Complex Systems Summer School 2012

SANTA FE INSTITUTE

One nature - One science

Turning Points

Course graining

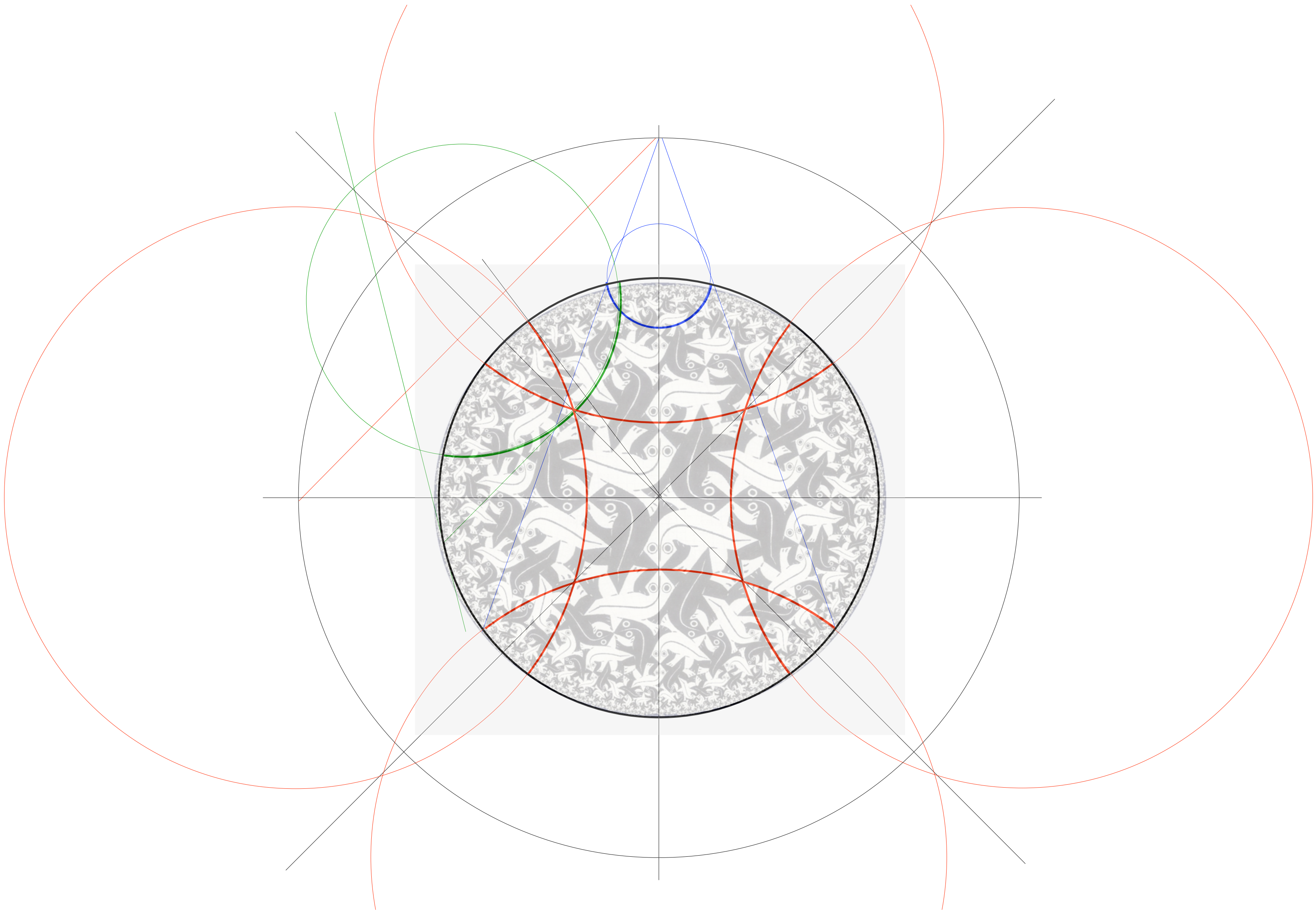
Subjects

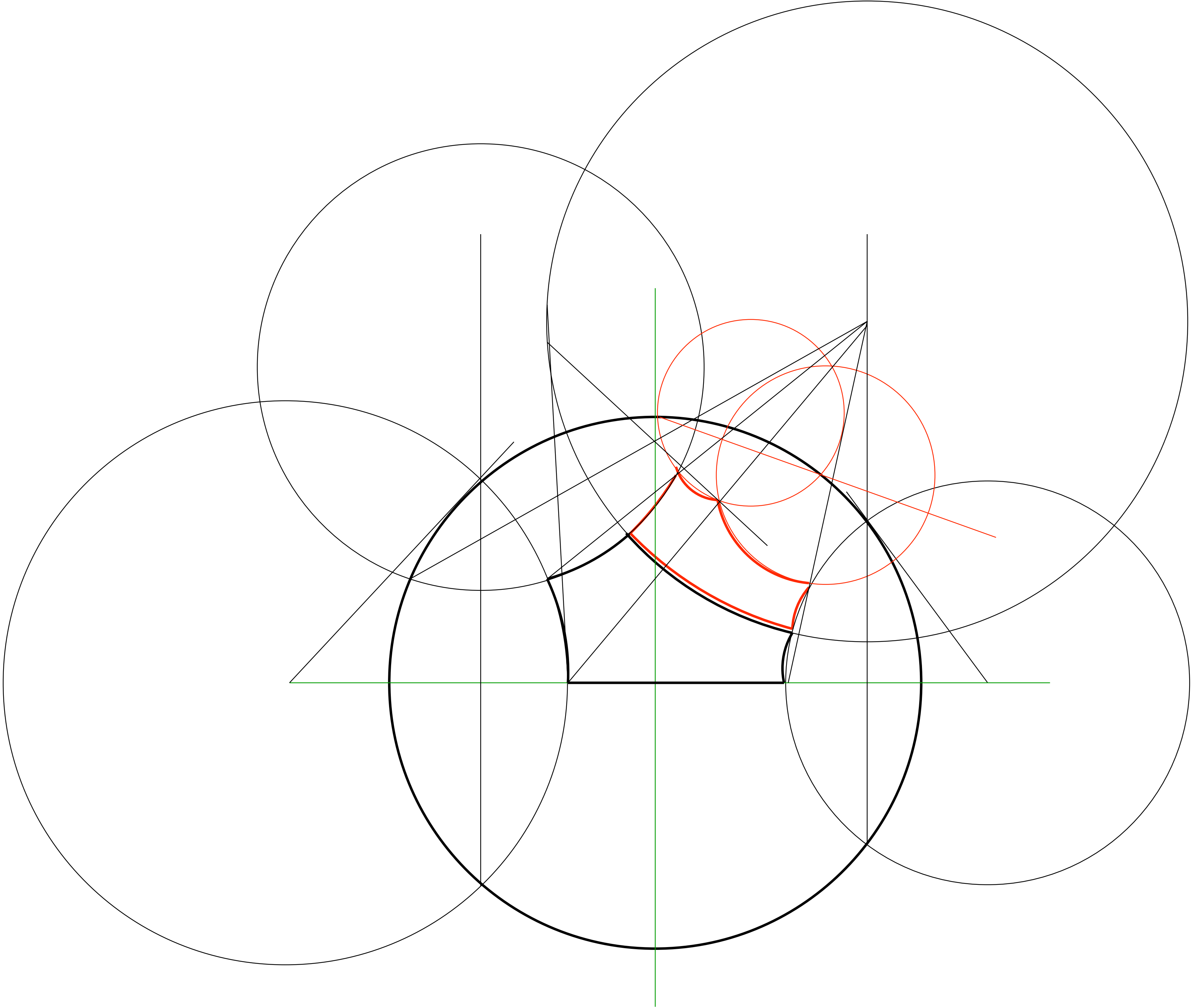
- From Art to Maps
- Entropy once more
- Maximal Entropy principle
- From microscopics to macroscopics and back
- Phase transitions/ symmetry breaking
- Nonlinear dynamics and information
- Food for thought

