Eigenvalues & Eigenvectors

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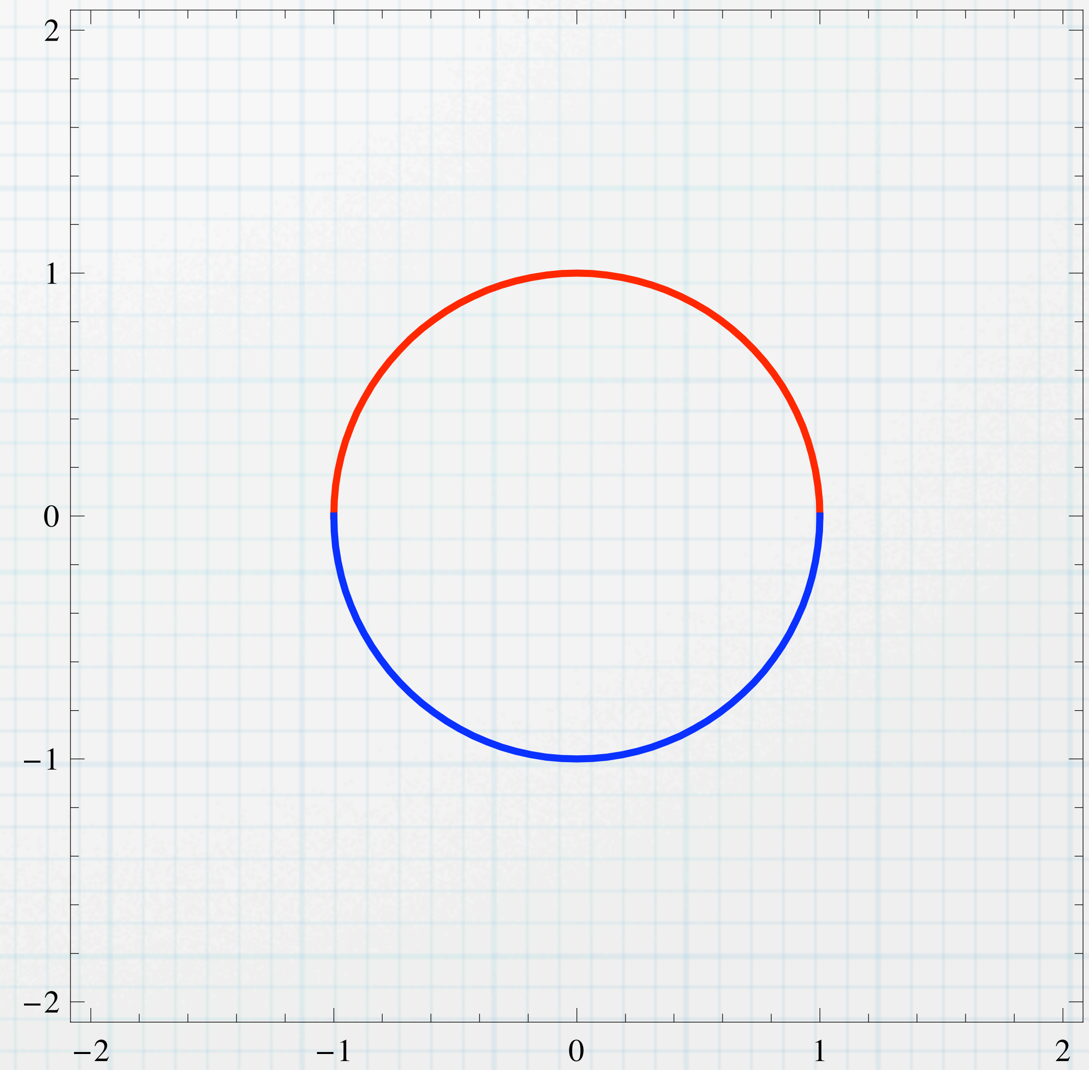
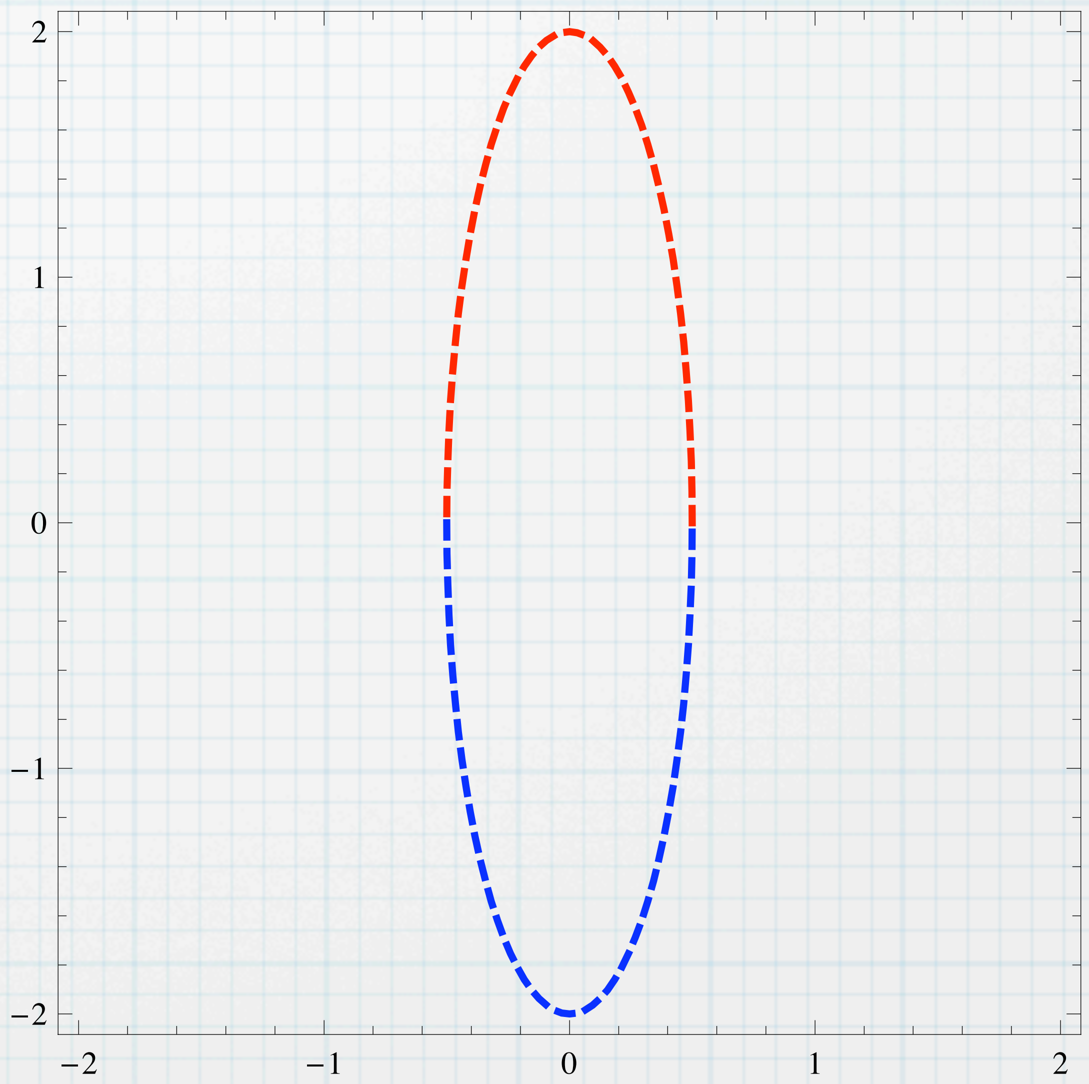
Linear transformations

