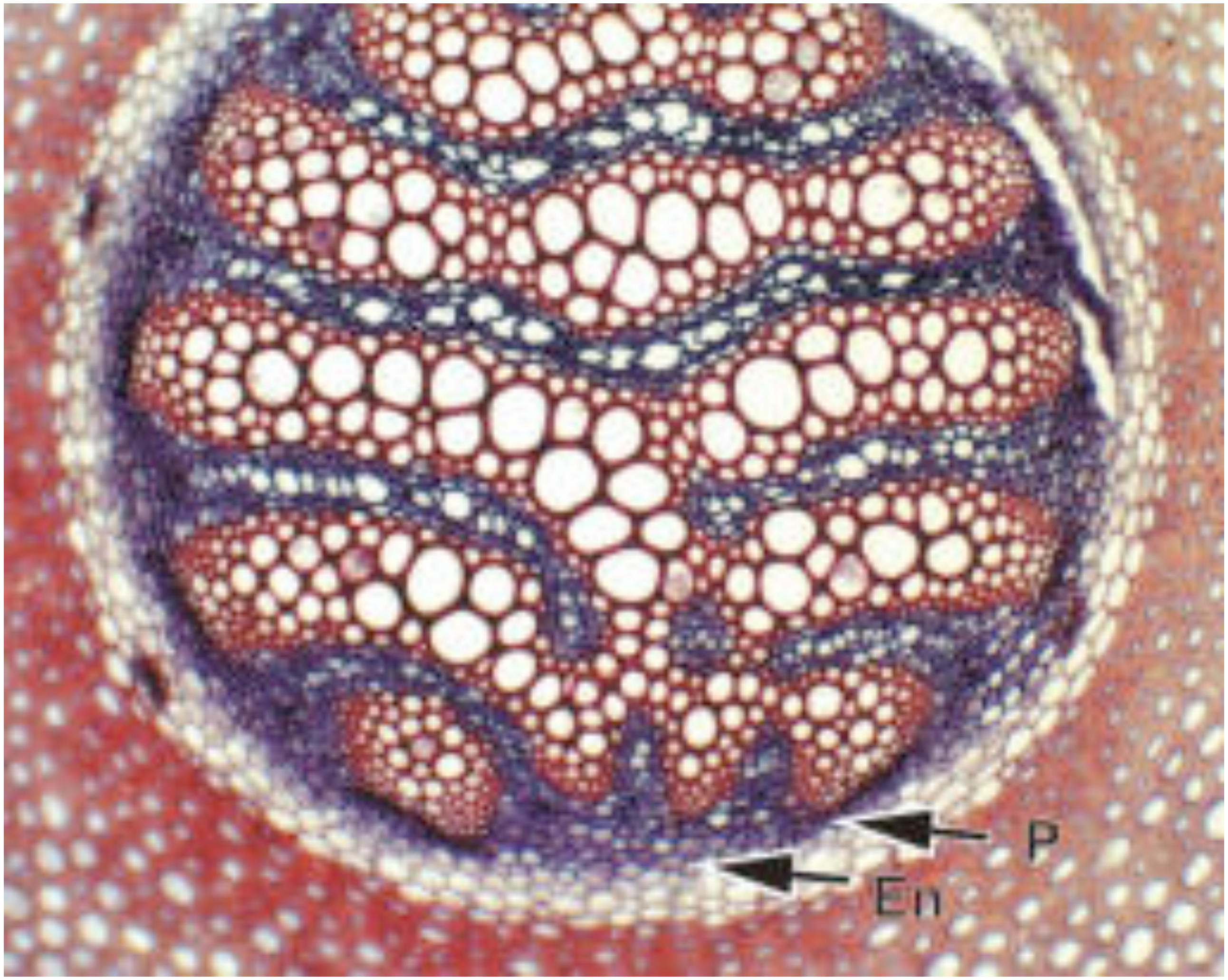


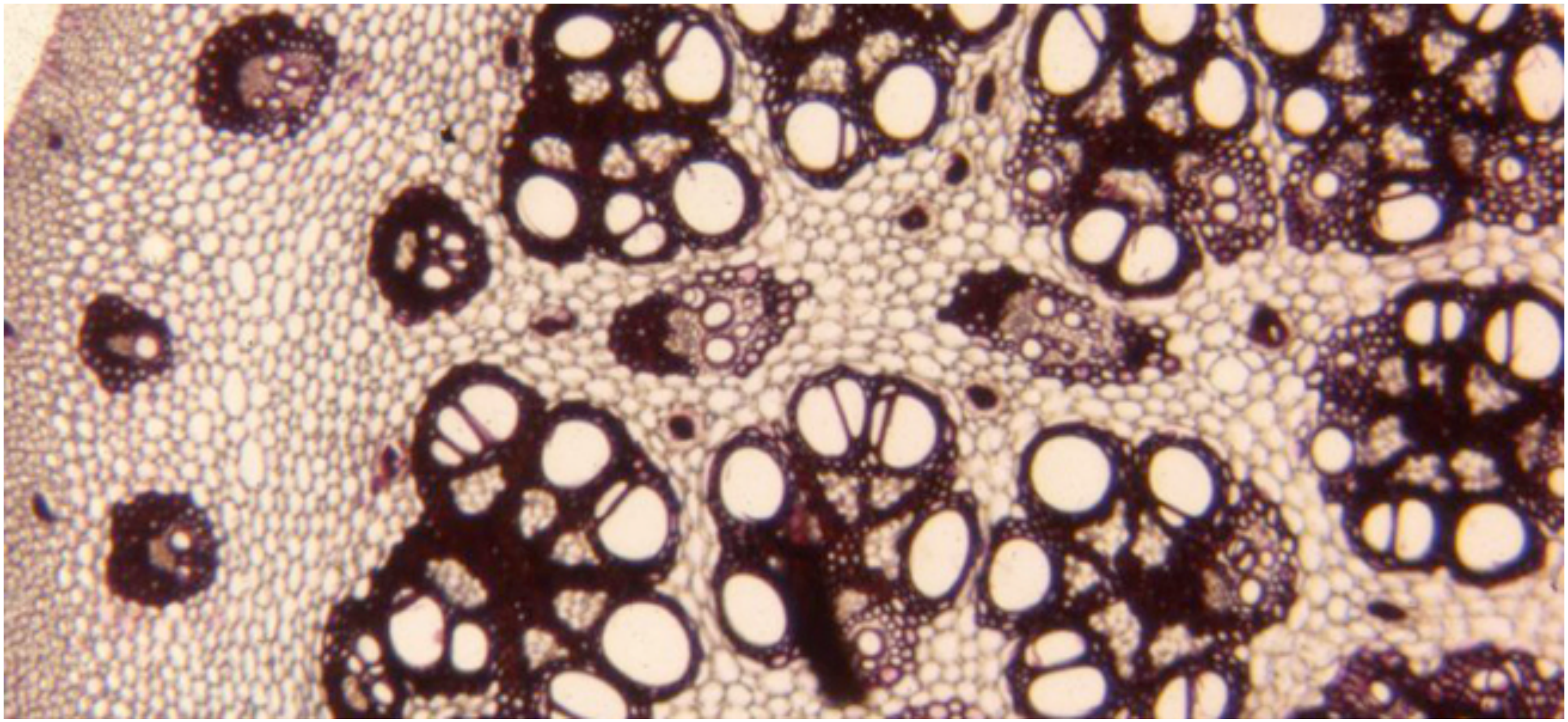
Katherine Du Tiel, 1994

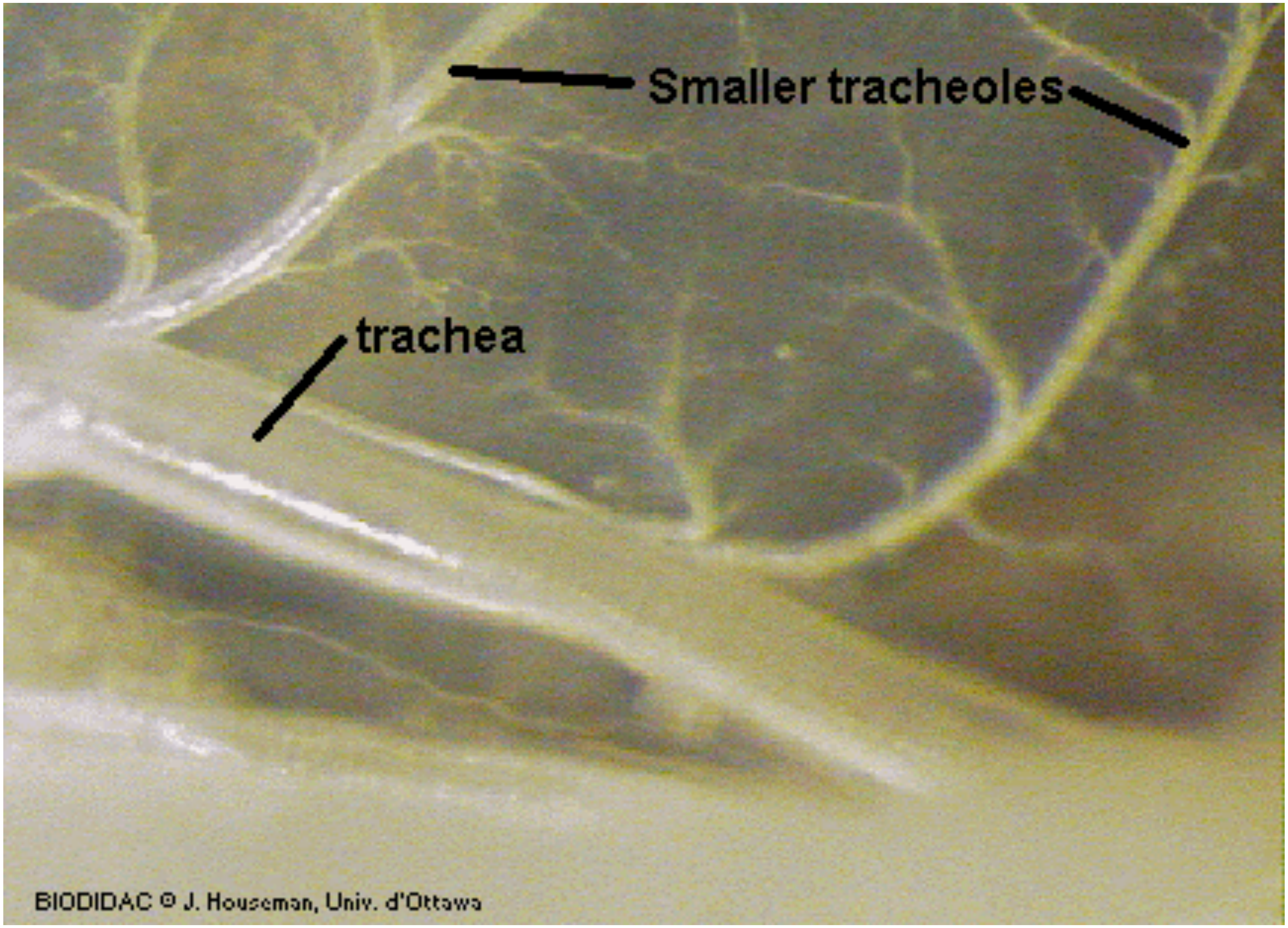
Casts of coronary artery

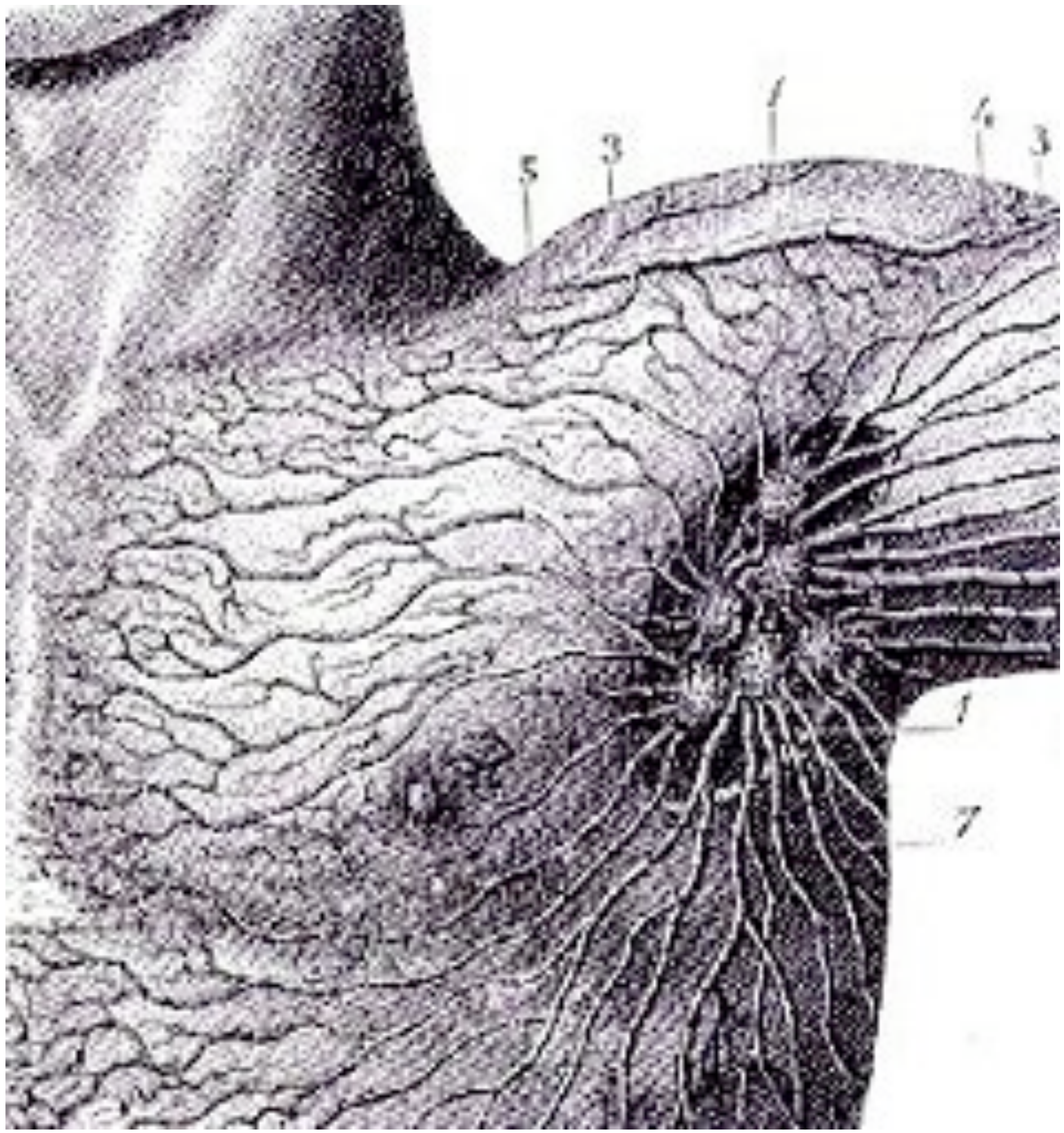


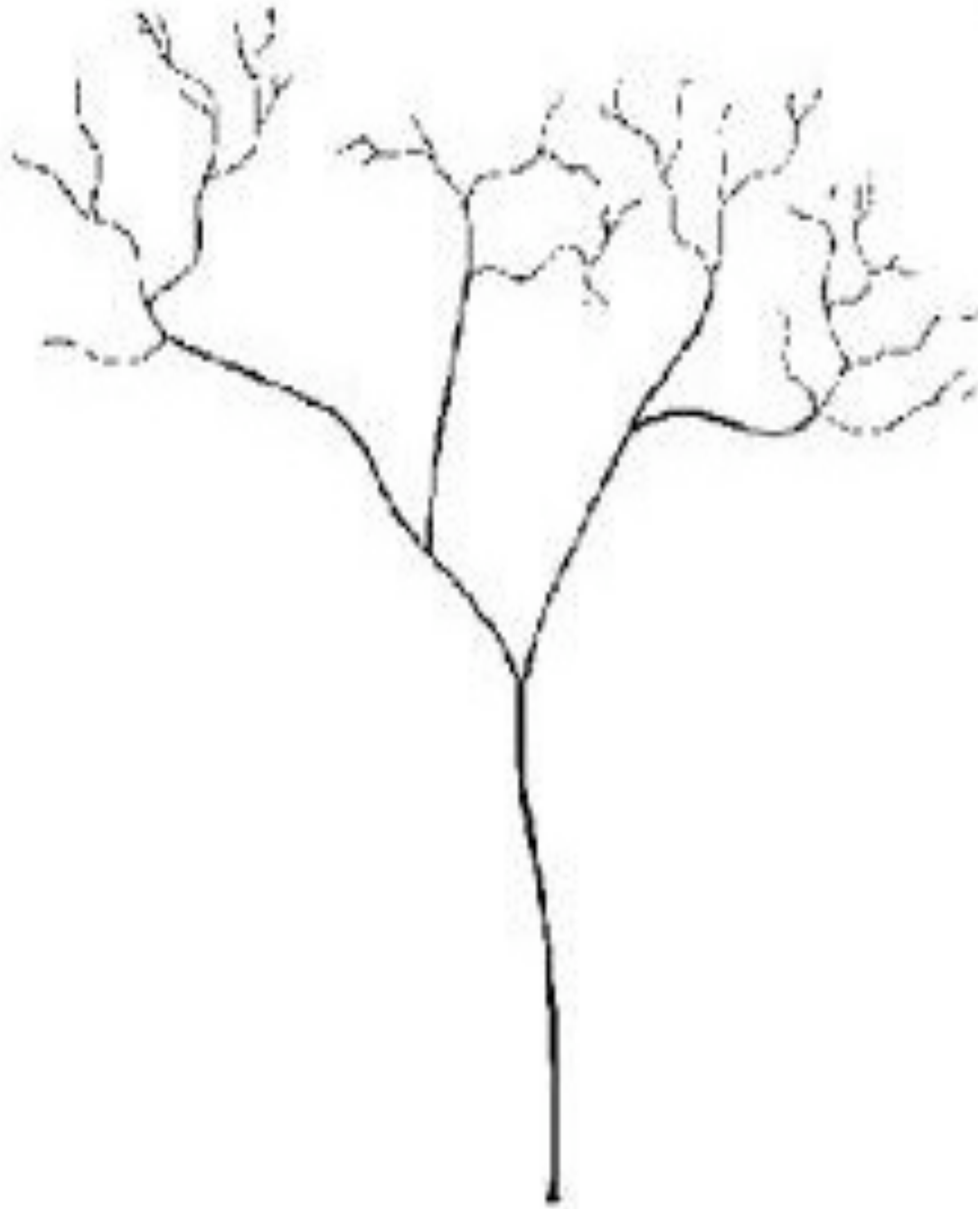

S


P
En




Smaller tracheoles
trachea
BIODIDAC © J. Houseman, Univ. d'Ottawa


5
3
1
4
3
1
7



Compare and contrast these networks

Similarities	Differences