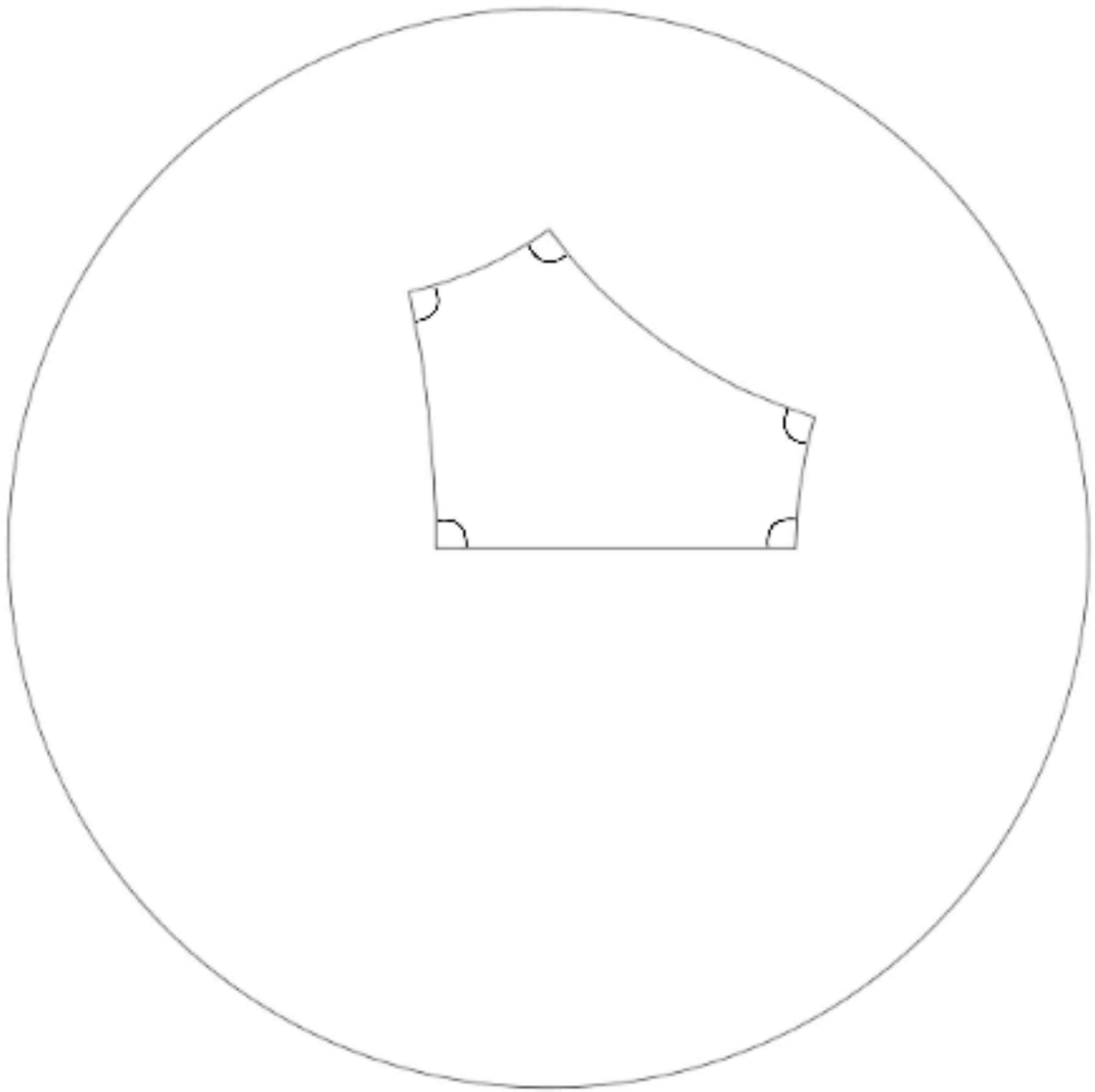
Escher

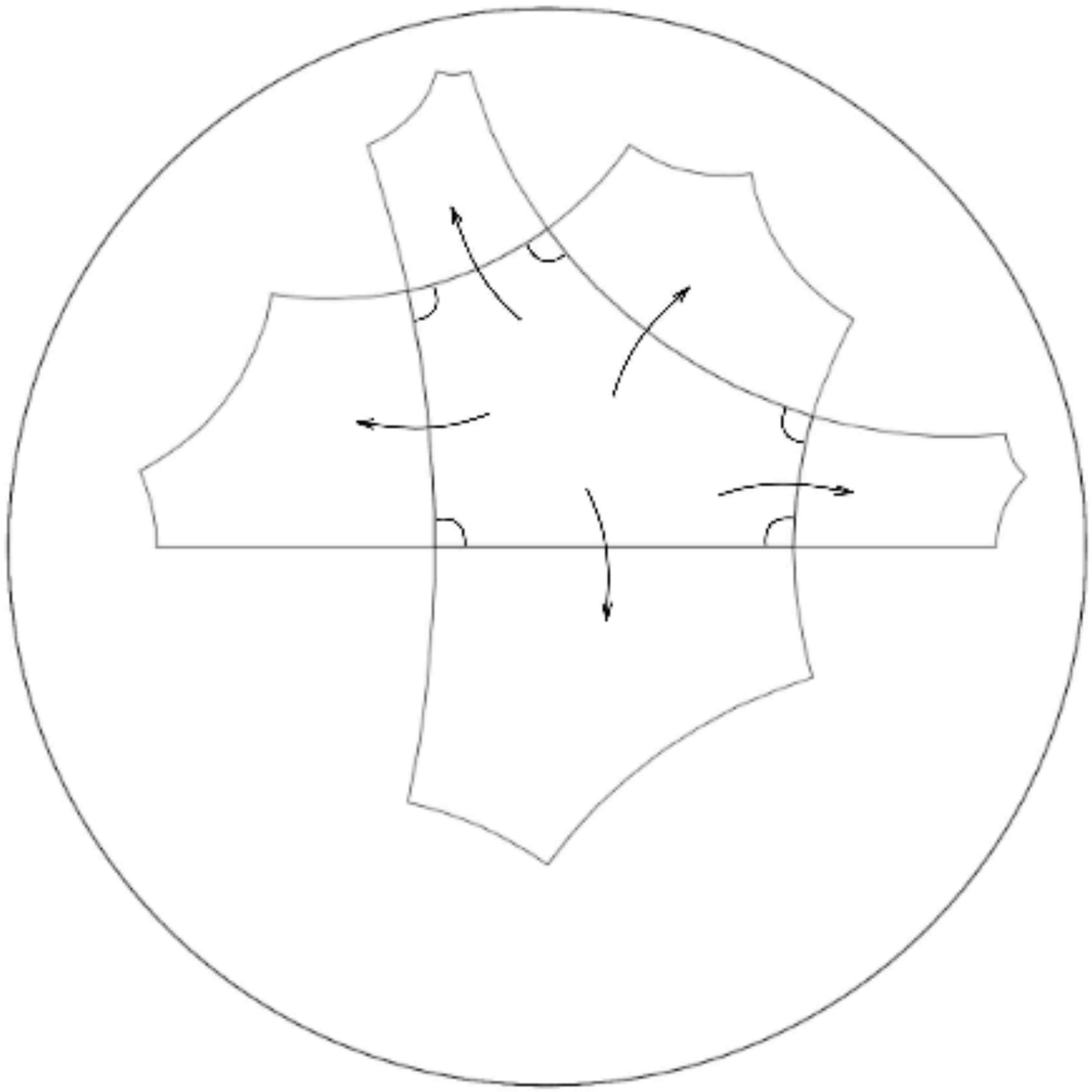
Escher

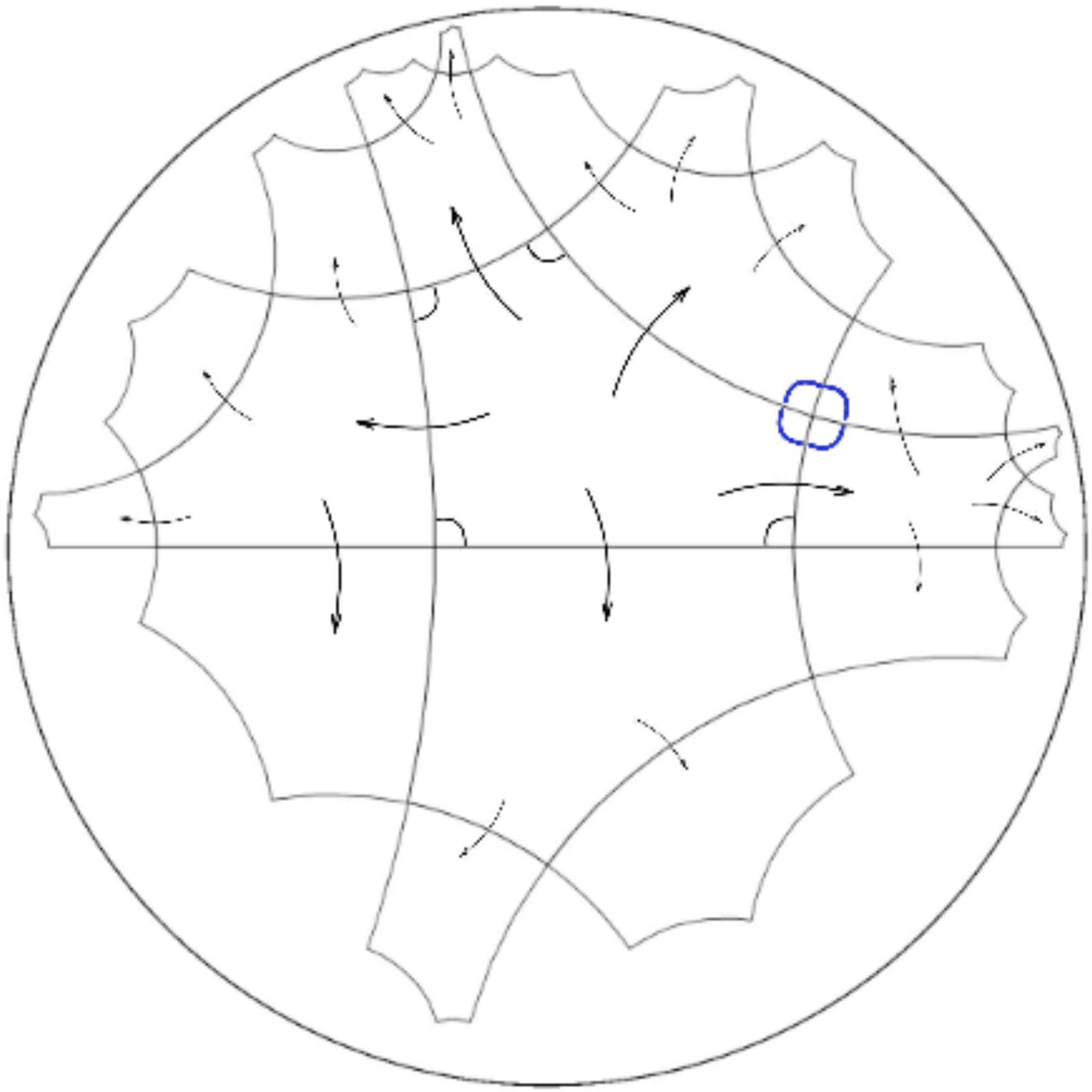
Escher

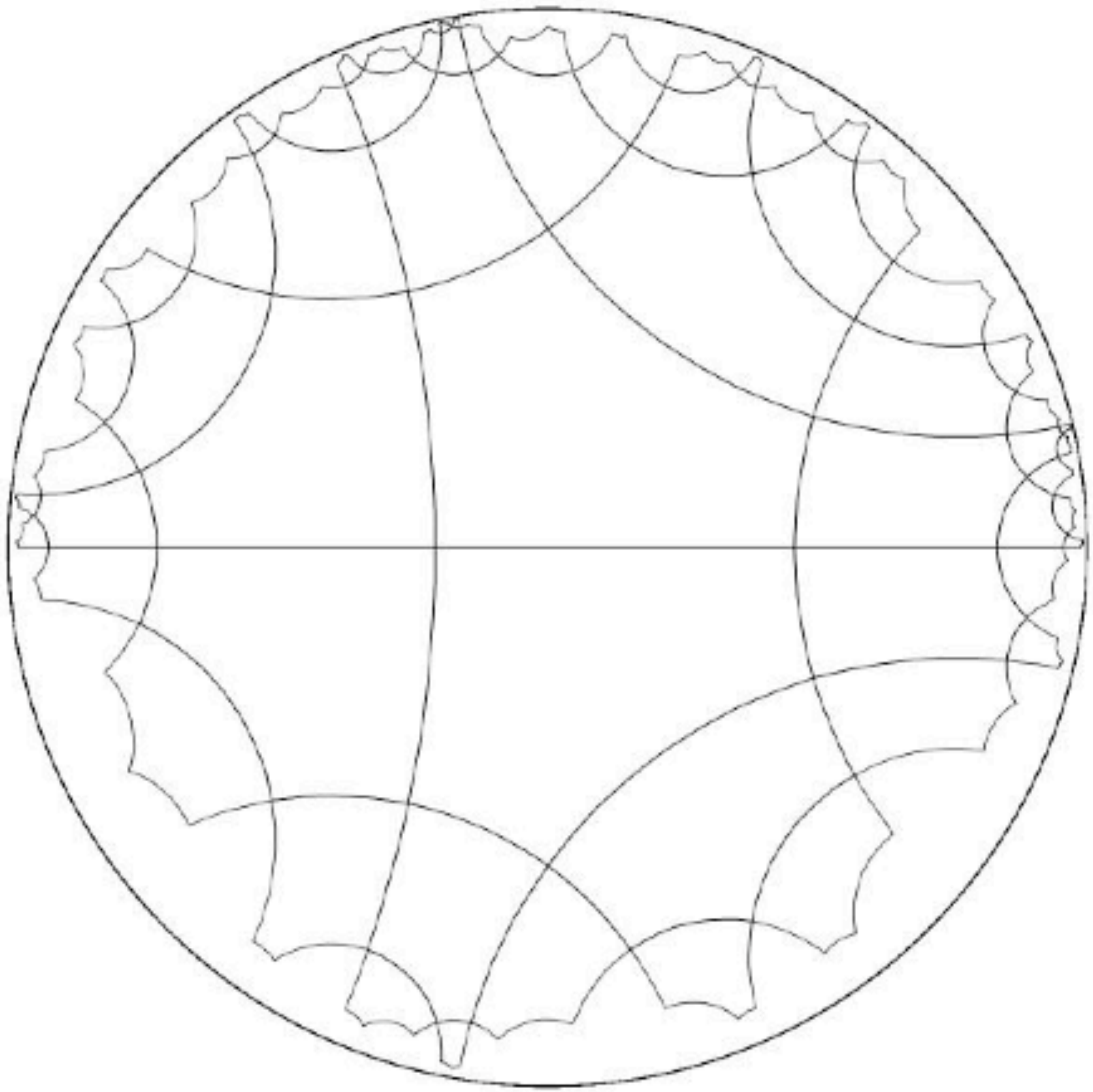


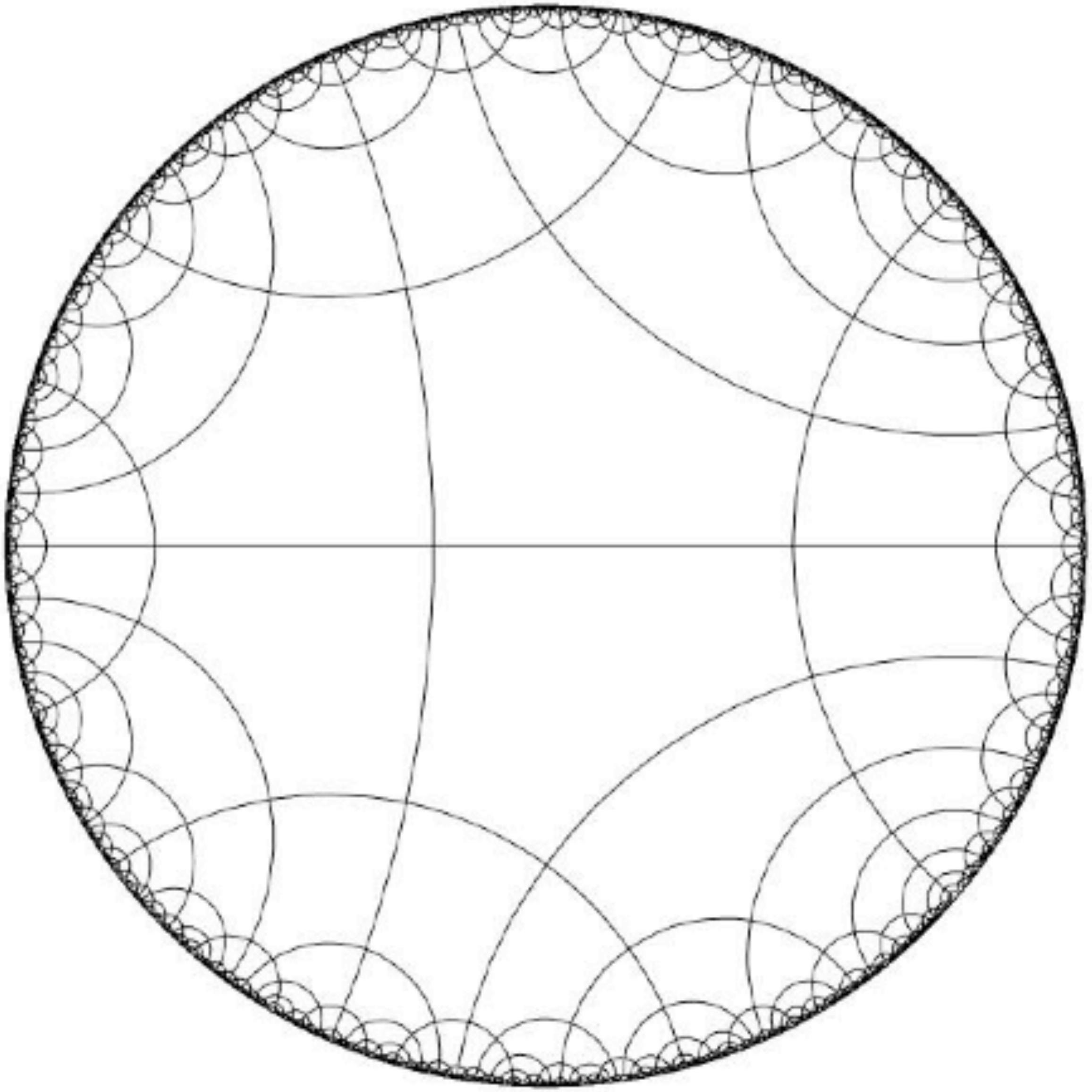






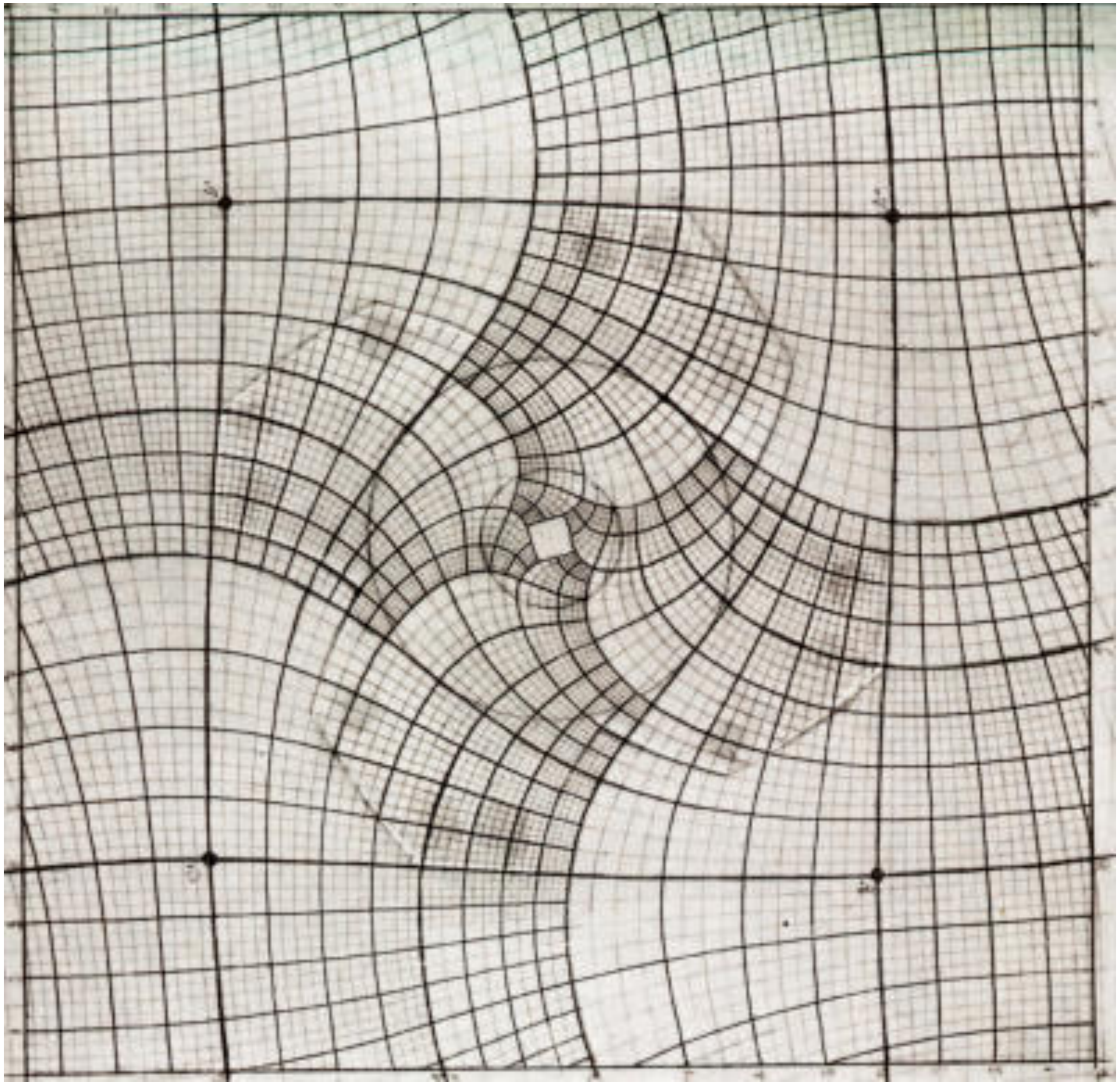






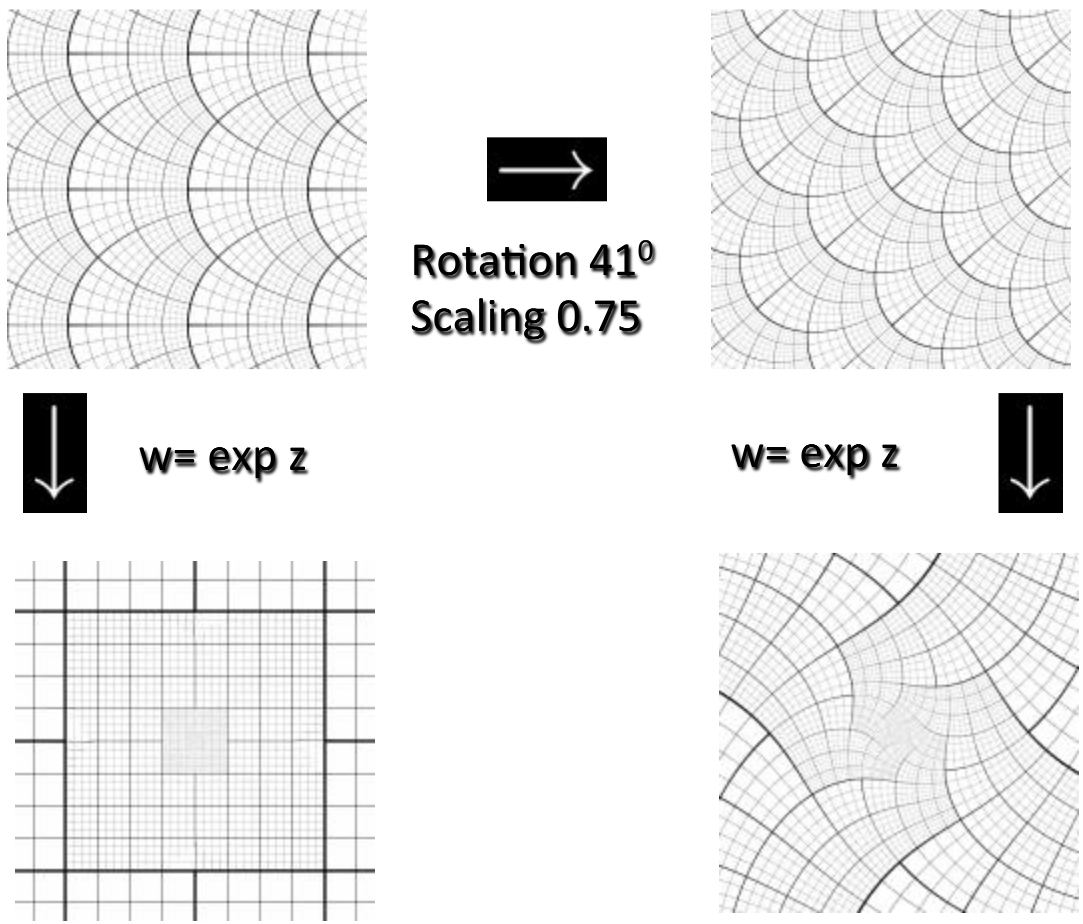
A distorted view

A distorted view

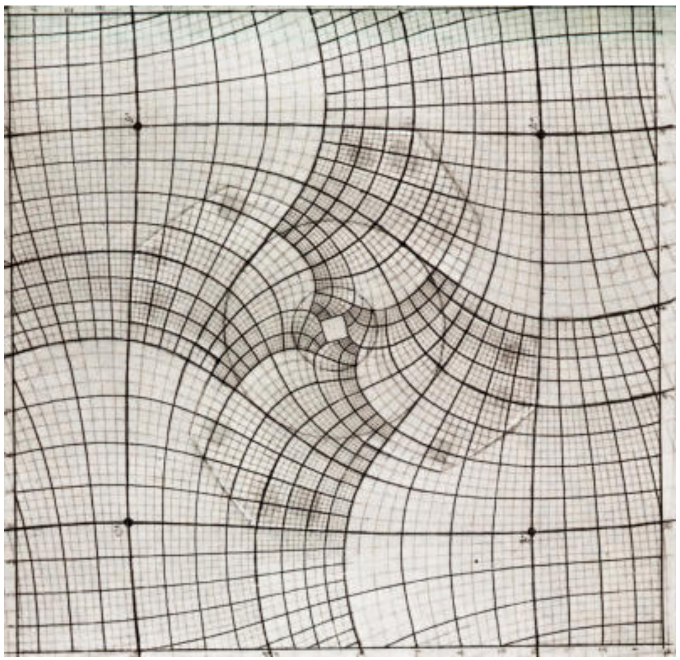


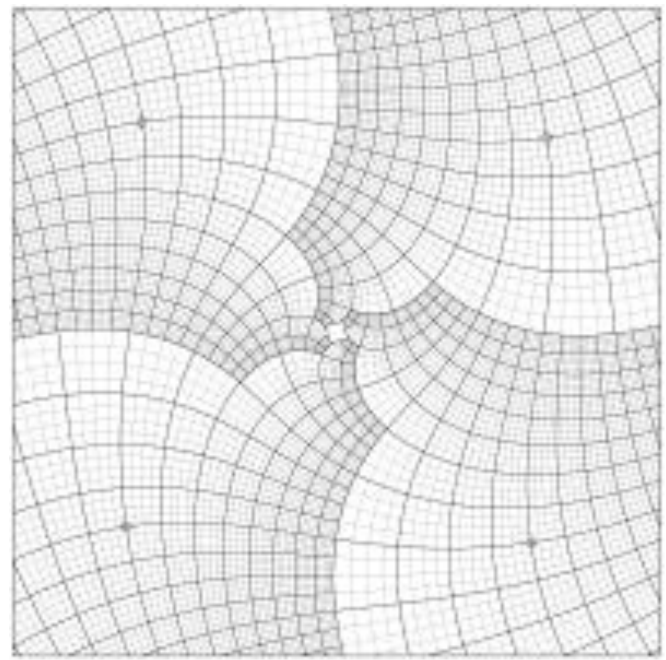
Hendrik Lenstra: A sequence of maps

http://escherdroste.math.leidenuniv.nl/index.php



How well did we/Escher do?





The perfect Escher …

The solution

So the story goes on and on and on …

The other Eschers… or next years T shirt…

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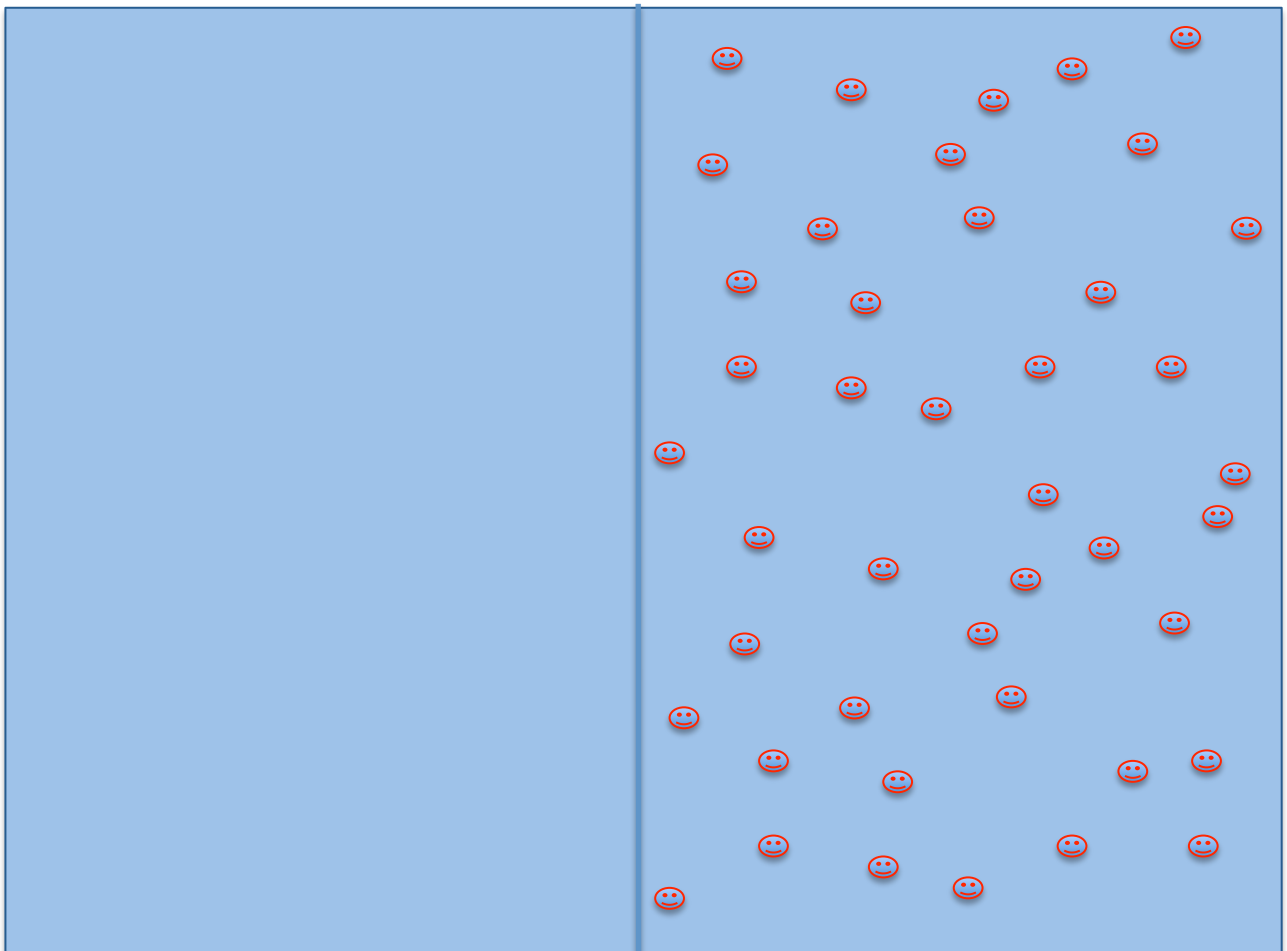
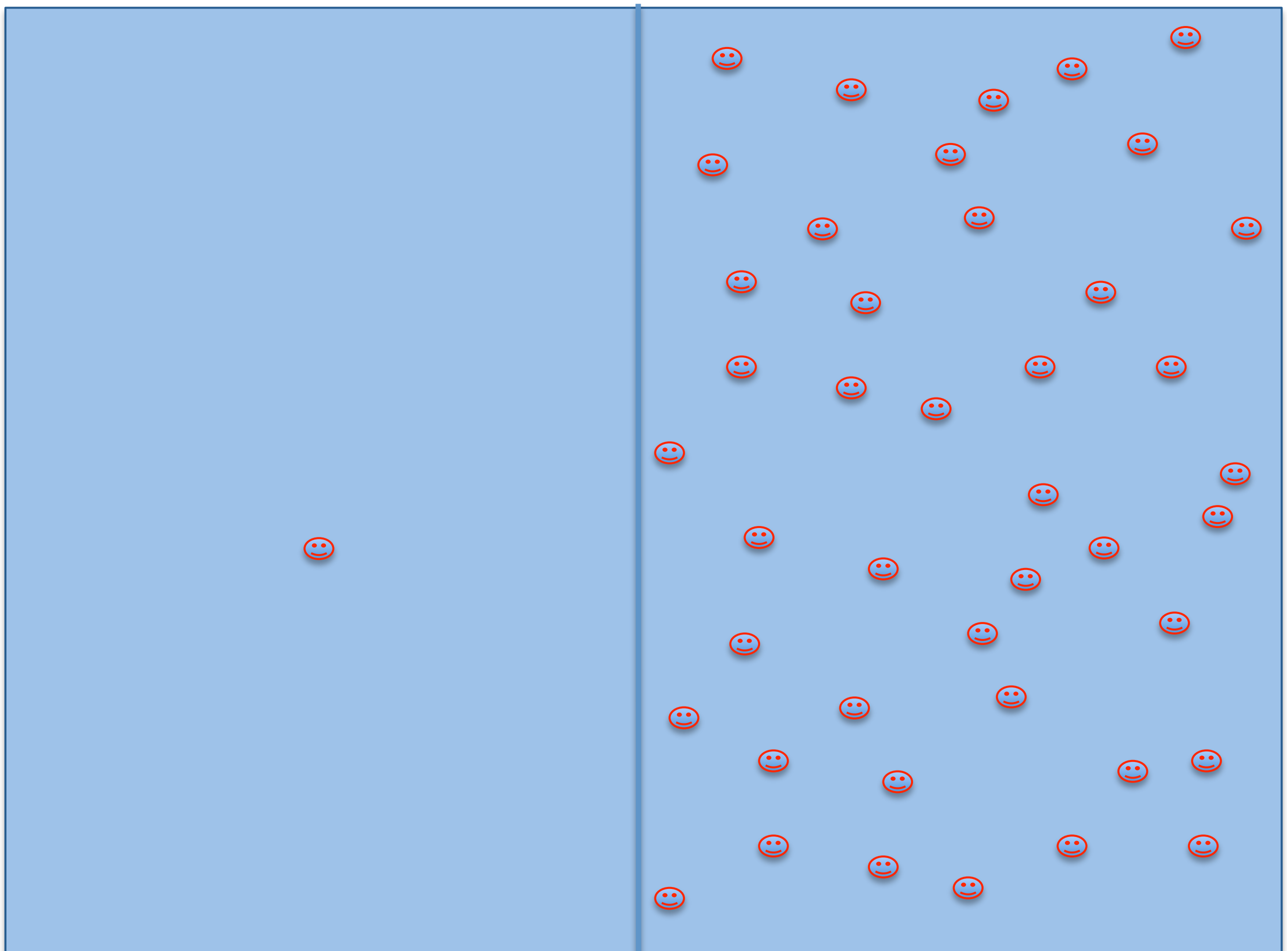
Entropy:



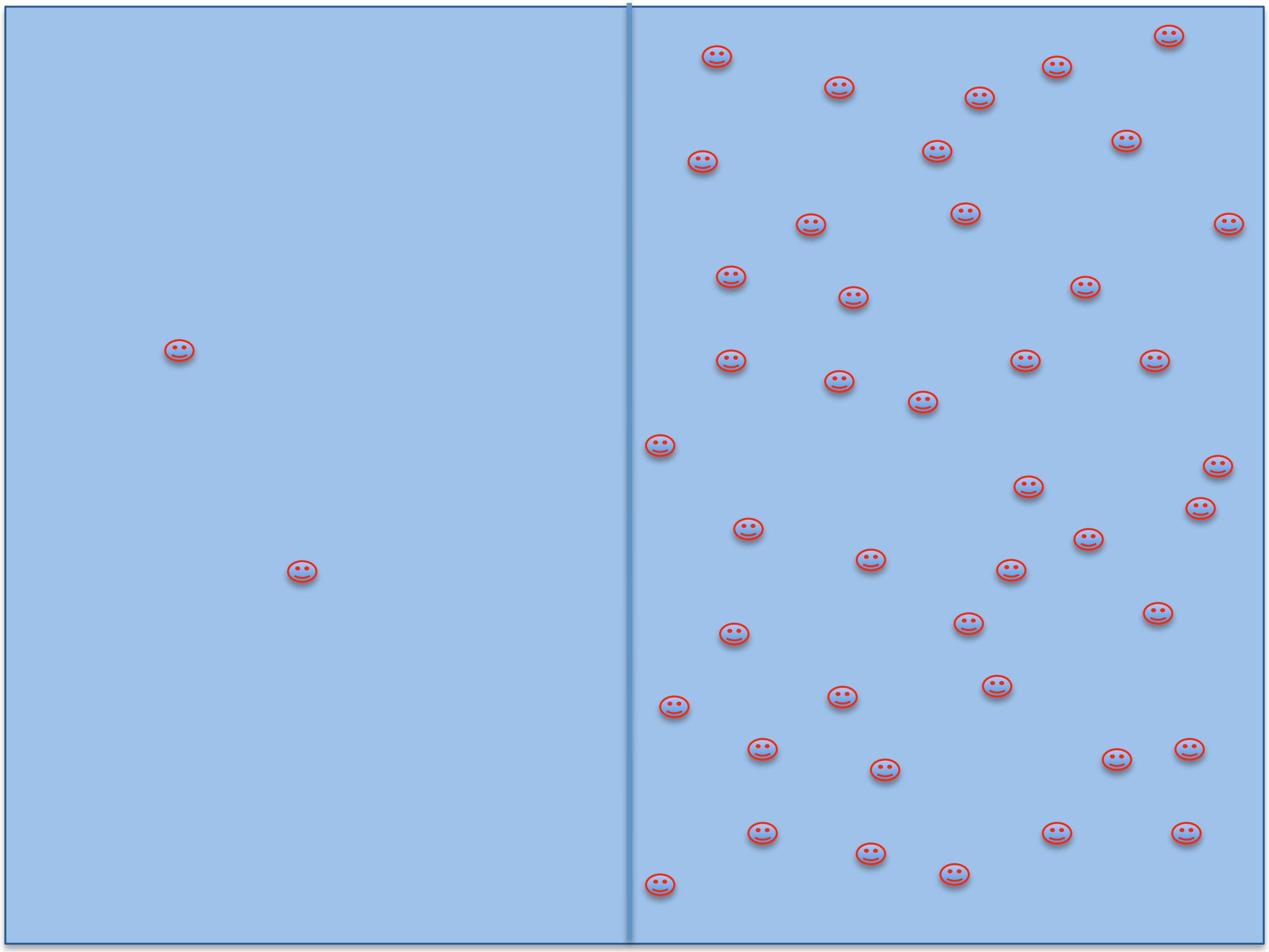
configurations = 1

Entropy:



configurations = N

Entropy:



configurations = N(N-1)

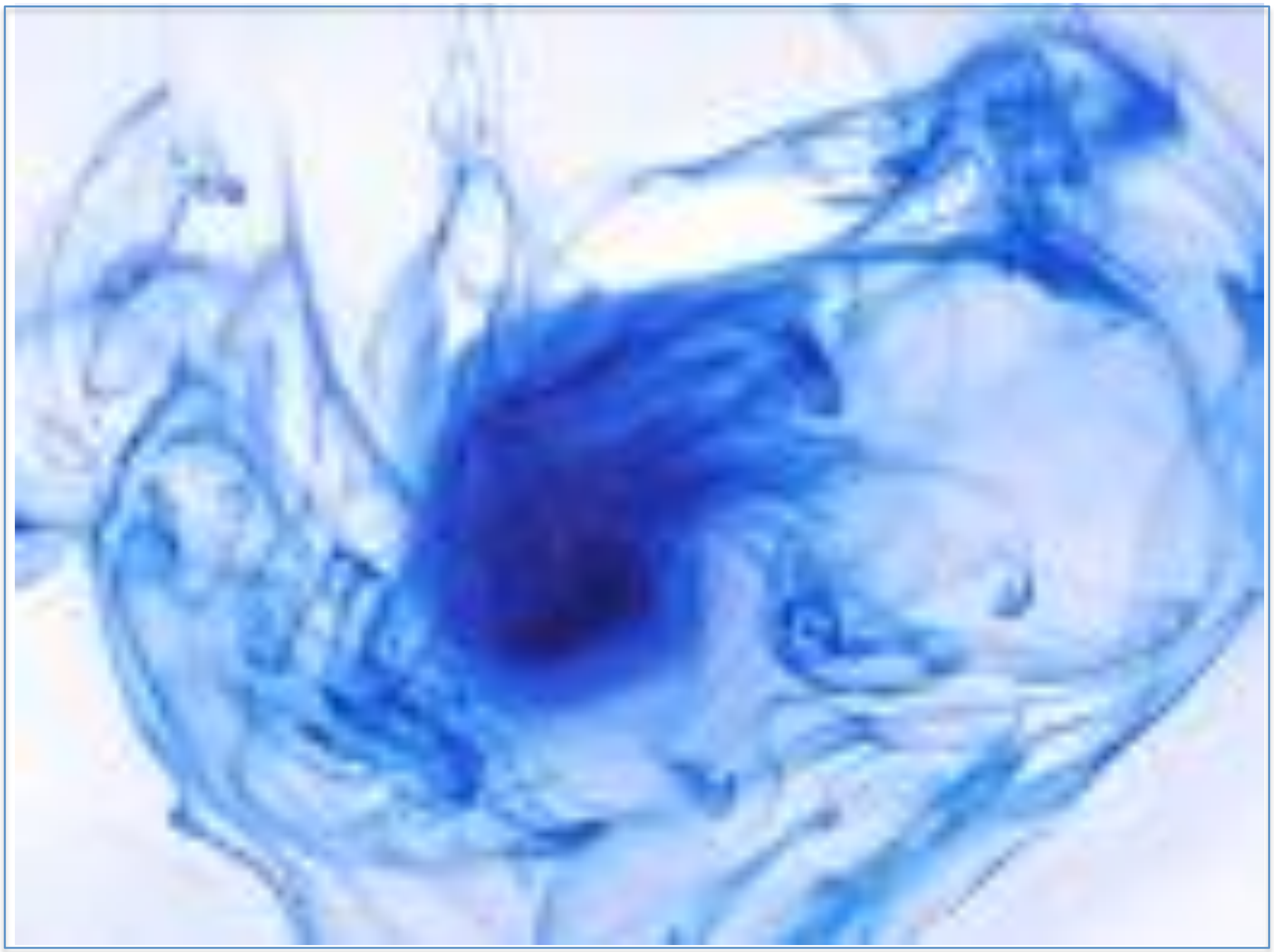
Ink Droplet

Constrained to droplet: # configurations = 1

Unconstrained: # configurations= $\pm (N)!/N_0!(N-N_0)! = \pm N^{No}/ N_0! = \pm (N/N_0)^{No}$ = 10^{9*10^{23})
=> Plenty of room at the bottom

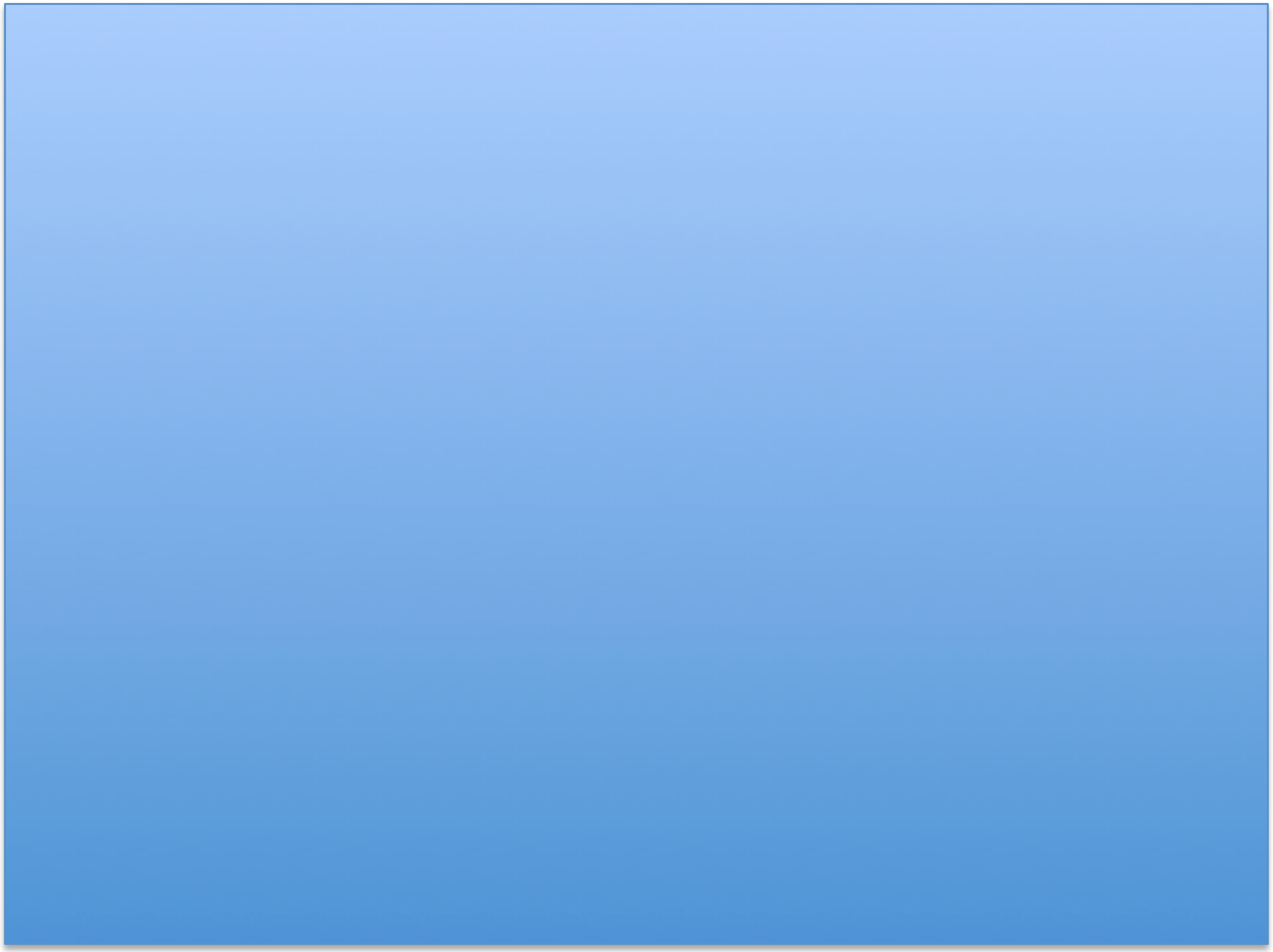
Phase space

Phase space



Entropy

Ink molecules heavier then water



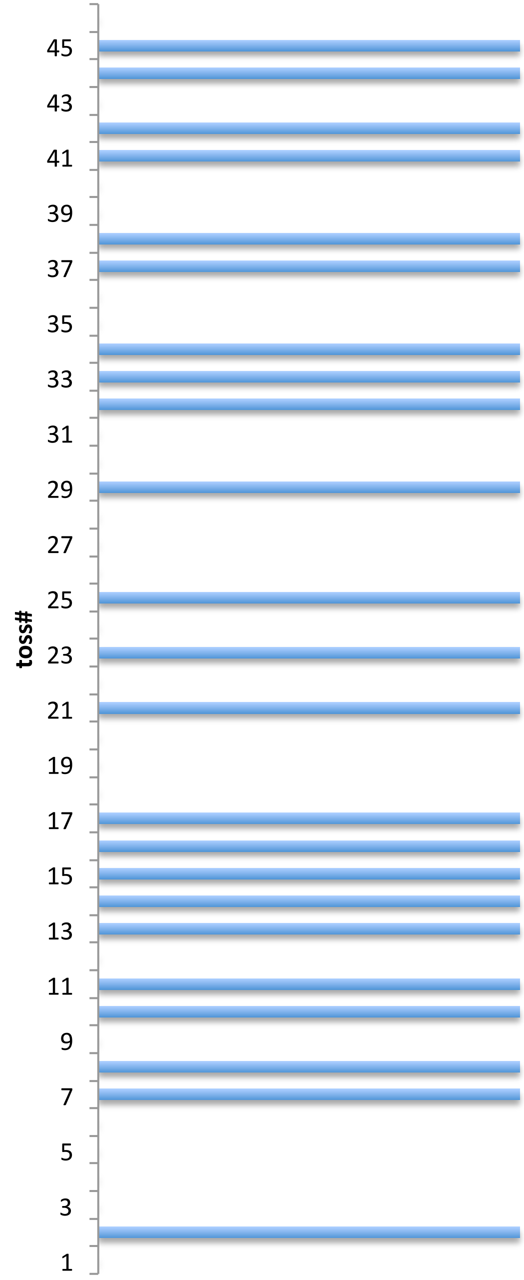
On entropy

- What have we learned (heuristically):
- Macroscopic configurations have a certain entropy, which equals the log of the # of microstates that correspond to the same macrostate.
- Natural tendency for entropy (of a closed system) to increase (2^{nd} law of thermodynamics).
- Entropy is a measure for disorder/mixing.
- The process of increasing entropy is irreversible.
- Equilibrium state is one of maximal entropy.
- State may depend on certain microscopic properties, i.e. the entropy is maximized under certain constraints.
- Very universal phenomenon.