$$y = Ax$$



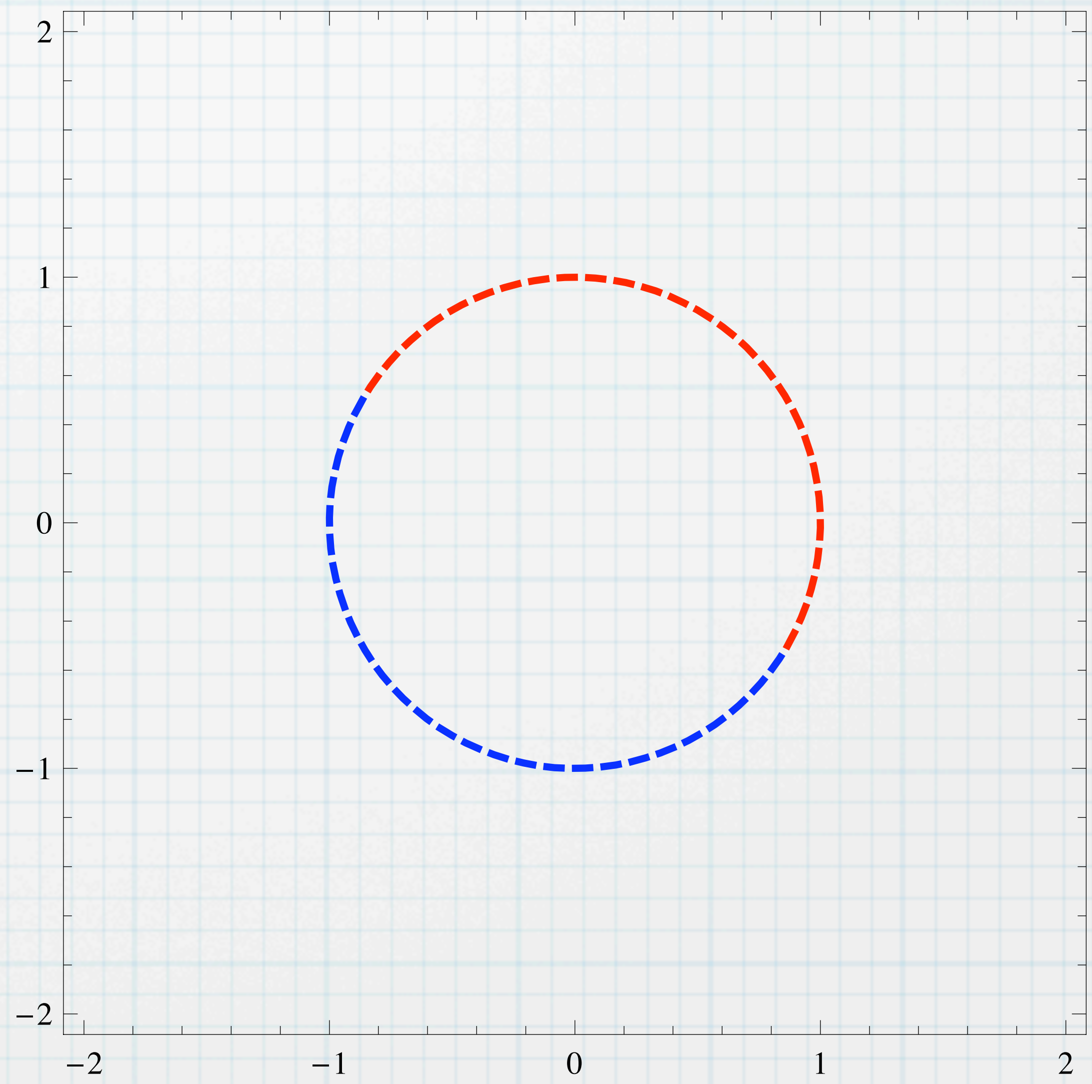
Scaling

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$



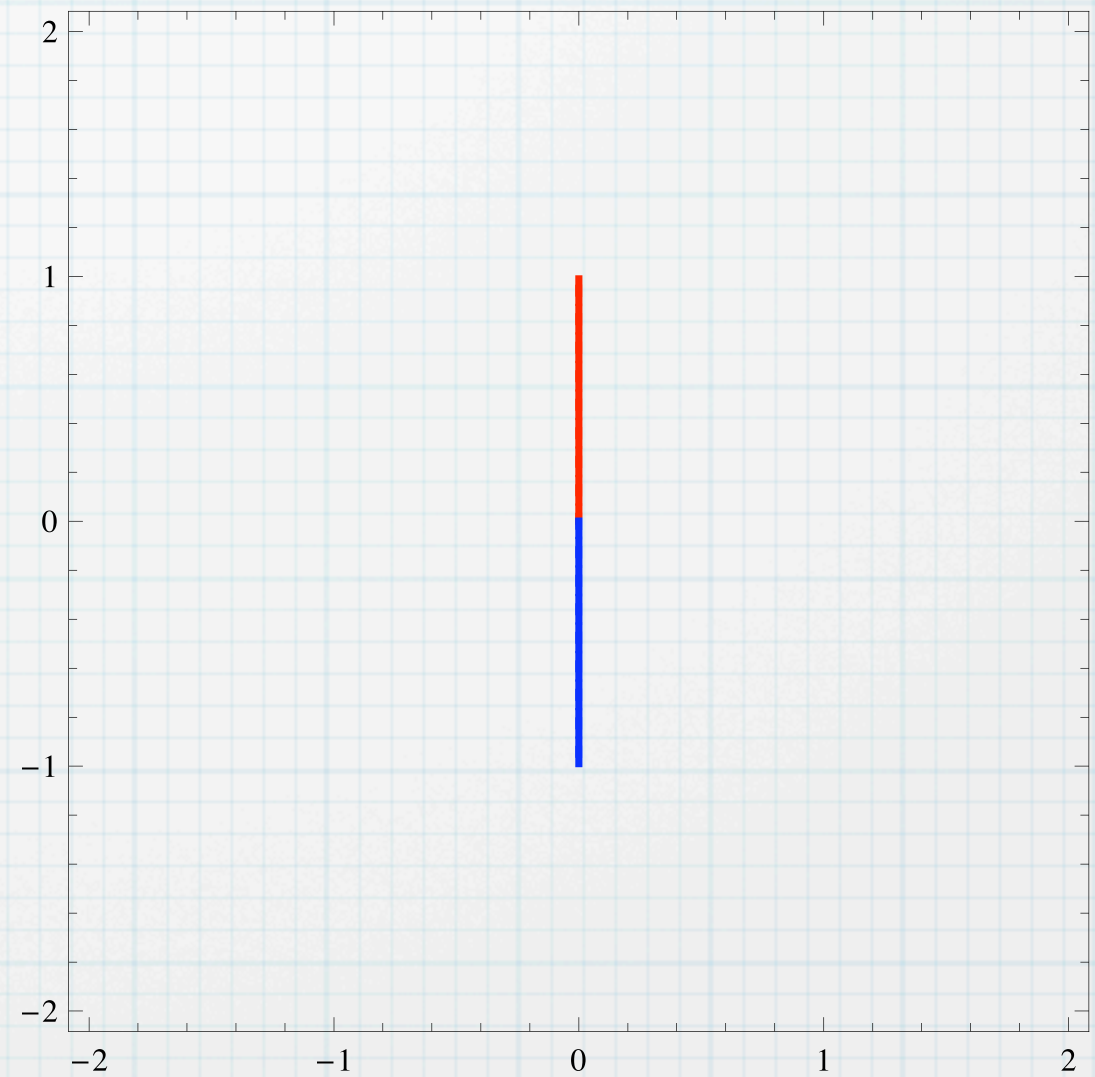
Rotation

$$A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$



Projection

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



Mixed rotation and scaling

$$A = \begin{pmatrix} \sqrt{3}/4 & 1 \\ -1/4 & \sqrt{3} \end{pmatrix}$$

But where are the eigenvalues and the eigenvectors?

The eigenvalues and eigenvectors of a linear operator are found by solving the equation system:

$$Ax = \lambda x$$

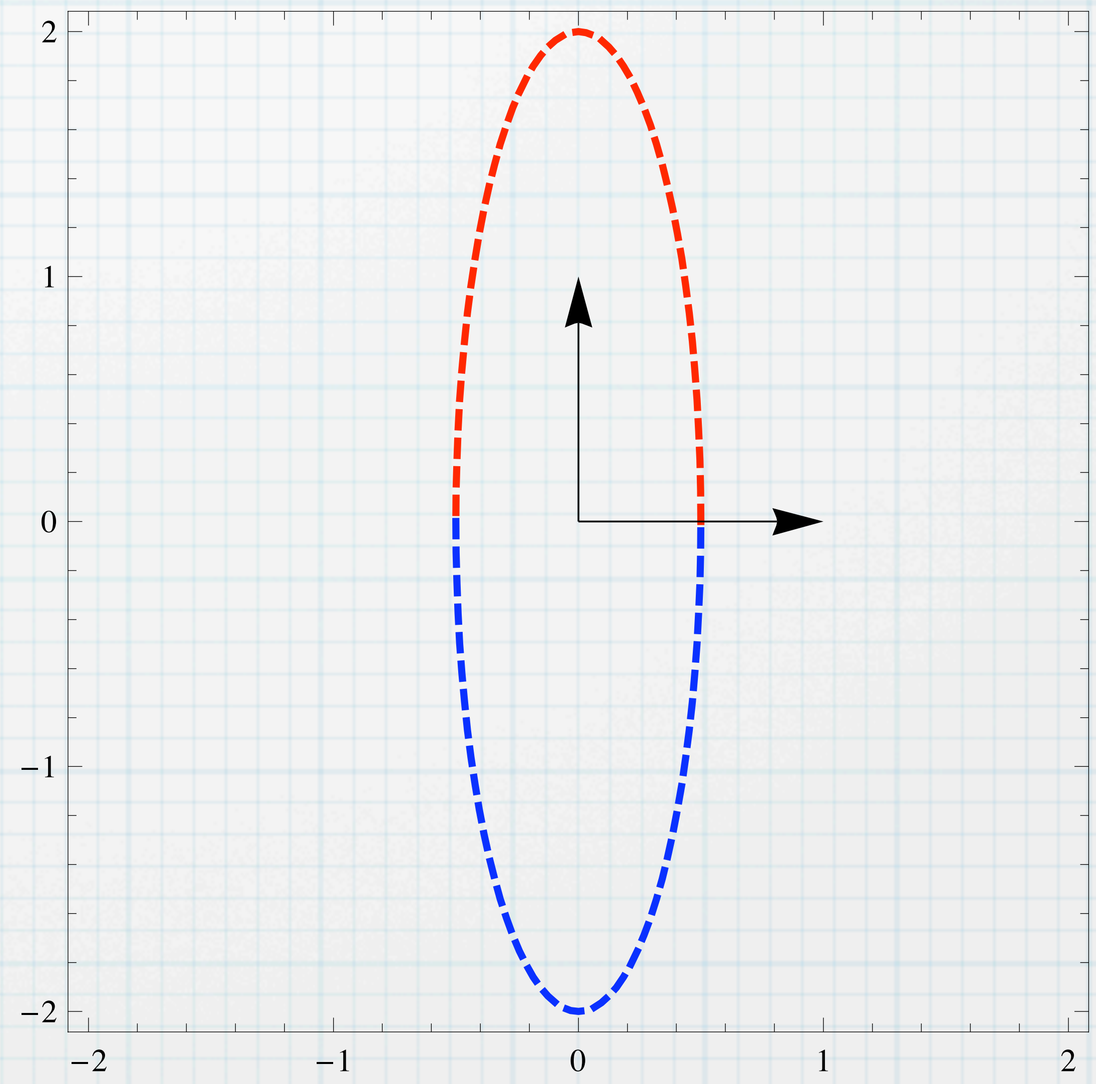
Scaling - Eigenvalues and eigenvectors

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\lambda_1 = 2, \quad \lambda_2 = 1/2$$