Fluid flows through networks	Centralized versus non-centralized
Delivery of resources	Hierarchical versus non-hierarchical
Vessels of differing size	Different types of “fluids” are flowing
Vessels of differing flow rates	Delivering different types of resources
Removal of wastes?	

Nevertheless, we can construct similar frameworks, mathematical tools, and principles for studying these.

Extended models for vascular systems

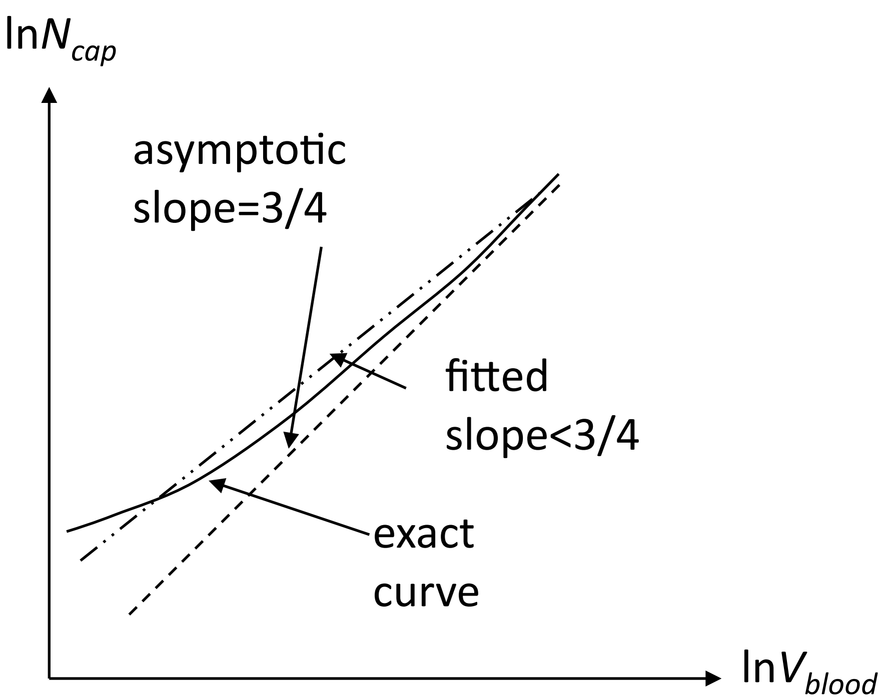
finite-size corrections
more realistic hydrodynamics
alternative geometries

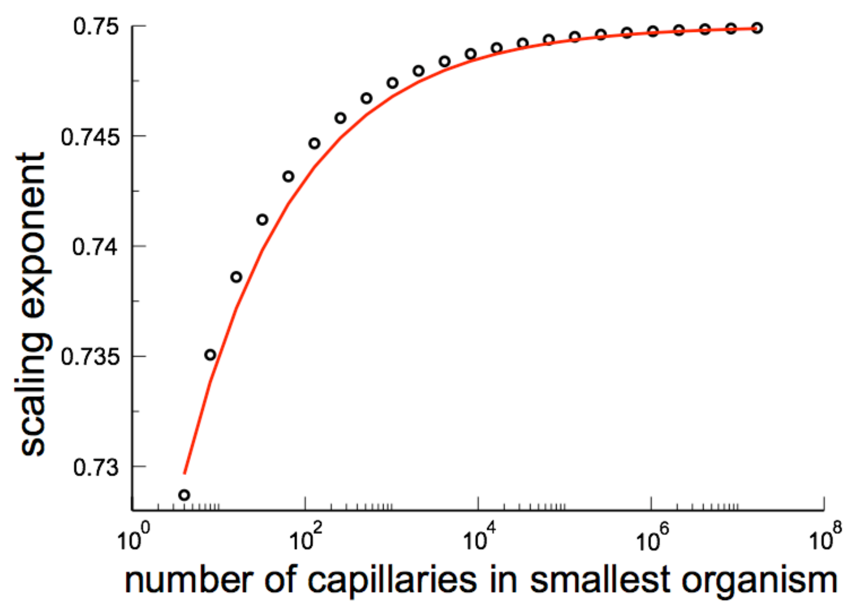
finite-size corrections

Finite size corrections: area preserving

tangent at each point of curve

$$\theta = \frac{d\ln B}{d\ln M} = \frac{d\ln N_{cap}}{d\ln V_{blood}} \sim \frac{3}{4}\left(1 - |C| N_{cap}^{-1/3}\right)$$