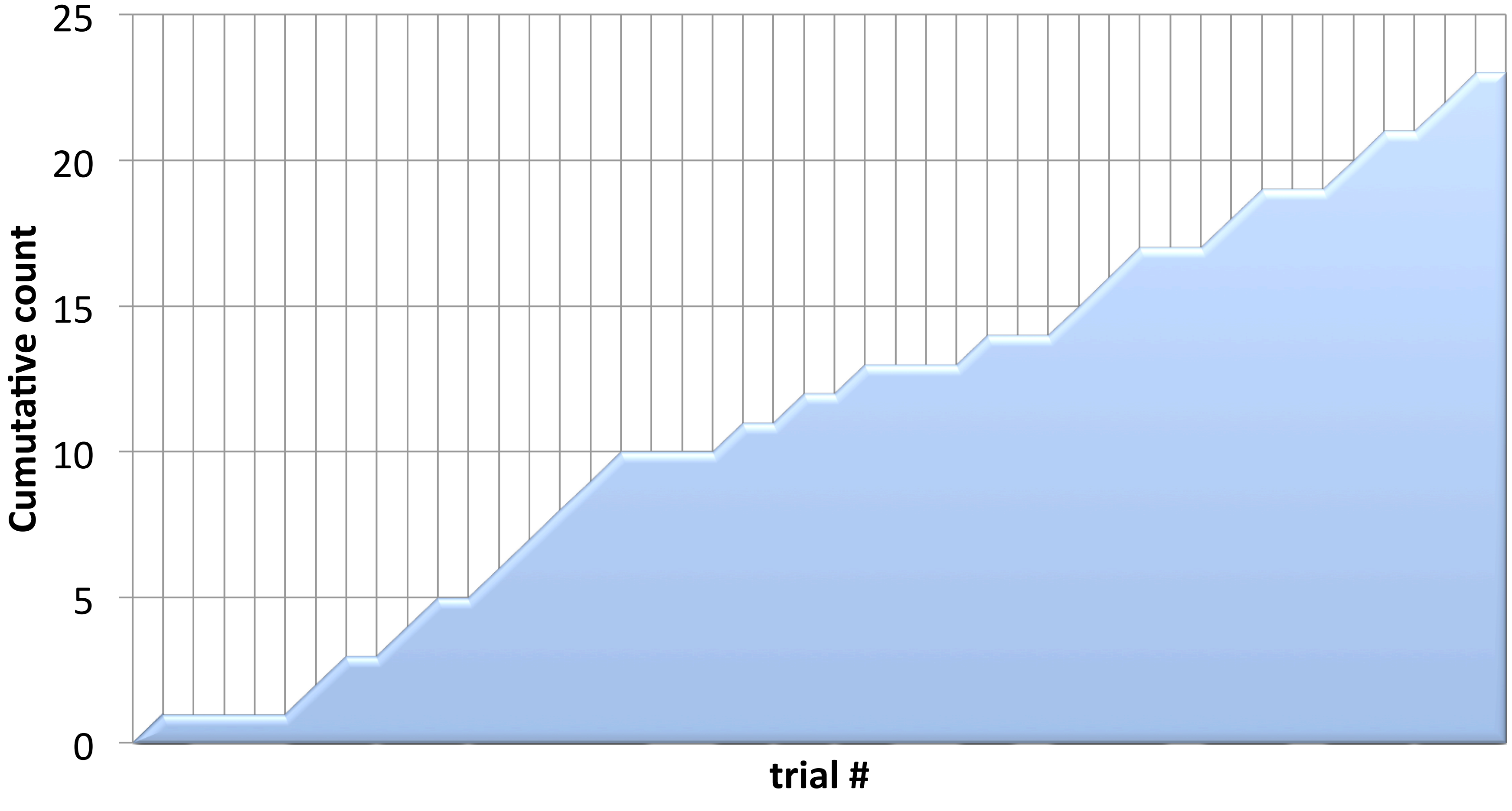
Tossing a coin 46 times

- Outcomes:

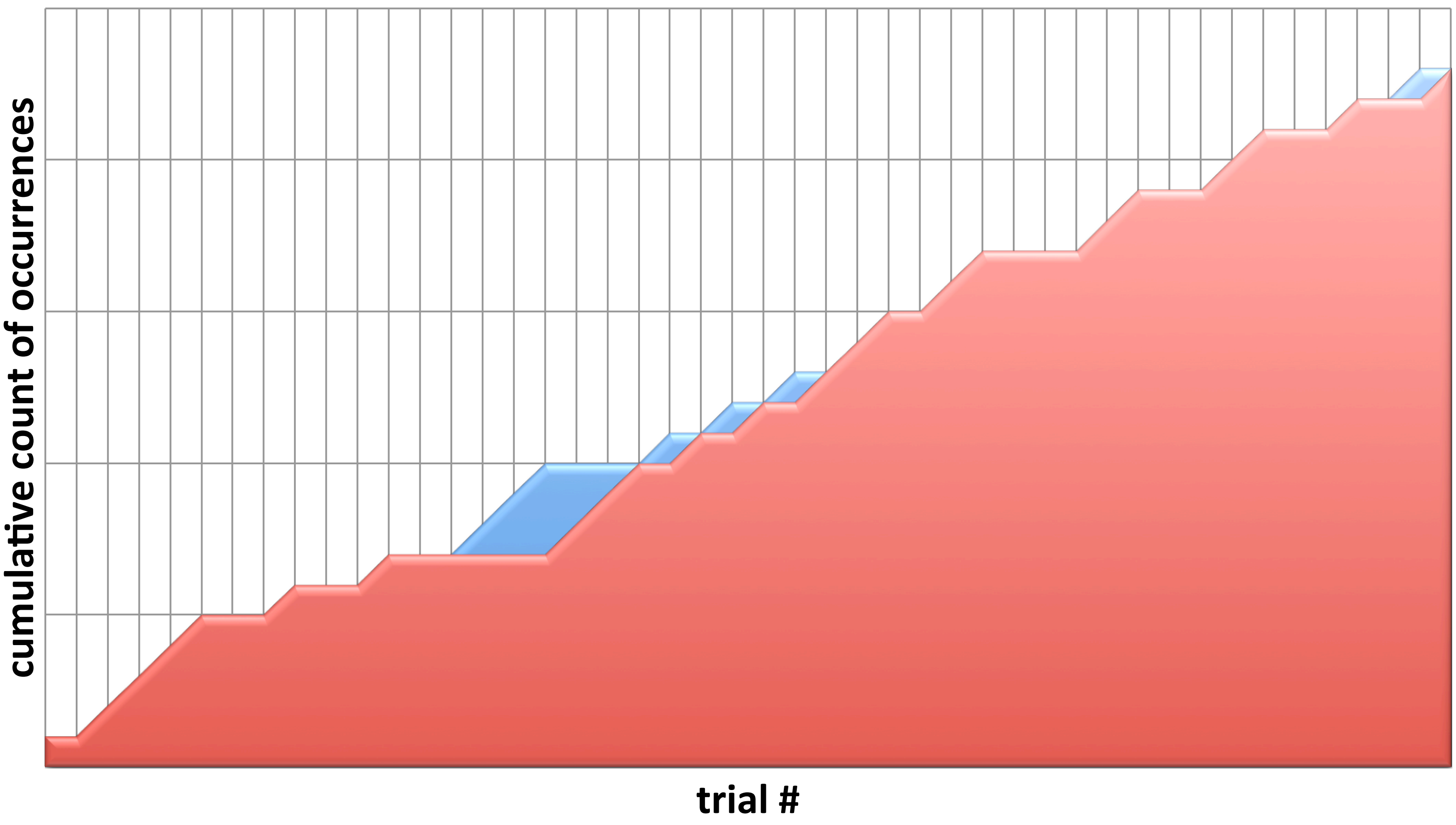
- 0100001101101111000101
- 0100010011100110011010110
- 110110..



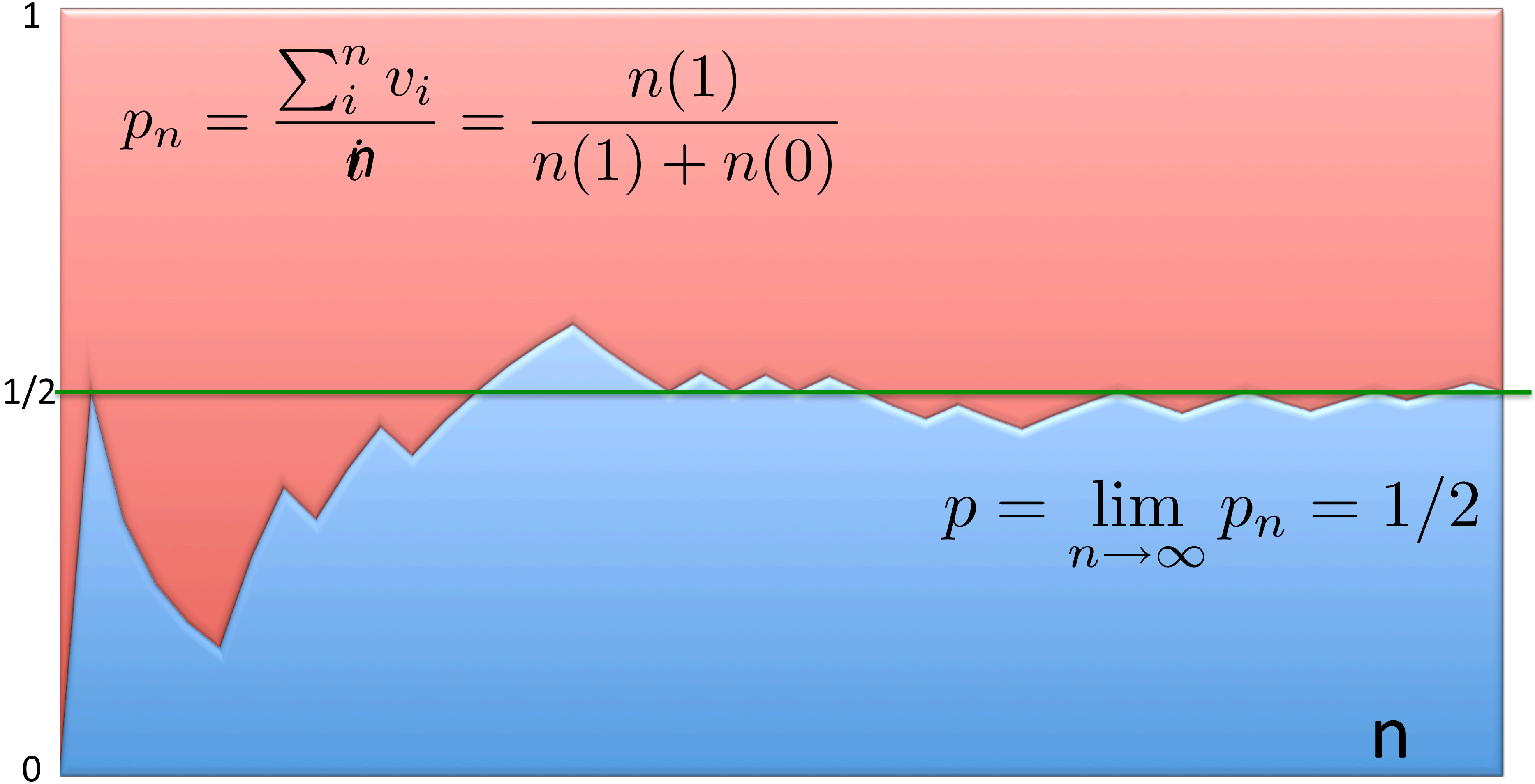
Cumulative count of the 46 tosses



Cumulative distribution



Limiting distribution

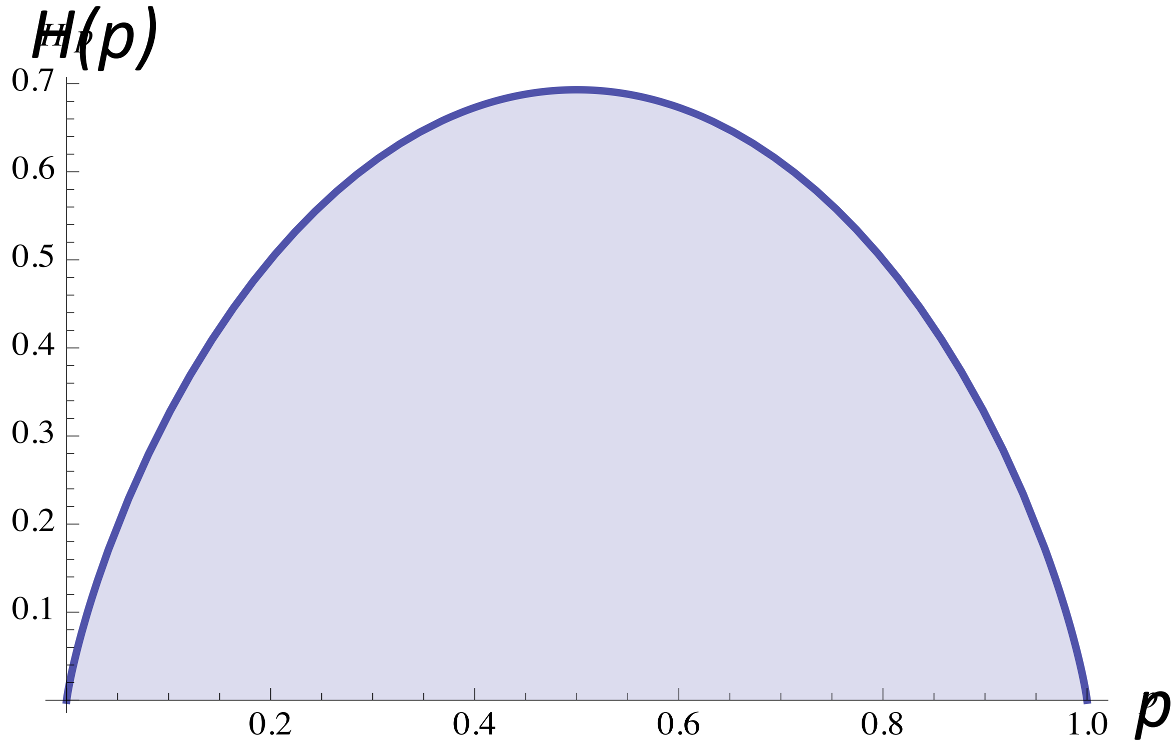


$$p_n = \frac{\sum_i^n v_i}{n} = \frac{n(1)}{n(1) + n(0)}$$

$$p = \lim_{n \to \infty} p_n = 1/2$$

Entropy function

p(x=1) = p
p(x=0) = 1-p



$$
\begin{aligned}
H(p) &= -\sum_{i=1}^{2} p_i \log p_i \\
&= -p \log p - (1-p) \log (1-p)
\end{aligned}
$$

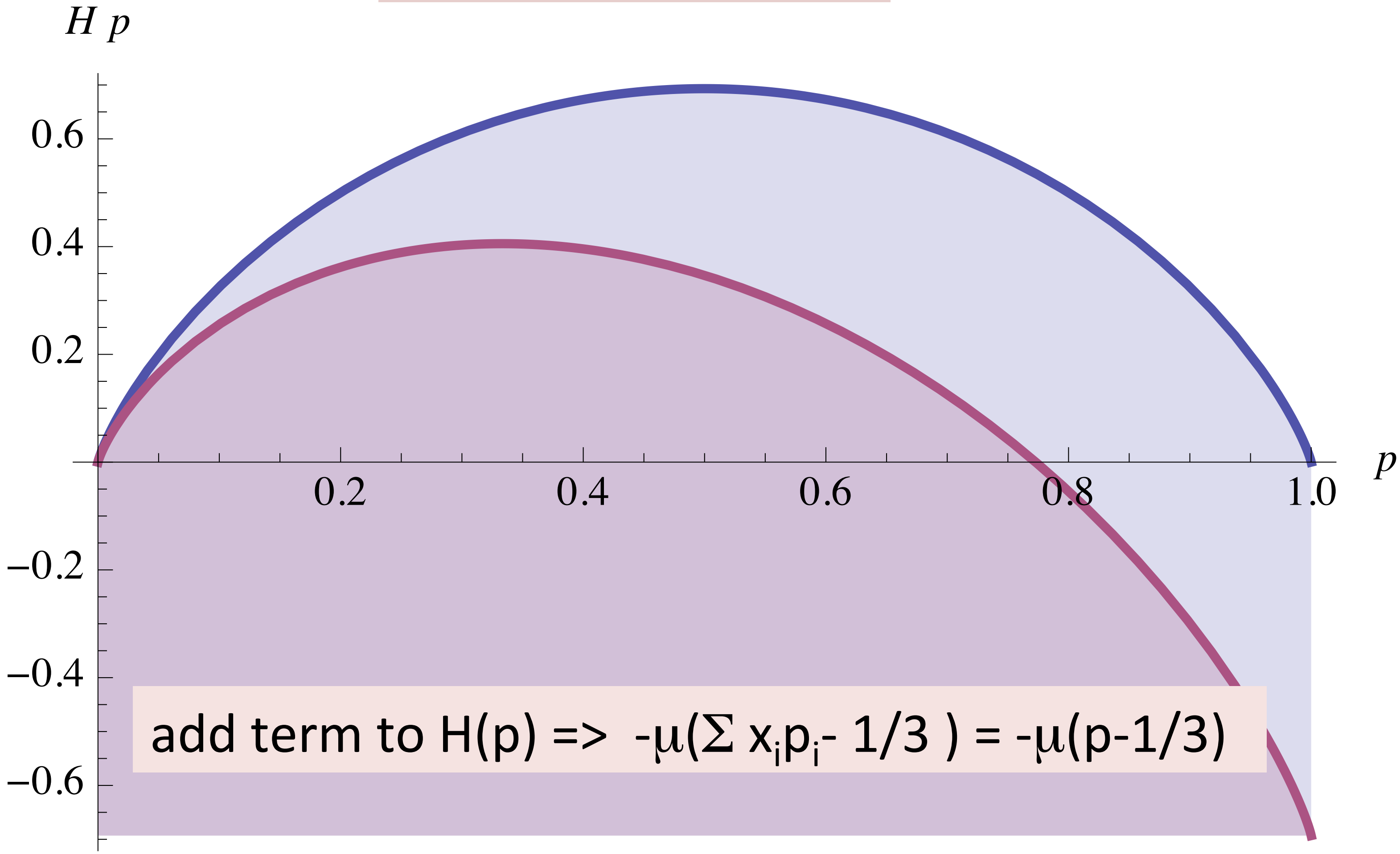
Maximum entropy principle

Limiting distribution maximizes the entropy function
(under certain constraints, here the sum of probabilities is one).
Determine points of vanishing slope (maxima and minima):

$$
\begin{aligned}
\frac{dH}{dp} &= 0 \\
&= -\log p - p(1/p) + \log(1-p) - (1-p)(\frac{-1}{1-p}) \\
&= -\log p - 1 + log(1-p) + 1 \\
\Rightarrow \quad & \log \frac{(1-p)}{p} = 0 \\
& 1 - p = p \Rightarrow p = 1/2
\end{aligned}
$$

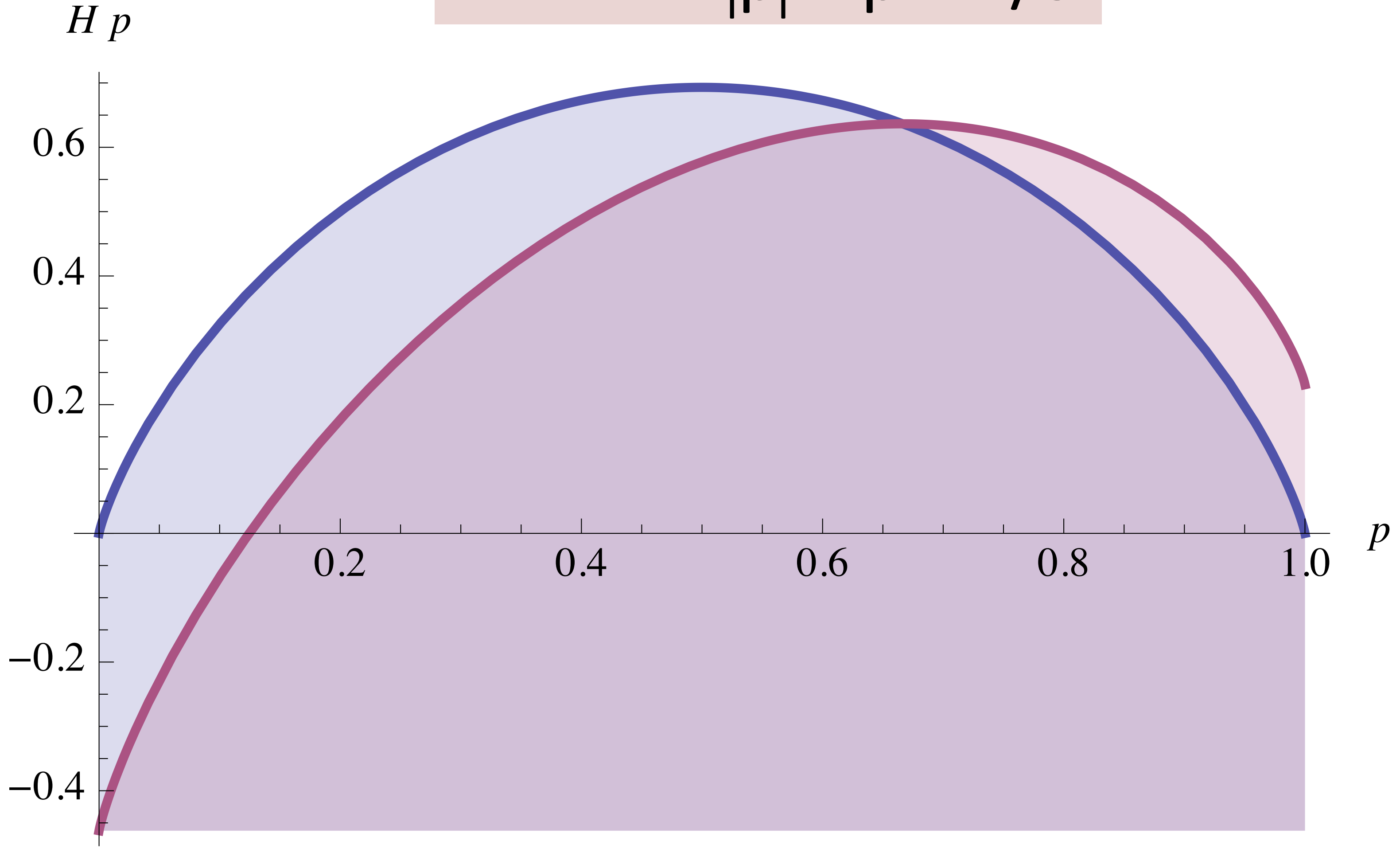
Adding a constraint on the average ⇔ adding information

$\langle x \rangle = \Sigma\, x_i p_i = p = 1/3$



Tossing with constraint

$\langle x \rangle = \Sigma\, x_i p_i = p = 2/3$



The Gaussian from maximal entropy principle

$$
\begin{aligned}
H(p_i, \lambda) &= -\sum_i^N p_i \log p_i - \lambda((\sum_i p_i) - 1) - \mu((\sum_i p_i i^2) - \sigma^2) \\
\frac{\partial H}{\partial p_i} &= 0 \quad (i = 1, ..., N) \\
\Rightarrow & \quad -\log p_i - 1 - \lambda - \mu i^2 = 0 \\
\Rightarrow & \quad p_i = \gamma e^{-u i^2} \\
\frac{\partial H}{\partial \lambda} &= 0 \\
\Rightarrow & \quad \sum_i p_i - 1 = 0 \\
\frac{\partial H}{\partial \mu} &= 0 \\
\Rightarrow & \quad \sum_i p_i i^2 - \sigma^2 = 0
\end{aligned}
$$

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Relation with Thermodynamics

Partition function:

$$Z \equiv \sum_i e^{-\varepsilon_i/T}$$

Free energy:

$$F = -T \ln Z$$

$$F \equiv U - TS \qquad dF = -PdV - SdT$$

Non-equilibrium situation: Kinetic Bolzmann equation

$$f(\mathbf{r}, \mathbf{p}, t)$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{\mathrm{coll}}$$

→Hydrodynamics, diffusion etc

Hydrodynamics equations

$$\rho(x,t) = m \int f(x,v,t)dv$$
$$\vec{v}(x,t) = \int v f(v,x,t)dv$$
$$\epsilon(x,t) = \tfrac{1}{2} m \int v^2 f(x,v,t)dv$$

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

$$\overbrace{\rho\Big(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{Unsteady acceleration}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{\text{Convective acceleration}}\Big)}^{\text{Inertia (per volume)}} = \overbrace{\underbrace{-\nabla p}_{\text{Pressure gradient}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\text{Viscosity}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{f}}_{\text{Other body forces}}$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right)\right] = \rho T \frac{ds}{dt}$$

Equations vs solutions

- Obtaining the equations
  - Underlying equations for constituents and their interactions and
  - Identify symmetries and conservation laws
- Obtaining solutions
  - Math
  - Sensible approximation schemes
  - Computational
- Intodude Coarse grained variables
  - Continuity equations for the conserved currents and charges

$$\frac{dq(x,t)}{dt} = \nabla q(x,t)$$

Evolutionary dynamics

I do not know what the ultimate understanding of biology will look like, but one thing is clear: it will be based on precise mathematical descriptions of evolutionary processes. Mathematics is the proper language of evolution, because the basic evolutionary principles are mathematical in nature. Though verbal in origin, the theory has become more and more mathematical over time.

