Coarse-Graining, Scaling, and Hierarchies

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1 INTRODUCTION

We present a scenario that is useful for describing hierarchies within classes of many-component systems. Although this scenario may be quite general, it will be illustrated in the case of many-body systems whose space-time evolution can be described by a class of stochastic parabolic nonlinear partial differential equations. The stochastic component we will consider is in the form of additive noise, but other forms of noise such as multiplicative noise may also be incorporated. It will turn out that hierarchical behavior is only one of a class of asymptotic behaviors that can emerge when an out-of-equilibrium system is coarse grained. This phenomenology can be analyzed and described using the renormalization group (RG) [6, 15]. It corresponds to the existence of complex fixed points for the parameters characterizing the system.

As is well known (see, for example, Hochberg and Perez-Mercader [8] and Onuki [12] and the references cited there), parameters such as viscosities, noise couplings, and masses evolve with scale. In other words, their values depend on the scale of resolution at which the system is observed (examined). These scale-dependent parameters are called *effective parameters*. The evolutionary changes due to coarse graining or, equivalently, changes in system size, are analyzed using the RG and translate into differential equations for the probability distribution function [8] of the many-body system, or the n-point correlation functions and the effective parameters. Under certain conditions and for systems away from equilibrium, some of the fixed points of the equations describing the scale dependence of the effective parameters can be complex; this translates into complex anomalous dimensions for the stochastic fields and, therefore, the correlation functions of the field develop a complex piece. We will see that basic requirements such as reality of probabilities and maximal correlation lead, in the case of complex fixed points, to hierarchical behavior.

This is a first step for the generalization of extensive behavior as described by real power laws to the case of complex exponents and the study of hierarchical behavior.

A system may exhibit many possible asymptotic behaviors, including hierarchical and standard power-law behavior. Which behavior the system attains on coarse graining depends on where the initial values of the physical parameters are located in the basins of attraction of the fixed points. Thus, if their initial values are within the appropriate region of parameter space, one could start with (real) power-law behavior and, via coarse graining, enter into the domain where the complex fixed point dominates and the system develops a phase characterized by hierarchical behavior. Thus, the various behaviors of a system can *emerge* as scale-dependent collective phenomena.[1] In principle, for a given system, all possible combinations can occur: not only from real to complex, but from real to real, from complex to real or from complex to complex. What and how it happens depends on the microscopic dynamics that underlie the system, the nature of the stochastic component and, as already mentioned, on the initial values for the physical parameters at a particular scale.

We will conclude that hierarchical behavior is a class of *emergent* critical behavior in many-body systems where there is, *in addition to complex fixed-points, maximal correlation* between the components in the system.

After some required definitions and examples, we will dwell on the phenomenology of hierarchical behavior, develop the theoretical basis to describe it and its emergence, and illustrate this through some examples existing in nature. All our work is analytic, there are no computer simulations, and the results can be derived from first principles.

[1] The scale can be a scale of length, differences in temperatures, or any other suitable dimensional parameter, as in the applications of the renormalization group to critical phenomena.

2 HIERARCHIES: DEFINITIONS AND EXAMPLES

2.1 DEFINITIONS

A hierarchy is a manifestation of ordered behavior in a given structure or physical system. We find such behavior in many examples and, broadly speaking, it is characterized by the property that these systems organize into sequential subsystems. These subsystems are not isolated, but are contingent upon both a larger subsystem and a smaller subsystem. In a hierarchic system the whole system is "composed of interrelated subsystems, each of the latter being in turn hierarchic in structure until we reach some lowest level of elementary subsystem." This definition, from Simon, is the one we will use for a hierarchy and a hierarchical system.[2]

Hierarchies are present in social organizations such as an army, a corporation, or a university; in biological systems such as cells or ecologies; and in the realm of astrophysical objects or in the forces known to control the universe as we know it today. They are commonplace in *complex systems*. From the presence of hierarchical organization in nature ranging from nuclei to atoms, molecules, macromolecules, cells, organisms, ecologies, planets, planetary systems, interstellar clouds, groups of stars, clusters of stars, galaxies, clusters of galaxies, and the whole universe, we infer that hierarchical behavior must be related to *differentiation, evolution, and adaptation* of the system to a given environment, which can be random in its nature and which could, itself, be hierarchical!

2.2 EXAMPLES

Perhaps the simplest and most illustrative example of a hierarchy is provided by the traditional Russian dolls known as "matryushky" (fig. 1). This traditional toy consists of a succession of hollow lacquered wooden dolls which, when opened, contain inside a similar doll, which once opened contains yet another doll and so on. There is a largest doll (limited in size only by how big the craftsman can afford to make it) and a smallest doll (limited in size by the skill of the craftsman). Seen from far away each of the dolls "looks" like the others except that it is built at a different scale; however, on closer inspection, they can be different not only in size but in the *details* of their painted dress and so on. The "matryushky" clearly fit the definition given above for a hierarchical system and visually illustrate that the members of a hierarchy can be different in their *particular* details.

Another example of a hierarchy is provided by the known forces and their strengths: the strong, electromagnetic, weak, and gravitational forces have very different strengths and the realms where they apply do not coincide.