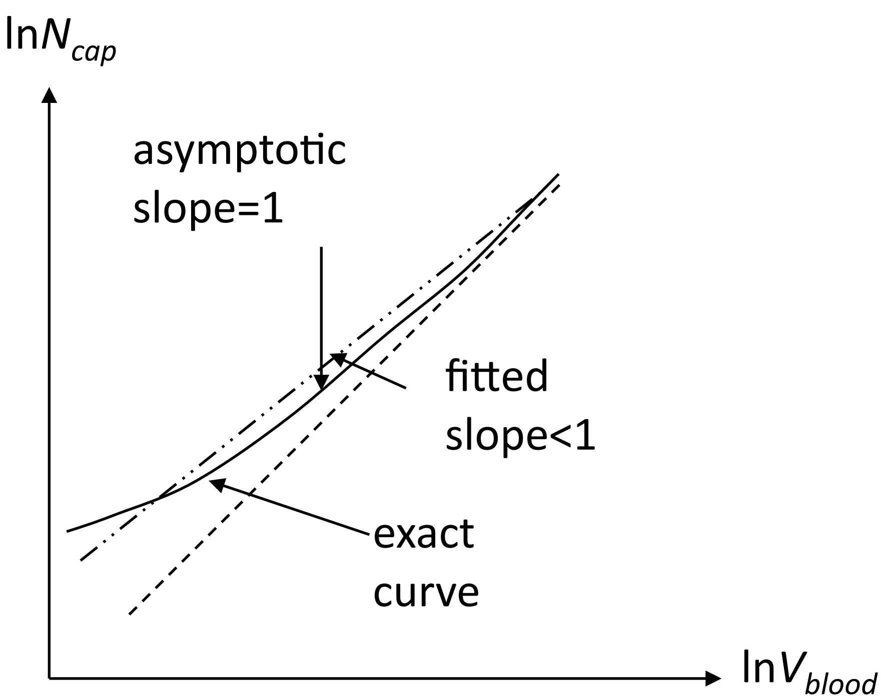


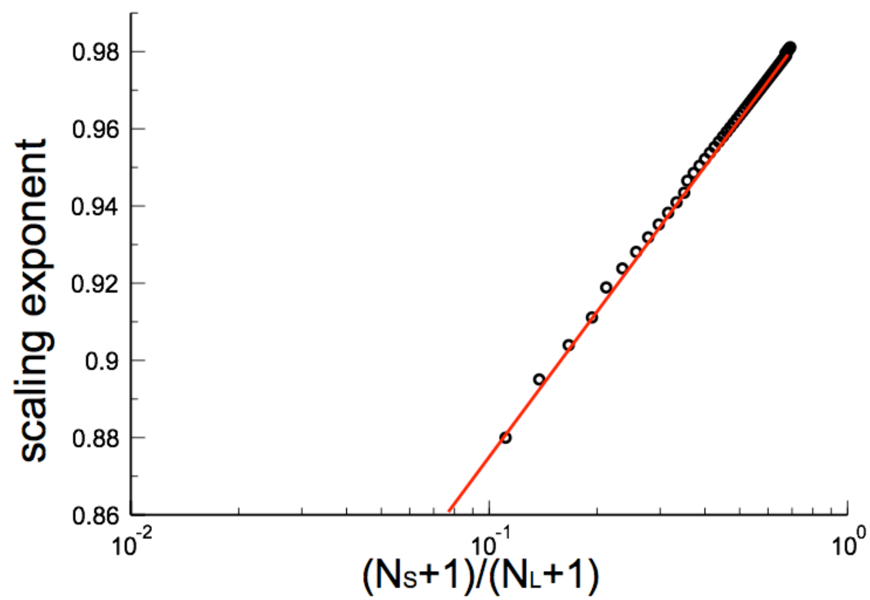
Savage et al., *PLoS. Comp. Biol.*, 2008

Finite size corrections: area increasing

tangent at each point of curve

$$\theta = \frac{d \ln B}{d \ln M} = \frac{d \ln N_{cap}}{d \ln V_{blood}} \sim 1 - \frac{1}{N+1}$$



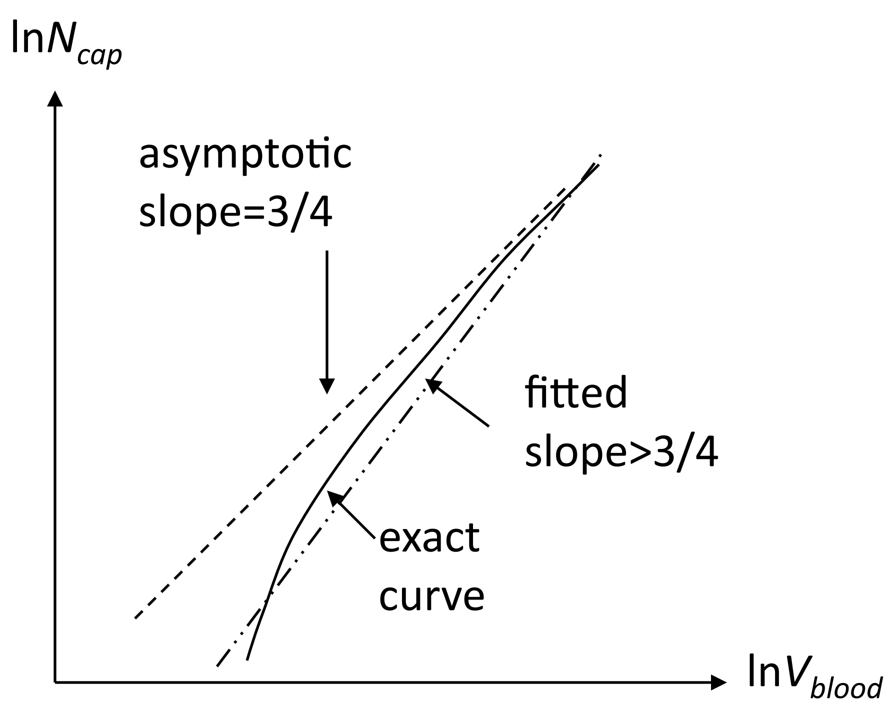


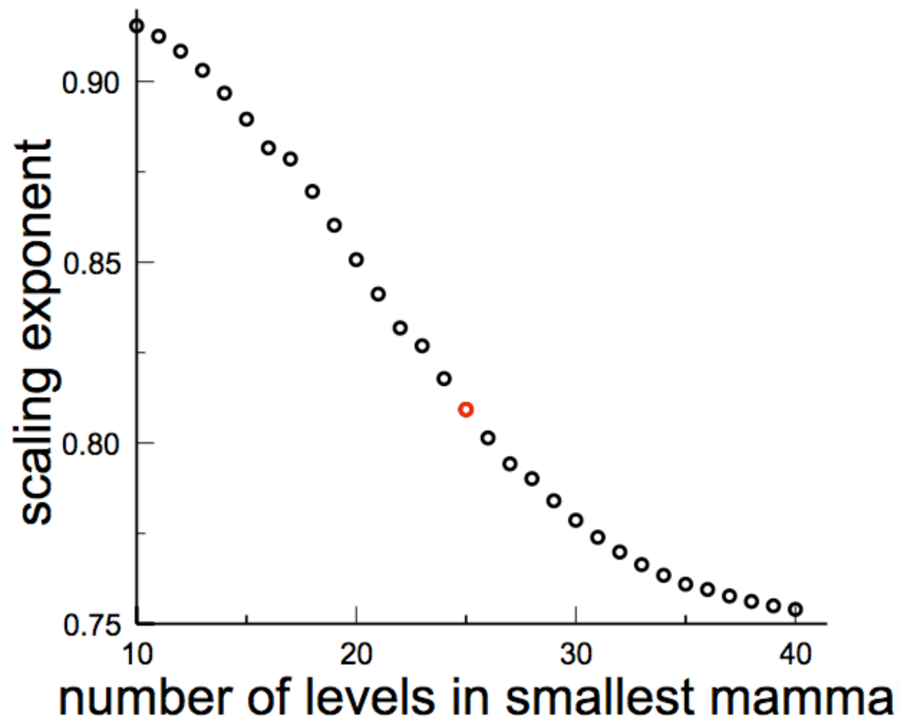
Savage et al., *PLoS. Comp. Biol.*, 2008

Finite-size corrections: mixed WBE model

tangent at each point of curve

$$\theta = \frac{d\ln B}{d\ln M} = \frac{d\ln N_{cap}}{d\ln V_{blood}} \sim \frac{3}{4}\left(1 + |C| N_{cap}^{-1/3}\right)$$

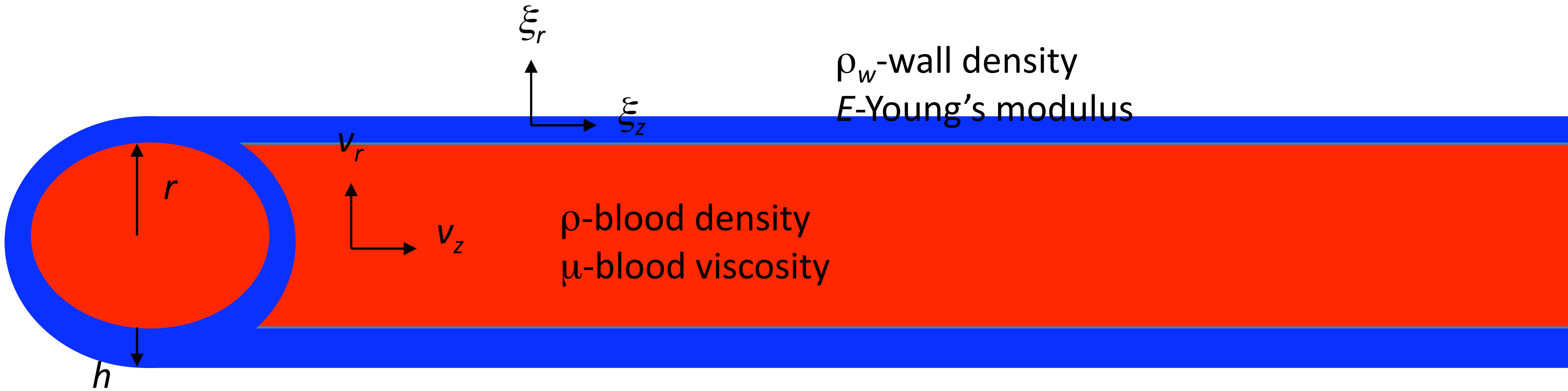




Savage et al., *PLoS. Comp. Biol.*, 2008

more realistic hydrodynamics

More realistic hydrodynamics



blood flow

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mu \nabla^2 \mathbf{v} - \nabla \mathbf{p}$$

can derive total impedance to flow

$$Z \sim \frac{c_o \rho i}{\pi r^2} \sqrt{\frac{J_0\left(i^{3/2}\sqrt{\frac{\omega\rho}{\mu}}r\right)}{J_2\left(i^{3/2}\sqrt{\frac{\omega\rho}{\mu}}r\right)}}$$