Martin A. Nowak

Ian Mc Ewan

For all recent advances, it's still not known how this well-protected one kilogram or so of cells actually encodes information, how it holds Experiences, memories, dreams and intentions. He doesn't doubt that in years to come the coding mechanism will be known, though it might in his lifetime. Just like the digital codes of replicating life held within DNA, the brains fundamental secret will be laid open one day,

…

Could it ever be explained how matter becomes conscious? He can't begin to imagine a satisfactory account, but he knows it will come, the secret will be revealed over decades, as long as the scientists and the institutions remain in place, the explanations will refine themselves into an irrefutable truth about consciousness

…

The journey will be completed. He is certain of it. That's the only faith he has. There's grandeur in this view of life.

(Saturday)

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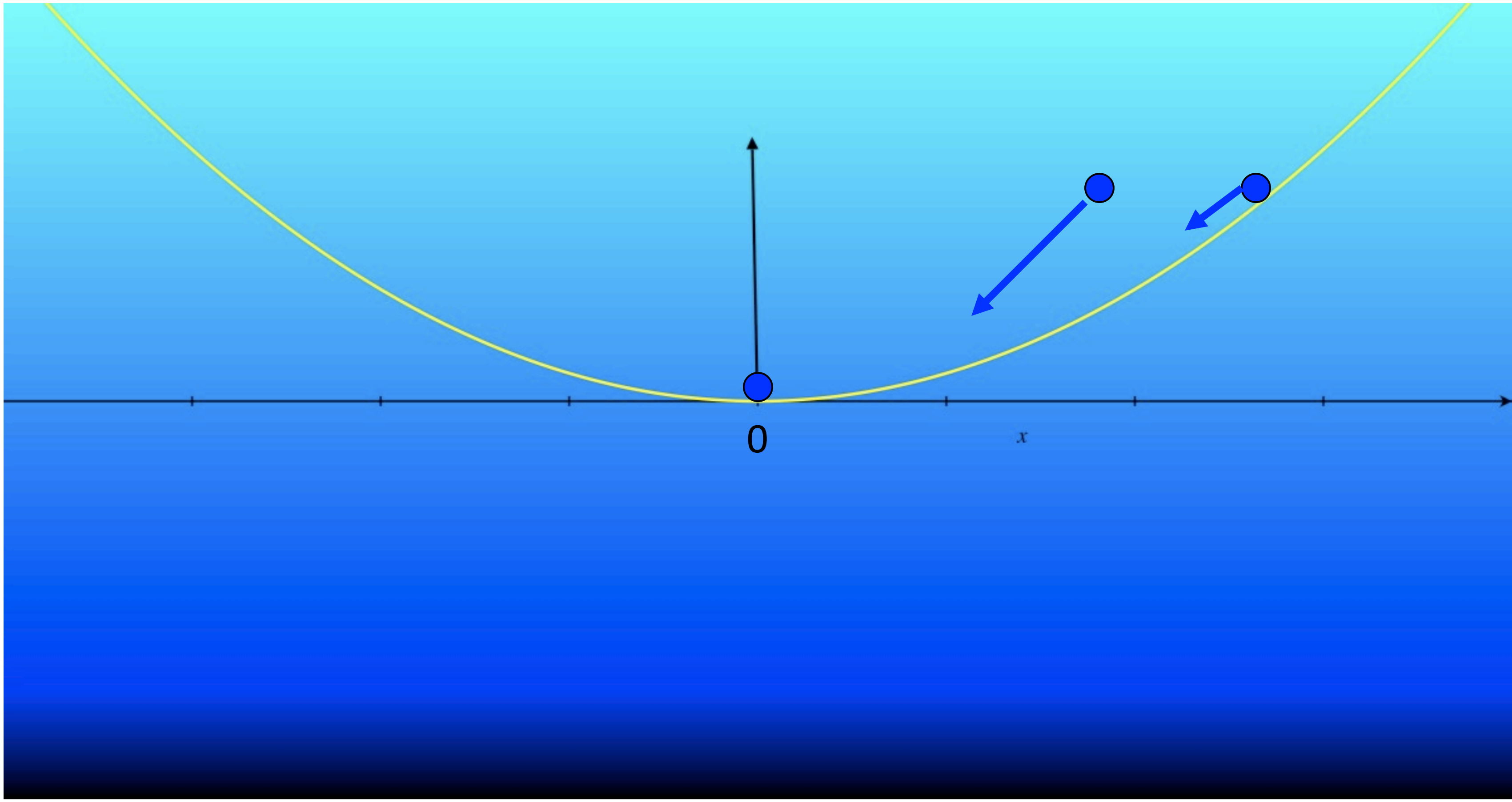
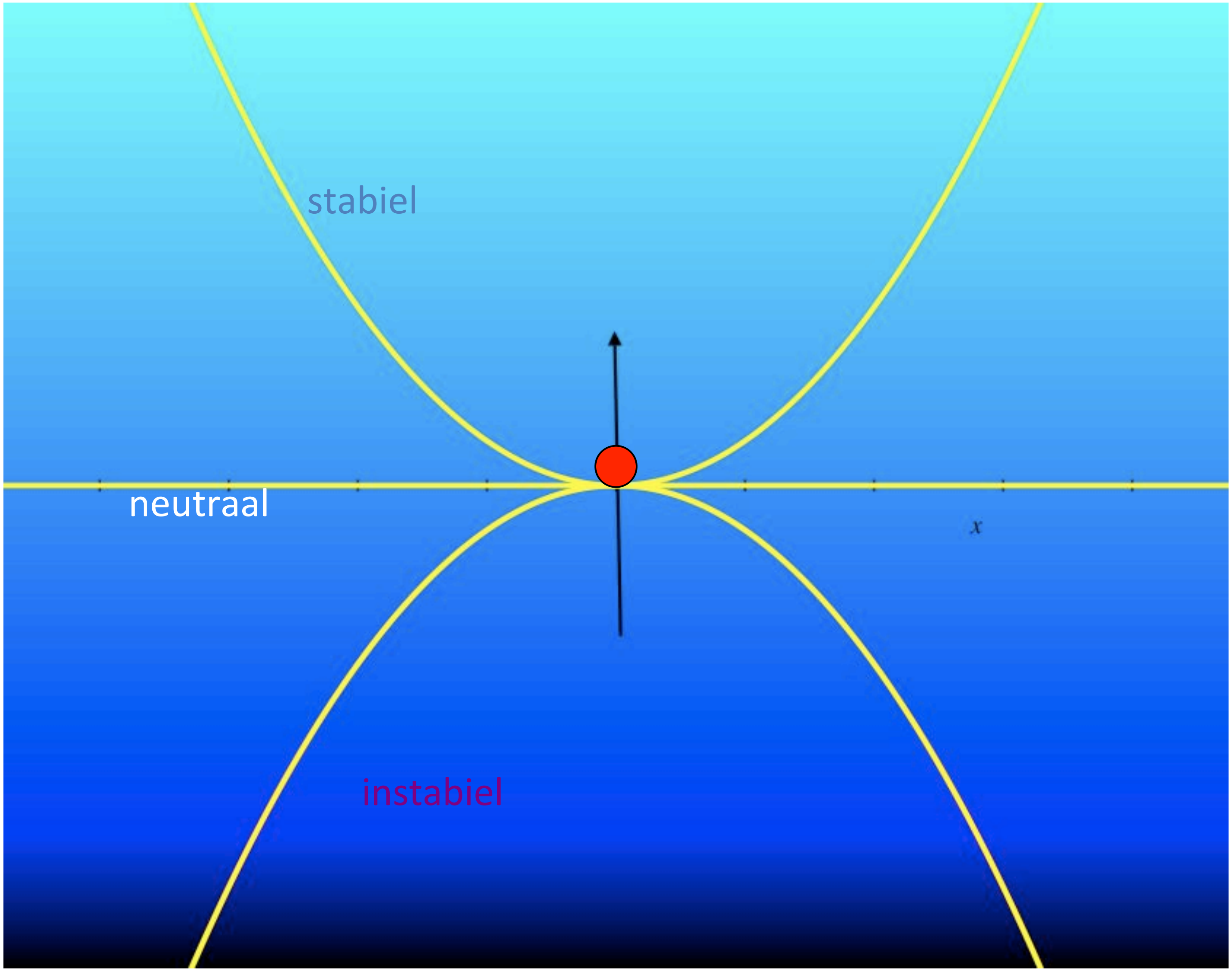
Vacuum = grondtoestand = laagste energie

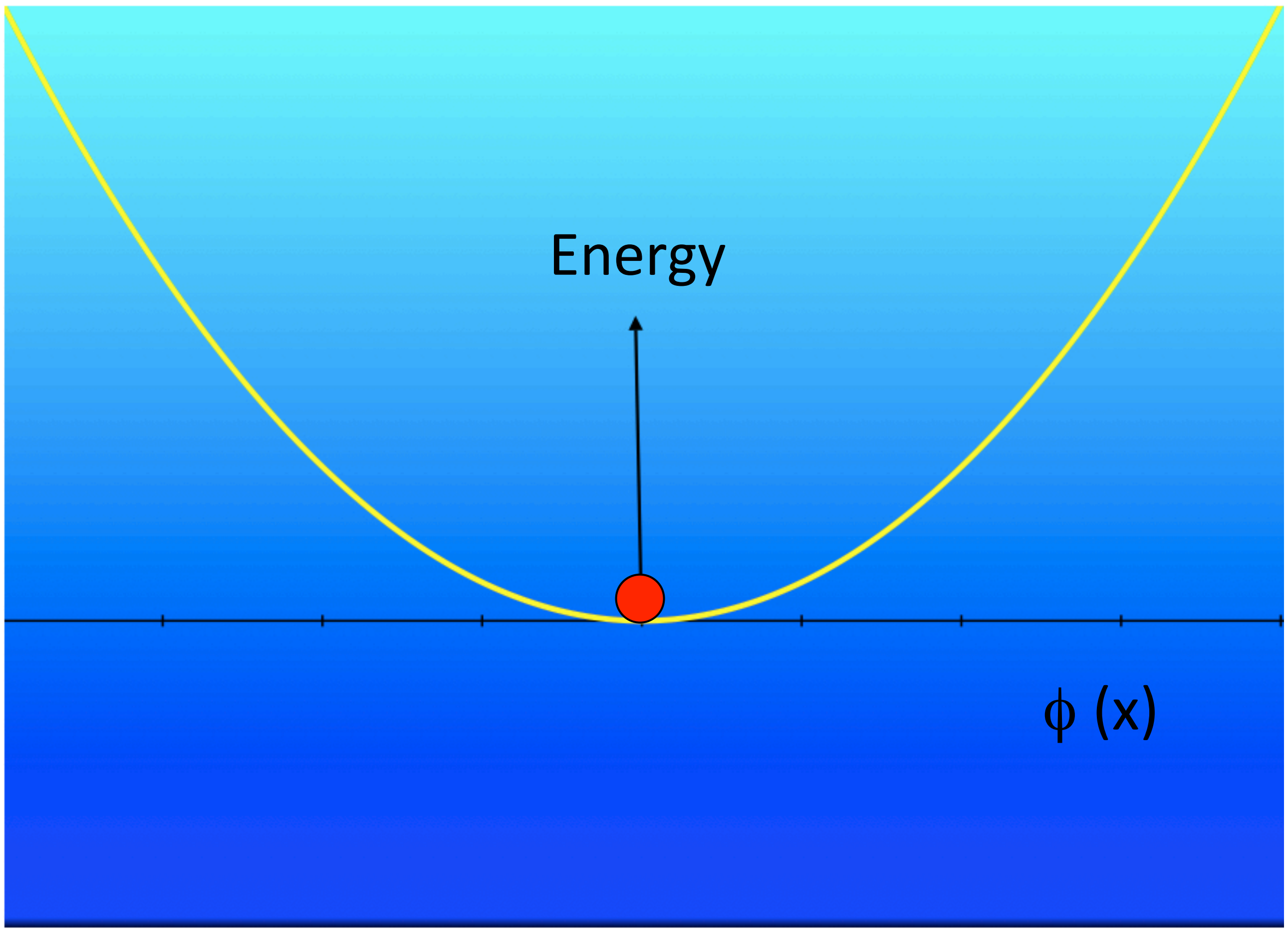


Stable <==> Unstable

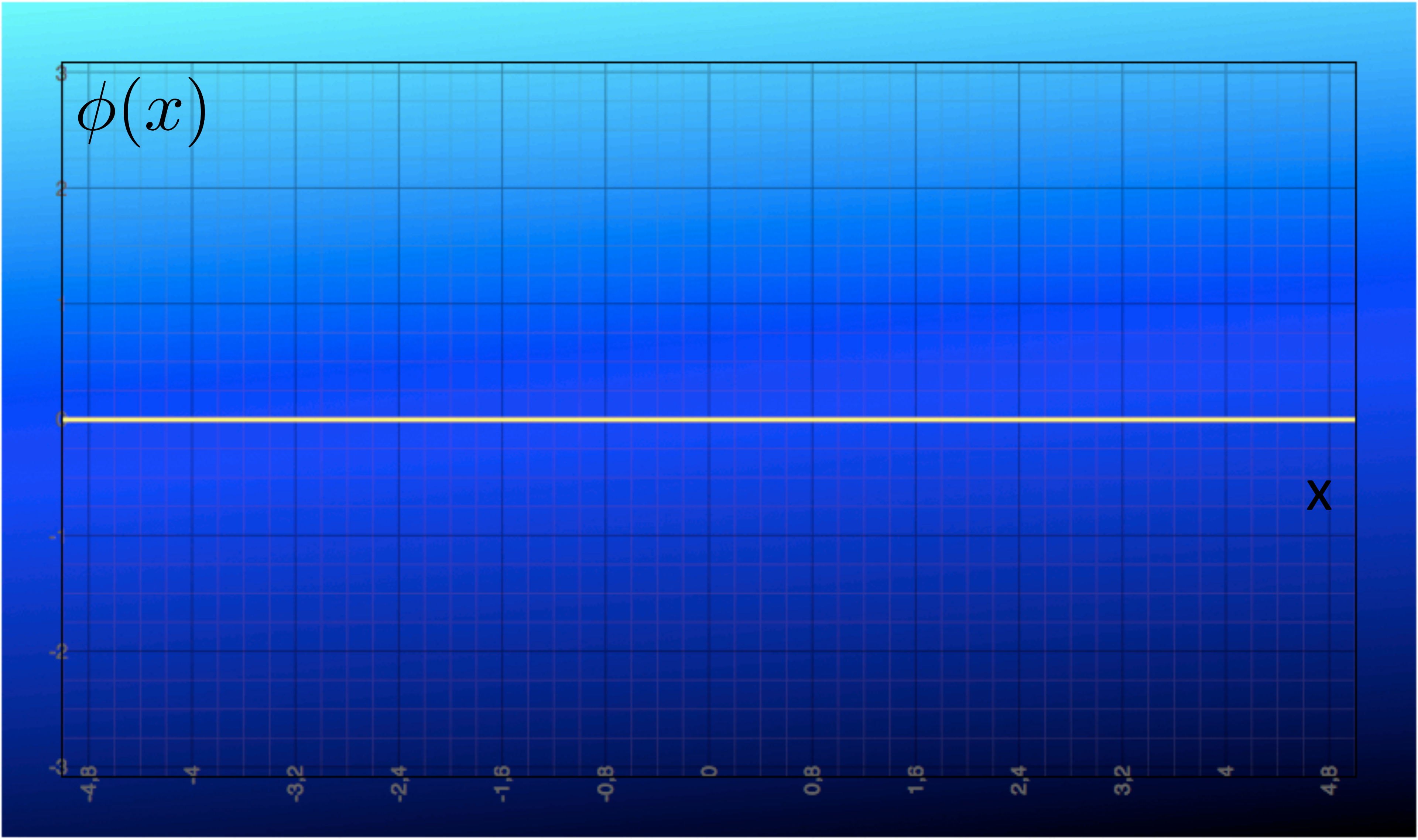


Phase transistions and symmetry breaking

$$E = (\frac{d\phi}{dx})^2 + a(\phi^2 - b)$$



The groundstate (vacuum)



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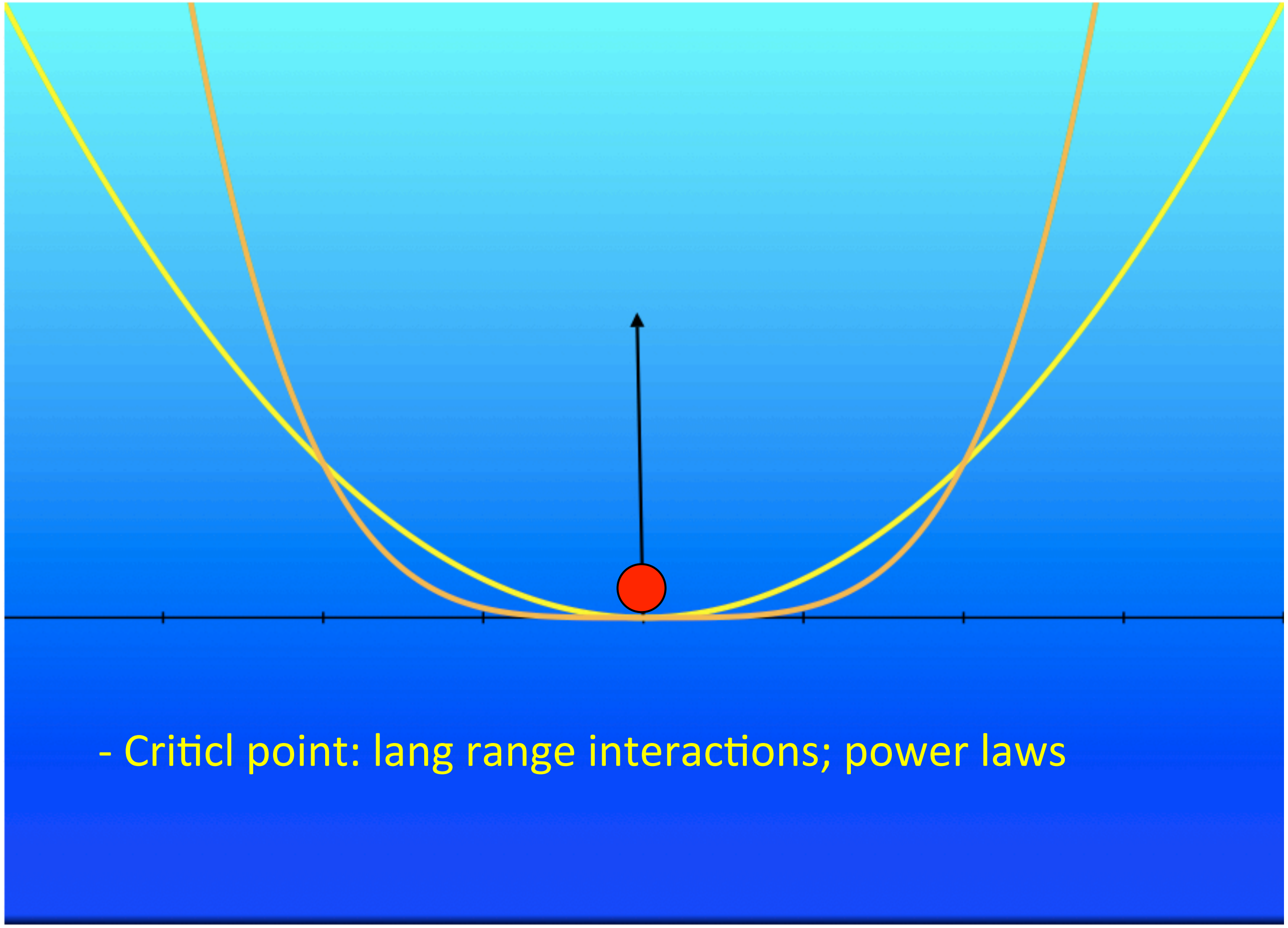
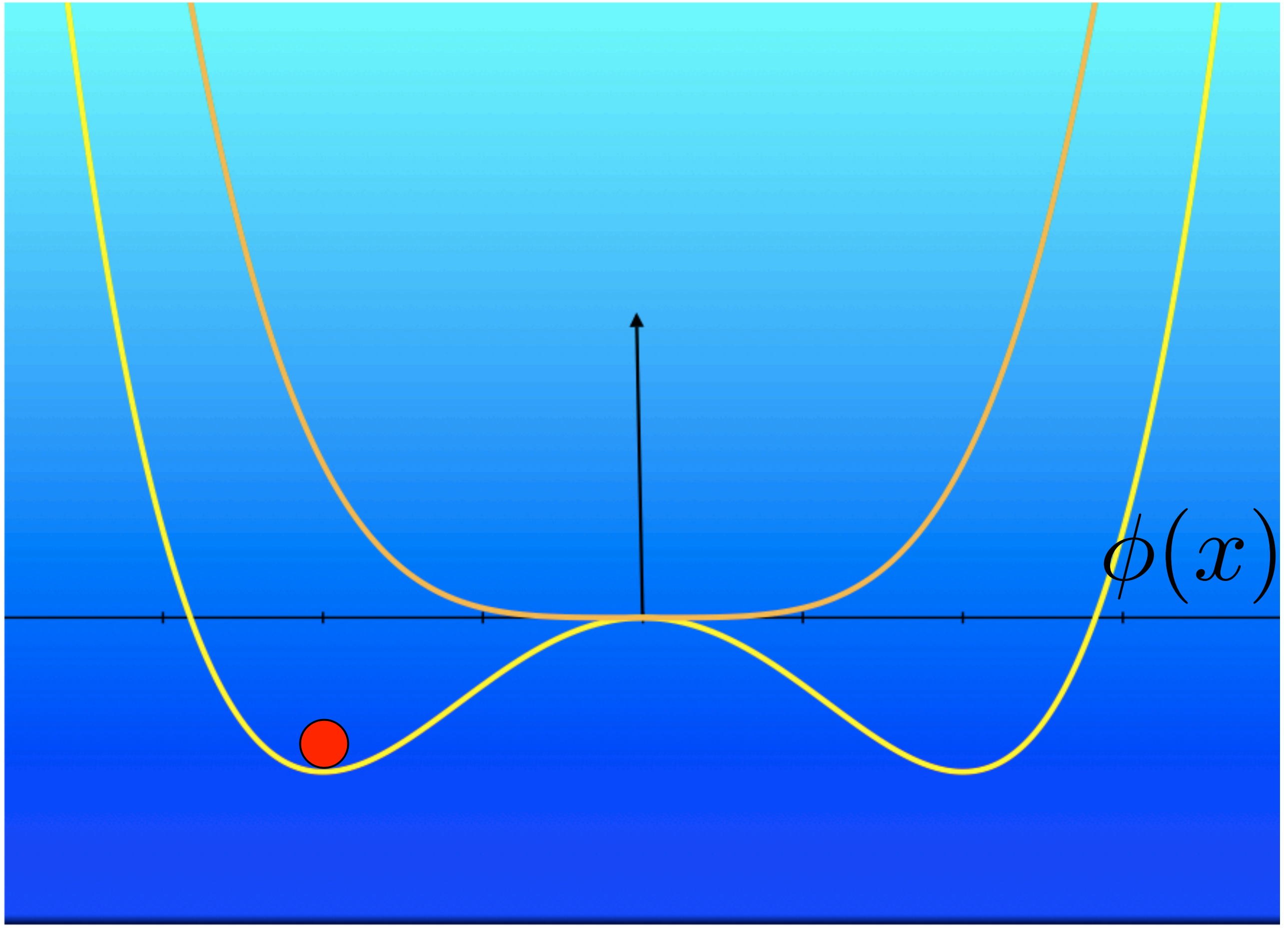