[2] For a classical description of what is understood by a hierarchy see Simon [13].

FIGURE 1 The traditional Russian toy "matryushky" illustrates the notion of hierarchy.

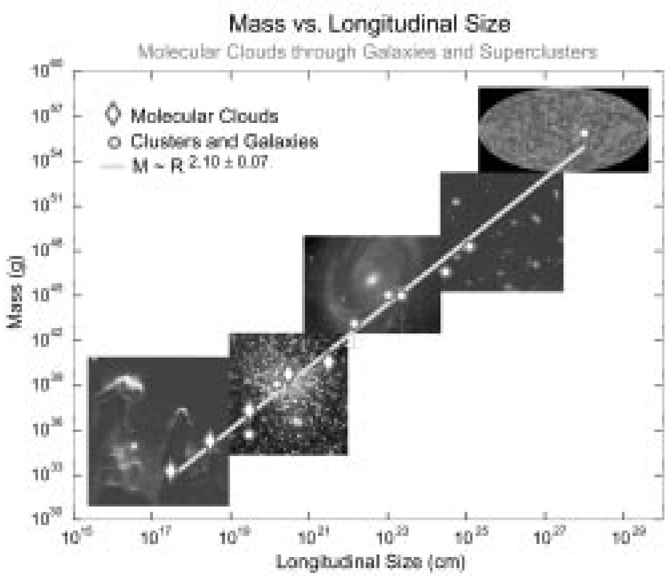

FIGURE 2 Plot of the data corresponding to the known classes of structures in the universe (larger than planetary systems). We show the average mass vs. average longitudinal size of structures from molecular clouds through galaxies to superclusters of galaxies and the cosmic microwave background radiation as well as their fit to a power law (exponent $= 2.10 \pm 0.07$).

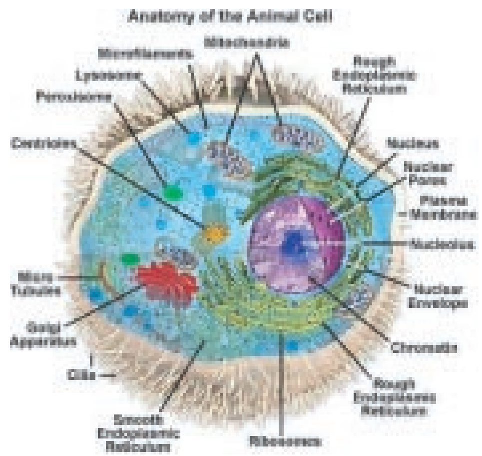

FIGURE 3 The eukaryotic cell. Today it is viewed as a complex network with multiple connections and functions, with hierarchical organization playing a basic role that could be fundamental to understanding its operation.

Yet they all are forces. In fact, their current observed properties suggest that they may all have derived from a unique unified force existing early in the history of the universe, and that the forces differentiated during the evolution of the universe. Their differentiation gave rise to the hierarchy observed today. The coarse graining in this example may have been provided (in partially understood ways) by the expansion of the universe together with the concomitant decoupling [9] of degrees of freedom.

But perhaps one of the clearest examples of hierarchical organization is provided by the universe at scales where gravity is important. We know that clusters of galaxies contain galaxies (of various ages and morphologies), which in turn contain huge molecular clouds, and globular and open clusters of stars. In some of these stars (although it is suspected for many of them) there are planetary systems which, as we will briefly discuss, are also hierarchically organized (fig. 2).

Many examples of hierarchical organization are known in biological systems: the eukaryotic cell itself (fig. 3) with its nucleus, ribosomes, and other corpuscles inside its cytoplasm, all contained within the cell membrane and wall, as well as the set of metabolic cycles and chemical reactions which take place to make it and keep it alive, are all magnificent examples of hierarchies of various types. And, of course, they provide extraordinary challenges when we attempt their mathematical/physical descriptions from first principles!

3 PHENOMENOLOGY OF HIERARCHIES IN CONDENSED SYSTEMS

Next we give a succinct description of the basic aspects of the phenomenology associated with hierarchies. These will help us identify the essential features present in hierarchical behavior. We will use them as guidelines to identify a theoretical framework appropriate for the description of systems and the conditions in which hierarchical behavior appears.

From the definition of hierarchy given above, it follows that they occur in many-body systems, in other words, in systems where there are many components interacting according to some physical rules. The rules summarize the effects of some force or they could describe the nature of some situation constrained by geometry or dynamical conditions, such as, in the case of nearest-neighbor interactions or near-to-nearest neighbor, or any other form of connection between system components that makes the set of many components into a many-body system.

Another generic feature of hierarchical behavior is that it manifests in systems where "the whole is more than the sum of its parts," that is, in systems experiencing *emergence*, a behavior that cannot be expected on the basis of the superposition (simple sum) of the individual properties attributable to each of the parts. For example, the emergence of the aroma in coffee comes from the interaction of the molecules of the alcaloids it contains.