vessel wall

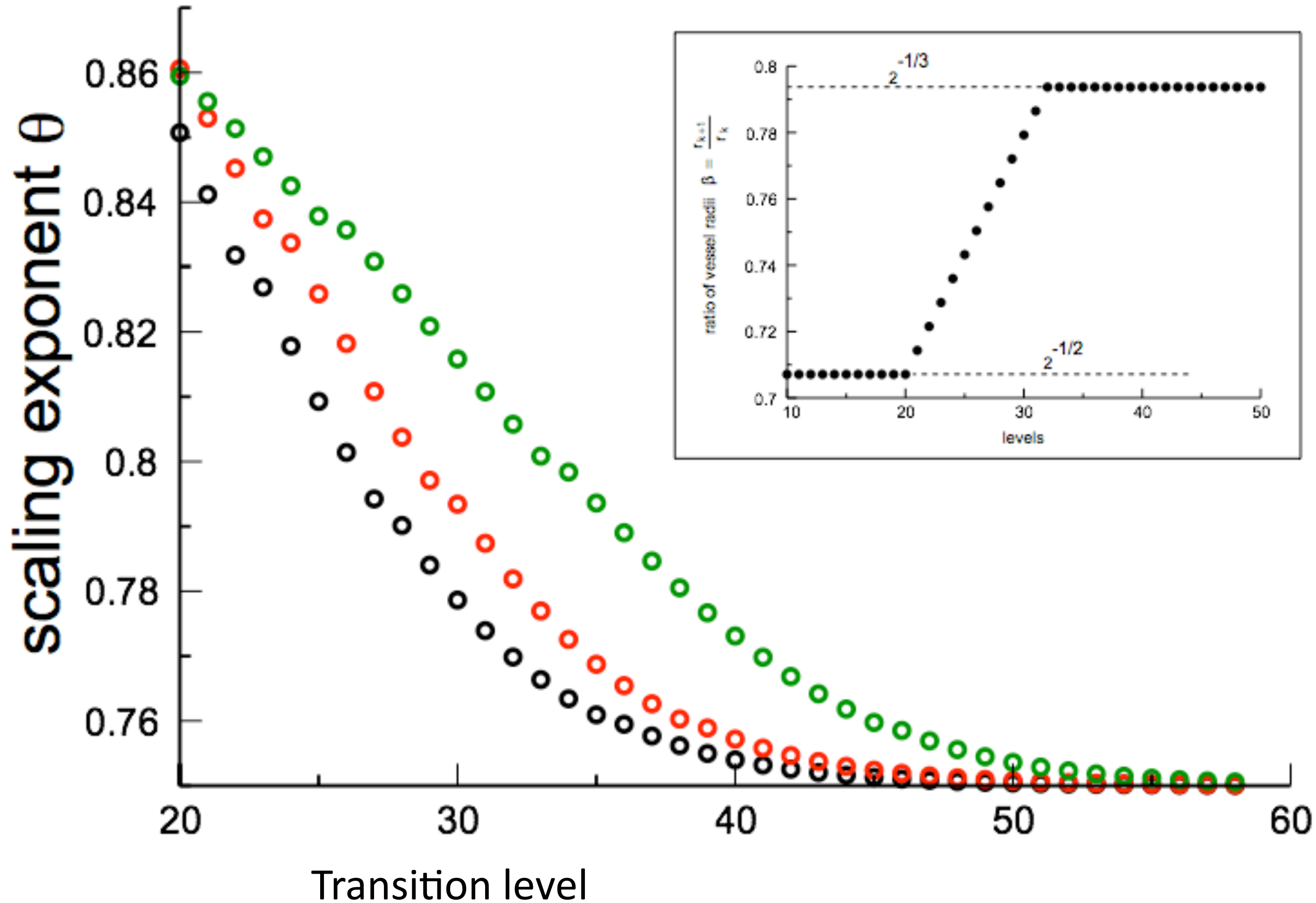
$$\rho_w \frac{\partial^2 \xi}{\partial t^2} = E \nabla^2 \xi - \nabla \mathbf{p}$$