$$x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



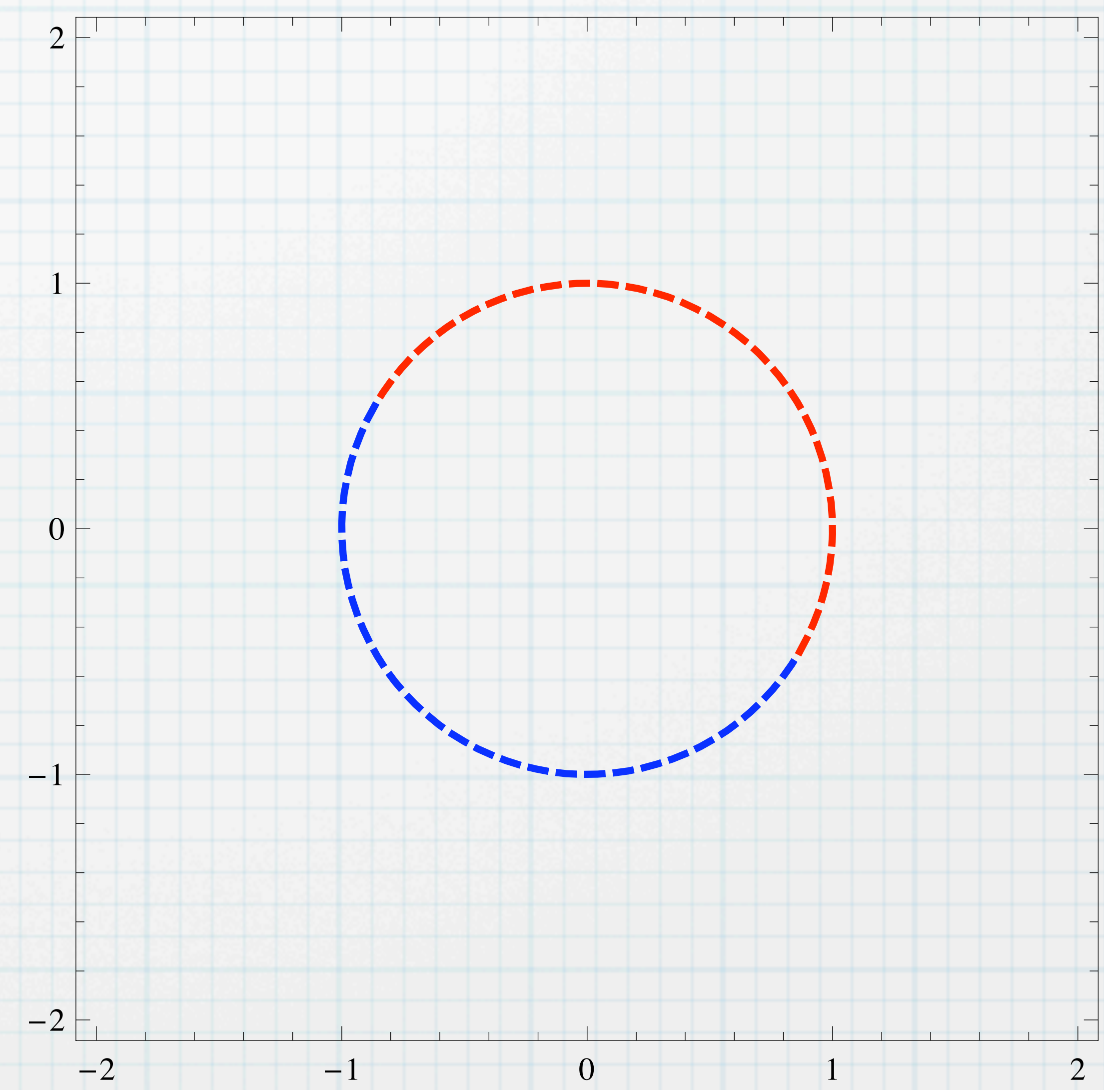
Rotation - Eigenvalues and eigenvectors

$$A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$\lambda_1 = \sqrt{3}/2 + 1/2\,\mathrm{i}, \quad \lambda_2 = \sqrt{3}/2 - 1/2\,\mathrm{i}$$

$$x_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}$$



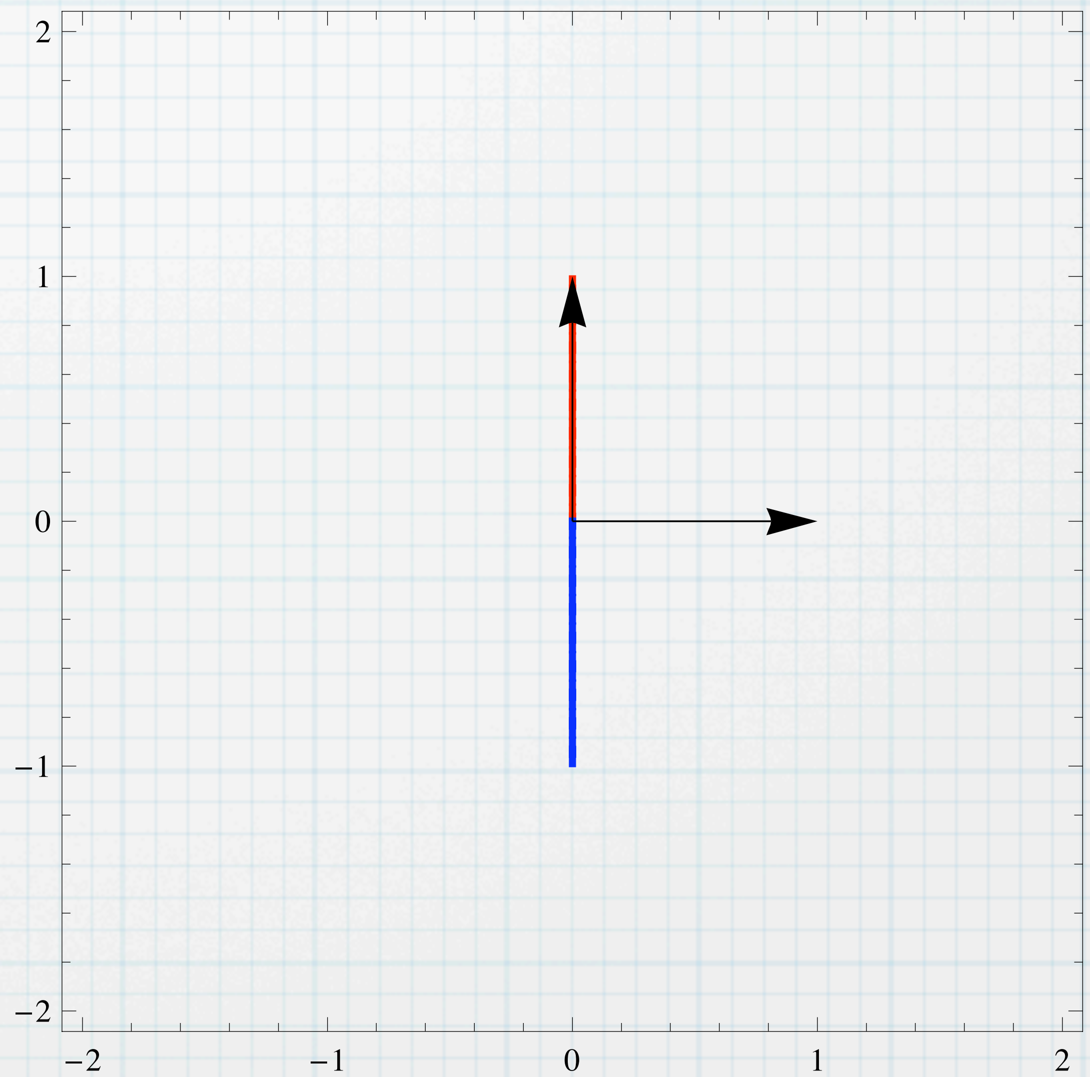
Projection - Eigenvalues and eigenvectors