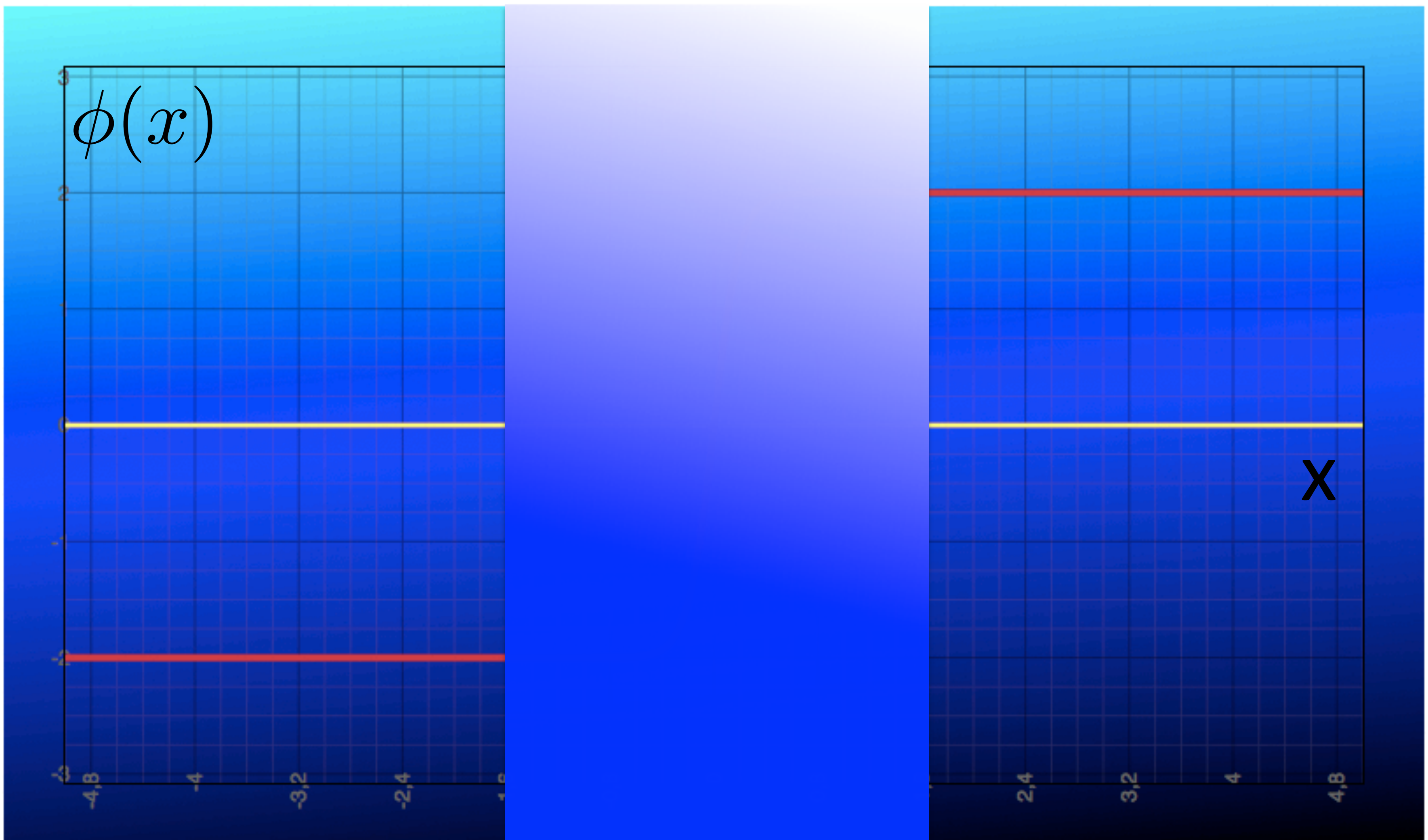
Symmetrybreaking

$$E = (\frac{d\phi}{dx})^2 + a(\phi^2 - b)$$

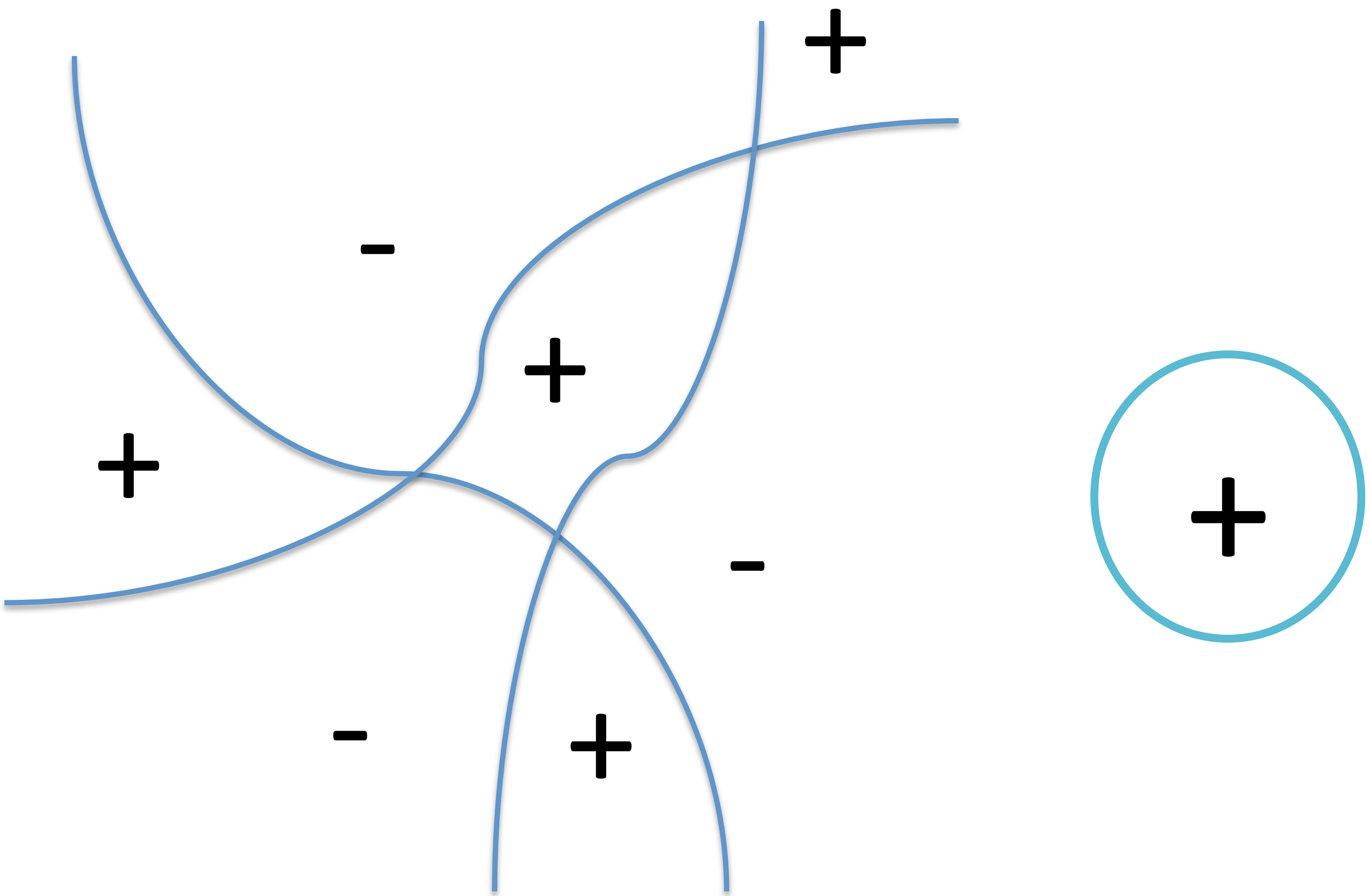


The topological defect: Kink or Domain wall

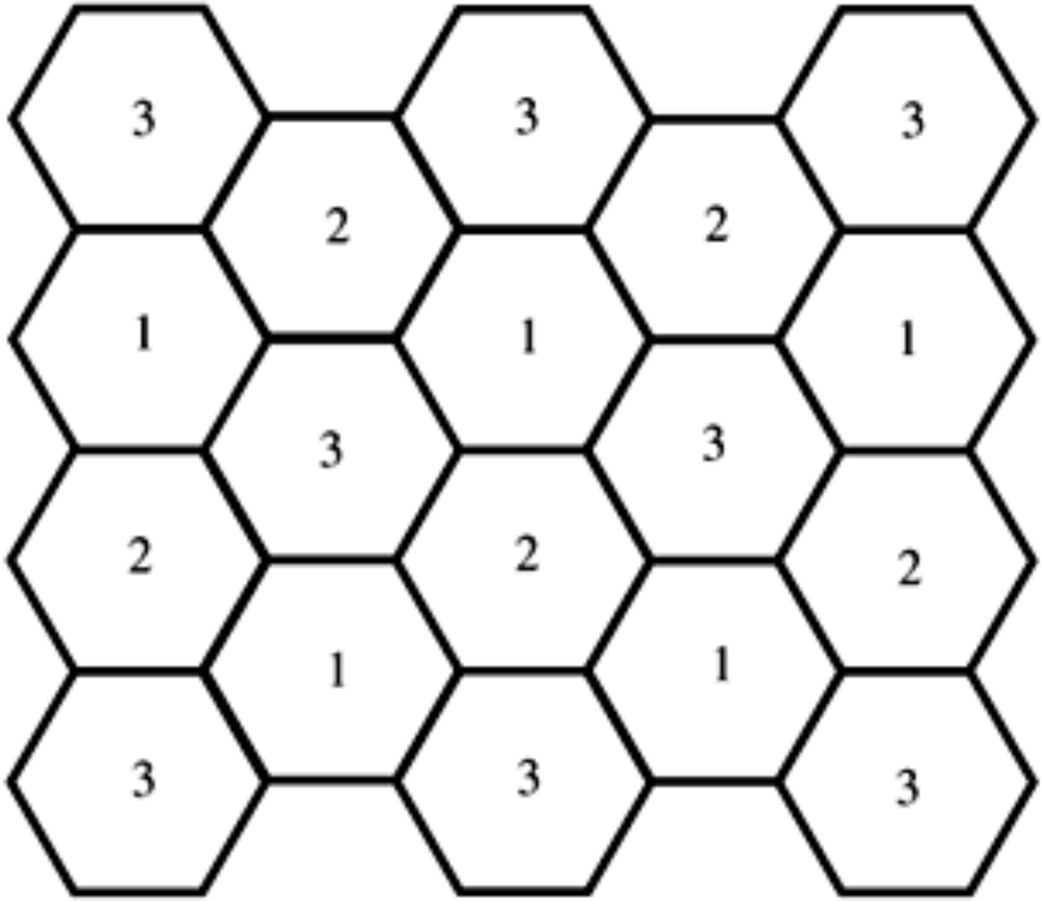
$$\phi(x) = 2\tanh(5x)$$



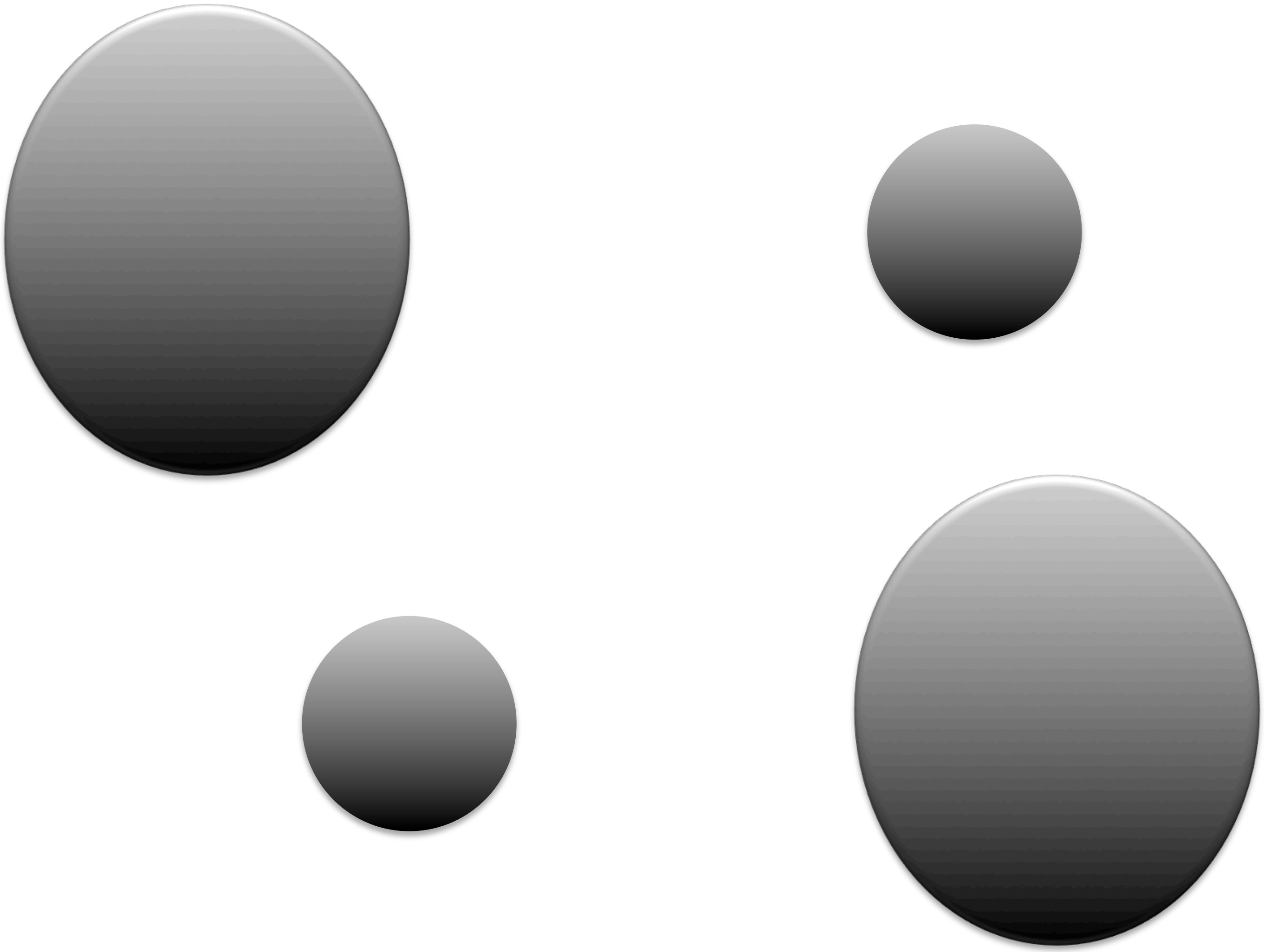
Domain structure => network



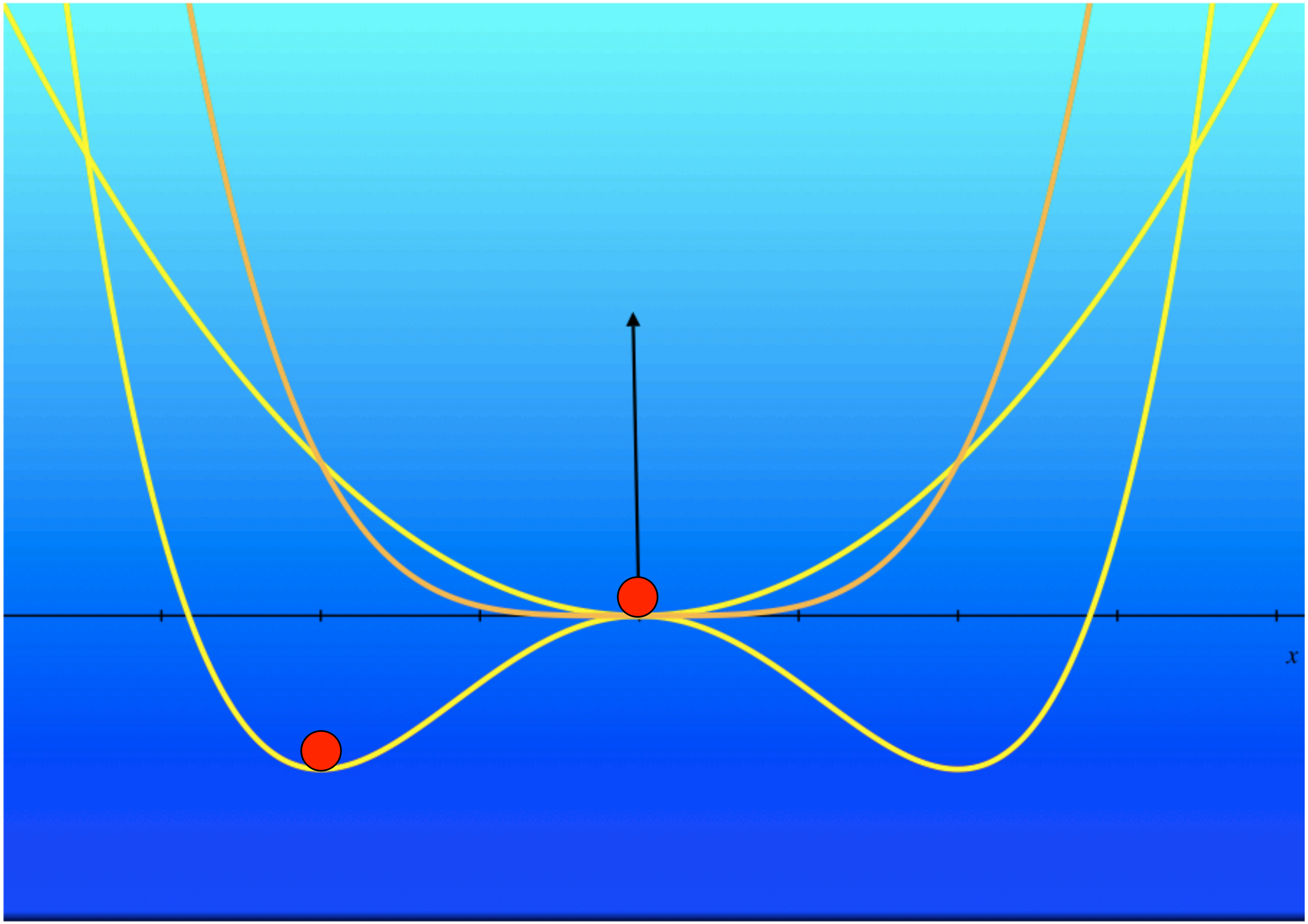
Domain structure => pattern formation



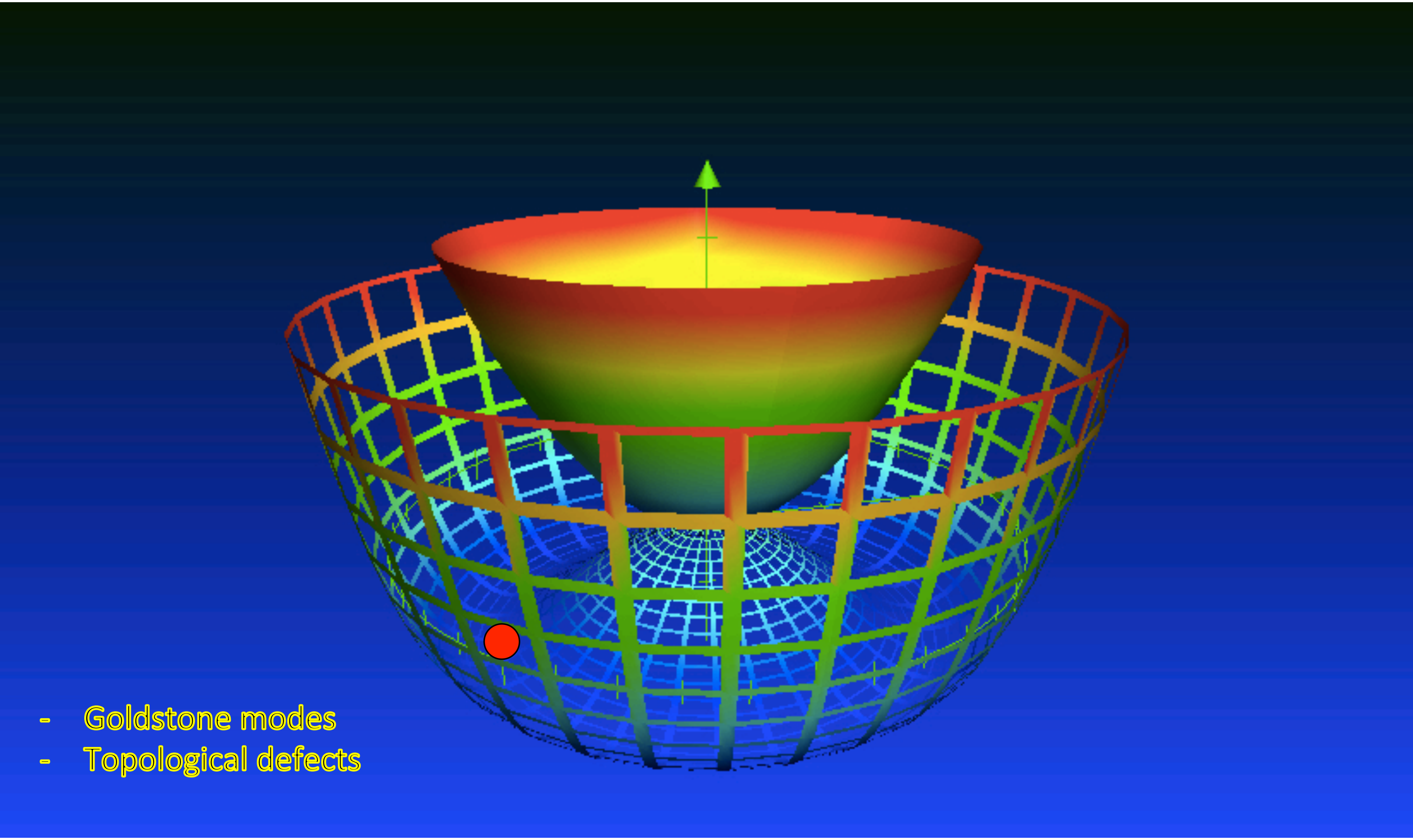
Domains in 3 dimensions



Symmetry breaking



Excitations in broken phase



Conformal invariance in d=2

Symmetry algebra infinite dimensional:

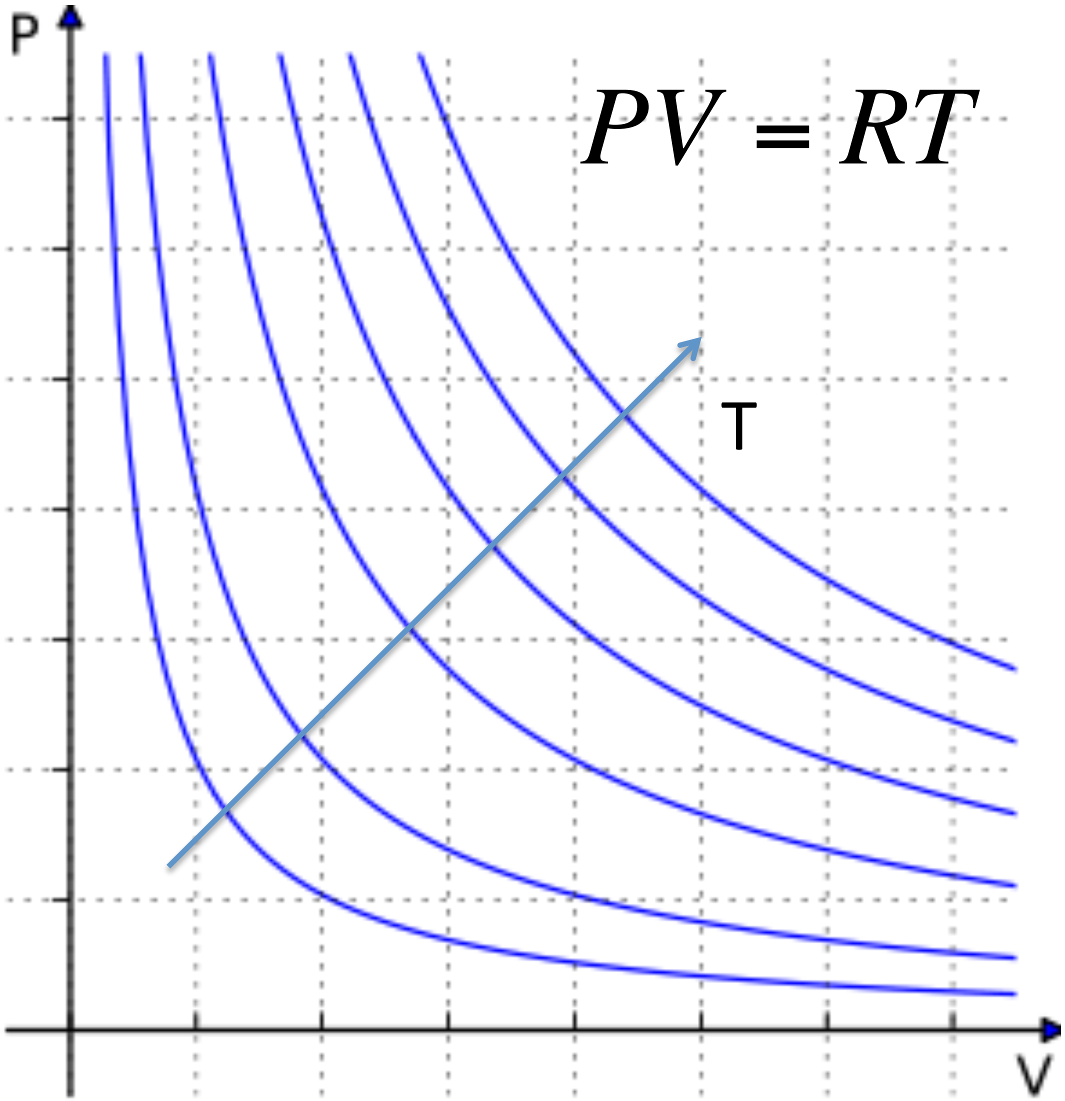
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}cm(m^2-1)$$

Fields are organized in representations of conformal algebra
And the lowest weight of such a representation corresponds to the
Critical exponents (powers):

$$c = c^{(m)} = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, 5, \ldots .$$

$$h_{p,q}^{(m)} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \qquad \begin{cases} 1 \leqslant p \leqslant m-1 \\ 1 \leqslant q \leqslant m \end{cases}$$