These two basic tenets are, of course, not unique to hierarchical systems. There are other *systematic* properties that one associates with hierarchical systems. More specifically, hierarchical organization appears in *evolving* dynamical systems where the system undergoes some kind of transition from a nonhierarchical to a hierarchical organization. This organization is the result of the reshuffling of system components during the evolutionary process, and results as free energy is exchanged between system parts [2]. When we put this together with the association with emergence, we infer that hierarchies are related to (i) the spatiotemporal evolution of the complete system both of its parts and of the system as a collective, when all the components are involved, even if in different degrees. In hierarchical behavior, slow and fast, small and large size scales are simultaneously involved, although they may be *decoupled*, in the sense that fast degrees of freedom at a lower level in the hierarchy are averaged out and the slow motions one level up are "constant." In other words, (ii) scale diversity and scale changes are important when hierarchical behavior is at work. (Note that over the above qualitative description hovers the notion of "nonequilibrium.")

The association of spatiotemporal evolution, the decoupling of degrees of freedom, and scale changes implies that hierarchies should arise as a system is "coarse grained" [15], that is, as we change the resolution scale at which we study the evolving system. When this happens, and as is well known from the application of the RG to critical phenomena, clusters form within the system. This implies the existence of interfaces which can, in some instances, be energy rich and lead to emergent behavior at the interfaces. For dynamical critical phenomena, the reordering associated with coarse graining can *in some instances* also be related to a form of hierarchical behavior. Next we will study how this comes about in situations where complex fixed points of the renormalization group are present.[3]

In order to be self-contained, we will briefly discuss scaling in a physical system (section 4); then we will focus on some consequences of renormalization group scaling (section 5), and complex renormalization group exponents and hierarchies (section 6), then we will consider some applications (section 7); and finally we offer some conclusions.

4 SCALING IN A PHYSICAL SYSTEM

We say that a system displays scaling behavior when basic physical variables of the system, such as the free energy, satisfy a relation of the form

$$f(x) = \frac{1}{a} f(\lambda x)\,, \tag{1}$$

where f represents the physical variable, and x collectively represents the variables and parameters characterizing the physical system such as temperature or size; λ and a are numbers (for simplicity we will assume that $a \neq 1$). The number λ defines a change of scale from x to λx. The above functional equation has the solution

$$f(x) = C\, x^{\sigma}\,. \tag{2}$$

That is, $f(x)$ has power-law behavior in x with exponent σ. Substituting eq. (2) into eq. (1) we see at once that

$$\lambda^{\sigma} = a = e^{2\pi i n} \cdot a, \ \text{ with } \ n = 0, \pm 1, \pm 2, \ldots \tag{3}$$

where we have simply rewritten the unit as $e^{2\pi i n}$. Hence,

$$\sigma = \frac{\ln a}{\ln \lambda} + i \frac{2\pi}{\ln \lambda} \cdot n \equiv \sigma_R + i\sigma_I\,. \tag{4}$$

In other words, the exponent σ can be complex, with a *discrete* ("quantized") imaginary part proportional to $n = 0, \pm 1, \pm 2, \ldots$.

This can happen in many-body systems through the combined effects that the "regular" (interactions and causal evolution) and the "random" (fluctuations and uncertainties of various kinds) have on the probability distribution function

[3]This happens for nonunitary systems, where the Wallace-Zia theorem does not apply. Dynamical critical systems provide an explicit example of this. The author thanks R. Stinchcombe for a discussion on this point.

(pdf) for the system under consideration. In a stochastic field theory, for example, interactions and fluctuations (a) modify the n-point correlation functions of the fields at different space-time points, and (b) lead to divergences in the original set of parameters such as diffusion constants, masses, coupling constants, or noise parameters which serve phenomenologically to characterize the system. It follows from (a) and (b) that the originally Gaussian pdf is modified in a way which is scale dependent [8].[4] It can be shown that the system enters asymptotic scaling regimes with exponents which are calculable from the original equations and boundary or initial conditions describing the regular, the random, and their interrelation with the environment where evolution takes place.

The pdf is related to $Z[J]$, the characteristic functional (also known as the "generating functional") for the n-point correlation functions. In field theory, this is obtained from the action $S[\phi]$ by performing a path integral over all field configurations:

$$Z[J(\vec{x},t)] = \int [d\phi] e^{iS[\phi]+i\int dx^d \, dt \phi(\vec{x},t)J(\vec{x},t)} \,. \tag{5}$$

Here $\phi(\vec{x},t)$ represents the field value at a space-time point with coordinates $(\vec{x},t)$ in a $(d+1)$-dimensional space time and $J(\vec{x},t)$ is the classical source for the field. When dealing with systems away from equilibrium, there are many technical complications for the application of these techniques, but the essence of this part of the procedure is completely captured by eq. (5). The correlation functions are essentially obtained by taking functional derivatives of $Z[J]$ [16].

