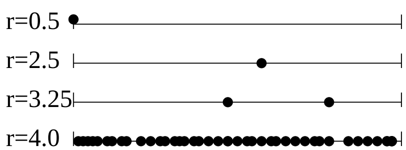
What else can the logistic equation do?

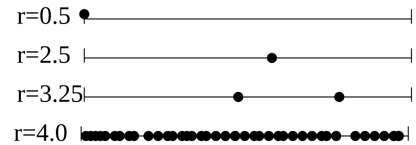
Thus far, we have seen several possible long-term behaviors for the logistic equation:

1. $r = 0.5$: attracting fixed point at 0.
2. $r = 2.5$: attracting fixed point at 0.6.
3. $r = 3.25$: attracting cycle of period 2.
4. $r = 4.0$: chaos.

Graphically, we can illustrate this as follows:

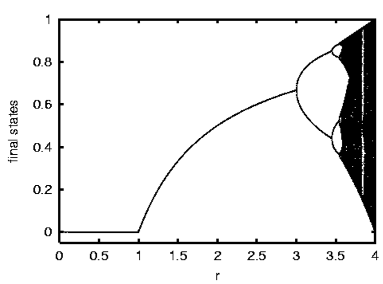


- I.e., for each r, iterate and plot the final x values as dots on the number line.



- Do this for more and more r values and "glue" the lines together.
- Turn sideways and ...

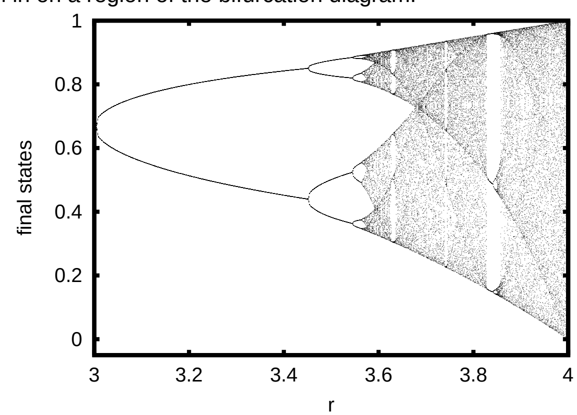
Bifurcation Diagram



- The bifurcation diagram shows the all the the possible long-term behaviors for the logistic map.
- $0 < r < 1$, the orbits are attracted to zero.
- $1 < r < 3$, the orbits are attracted to a non-zero fixed point.
- $3 < r < 3.45$, orbits are attracted to a cycle of period 2.
- Chaotic regions appear as dark vertical lines.

Bifurcation diagram, continued

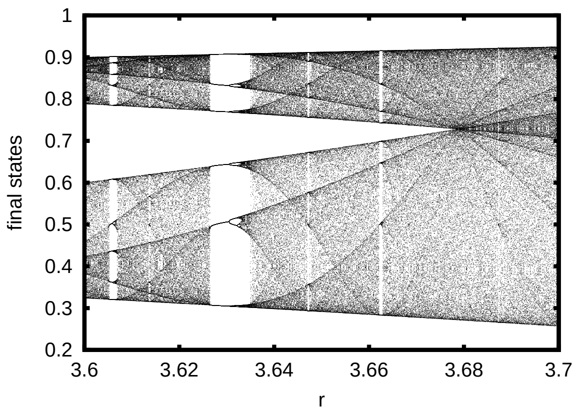
Let's zoom in on a region of the bifurcation diagram:



- The sudden qualitative changes are known as **bifurcations**.
- There are **period-doubling bifurcations** at $r \approx 3.45$, $r \approx 3.544$, etc.
- Note the window of period 3 near $r = 3.83$.

Bifurcation diagram, continued

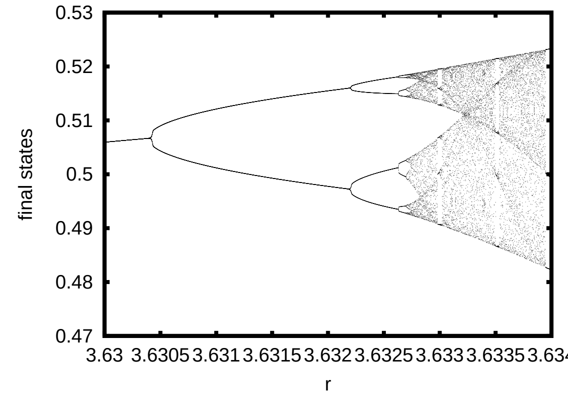
Let's zoom in again:



- Note the sudden changes from chaotic to periodic behavior.