ω-angular frequency of wave

c_0-Korteweg-Moens velocity

$$c_0 = \sqrt{\frac{Eh}{2\rho r}}$$

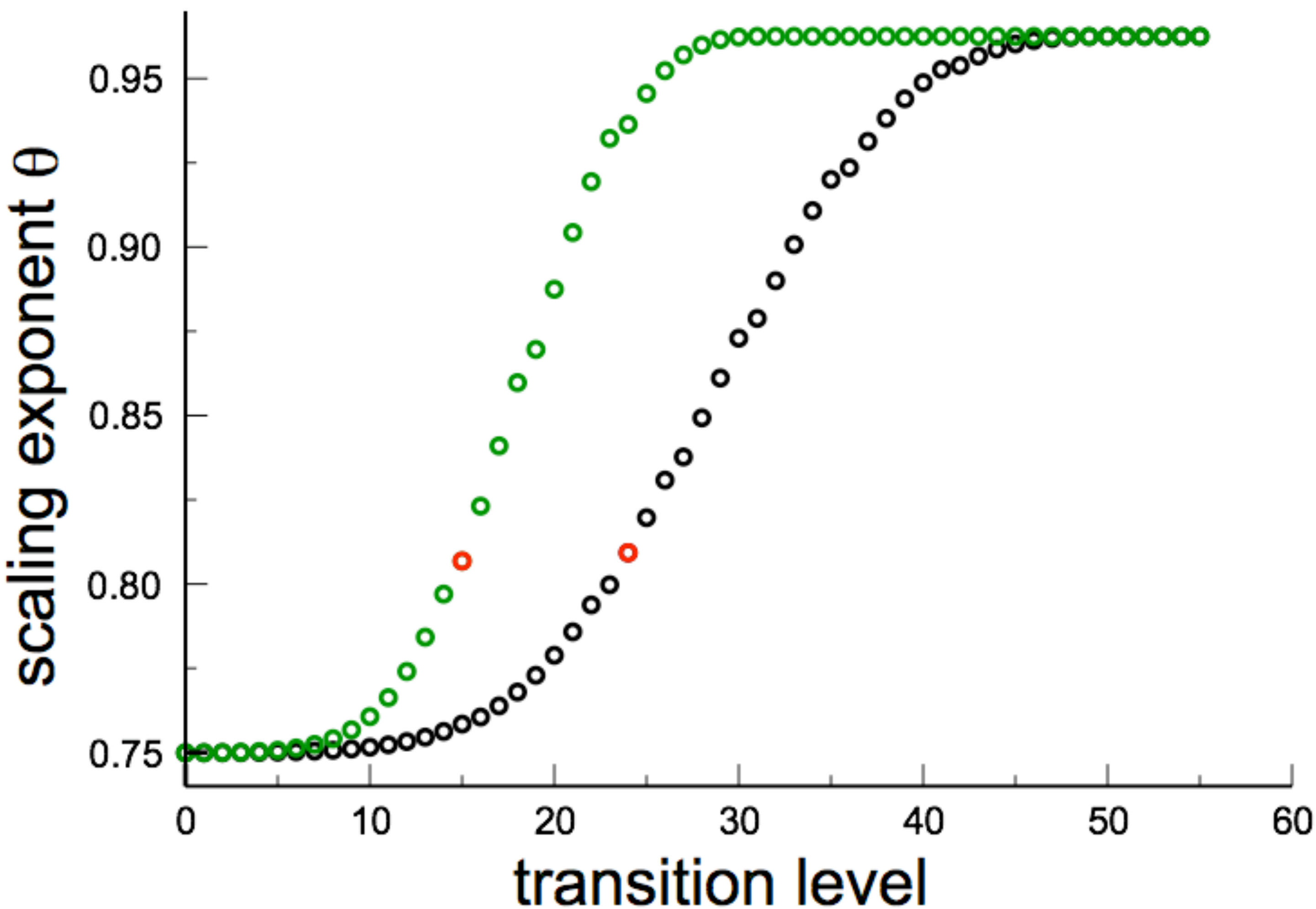
Changing width of transition region



black=0 levels; red=12 levels; green=24 levels

Savage et al., *PLoS. Comp. Biol.*, 2008

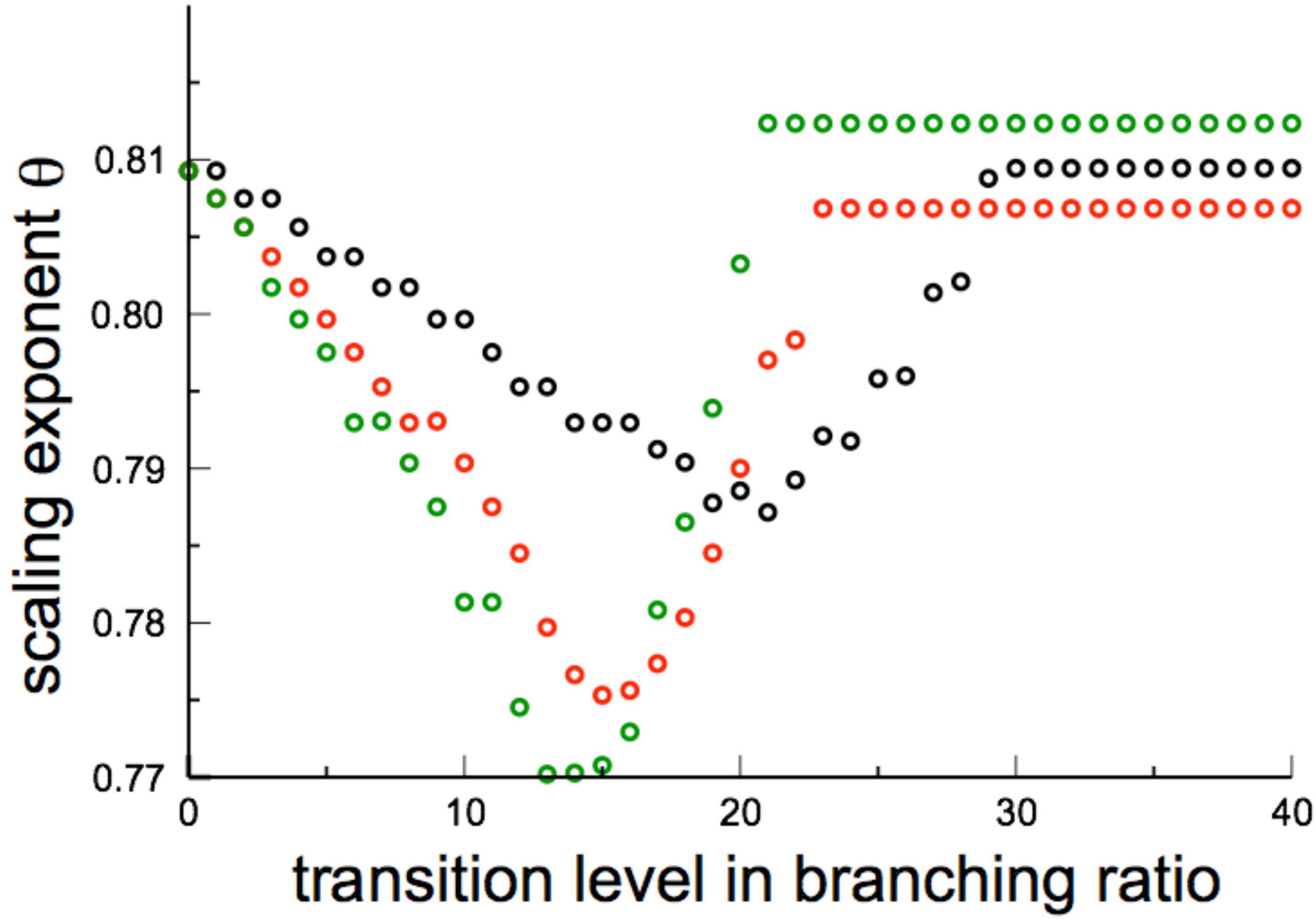
Changing transition level