4.1 SCALING AND THE RENORMALIZATION GROUP

Requiring independence of the physics from the choice of arbitrary length or momentum[5] scale [6], immediately leads to a partial differential equation satisfied by the pdf, and, therefore, by the Green's functions (see eq. (6) below) that describe the statistical properties of the many-body system. The coefficient functions of this partial differential equation are related to the ordinary differential equations satisfied by the effective parameters. The scale dependence induced on the parameters is precisely the one necessary to cancel the overall scale dependence of the Green's functions. The RG equation satisfied by the n-point correlation function $G^{(n)}(r_1,...,r_n;\alpha_j(\lambda);\lambda)$ of the stochastic field ϕ is

$$\left[-\lambda\frac{\partial}{\partial\lambda} + \sum_i \beta_{\alpha_i}\frac{\partial}{\partial\alpha_i} + \frac{n}{2}\gamma(\alpha_j)\right] G^{(n)}(r_1,...,r_n;\alpha_j(\lambda);\lambda) = 0 \tag{6}$$

[4] Divergences need to be subtracted. This introduces an arbitrary reference scale into the system. Requiring that the physics remains independent of this arbitrary scale is the statement mathematically represented by the Renormalization (semi) Group [6].

[5] One can also perform these operations in momentum space and instead use a momentum scale $\mu \sim 1/\lambda$. This is often more convenient. We are using λ in order to keep closer to the more intuitive notion of "scale" as identified with "size."

and the effective couplings α_i satisfy

$$\lambda \frac{d\alpha_i}{d\lambda} = -\beta_{\alpha_i}(\alpha_j)\,. \tag{7}$$

These β-functions (one per independent coupling α_i) are calculable in perturbation theory, and they summarize the effects of interactions and fluctuations on the couplings. The solutions to eq. (7) are the effective parameters. Just as the couplings become scale dependent, so do the fields themselves, which have a scale-dependent dimension and, therefore, deviate from the canonical value for free (noninteracting) fields. The quantity $\gamma(\alpha_j)$ in eq. (6) denotes that deviation, and for that reason receives the name of the *anomalous* dimension of the field ϕ. Put differently, γ is to the stochastic field ϕ what the $\beta_{\alpha_i}(\alpha_j)$ are to the couplings α_i.

The RG equation (6) together with eq. (7) can be immediately integrated using the method of characteristics. One finds that

$$G^{(n)}(r_1, r_2, ..., r_n; \{\alpha_j(\lambda)\}; \lambda) = \exp\left(-\frac{n}{2}\int_1^s \frac{d\lambda'}{\lambda'}\ \gamma_\phi(\lambda')\right) \times G^{(n)}(r_1, r_2, ..., r_n; \{\alpha_j(\lambda_0)\}; \lambda_0)\,. \tag{8}$$

This form of the solution will be important later, when we discuss the connection between evolution, emergence, and scaling behavior leading, in some cases, to hierarchical behavior.

Many out-of-equilibrium stochastic field theories contain various typical terms: a combination of reaction-diffusion terms and stochastic forcing. In other words, they contain a form of the standard parabolic diffusion partial differential equation modified by several local monomials of the field. The monomials can have stochastic couplings and the system is said to be subject to multiplicative noise. When the stochastic term is additive, the contribution from the noise can be interpreted as a stochastic forcing term. In a many-body system, where we do not have access to *all* the variables, one adopts the strategy of following the deterministic evolution of a small subset of all possible variables, but the evolution of these variables may depend *also* on the evolution of the variables over which we have averaged and given up following their deterministic evolution. Thus, the initial values of time $t = 0$ of the variables that we follow do not fully determine their evolution. The stochastic terms are a means of incorporating into the dynamics the effects from degrees of freedom that may not have fully decoupled from the dynamics. We do that by representing the noise through the statistical properties of *its* probability distribution function.[6] These equations, as is well known, describe an amazing range and variety of systems.

It is known that complex fixed points do appear in these systems. We will illustrate this using an important example.

[6]We note here that the effects of initial and boundary conditions can also be incorporated into the noise.

4.2 EXAMPLE: A FORCED FLUID AND DYNAMICAL CRITICAL PHENOMENA

Many of the features just mentioned are represented in some regimes of the Navier-Stokes equations for the hydrodynamics of a stirred fluid subject to damping proportional to the velocity. A particularly important example is provided in the case of stirred and damped potential flow. Here, the Navier-Stokes equations take the form

$$\frac{\partial \vec{v}}{\partial t} + \lambda \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = \nu \nabla^2 \vec{v} + \left(\frac{\zeta}{\rho} + \frac{1}{3}\nu \right) \vec{\nabla}(\vec{\nabla} \cdot \vec{v}\,) - \frac{1}{\rho}\vec{\nabla} p + \vec{f}(\vec{x}, t)\,, \tag{9}$$

with the standard terms describing the evolution of the velocity field, but with an extra term $\vec{f}(\vec{x}, t)$ containing both linear damping and a stirring force according to

$$\vec{f} = -m^2 \vec{v}(\vec{x}, t) - \vec{\nabla}\xi(\vec{x}, t)\,. \tag{10}$$

(The convective coupling λ in eq. (10) gets modified under renormalization, and should not be confused with the length scale λ introduced above.) Here ρ is the density, m is a masslike damping coefficient (which introduces an explicit length scale in the problem), and $\xi(\vec{x}, t)$ is the stirring potential, which is a Gaussian stochastic field defined by its correlation function which, in this example, we take as (using its Fourier decomposition)

$$\langle \xi(\vec{k}, \omega) \rangle = 0\,, \tag{11}$$

and

$$\langle \xi(\vec{k}, \omega)\xi(\vec{k'}, \omega') \rangle = 2\tilde{D}(\vec{k}, \omega)(2\pi)^{d+1}\delta^{(d)}(\vec{k} + \vec{k'})\delta(\omega + \omega') \tag{12}$$

together with

$$\tilde{D}(\vec{k}, \omega) = D_0 + D_\theta \left(\frac{k}{\Lambda} \right)^{-2\rho} \left(\frac{\omega}{\nu\Lambda^2} \right)^{-2\theta} . \tag{13}$$

The two couplings D_0 and D_θ control the amplitude of the noise, which we have taken as colored in space and in time with exponents ρ and θ. The quantity Λ is a momentum cutoff.