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = 0$$

$$x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



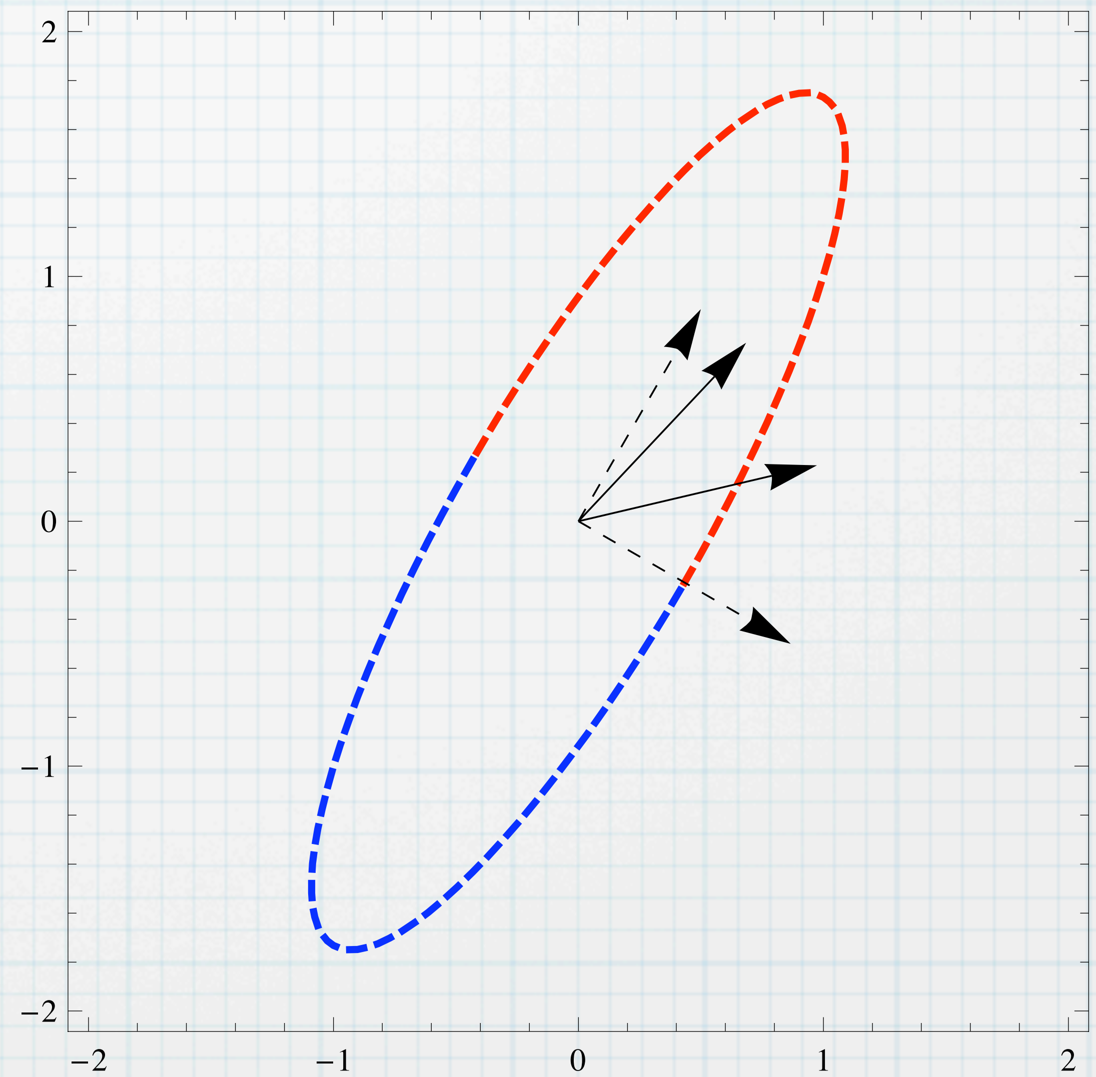
Mixed rotation and scaling - Eigenvalues and eigenvectors

$$A = \begin{pmatrix} \sqrt{3}/4 & 1 \\ -1/4 & \sqrt{3} \end{pmatrix}$$

$$\lambda_1 = 1.497, \quad \lambda_2 = 0.668$$

$$x_1 = \begin{pmatrix} 0.685 \\ 0.729 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 0.973 \\ 0.229 \end{pmatrix}$$



Why are eigenvalues and eigenvectors EXTREMELY important?

In physics, all dynamics are formulated in terms of operators. The eigenvalues and eigenvectors characterize a system.

In general in science, all kind of data are represented in matrices (images, dna-data, climate measurements etc). The eigenvalues and eigenvectors reveal important structures of the system.

Eigenvalues and eigenvectors in (non-)linear dynamical systems

A linear dynamical system is given by the differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$$

In many situations, the matrix A can be diagonalized:

$$\Lambda = U^{-1}AU$$

Example: Wave equation

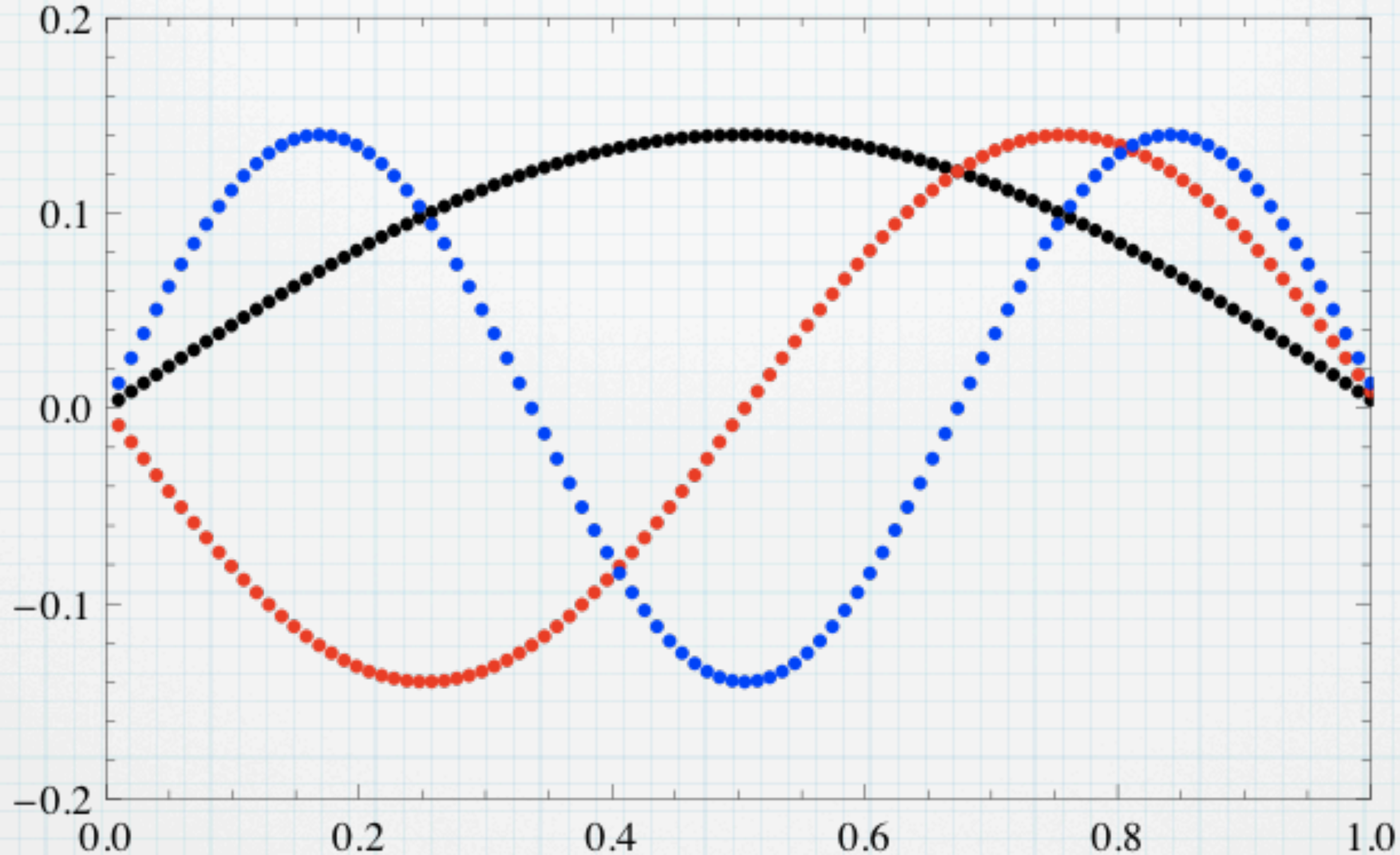
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Example: Wave equation

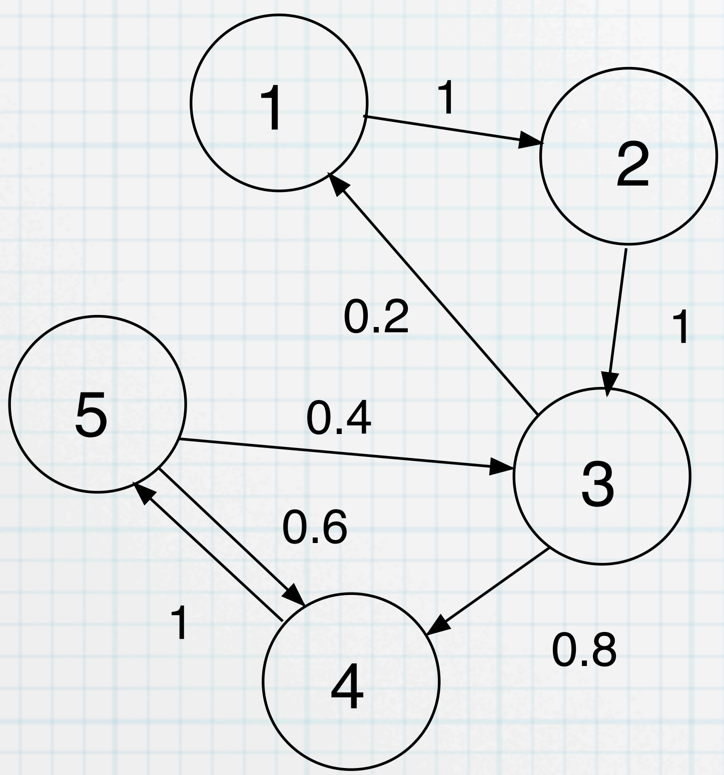
Example: Wave equation

$$\begin{pmatrix}
-2. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
1. & -2. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 1. & -2. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 1. & -2. & 1. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 1. & -2. & 1. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 1. & -2. & 1. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 1. & -2. & 1. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 1. & -2. & 1. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -2. & 1. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -2. & 1. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -2.
\end{pmatrix}$$