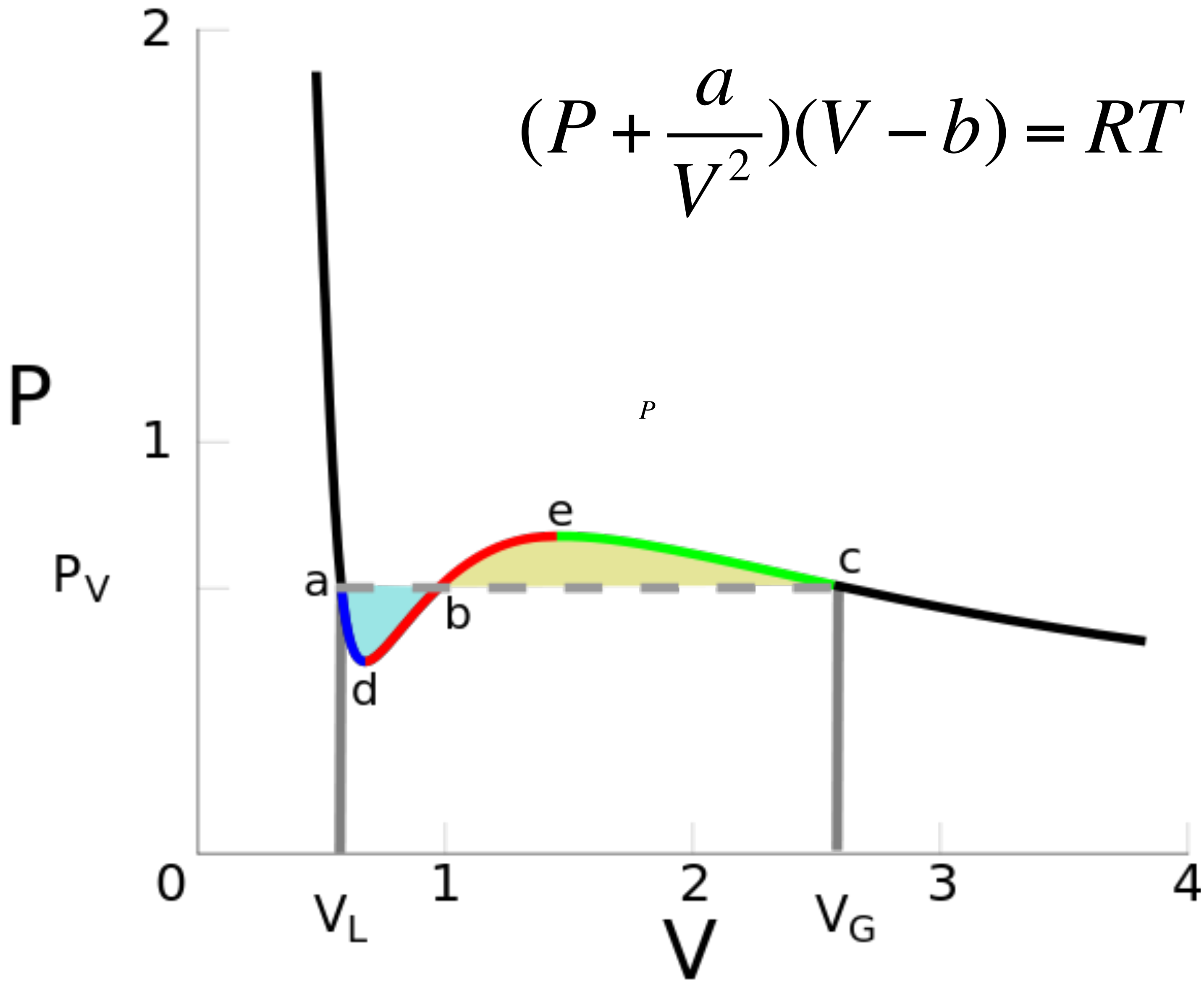
Equation of state: Ideal gas law



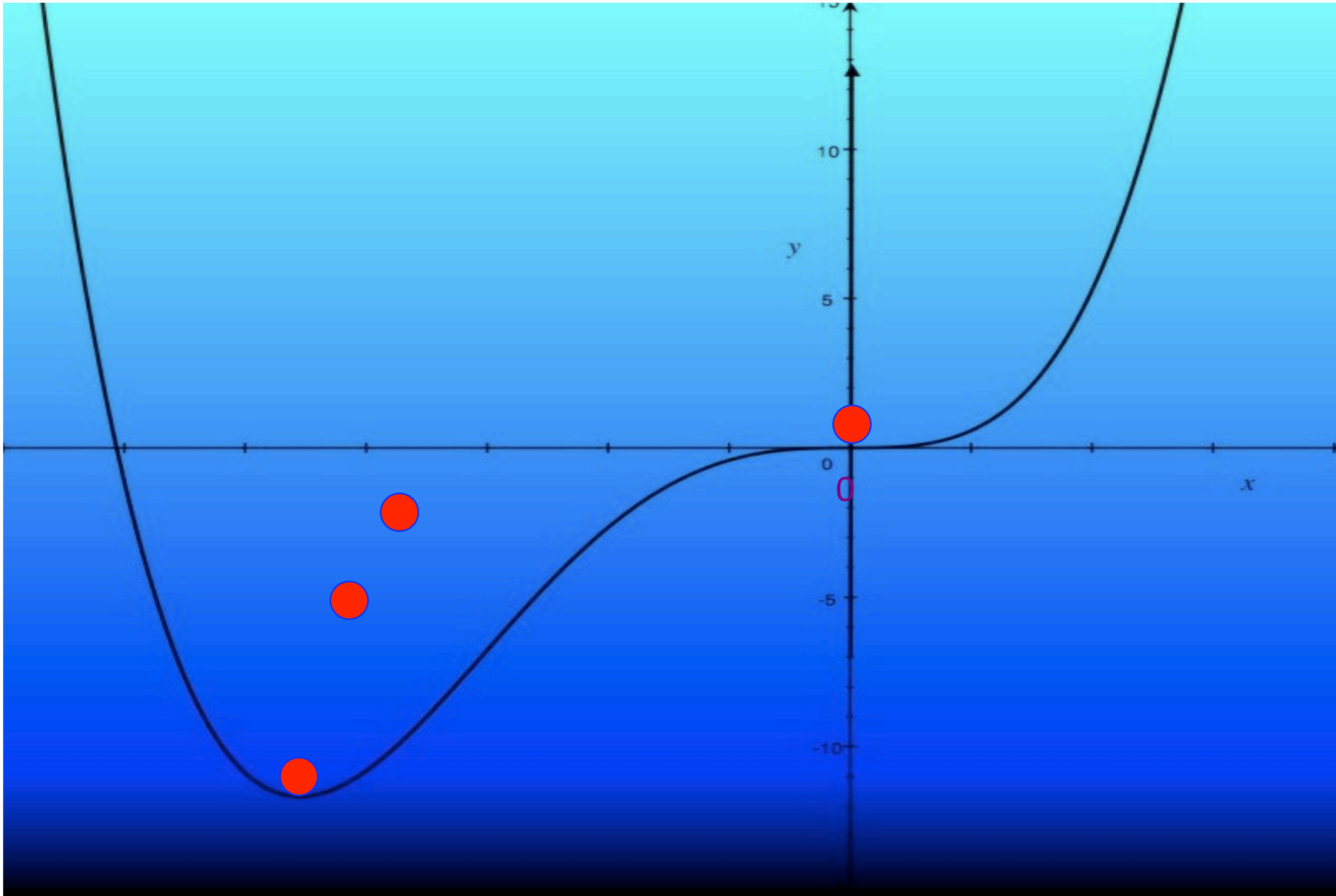
$$PV = RT$$

Van der Waals equation of state

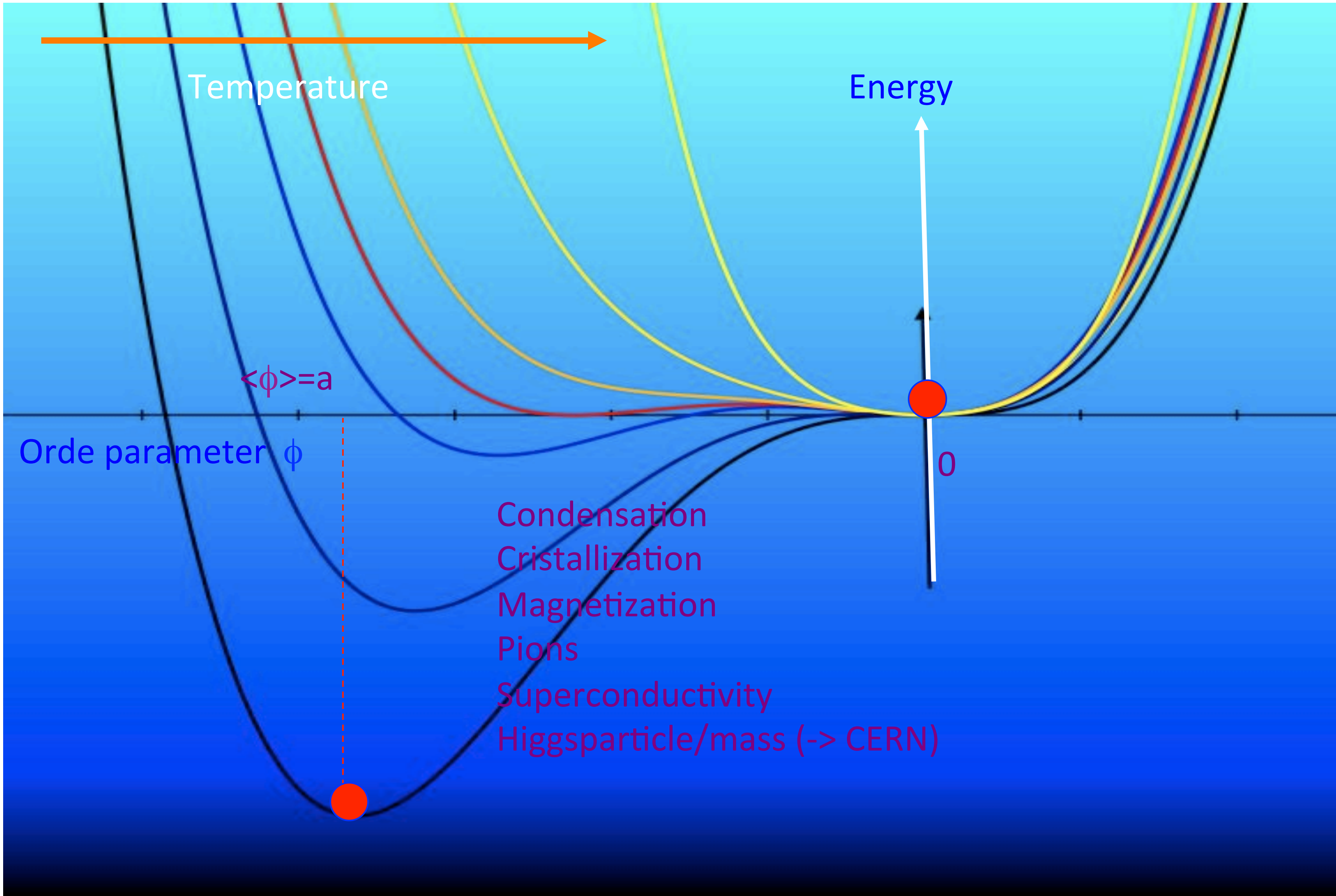
$$(P + \frac{a}{V^2})(V - b) = RT$$



1st order transitions



Potentials en phase transitions (bubble formation)



Ron Suskind (NYT, 2002)

In the summer of 2002...I had a meeting with a senior adviser to Bush.

…

The aid said that guys like me were ``in what we call a reality based community," which he defined as people who ``believe that solutions emerge from judicious study of discernible reality." I nodded and murmured and murmured something about enlightenment principles and empiricism. he cut me off. ``That's not the way things work anymore," he continued.``We are an empire now, and when we act, we create our own reality. And while you're studying that reality--judiciously,as you will--we'll act again, creating other new realities, which you can study too, and that's how things will sort out. We're history's actors...and you, all of you, will be left to just study what we do."

[the aide was presumable Karl Rove]

George Soros (2008)

The public is now awakening, as if from a bad dream. What can it learn from the experience? That reality is a hard task master, and we manipulate it at our peril: The consequences of our actions are liable to diverge from our expectations. However powerful we are, we cannot impose our will on the world: we need to understand the way the world works. Perfect knowledge is not within our reach; but we must come as close to it as we can. Reality is a moving target, yet we need to pursue it. In short, understanding reality ought to take precedence over manipulating it.

As things stand now, the pursuit of power tends to take precedence over the pursuit of truth. Popper and his followers--including me--made a mistake when we took the pursuit of truth for granted.

...

In addition to the familiar attributes of a liberal democracy--free elections, individial liberties, division of powers, the rule of law, etc.--it entails an electorate that insists on certain standards of honesty and truthfulness.

[George Soros (in *The new paradigm for financial markets, 2008*)]

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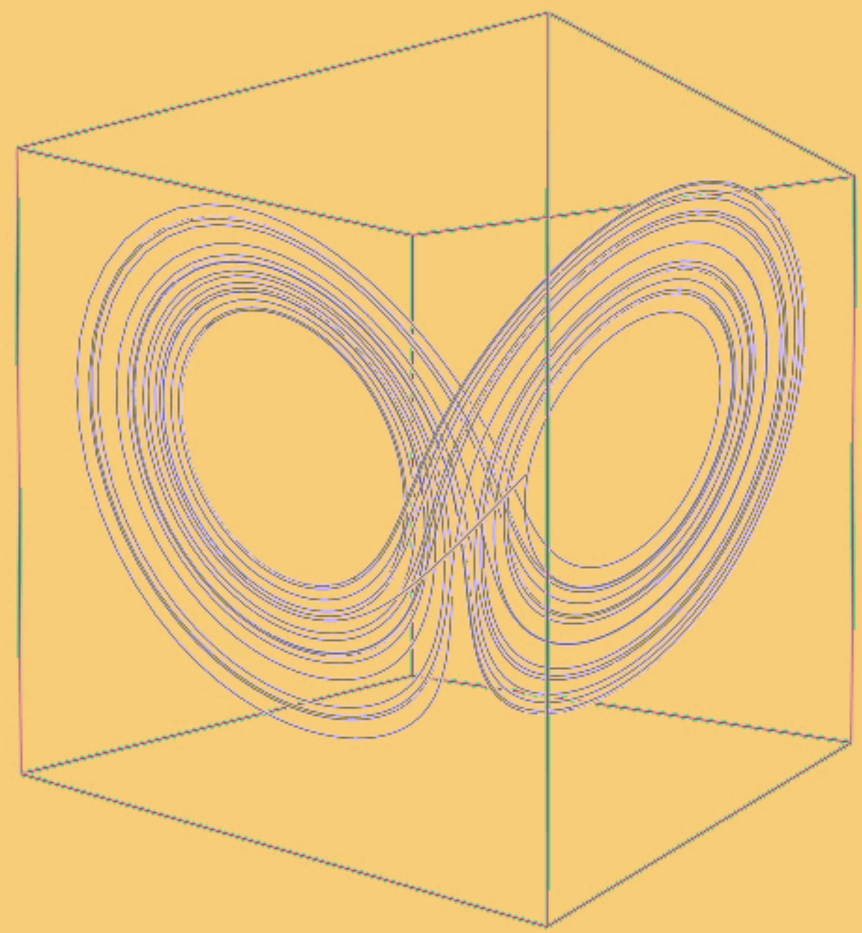
Lorentz attractor: icon of unpredictability



The nonlinear equations:

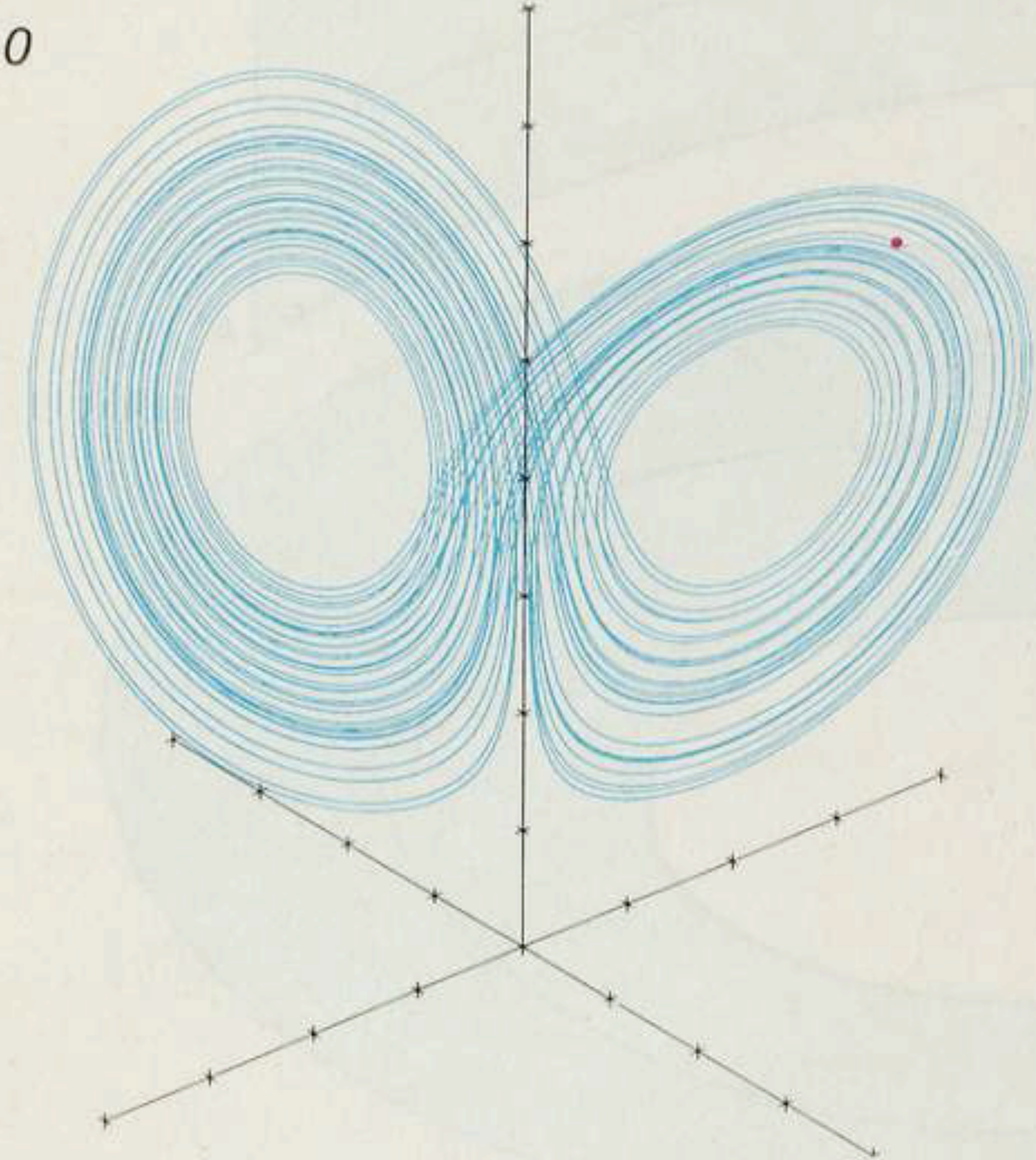
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10(y-x) \\ 28x-y-xz \\ -\frac{8}{3}z+xy \end{bmatrix}$$

The initial; condition:

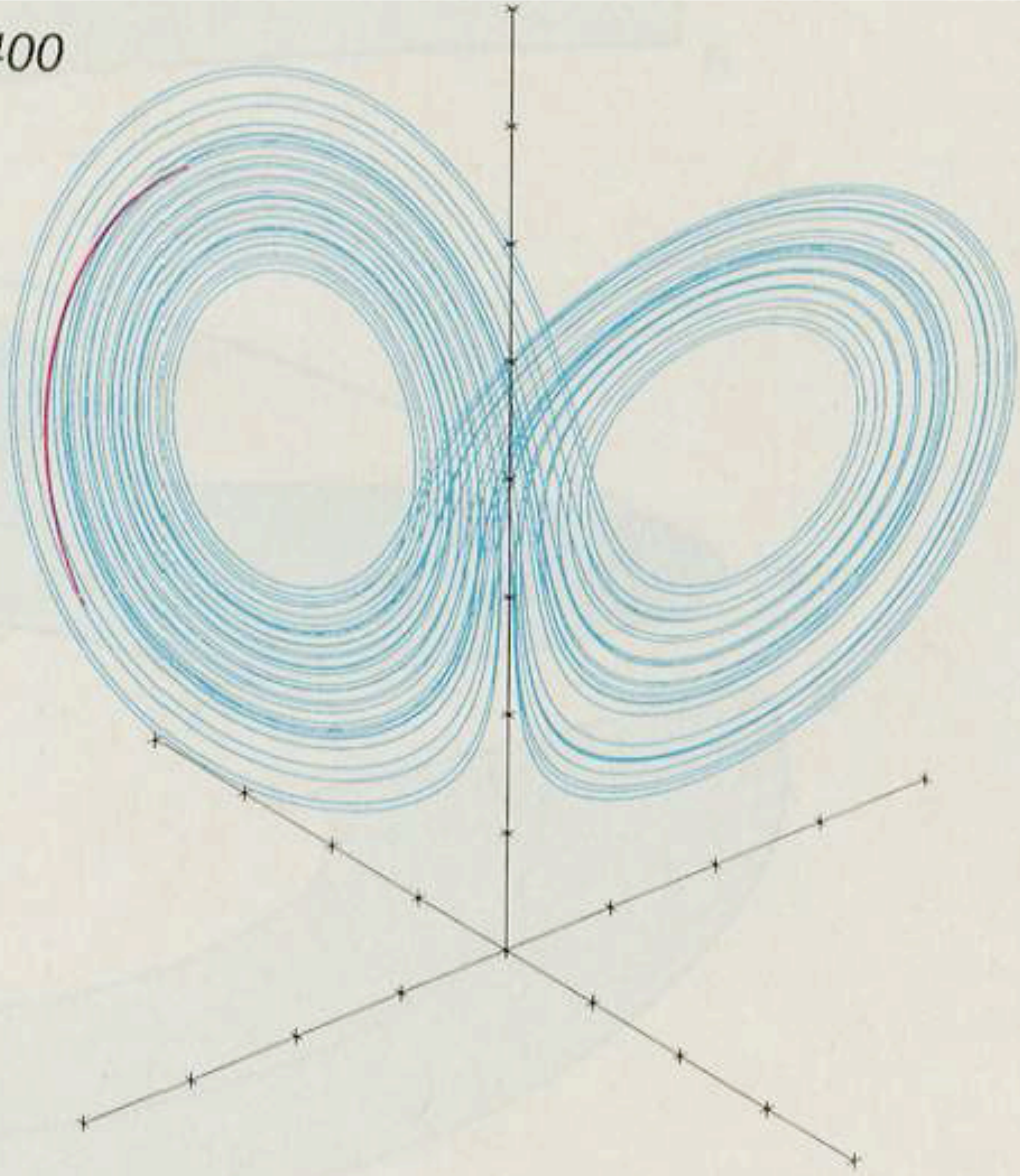
$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 25 \end{bmatrix}$$