These equations play an important role in many problems, ranging from astrophysics, the chemistry of the origin of life, to directed percolation to polymers. They can be coarse grained [3, 4] using the standard techniques of the dynamical renormalization group (DynRG) [12]. This allows one to follow the changes in scale of the various parameters appearing in the above equations.

Even for the simplest case of generalized potential flow ($\vec{v} \propto \nabla\psi$) and a simple equation of state relating p and ρ, and relatively simple densities, these equations find application in many problems. For the purpose of applying the RG to the resulting equations it is convenient to select as variables the following dimensionless combinations of the couplings ($K_d = S_d/(2\pi)^d$ and S_d is the volume of the d-dimensional sphere)

$$V = 1 + \frac{m^2}{\nu\Lambda^2}\,, \tag{14}$$

$$U_0 = \lambda^2 D_0 K_d \frac{\Lambda^{d-2}}{\nu^3}\,, \tag{15}$$

$$U_\theta = \lambda^2 D_\theta K_d \frac{\Lambda^{d-2-2\rho-4\theta}}{\nu^{3+2\theta}}\,, \tag{16}$$

which become the effective parameters satisfying the RG equations from which one can examine the behavior of the system on coarse graining. Then one can determine the fixed points toward which the system tends asymptotically.

The equations for the evolution of the couplings as functions of the scaling variable s defined through $\vec{x} \to s\vec{x}$, are given in this example by

$$\frac{\mathrm{d}\nu}{\mathrm{d}\log s} = \nu\left[-\frac{d-2}{4d}U_0 - \frac{d-2-2\rho}{4d}U_\theta(1+2\theta)\sec(\theta\pi)\right]\,, \tag{17}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\log s} = \lambda\left[-\frac{U_\theta}{d}\theta(1+2\theta)\sec(\pi\theta)\right]\,, \tag{18}$$

$$\begin{aligned}\frac{\mathrm{d}U_0}{\mathrm{d}\log s} &= (2-d)U_0 + \frac{U_0^2}{2d}(2d-3) + \frac{U_\theta^2}{4}(1+4\theta)\sec(2\pi\theta) \\ &\quad + \frac{U_0 U_\theta}{4d}(5d-8\theta-6\rho-6)(1+2\theta)\sec(\pi\theta)\,,\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\mathrm{d}U_\theta}{\mathrm{d}\log s} &= (2-d+2\rho+4\theta)U_\theta + \frac{U_0 U_\theta}{4d}(d-2)(3+2\theta) \\ &\quad + \frac{U_\theta^2}{4d}[-8\theta + (d-2-2\rho)(3+2\theta)](1+2\theta)\sec(\pi\theta)\,.\end{aligned} \tag{20}$$

For some ranges of values of the noise parameters this system of equations has both real and complex solutions [3, 4].

It follows from eq. (8) that when approaching a fixed point the exponential prefactor in the n-point function becomes

$$\begin{aligned}\exp\left(-\frac{n}{2}\int_1^s \frac{d\lambda'}{\lambda'}\ \gamma_\phi(\lambda')\right) &\longrightarrow \exp\left(-\frac{n}{2}\gamma_\phi^*\int_1^s \frac{d\lambda'}{\lambda'}\right) \\ &= \exp\left(-\frac{n}{2}\gamma_\phi^* \ln s\right) = \left(\frac{\lambda}{\lambda_0}\right)^{-\frac{n}{2}\gamma_\phi^*}\end{aligned} \tag{21}$$

where γ_ϕ^* is the value of the anomalous dimension of the stochastic field at the fixed point. In the case of a complex fixed point, γ_ϕ^* is a complex number.

As will be seen below, this limit is important in understanding how hierarchies are generated through coarse graining.

5 SOME CONSEQUENCES OF RENORMALIZATION GROUP SCALING

The fixed points follow from eqs. (17)–(20) by simply setting them equal to zero and finding the roots of the resulting system of nonlinear algebraic equations. The physical parameters characterizing the physical system will be attracted to fixed points in the UV-regime (short distance or $s \to 0$) or in the IR-regime (long distance or $s \to \infty$) depending on whether the fixed point is UV attractive or IR attractive. As with any autonomous set of differential equations, other possibilities do exist. From the fixed points and their asymptotic behaviors follows the behavior of the correlation function as it is implicit in eq. (8).

Which fixed point the system is attracted to or repelled from depends on the properties of the basin of attraction where the initial conditions for the DynRG equations are located in parameter space.