Savage et al., *PLoS. Comp. Biol.*, 2008

alternative geometries

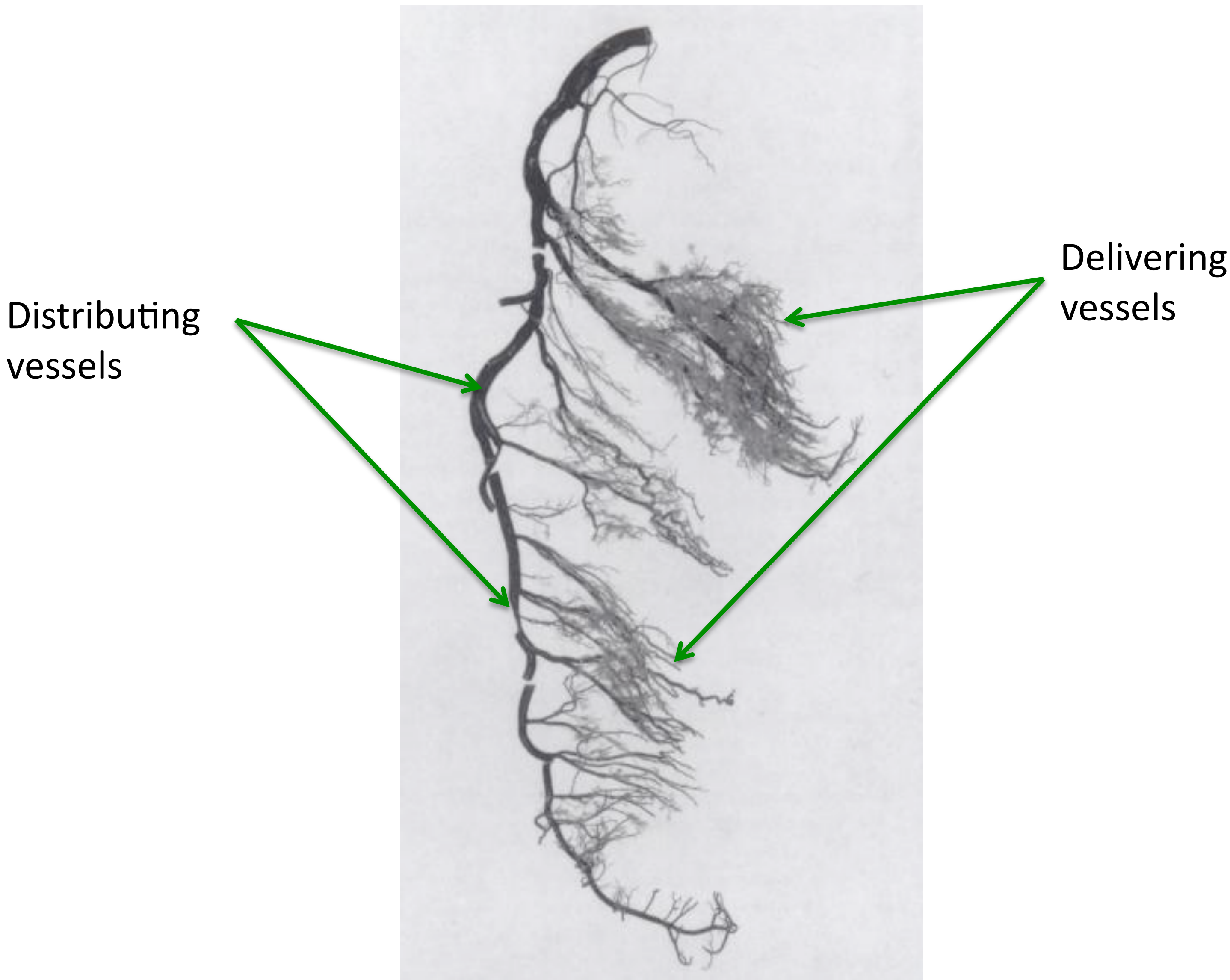
Changing branching ratio



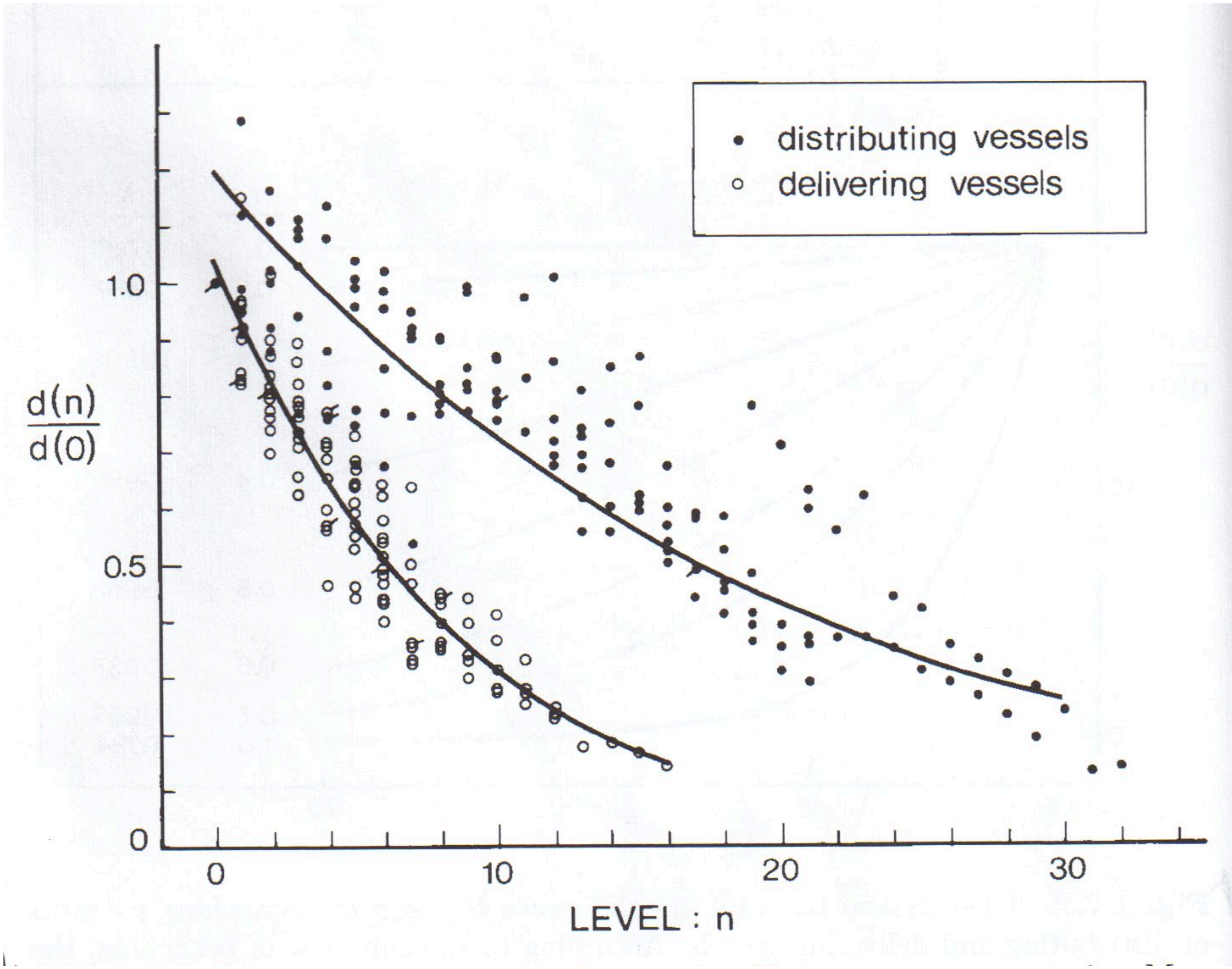
Savage et al., *PLoS. Comp. Biol.*, 2008

Asymmetric branching in left coronary artery

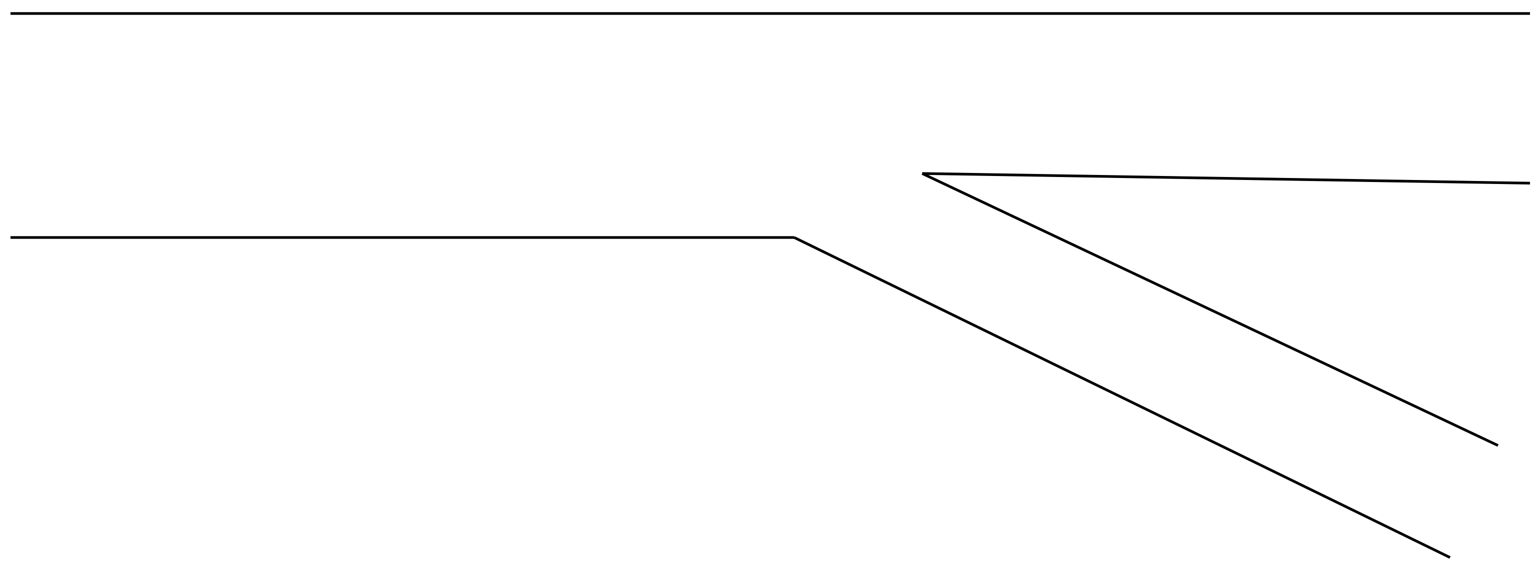
Distributing vessels versus delivering vessels: a type of asymmetry