Bifurcation diagram, continued

Let's zoom in once more:



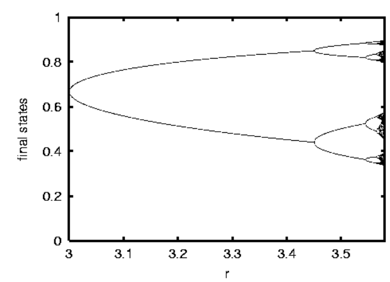
- Note the small scales on the vertical axis, and the tiny scale on the y axis.
- Note the self-similar structure. As we zoom in we keep seeing sideways pitchforks.

Bifurcation Diagram Summary

- As we vary r, the logistic equation shuffles suddenly between chaotic and periodic behaviors, but the bifurcation diagram reveals that these transitions appear in a structured, or regular, way.
- In the next few slides we'll examine one of the regularities in the bifurcation diagram: The **period-doubling route to chaos**.

Period-Doubling Route to Chaos

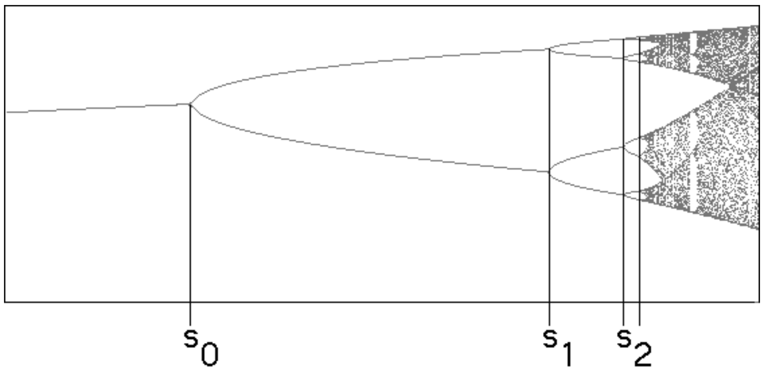
- As r is increased from 3, a sequence of period doubling bifurcations occur.



- At $r = r_\infty \approx 3.569945672$ the periods "accumulate" and the map becomes chaotic.
- For $r > r_\infty$ it has SDIC. For $r < r_\infty$ it does not.
- This is a type of **phase transition**: a sudden qualitative change in a system's behavior as a parameter is varied continuously.

Period-Doubling Route to Chaos: Geometric Scaling

- Let's examine the ratio of the lengths of the pitchfork tines in the bifurcation diagram.

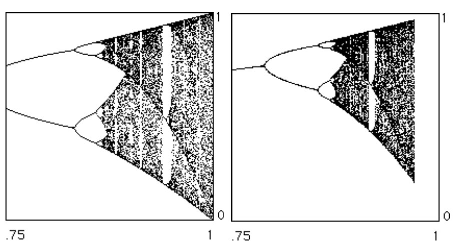


- The first ratio is: $\delta_1 = \frac{s_1 - s_0}{s_2 - s_1}$.
- The n$^{\text{th}}$ ratio is: $\delta_n = \frac{s_n - s_{n-1}}{s_{n+1} - s_n}$.

Feigenbaum's Constant

- This ratio approaches a limit: $\lim_{n\to\infty} \delta_n = 4.669201609\ldots$. This is known as **Feigenbaum's constant** δ.
- This means that the bifurcations occur in a regular way.
- Amazingly, the value of δ is **universal**: it is the same for any period-doubling route to chaos!
- Figure Source: `http://classes.yale.edu/fractals/Chaos/Feigenbaum/Feigenbaum.html`

Universality



- The figure on the left is the bifurcation diagram for $f(x) = r\sin(\pi x)$.
- The figure on the right is the bifurcation diagram for $f(x) = \frac{27}{4} r x^2 (1-x)$.
- The bifurcation diagrams are very similar: **both have** $\delta \approx 4.6692$.
- Mathematically, things are constrained so that there is, in some sense, only one possible way for a system to undergo a period-doubling to chaos.
- Figure Source: `http://classes.yale.edu/fractals/Chaos/LogUniv/LogUniv.html`

Experimental Verification of Universality

- Universality isn't just a mathematical curiosity. Physical systems undergo period-doubling order-chaos transitions. Almost miraculously, these systems also appear to have a universal δ.
- Experiments have been done of fluids, circuits, acoustics:
  - Water: $4.3 \pm .8$
  - Mercury: $4.4 \pm .1$
  - Diode: $4.5 \pm .6$
  - Transistor: $4.5 \pm .3$
  - Helium: $4.8 \pm .6$

  Data from Cvitanović, *Universality in Chaos*, World Scientific, 1989.
- A very simple equation, the logistic equation, has produced a quantitative prediction about complicated systems (e.g., fluid turbulence) that has been verified experimentally.
- Nature is somehow constrained.