For example, one may start at a given scale with a particular behavior at this system size; as the system is coarse grained, the parameters can evolve into a completely different set of values which correspond not only to a completely different quantitative behavior, but also to a completely different qualitative behavior. In this case we can talk of the *emergence* of "novel" behavior.

How can this *emergent* behavior be phenomenologically detected? Since the values of the couplings at the fixed points determine the anomalous dimensions, a strategy is to measure the correlation functions in the scaling regime associated with the approach to the fixed point. As is seen from eq. (8) and the discussion following (2), if the anomalous dimensions are complex, then there is some form of "wiggly" behavior (log-periodic) in the scale dependence of the correlation function. Thus the various *local maxima* of the correlation function would correspond to the regions where the physical system has maximal correlation. These maxima would happen at *discrete scales* which could be classified as members of a discrete *sequence*. We also note that the "wiggly" behavior is modulated by a power law. We now explore this in some detail.

6 COMPLEX RENORMALIZATION GROUP FIXED POINTS, THE TWO-POINT CORRELATION FUNCTION, AND HIERARCHIES

Let us now focus our attention on the two-point correlation function. This correlation function is the easiest to measure in a many-body system. Its phenomenological interpretation is that it represents *the joint probability of finding two objects located in two independent volume elements*. It is important to note that since the two-point correlation function is interpreted as a probability, it *must* then be *real*.

6.1 ASYMPTOTIC BEHAVIOR OF THE TWO-POINT CORRELATION FUNCTION

We ask the question, "how does the two-point correlation function behave when, under coarse graining, the system enters the basin of attraction of a complex fixed point?" From eqs. (8) and (21), in the neighborhood of a fixed point the two-point correlation function has the limit

$$G^{(2)}(r_1, t_1; r_2, t_2) \propto |r_1 - r_2|^{2\chi} F\left(\frac{|t_1 - t_2|}{|r_1 - r_2|^z}\right), \tag{22}$$

where the so-called "roughening" and "dynamic" exponents χ and z are simple arithmetic combinations of anomalous, engineering and the number of spatial dimensions [1]. The function $F(u)$ is a universal function with the following behavior as its argument approaches the indicated limits

$$\lim_{u\to 0} F(u) \propto \text{constant}, \tag{23}$$

$$\lim_{u\to\infty} F(u) \propto u^{2\chi/z}. \tag{24}$$

To simplify our discussion let us take the limit when $|r_1 - r_2| \to \infty$ for a fixed time interval (the $u \to 0$ limit in the above equations). Then the two-point correlation function becomes

$$G^{(2)}_\infty(r, t; r', t) \propto |r - r'|^{2\chi}. \tag{25}$$

Assuming now that $\chi = \alpha + i\zeta$ we get

$$\begin{aligned} G^{(2)}_\infty(r, t; r', t) &= \mathrm{Re}\, \tilde{c}\, e^{i\beta} \left|\frac{r - r'}{r_0}\right|^{2(\alpha+i\zeta)} \\ &= c \cdot |r - r'|^{2\alpha} \cdot \cos\left[\beta + 2\zeta \log\left(\frac{|r - r'|}{r_0}\right)\right]. \end{aligned} \tag{26}$$

In other words, and as advertised, asymptotically the two-point correlation function is a log-periodic function modulated by a power law [5] and [14]. The various constants c, r_0, and β depend on the specific problem that we are considering. However, the two exponents α and ζ are calculable through the renormalization group and depend on *general* features of the system and, perhaps, they are *universal*, although a classification of dynamical critical phenomena in terms of classes of universality to date is not available [12].

6.2 MAXIMAL CORRELATION

Where does maximal two-point correlation occur? This happens for region sizes given by

$$|r_{(n)} - r'| = r_0 \cdot \exp\left\{ \frac{1}{2\zeta} \left[\tan^{-1}(\alpha/\zeta) - \beta\right] \right\} \cdot (e^{\frac{\pi}{2\zeta}})^n \tag{27}$$

or

$$|r_{(n)} - r'| \equiv A \cdot b^n \,, \tag{28}$$

with $n = 0, \pm 1, \pm 2, \ldots$. Here A is not fully predictable by the RG, but the "base" b is fully calculable, and both its presence and specific value are a result of the coarse graining.

Equation (27) is our main result. It shows that in the presence of complex fixed points of the renormalization group there are regions of finite size within finite-sized regions, all of which are calculable once the size of any one region is known. The separation between the regions depends only on general properties of the dynamics and noise.

In short,

1. there is a *hierarchy* of regions of average linear sizes $R_{(n)} \equiv |r_{(n)} - r'|$ where correlation on that scale is favored; and
2. the hierarchy can be *classified* according to an *integer* n, with each member of the hierarchy having a unique number.

7 SOME APPLICATIONS

In this section we explicitly illustrate the above using examples occurring in nature. We will consider only two examples from astrophysics, where the equations given in section 4.2 do apply [3, 4], and will mention a few examples where we could conceivably extend the methods described here.