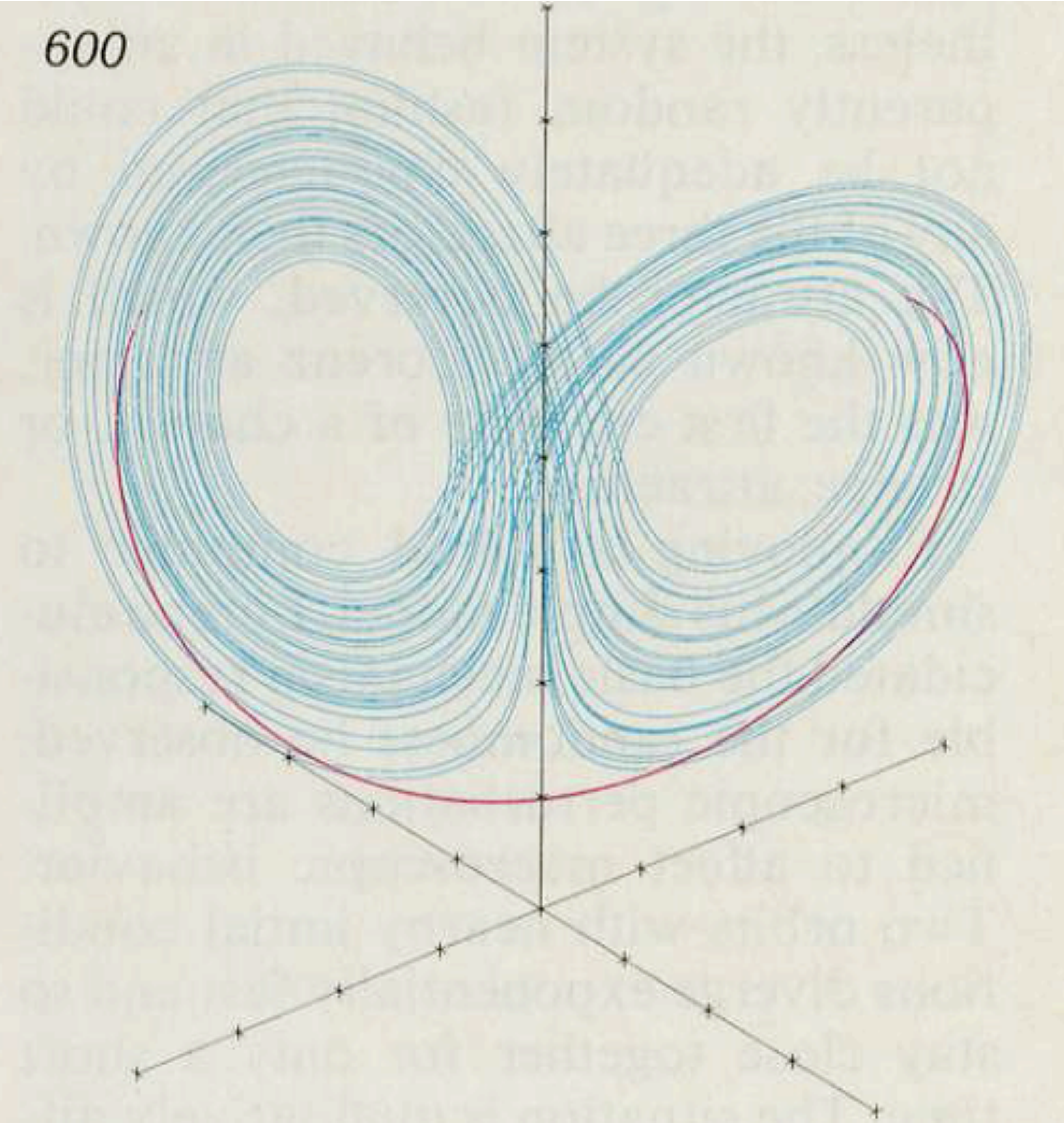
Let it run: from t = 0,…,30

0



400

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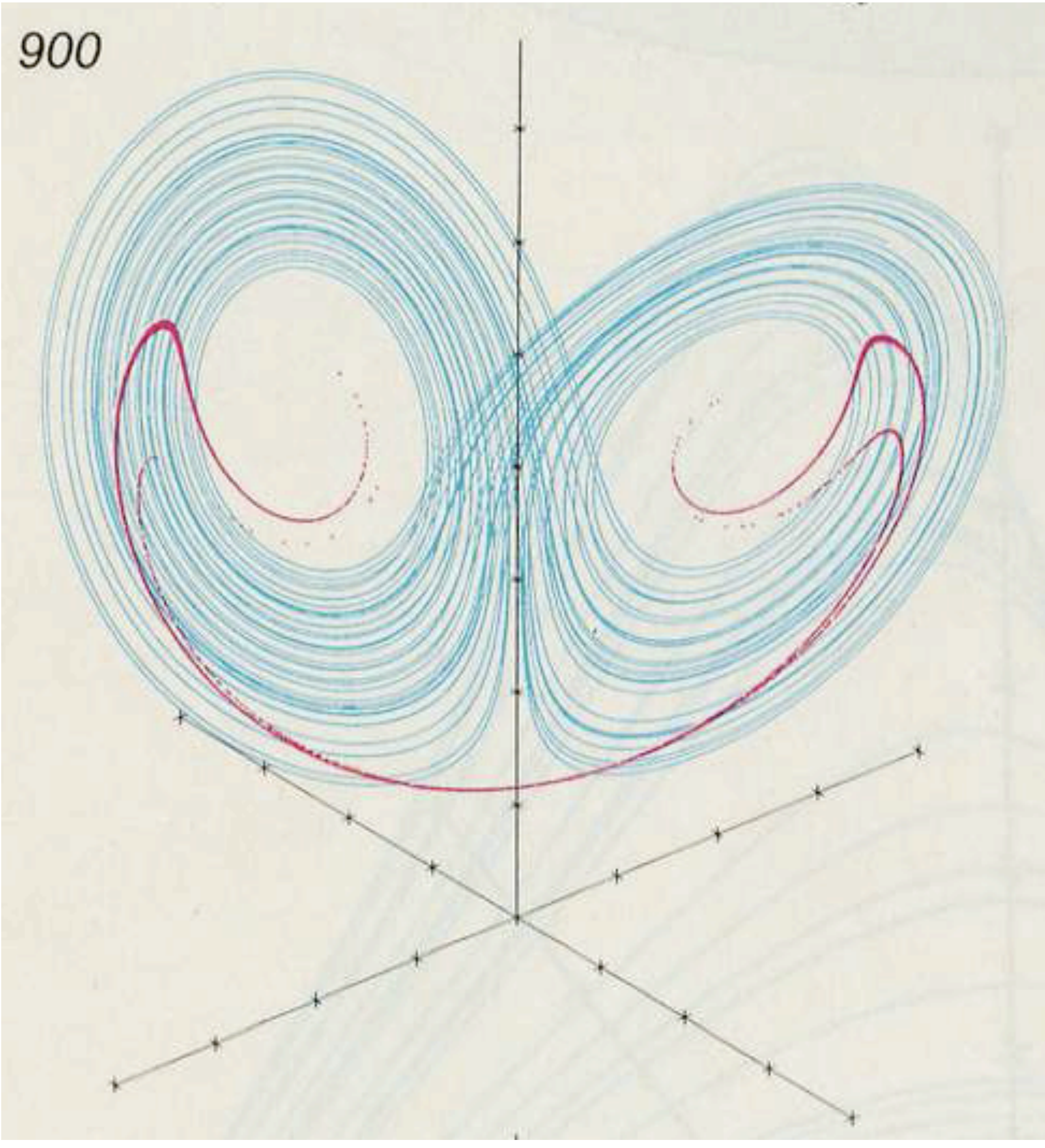
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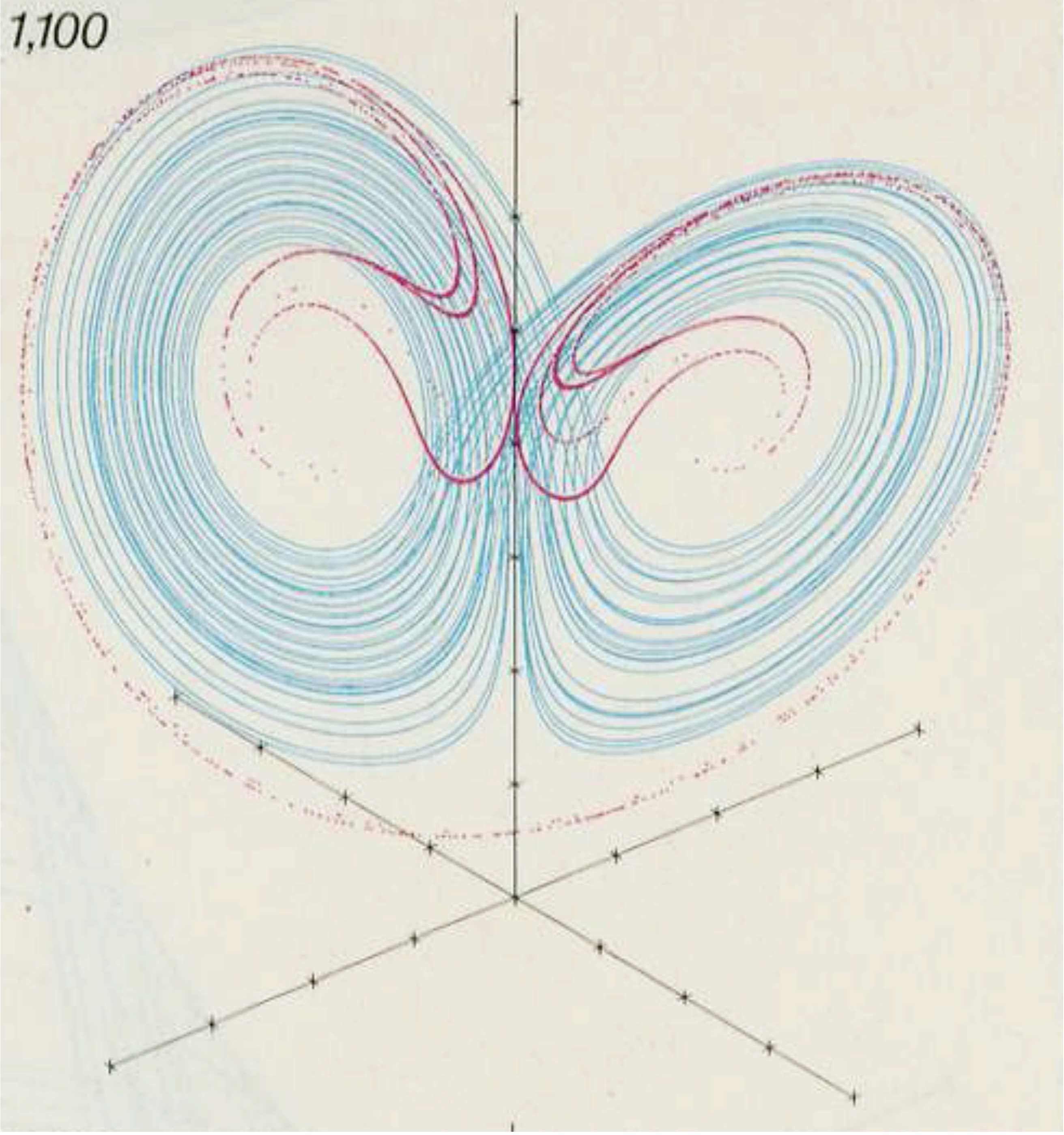
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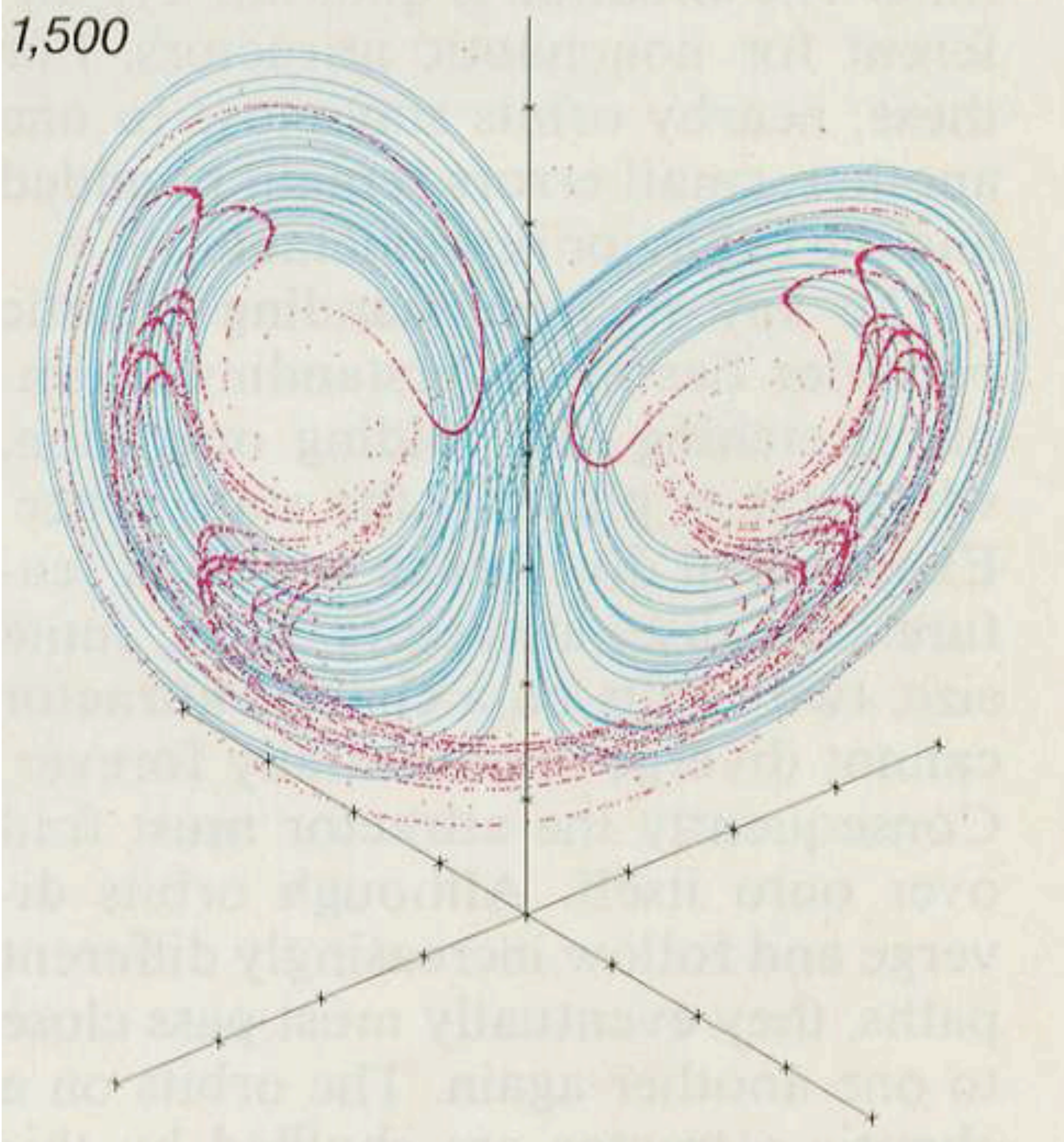
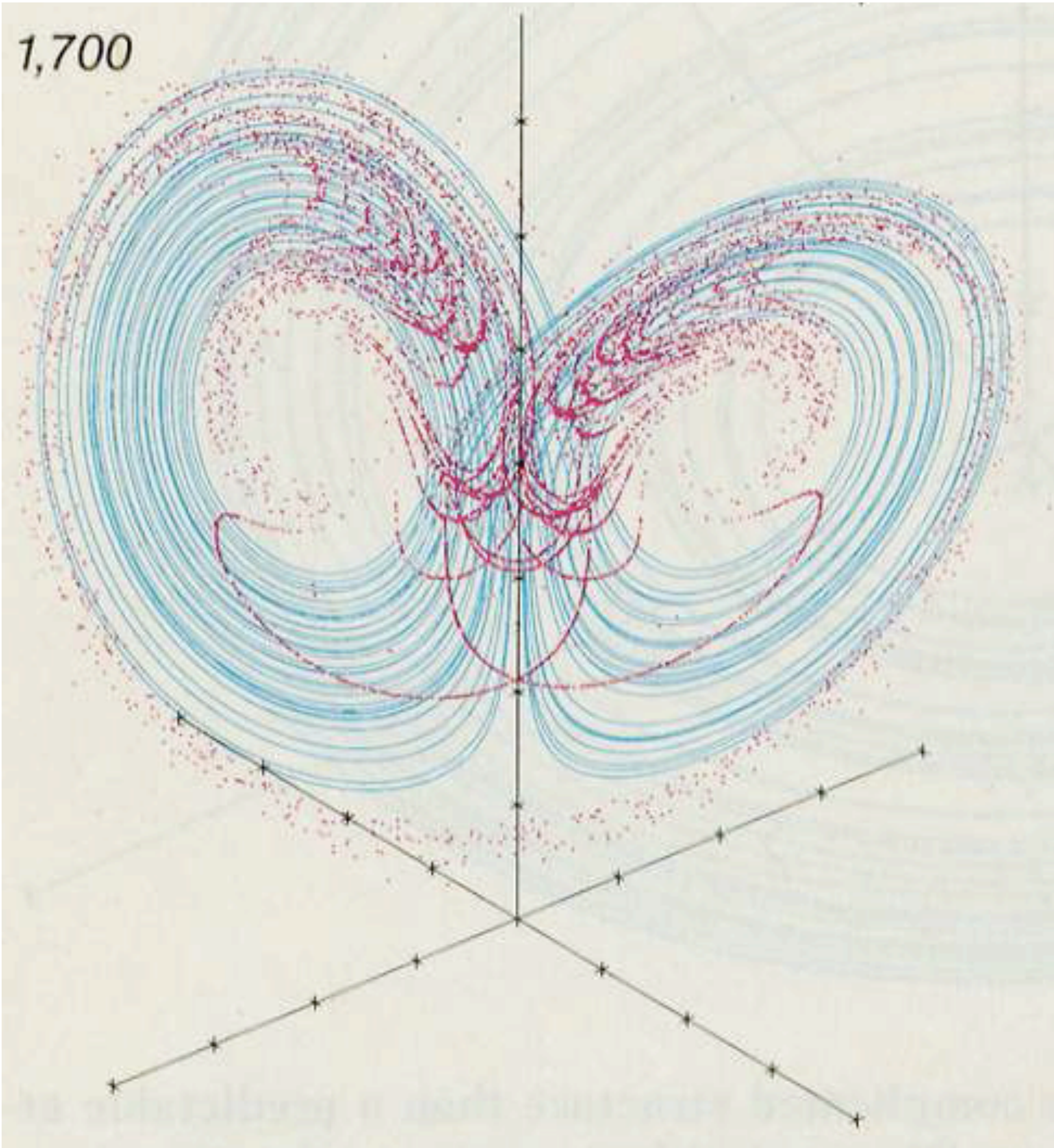
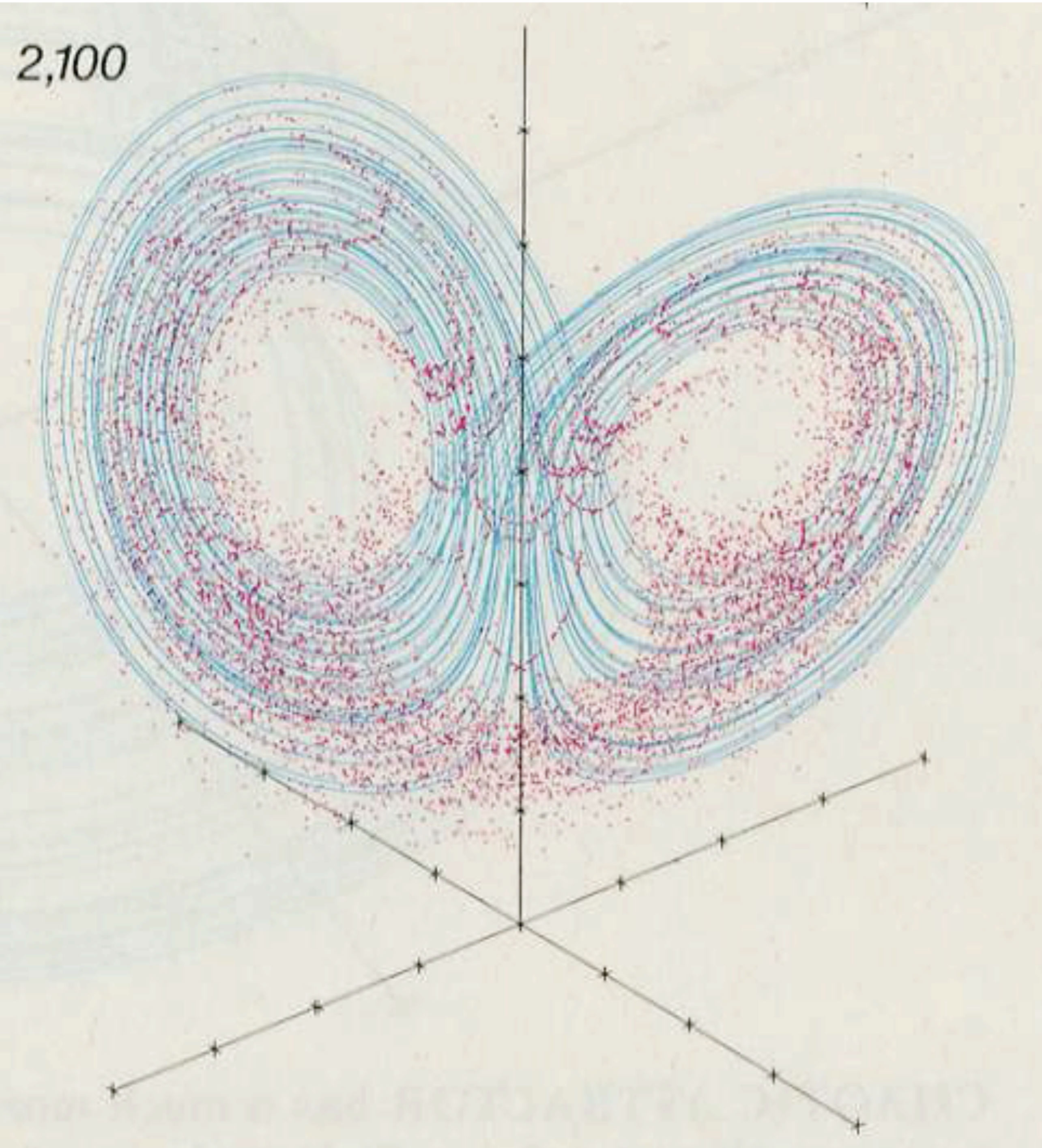
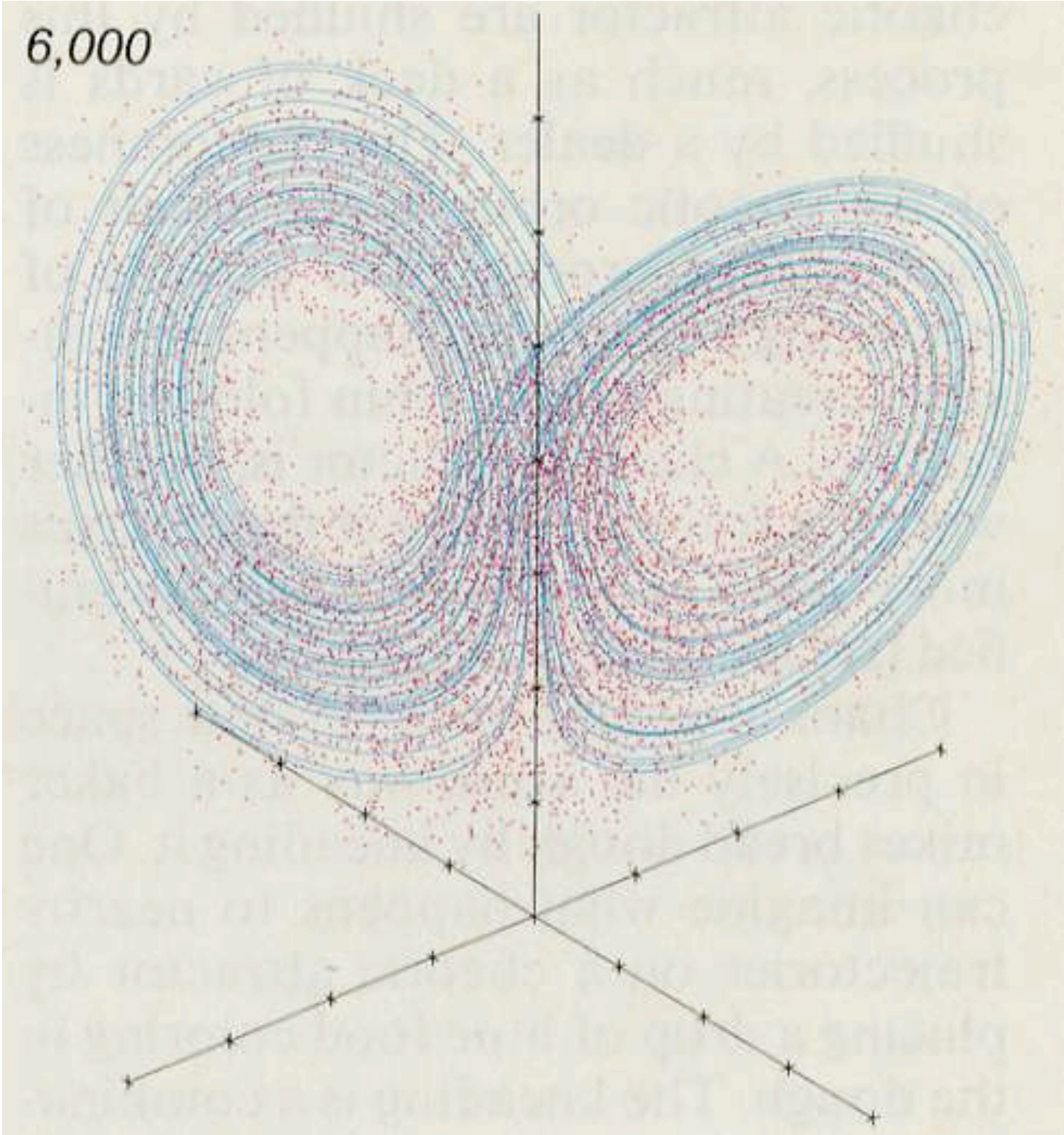

1,500

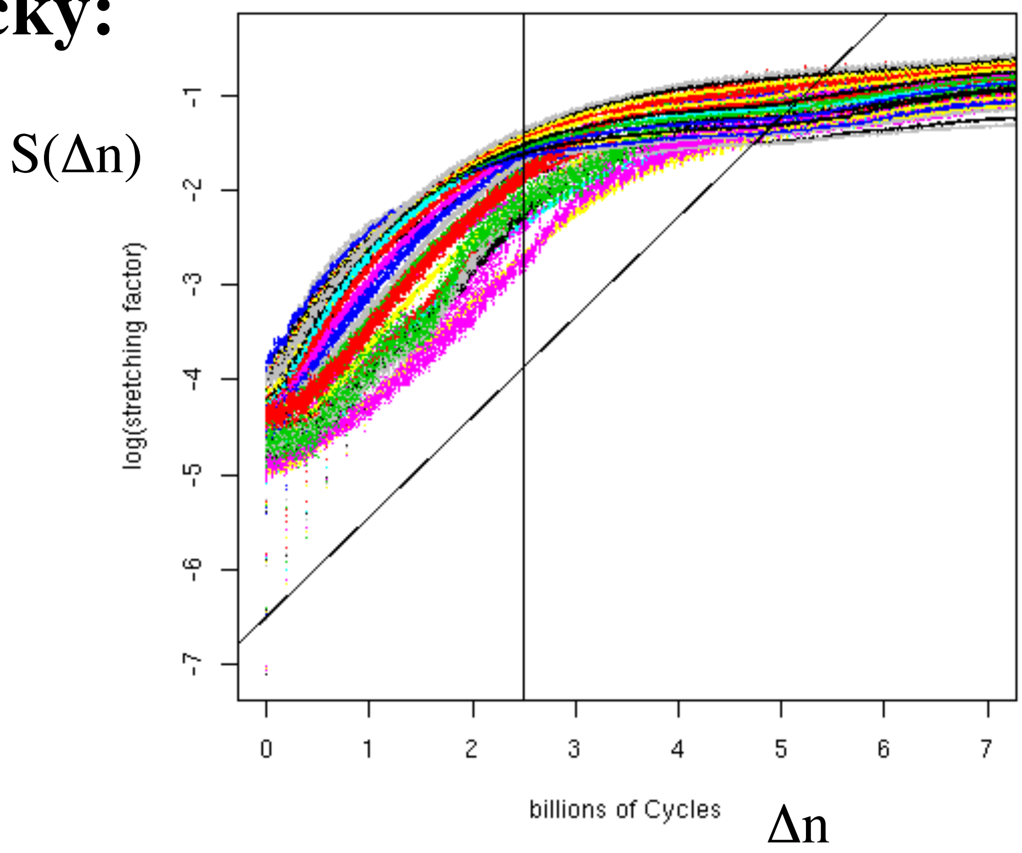

1,700


2,100


6,000

Indicator of chaos

If you're lucky:



The slope of the scaling region—iff one exists—is the λ_1

Limiting distributions

- In complex systems, we wish to understand how a large-scale pattern arises from the aggregation of small-scale processes.
- A single dominant principle sets the major axis from which all explanation of aggregation and scale must be developed. This dominant principle is the existence of a *limiting distribution*.

Sum variables and their distribution

Convolution: X_1 with p.d.f. $f(x)$

X_2 with p.d.f. $g(x)$

Then $X = X_1 + X_2$ has p.d.f.

$h(x) = \int f(u)g(x-u)du = f * g$

$h(x)$ is the convolution $f * g$ of f and g

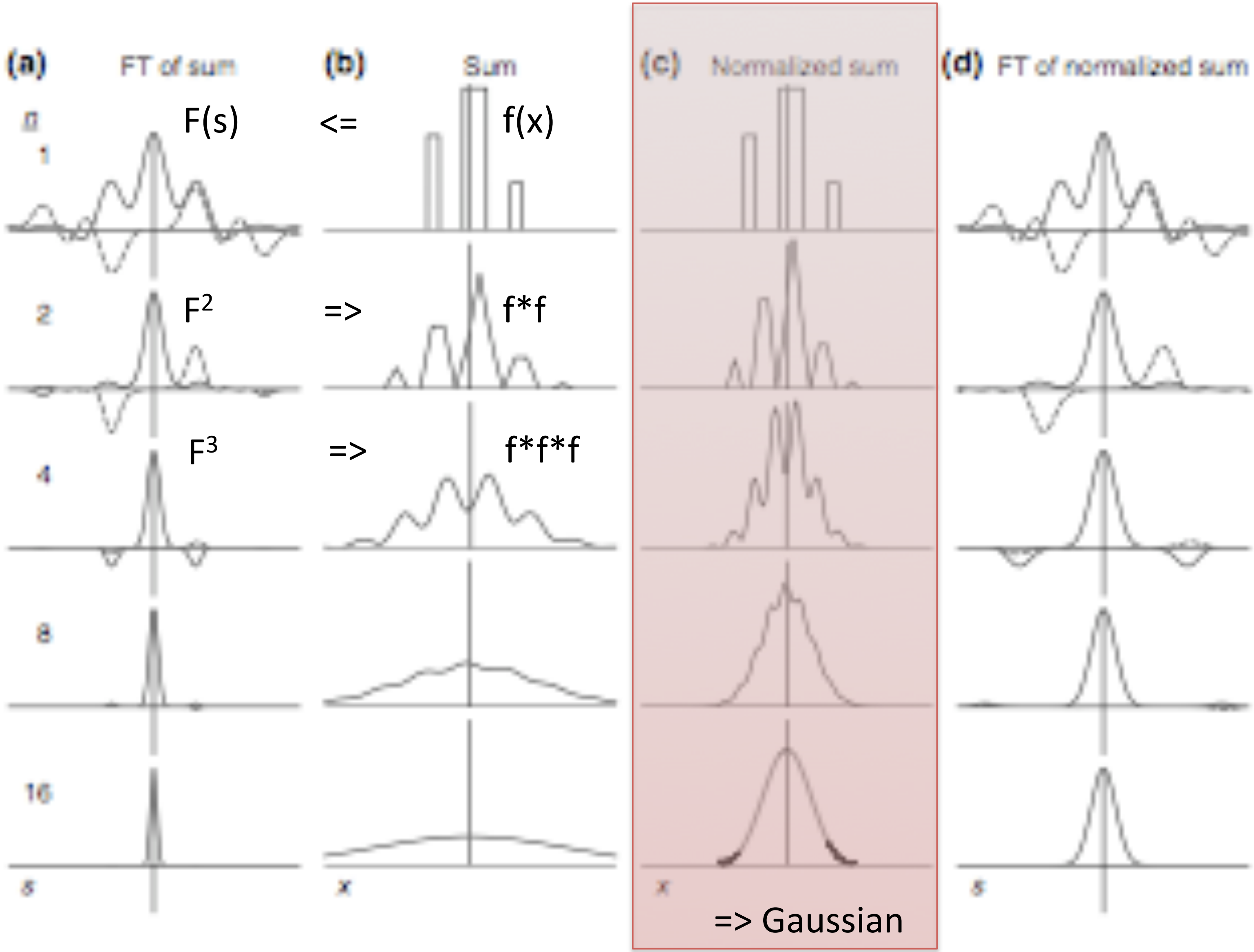
Fourier transform:

$F(s) = \mathcal{F}(f) = \frac{1}{2\pi} \int f(x)e^{isx}dx$

Then convolution becomes ordinary product:

$H(s) = \mathcal{F}(h) = F(s)G(s) = \mathcal{F}(f * g)$

Aggregation: Central limit theorem



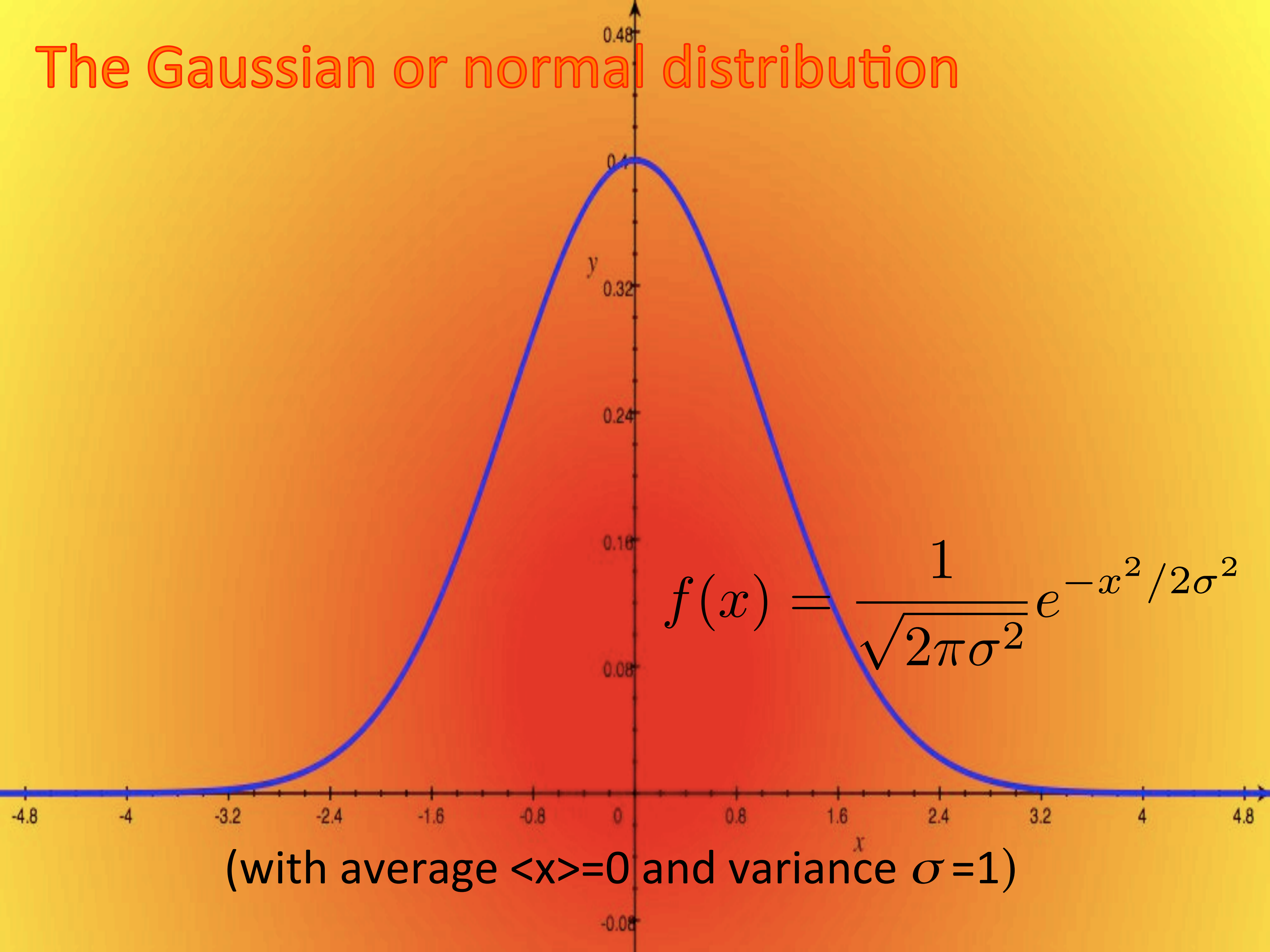
The essence of aggregation

- The individual, small-scale fluctuations caused by each contributing process rarely follow the Gaussian curve. But, with aggregation of many partly uncorrelated fluctuations, each small in scale relative to the aggregate, the sum of the fluctuations smoothes into the Gaussian curve – the limiting distribution in this case.
- One might say that the numerous small fluctuations tend to cancel out, revealing the particular form of regularity or information that characterizes aggregation and scale for the process under study.