Detour: A Little Bit More About Universality

- The order-disorder phase transition in the logistic map is not the only sort of phase transition that is universal.
- Second order (aka continuous) phase transitions are also universal.
- There are several different universality classes, each of which has different values for quantities analogous to δ.
- The symmetry of the order parameter and the dimensionality of the space of the system determine the universality class.
- The order parameter is a quantity which is zero on one side of the transition and non-zero on the other.
- The theory of critical phenomena does not tell one how to find order parameters. (Crutchfield, Shalizi, and I will each discuss this point further.)

A Little Bit More About Universality, continued

- At the transition point, or **critical point**, some quantities (e.g., specific heat) usually diverge. The divergence is described by a power law. The exponents for these power laws are called **critical exponents**.
- At the critical point, the correlations between components of the system usually decay with a power law as the distance increases. Away from the critical point, the decay is exponential—much faster.

A Little Bit about Power Laws

- At critical points, functions like the specific heat diverge with a power law.
- This divergence arises because the correlations between the system's parts is long range—the corrections decay with a power law, not an exponential.
- Power-law decay of correlations is an indication that the system is organized or complex.
- However, this does not mean that the only way that power law distributions can be formed is via long-range order or correlations or complexity.
- In fact, there are very simple mechanisms for producing power laws.
- For much more on this topic, see Cosma's lectures on power laws.

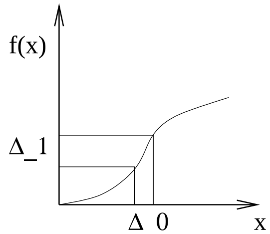
A few Power Law References

- M.E.J. Newman, Power laws, Pareto distributions and Zipf's law, *Contemporary Physics* **46**, 323-351 (2005). An excellent review paper. Highly recommended. `arXiv.org/cond-mat/0412004`.
- Reed and Hughes, Why power laws are so common in nature. Physical Review E 66:067103. 2002. `http://www.math.uvic.ca/faculty/reed/`.
- Sornette, Multiplicative processes and power laws. `arXiv.org/cond-mat/9708213`, 1998. (How to get power laws by multiplying random variables.)

Measuring Sensitive Dependence: Lyapunov Exponent

SDIC arises because the function pushes nearby points apart. The Lyapunov exponent measures this pushing.

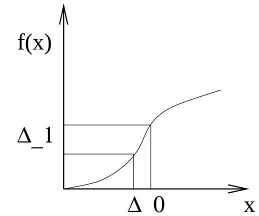
- Consider an initial small interval Δ_0 of initial conditions centered at x_0.



- After one iteration, this interval becomes $\Delta_1 \approx |f'(x_0)|\Delta_0$.
- $|f'(x_0)|$ is the local stretch (or shrink) factor.
- After n iterations, $\Delta_n = \prod_n |f'(x_n)|\Delta_0$.
- The idea is that for the n$^{\rm th}$ iterate interval is getting stretched (shrunk) by the stretch factor $f'(x)$ evaluated at the x_n, the location of the n$^{\rm th}$ iterate of x_0.

Lyapunov Exponent, continued

- We expect the growth of the interval Δ_0 to be exponential, since we're multiplying the interval at each time step.



- That is, we expect that $\frac{\Delta_n}{\Delta_0} = e^{\lambda n}$, where λ is the exponential growth rate.
- The exponential growth is just the the product of all the local stretch factors along an itinerary:

$$e^{\lambda n} = \prod_n |f'(x_n)| \ .$$

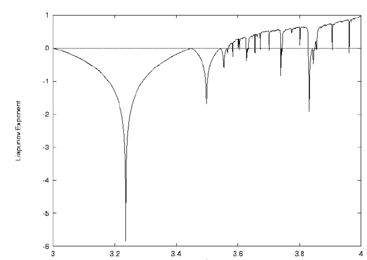
Lyapunov Exponent, continued

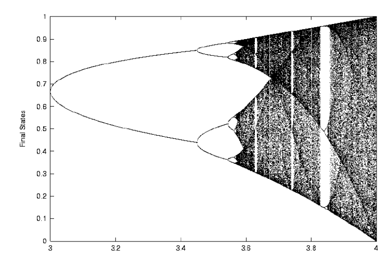
- Solving for λ:

$$\lambda \equiv \lim_{N\to\infty} \left[\frac{1}{N} \sum_n \ln |f'(x_n)| \right] \ . \qquad (1)$$

- λ is the **Lyapunov exponent**. It measures the degree of SDIC.
- If $\lambda > 0$, the function has SDIC.