7.1 THE UNIVERSE: FROM MOLECULAR CLOUDS IN THE INTERSTELLAR MEDIUM TO THE HORIZON

Matter in the universe clumps in a variety of structures which are contained within larger structures which, in turn, are contained within even larger structures. This observational property leads us to think of the universe as "hierarchical," in other words, as made up of structures contained within structures (cf. Goldman and Perez-Mercader [7] and references cited there). For astrophysical objects, this property manifests itself for at least 10 orders of magnitude in longitudinal size and more than 20 orders of magnitude in mass. Such regularity, from

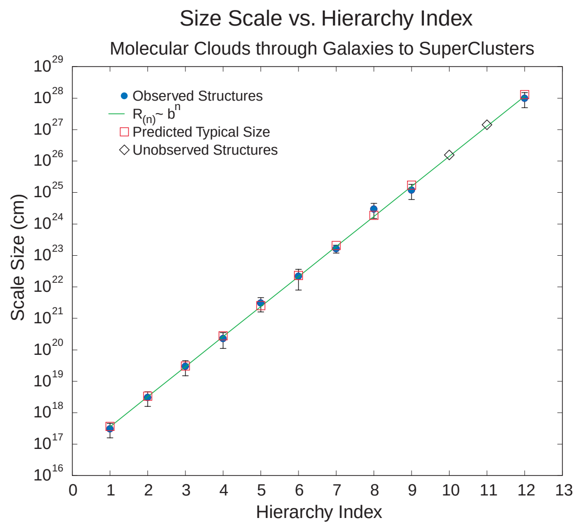

FIGURE 4 Longitudinal size scale vs. hierarchy index n for discrete structures in the universe from molecular clouds through galaxies to superclusters of galaxies and predictions from eq. (28). The value of b is 9.02 ± 0.24.

molecular clouds in the interstellar medium (ISM) to superclusters of galaxies and beyond, is remarkable (see fig. 2).

It is also known that *within each class* in this hierarchy, the two-point density fluctuation correlation function displays power-law behavior in object separation with an exponent value which, within observational errors, is the same for all the classes and, therefore, perhaps "universal," in spite of the different specific mechanisms that may be at work in the formation of each of the classes. For details see Goldman and Perez-Mercader [7]. In systems where gravity plays a dominant role, the appropriate order parameter is the local fluctuations in matter density, denoted by $\delta\rho(\vec{r}, t)$. Assuming that under coarse graining the gas and dust system develops a complex fixed point (this assumption can be made on the basis of the fact that the equations describing the evolution of the system at large scales can be written in the form (9) which is known to have them), then the two-point correlation function acquires a complex exponent.

Fitting the data to all the known structures gives the results shown in figures 2, 4, and 5. We can see that the results are in excellent agreement with observations.

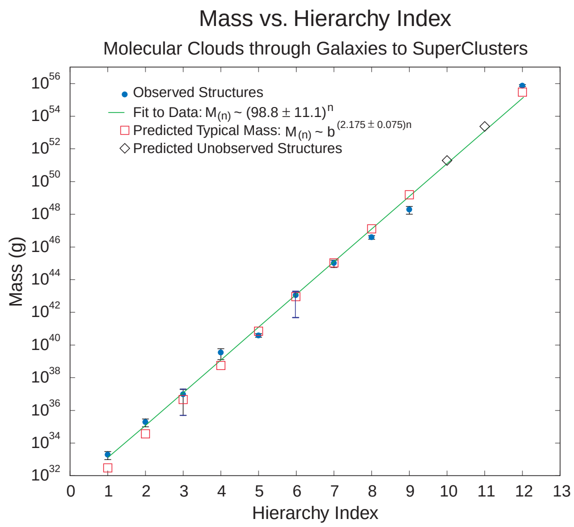

FIGURE 5 Mass vs. hierarchy index n for discrete structures in the universe from molecular clouds through galaxies to superclusters of galaxies and predictions from eq. (28). The value of b is 9.02 ± 0.24.

7.2 PLANETARY SYSTEMS: THE TITIUS-BODE LAW IN THE SOLAR SYSTEM

In its original form, the famous Titius-Bode law of planetary distances was expressed so that the radius of the orbit, $r(n)$, of a given planet in units of 0.1 AU, is given by eq. (2), (4), and (5)

$$r(n) = 4 + 3 \times 2^n \tag{29}$$

where $n = -\infty$ for Mercury, and $0, 1, 2, \ldots$ for the other planets ordered according to the sizes of their orbits around the Sun [10].

The "law" is phenomenological and in modern times is restated as follows: the (reduced) distance $d(n)$ to the nth planet from the Sun[7] is given by the power law:

$$d(n) = A \times B^n \times f(n) \tag{30}$$

where $A = 44$ (which is equivalent to a scale of 0.205 if distances are expressed in astronomical units instead of the size of the central object), $B = 1.73$, and

[7]Here $d(n) \equiv (r(n) - R_0)/R_0$, where $r(n)$ is the radius of the orbit of the nth object and R_0 the radius of the central object.