Look for different scaling exponents: coronary artery

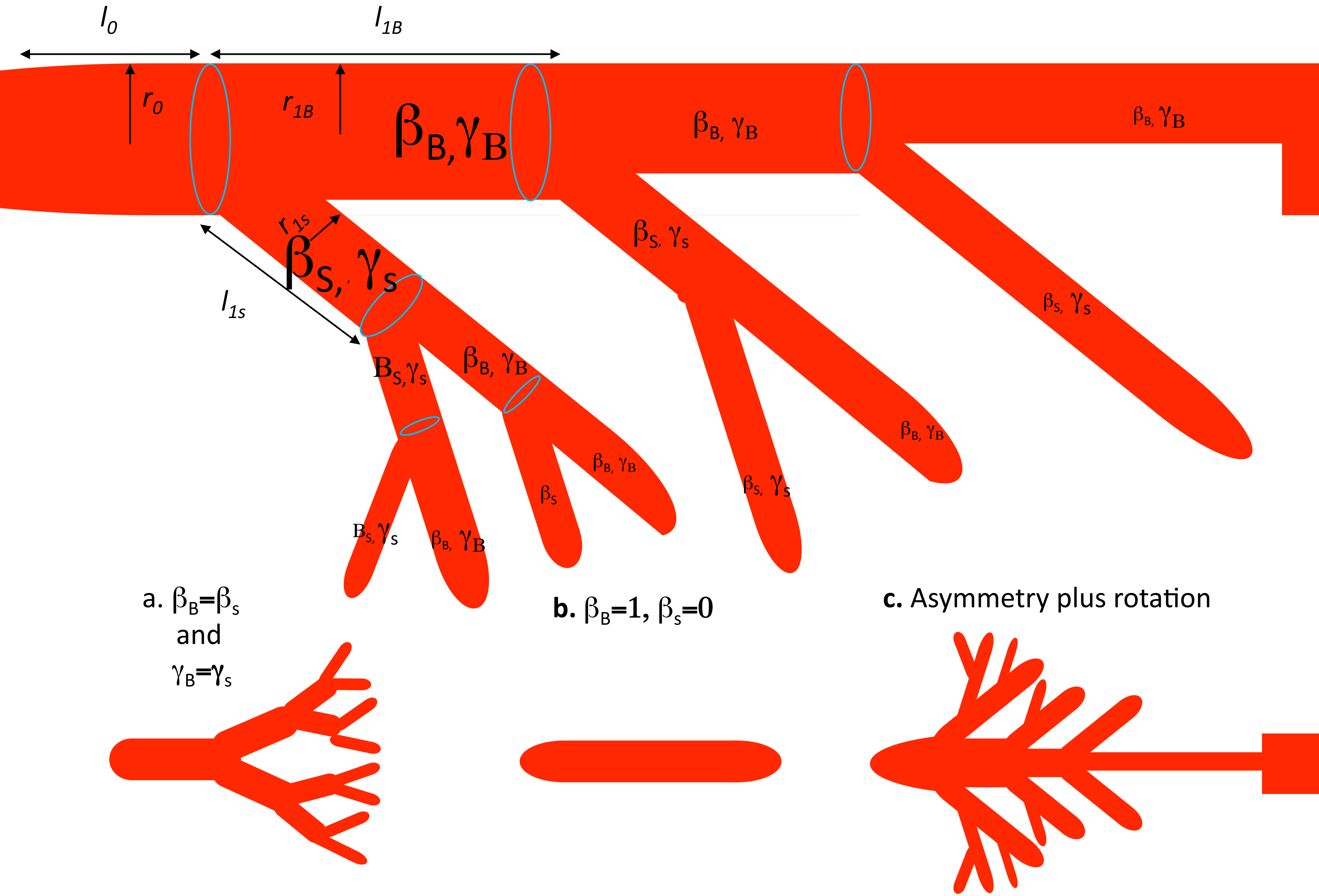


Asymmetric branching and calculations for whole networks

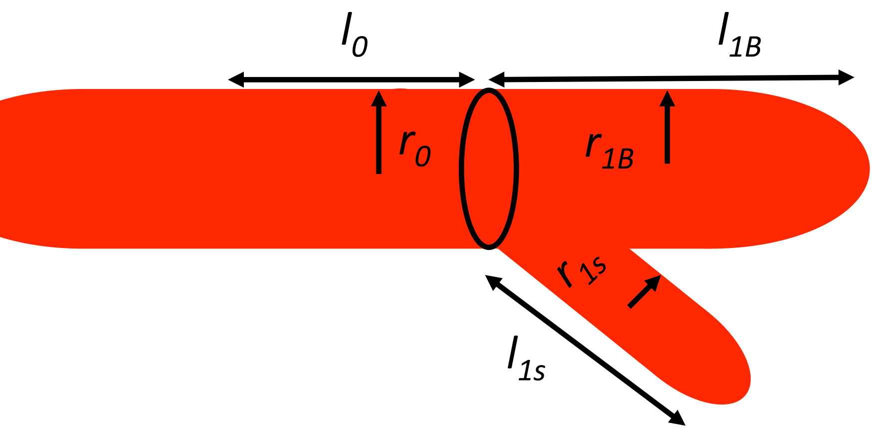


-Branching can still be area preserving or area increasing even when branching is asymmetric.

-If asymmetry occurs in a systematic or fixed way, this represents a type of self similarity, so should be able to derive scaling relations, do sums, and repeat similar analyses


l_0
l_{1B}
r_0
r_{1B}
β_B, γ_B
β_B, γ_B
β_B, γ_B
r_{1s}
β_S, γ_s
l_{1s}
β_S, γ_s
β_S, γ_s
B_S, γ_s
β_B, γ_B
β_B, γ_B
β_B, γ_B
β_s
β_S, γ_s
B_S, γ_s
β_B, γ_B
a. $\beta_B = \beta_s$
and
$\gamma_B = \gamma_s$
b. $\beta_B = 1, \beta_s = 0$
c. Asymmetry plus rotation

Can also do this for whole network for self similarity within levels without self similarity across levels



$A_{s,k}=\beta^2_{s,k-1}A_{k-1}$ and $l_{s,k}=\gamma_{s,k-1}l_{k-1}$

$A_{B,k}=\beta^2_{B,k-1}A_{k-1}$ and $l_{B,k}=\gamma_{B,k-1}l_{k-1}$

scaling ratios are constant within level k

volume at kth level:

$$V_k^{TOT} = \prod_{i=0}^{k} (\beta^2_{s,i}\gamma_{s,i} + \beta^2_{B,i}\gamma_{B,i})V_0$$

network volume:

$$V_{net} = \sum_{k=0}^{N} V_k^{TOT}$$

What do space filling and area preserving mean in this notation?

area preserving $$\beta^2_{S,k} + \beta^2_{B,k} = 1$$

space filling $$\gamma^3_{s,k} + \gamma^3_{B,k} = 1$$

Area preserving with arbitrary asymmetry in radii and symmetric space filling in length

$$A_0^{TOT} = A_k^{TOT} = A_N^{TOT}$$

For the symmetric case with area preserving

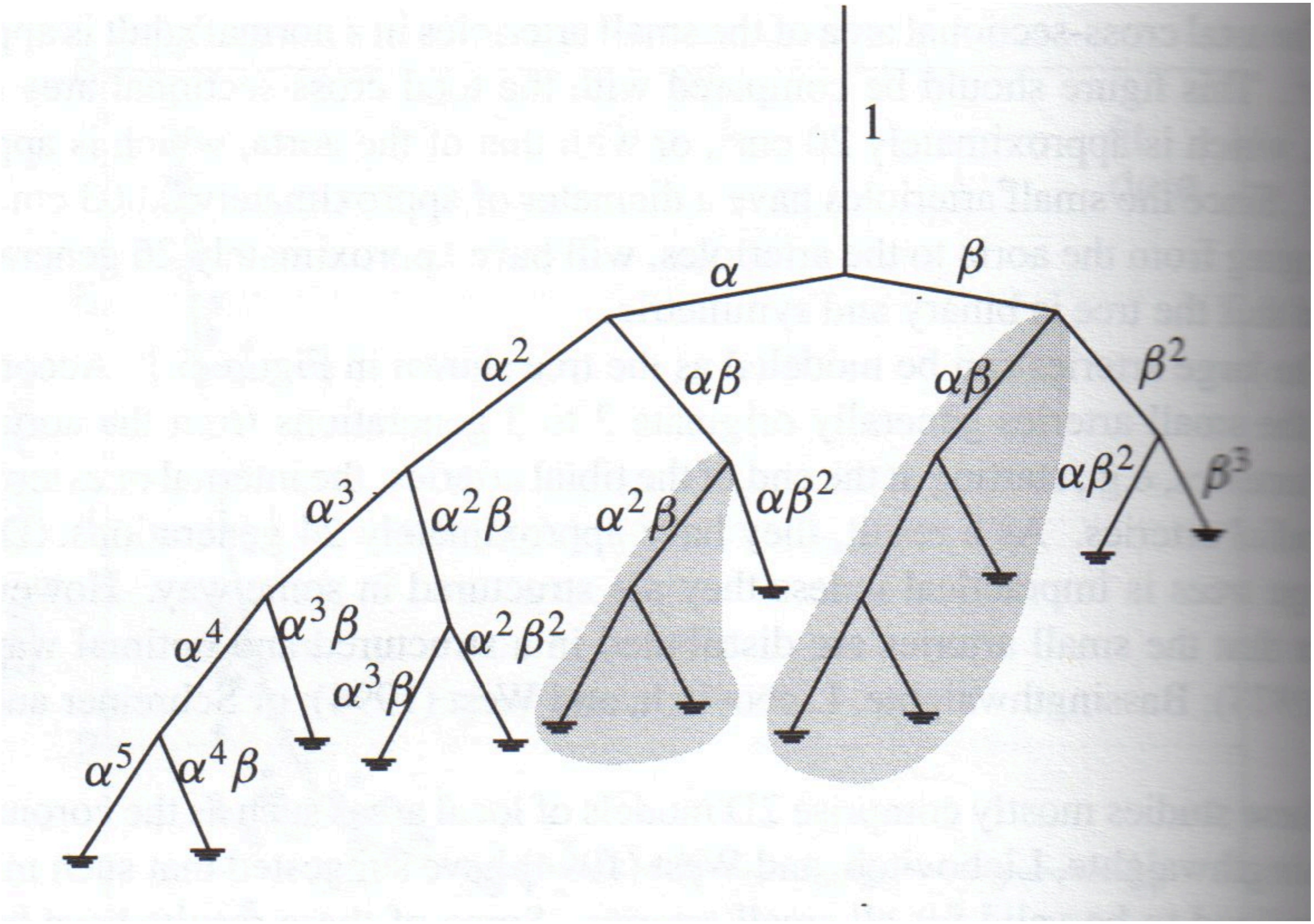
$$A_k^{TOT} = n^k \pi r_k^2 = n^k \pi \beta^{-2(N-k)} r_N^2 = n^k \pi n^{(N-k)} r_N^2 = n^N A_N$$

For the asymmetric case this becomes

$$A_0^{TOT} = A_k^{TOT} = A_N^{TOT} = n^N \overline{A_N}$$

You will still get the ¾ exponent. As long as area-preserving holds, this ¾ result is very robust to changes in geometry and asymmetry.

Invariant terminal units?



If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is. -John von Neumann

IV. Conclusions

1. Power laws are common in nature due to self similarity and behavior near critical points.

2. Be careful to make sure you have a power law.

3. Can often explain and predict a lot without knowing *details* of the problem

4. Power laws are common in biology (and elsewhere)

5. Dynamical model based on distribution of resources makes many predictions that match data of at level of vessels and whole organism.