Lyapunov Exponent for the Logistic Equation

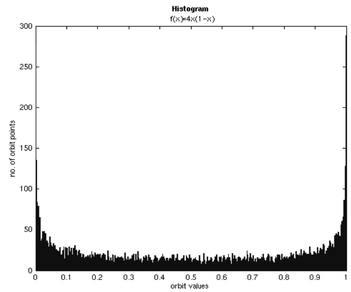




- The top graph shows the Lyapunov exponent as a function of r.
- Note that $\lambda > 0$ in the chaotic regions of the bifurcation diagram.

Initial Conditions?

- It seems like the definition of λ depends on the initial condition. If so, λ is a property of x_o, and not a global property of f.
- It turns out that for many dynamical systems you will get the same λ for almost all x_0. Why is this?
- Imagine building a histogram for orbits. For $r = 4$, this will look like:



- `http://www-m8.mathematik.tu-muenchen.de/personen/hayes/chaos/Hist.html`

Natural Invariant Densities and Ergodicity

- The distribution in this histogram $\rho(x)$ will be obtained by iterating almost any initial condition x_0.
- This distribution is known as the **Natural Invariant Density**.
- If we can figure it out, we can determine the Lyapunov exponent by integrating:

$$\lambda = \int \ln(|f'(x)|)\rho(x)\,dx\ .$$

- In general, if a dynamical property like the lyapunov exponent can be determined by integrating over x instead of performing a dynamical average, the system is **ergodic**.
- Proving that a system is ergodic is usually very hard.
- Trivia: for $r = 4$, $\rho(x) = \frac{\pi}{\sqrt{x(1-x)}}$.
- For other r values, an expression for $\rho(x)$ is not known. Generally, $\rho(x)$ is non-smooth.

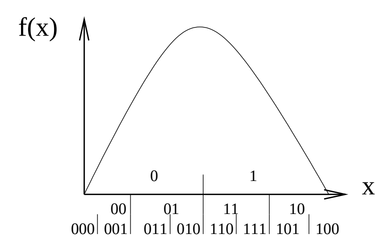
Symbolic Dynamics

- It is often easier to study dynamical systems via symbolic dynamics.
- The idea is to encode the continuous variable x with a discrete variable in some clever way that doesn't entail a loss of information.
- For the logistic equation:

$$s_i = \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} .$$

- Why is this ok? It seems that we're throwing out a lot of information!
  - The the function is deterministic, the initial condition contains all information about the itinerary.
  - For the coding above, longer and longer sequences of 1's and 0's code for smaller and smaller regions of initial conditions.
  - Codings that have this property are known as **generating partitions**.

Symbolic Dynamics, continued



- If we find a generating partition, we can use the symbols to explore the function's properties.
- The symbol sequences are "the same" as the orbits of x: they have the same periodic points, the same stability, etc.
- For $r = 4$, the symbolic dynamics of the logistic equation produce a sequence of 0's and 1's that is indistinguishable from a fair coin toss.

Chaos Conclusions

- Deterministic systems can produce random, unpredictable behavior. E.g., logistic equation with $r = 4$.
- Simple systems can produce complicated behavior. E.g., long periodic behavior in logistic equation.
- Some features of dynamical systems are universal—the same for many different systems.
- More generally, dynamics are important. Considering only static averages can be misleading.

Chaos $\Rightarrow$ Complex Systems

Some of the roots of complex systems are in chaos:

- Universality gives us some reason to believe that we can understand complicated systems with simple models.
- Appreciation that complex behavior can have simple origins.
- Awareness that there's more to dynamical systems than randomness. These systems also make patterns, organize, do cool stuff.
- Is there a way we can describe or quantify these patterns?
- Is there a quantity like the Lyapunov exponent that measures complexity or pattern?
- What is a pattern?