The normal distribution

- The best known of the limiting distributions, the Gaussian (normal) distribution, follows from the central limit theorem.
- If an outcome, such as height, weight or yield, arises from the summing up of many small-scale processes, then the distribution typically approaches the Gaussian curve in the limit of aggregation over many processes.

The Gaussian or normal distribution



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

(with average <x>=0 and variance σ =1)

The Gaussian distribution

- I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the 'law of frequency of error' [the normal or Gaussian distribution]. Whenever a large sample of chaotic elements is taken in hand and marshaled in the order of their magnitude, this unexpected and most beautiful form of regularity proves to have been latent all along. The law … reigns with serenity and complete self-effacement amidst the wildest confusion. The larger the mob and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of unreason.

  - (Galton, 1889)

Logistic map

A useful graphical solution technique

$$x_{n+1} = rx_n(1 - x_n)$$

- "cobweb" diagram
- *aka* return map
- *aka* correlation plot

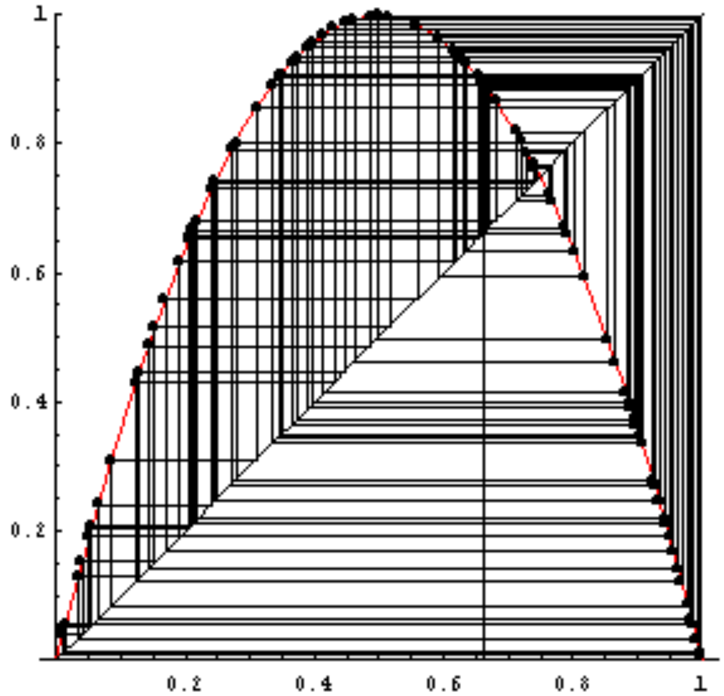
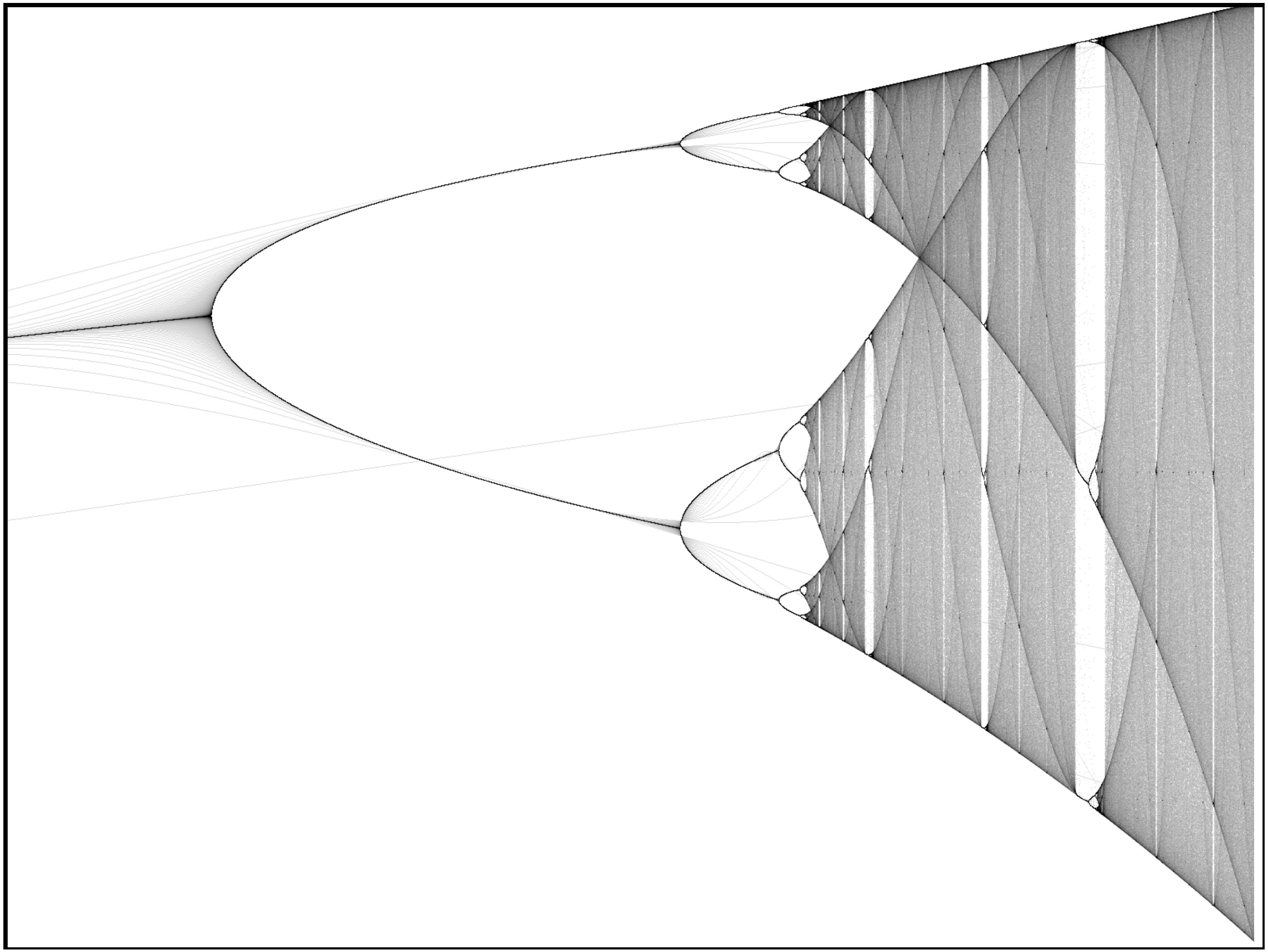
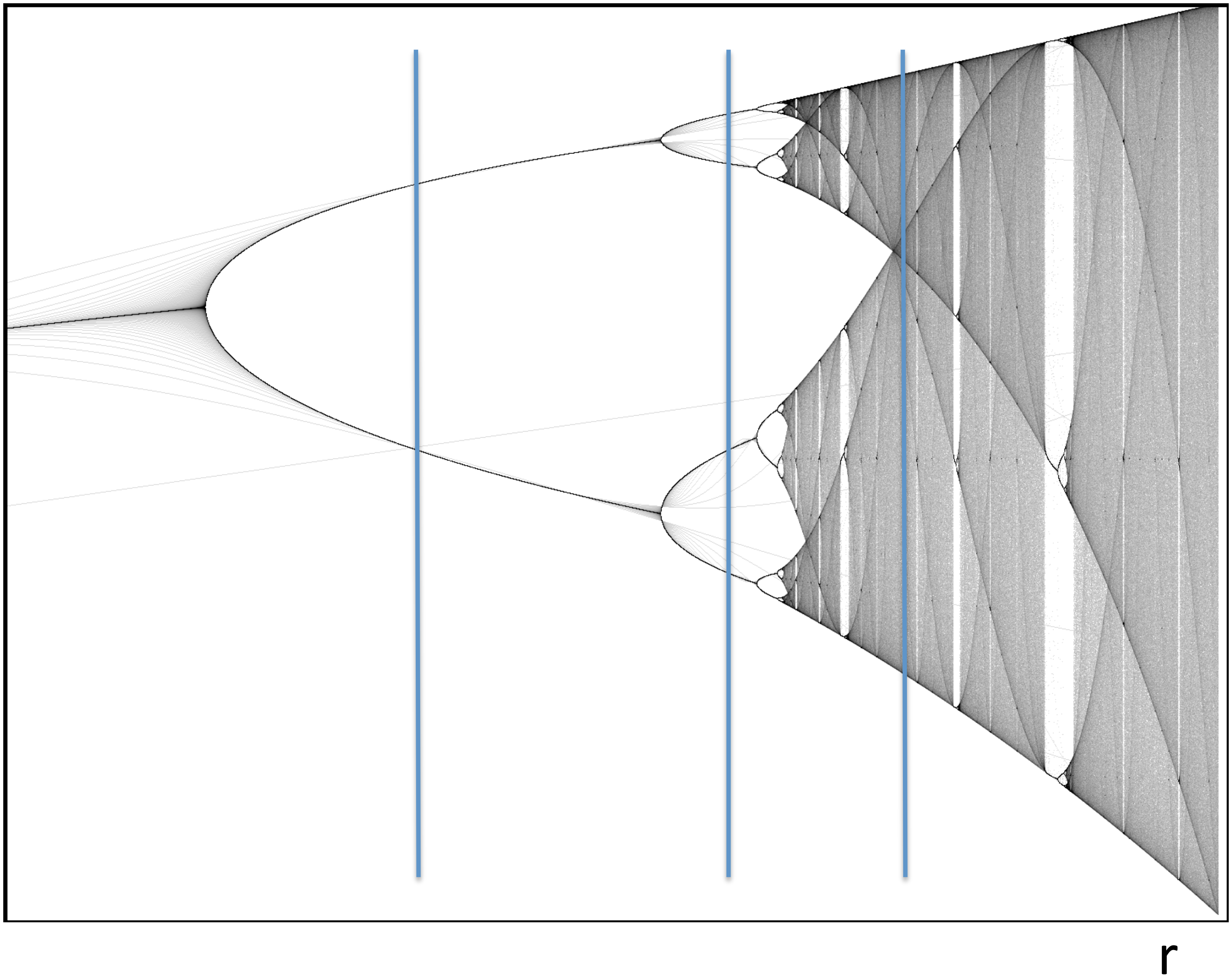


Image from Doug Ravenel's website at URochester

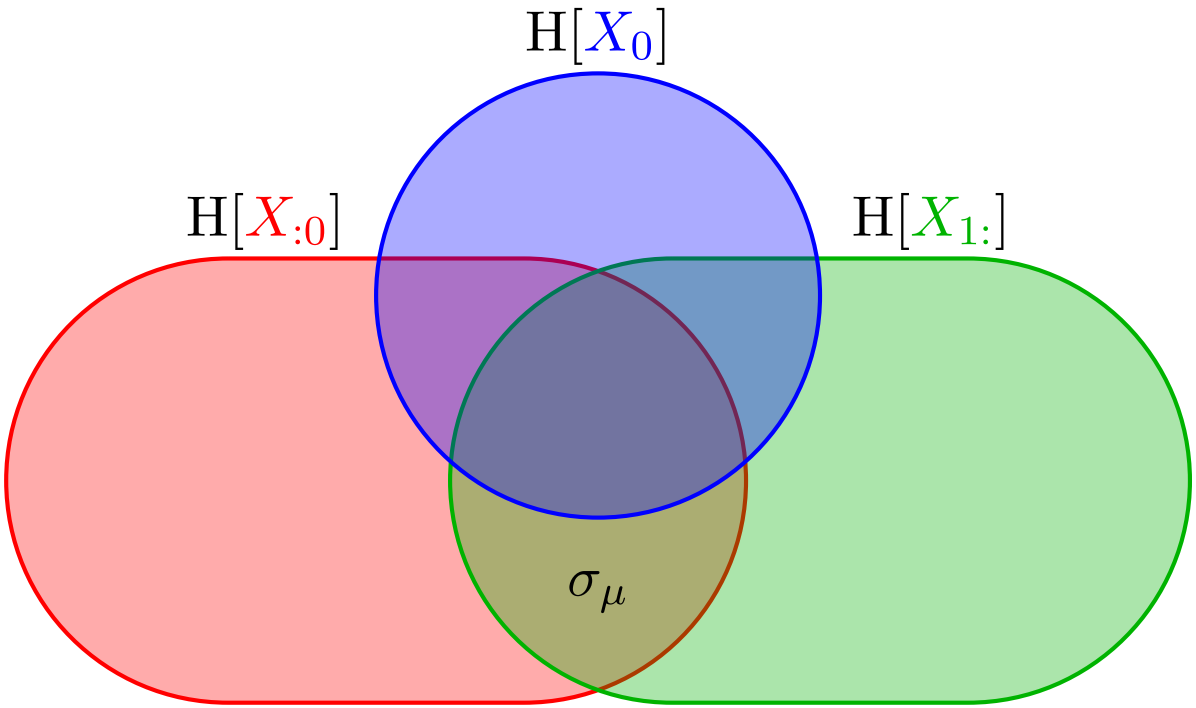
Period doubling → chaos



State space reconstruction for different r values



Jim Crutchfield (Ryan)



- $\mathrm{H}[X_0]$ is *partitioned* by the past and the future
- $\sigma_\mu = \mathrm{I}[X_{:0}; X_{1:} | X_0]$: evidence of internal states

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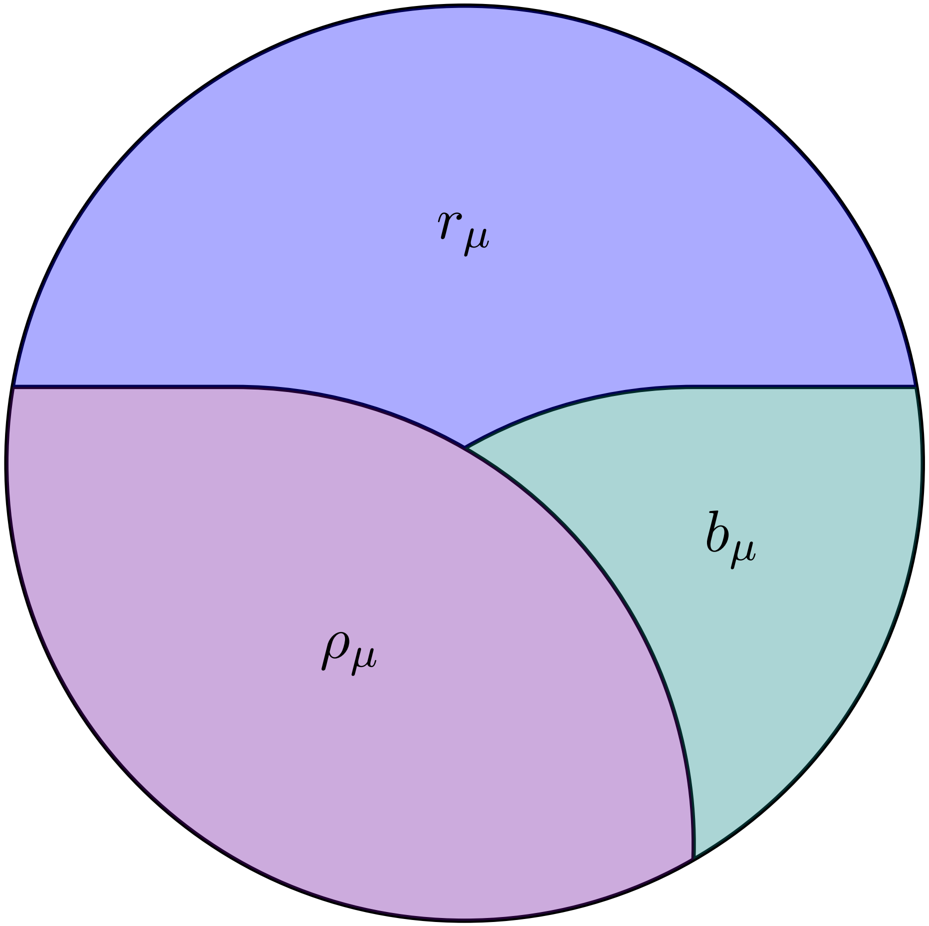
Jim Crutchfield (Ryan)

- $\rho_\mu = \mathrm{I}[X_{:0}; X_0]$: anticipated information
- $b_\mu = \mathrm{I}[X_0; X_{1:} | X_{:0}]$: unanticipated and relevant
- $r_\mu = \mathrm{H}[X_0 | X_{:0}; X_{1:}]$: unanticipated and irrelevant



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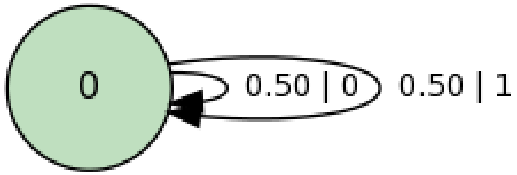
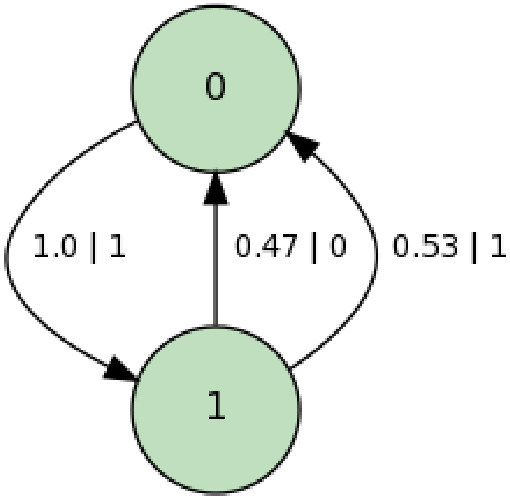
Epsilon machines with causal states



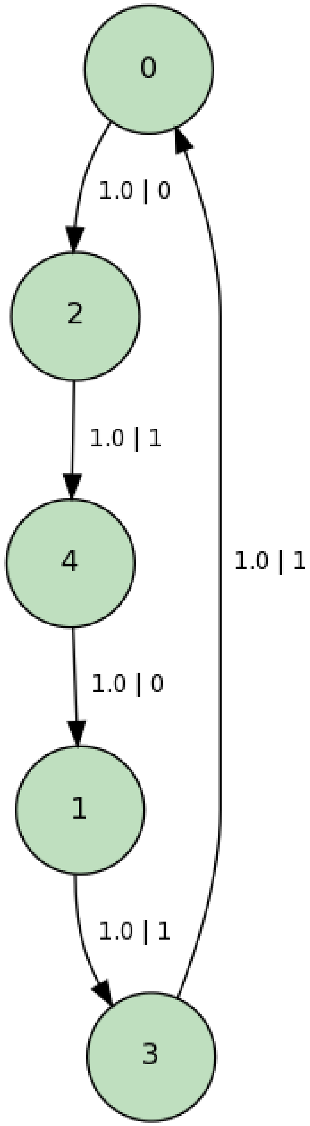
	Cμ	E	ρμ	bμ	rμ
r = 4.000	0.00000	0.00000	0.00000	0.00000	0.99998

R = 3.678



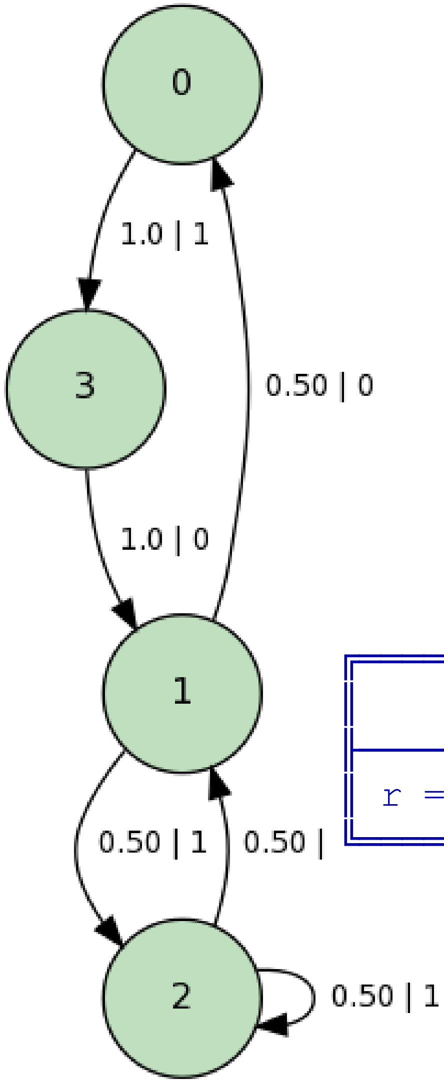
	Cμ	E	ρμ	bμ	rμ
r = 3.678	1.00000	1.00000	0.28425	0.00000	0.49825

R= 3.738



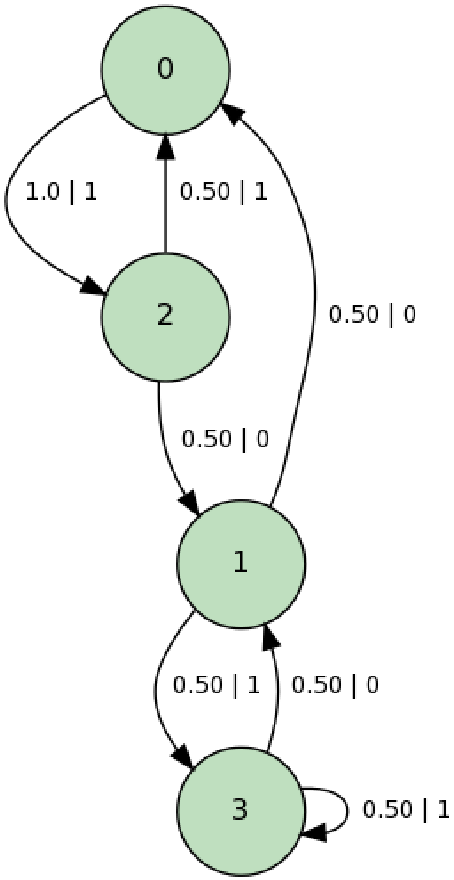
	Cμ	E	ρμ	bμ	rμ
r = 3.738	2.32193	2.32193	0.97095	0.00000	0.00000

R=3.9



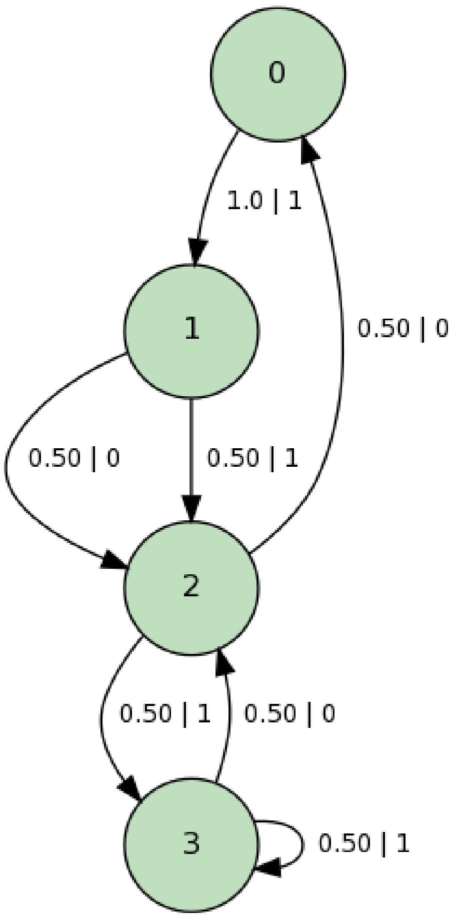
	Cμ	E	ρμ	bμ	rμ
r = 3.900	1.91830	0.68872	0.33333	0.27440	0.39227

R= 3.927



	Cμ	E	ρμ	bμ	rμ
r = 3.927	2.00000	1.00000	0.20443	--	--

R=3.95



	Cμ	E	ρμ	bμ	rμ
r = 3.950	1.91830	0.95915	0.14654	--	--

Questions on applicability

- What if not Markovian?
- Role of noise?
- How does the epsilon machine do for a time series measured in economy?
- What if there is a hierarchy in the underlying dynamics?
- How this combine with the Max Entropy principle?
- Is there an application horizon for networks?

Subjects

- From Art to Maps
- Entropy once more
- Maximal Entropy principle
- From microscopics to macroscopics and back
- Phase transitions/ symmetry breaking
- Nonlinear dynamics and information
- Food for thought

Albert Einstein 1918

“The most important task for scientists is to search for the most fundamental laws, from which a picture of the world can be deduced. There is no logical path that leads to these elementary laws, only an intuitive one, based on creativity and experience.

With such a methodological uncertainty one would think that an arbitrary number of equally valid systems would be possible. However, history shows that of all conceivable constructions always a single one did stand out as absolutely superior to all others.”

Max Perutz 1994

“Discovering its structure was wonderful. You must imagine the time when proteins were black boxes. Nobody knew what they looked like. There I was , having worked on this vital problem for twenty-two years, trying to find out what this molecule looked like, and eventually how it worked. When the result emerged from the computer one night and we suddenly saw it, it was like reaching the top of a difficult mountain after a hard climb and falling in love at the same time. It was an incredible feeling to see this molecule for the first time and to realize that my work had not been in vain: because during those long years I feared that I was wasting my life on a problem that would never be solved.”

Richard Feynman 1981

"You see, one thing is, I can live with doubt and uncertainty and not knowing. I think it's much more interesting to live not knowing than to have answers which might be wrong. I have approximate beliefs and different degrees of certainty about different things, but I'm not absolutely sure of anything and there are many things I don't know anything about, such as whether it means anything to ask why we're here…

I don't know the answer. I don't feel frightened by not knowing things, by being lost in a mysterious universe without any purpose, which is the way it really is as far as I can tell, it doesn't frighten me."