Example: Wave equation



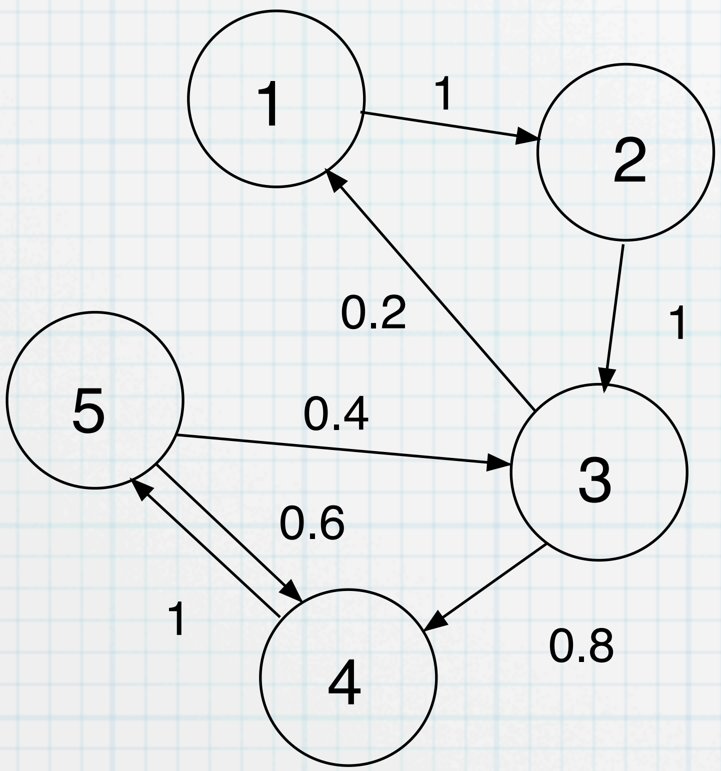
Example: Long term behavior of transition matrix



$$\pi P = \pi$$

Eigenvalue problem with eigenvalue 1. This is ok - a transition matrix has always 1 as the largest eigenvalue.

Example: Long term behavior of transition matrix

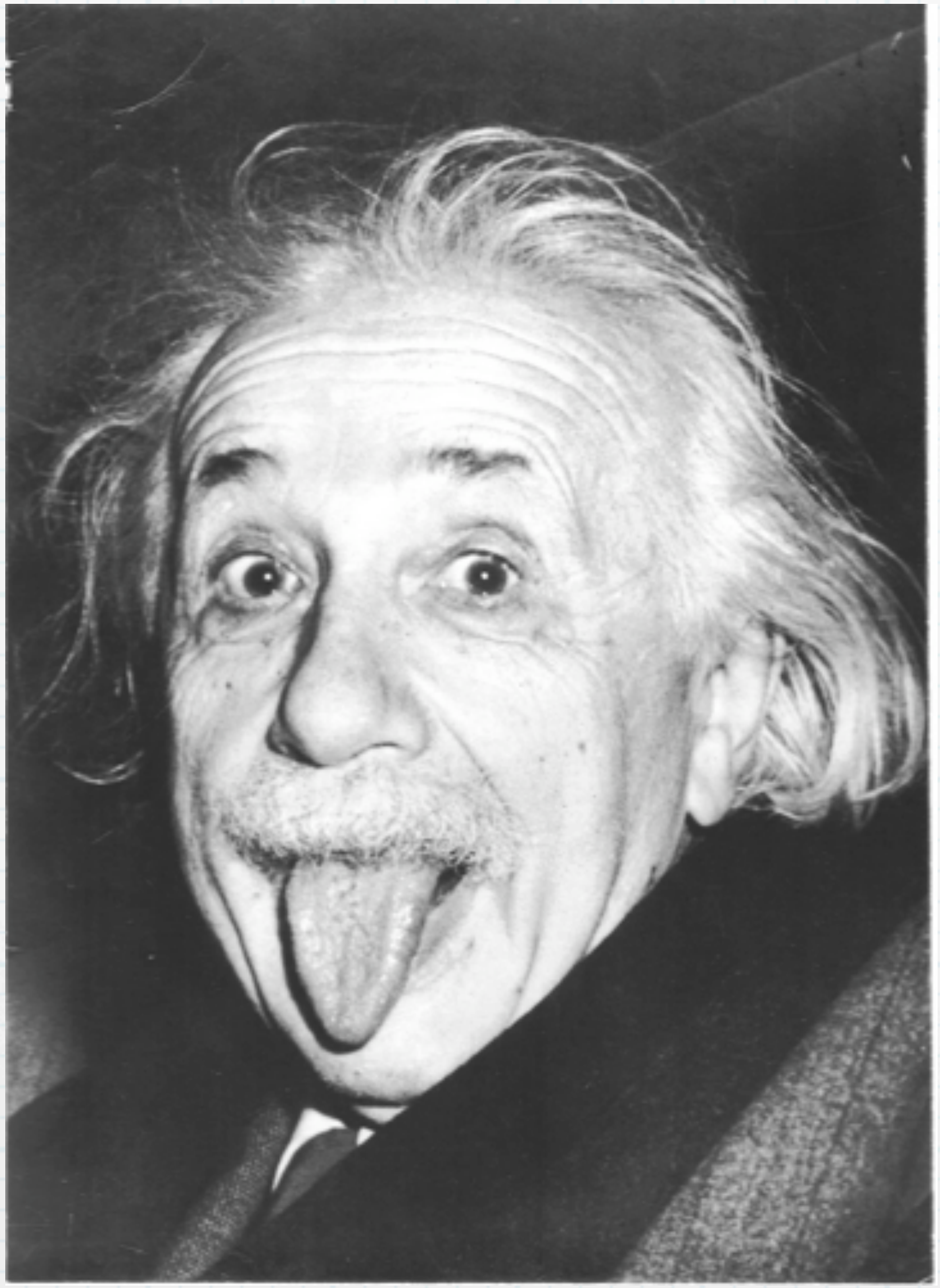


$$\pi P = \pi$$

Eigenvalue problem with eigenvalue 1. This is ok - a transition matrix has always 1 as the largest eigenvalue.

$$\pi = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.24 \\ 0.33 \\ 0.33 \end{pmatrix}$$

Singular value decomposition



Singular value decomposition