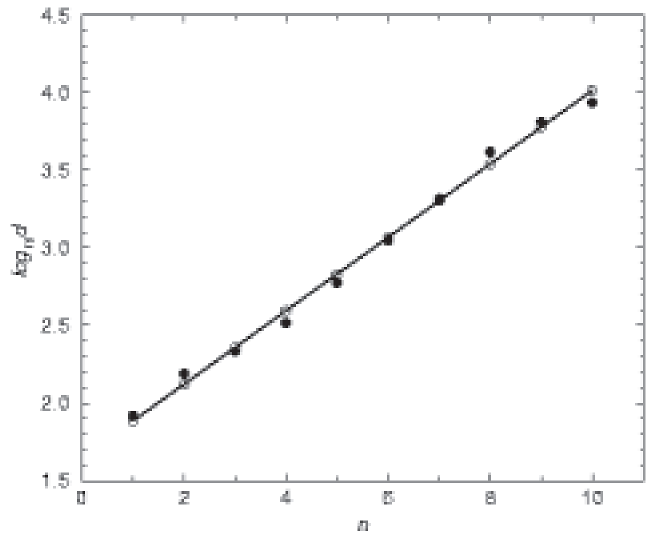

FIGURE 6 The Titius-Bode law. Predicted (open circles) and observed (filled circles) positions of the planets in the solar system. Predicted positions (and the straight line) are computed using the power law with $B = 1.73$, as derived without any object in location $n = 5$ (Asteroid Belt) and $n = 10$ (Pluto). The remaining points represent the other planets.

$f(n)$ oscillates around 1 with an amplitude of about 0.1.[8] The power-law piece of the Titius-Bode "law," $A \times B^n$, is of the form obtained in eq. (28) from our consideration of the properties of the two-point correlation function. The same phenomenology also holds for the average radii of the orbits of the satellites of Jupiter and Saturn. The values of the best fitted B are different than in the case where the central body is the Sun, but they are of the *same* order of magnitude.

The observed positions of the planets and the predictions of the power law in eq. (30) are shown in figure 6.

We see that the kind of hierarchical behavior obtained from maximal correlation in nonequilibrium phenomena, is very good for describing data at the largest scales in the universe, as well as in the solar system. In fact, one can show that for the only multiplanetary system known to date, υ-Andromedæ, this law also holds and predicts the existence of planets in the system at particular orbits.

[8]The oscillatory behavior can be ascribed [10] to "a point-gravitational or tidal evolution starting from *after* the planets were formed" (our italics); this has been interpreted as a result of the chaotic evolution of the solar system through its history.

7.3 OTHER POSSIBLE APPLICATIONS

Of course, many other hierarchical systems are known: metabolic pathways, reaction networks, proteins, cells and ecologies in biology, fault systems and rivers in geology or corporations and urban areas in socioeconomy, to just mention a few examples. But one particularly interesting and important system where these ideas may find application is in the context of the random networks associated with life.

Recent discoveries made in biology indicate that living systems are organized into a genome, transcriptome, proteome, and metabolome. There is strong evidence that each of these systems is hierarchical by itself, and, in addition, there are modes of interaction among them characterized by very different time scales. This has led to the proposal of a "Life's Pyramid of Complexity" [11], where hierarchical organization is the key to providing the functionality for life to exist and survive. At the same time, hierarchical organization is essential to help explain the observed "bottom-up" universality of life as it uses motifs and pathways assembled into functional modules and large-scale organizational systems such as the cell itself. But hierarchical organization is also the mechanism that seems to operate the "top-down" organism specificity that is observed in the living world.

Is there a way of applying the notions of hierarchical organization emerging during coarse graining, as discussed here, to what is observed in living systems? It would be the manifestation of some coarse-grained dynamics occurring in some particular environment.

This is, of course, an extremely ambitious challenge and a formidable problem, but perhaps some of the notions discussed here could help: the unification of nonlinear stochastic partial differential equations and the renormalization group is a powerful recipe of two separate ideas, each of which has its own trandition of success in science. Together they are more than their simple sum.

8 CONCLUSIONS

We have seen that coarse graining a nonequilibrium many-body system leads to several classes of scaling behavior. The phase transitions that take place in the coarse-graining process can be interpreted as manifestations of emergence. Emergence occurs as the system transits from one fixed point of the RG to another one. In particular, and as is well known, scaling behavior will be reached as one approaches the fixed points. In addition, the type of behavior depends crucially on the properties of the fixed point: for real fixed points one has the usual phenomenology also seen in equilibrium phase transitions.

In out-of-equilibrium systems, the presence of complex fixed points leads to a far richer phenomenology, and this phenomenology naturally describes hierarchical systems. In fact, one sees power-law behavior modulated by log-periodic corrections in the correlation functions. The modulation leads to "gaps" where the correlation is minimized, and "clumps" where the correlation is maximized. Because of this mathematical form of the correlation functions, one has clumps within clumps separated by gaps, which is precisely the structure corresponding to hierarchical behavior. Furthermore, given the properties (such as size) of just one of the members in this hierarchy, one can characterize the full hierarchy.

The above can be shown to describe reasonably well some natural phenomena, which we have illustrated by examining what is known about the universe at scales where gravity plays an important role. In planetary systems where gravitational effects are important but they are in competition with other phenomena like magnetic fields and collective effects such as turbulence, these quantitative arguments also apply, as in the case of the famous Titius-Bode law.

The scenario presented here can be generalized to any extended system where coarse graining can be applied and where complex exponents are present. This includes geophysical, biochemical, biological, and ecological phenomena.

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