Different timestep

Lorenz, *Physica D* **35**:229
Different arithmetic
Different solver algorithm…
symmetric multistep, $h = 100$
Adams explicit, \( h = 100 \)
Moral: numerical methods can run amok in “interesting” ways…

• can cause distortions, bifurcations, etc.
• and these look a lot like real, physical dynamics…
• source: algorithms, arithmetic system, timestep, etc.
• Q: what could you do to diagnose whether your results included spurious numerical dynamics?
Moral: numerical methods can run amok in “interesting” ways…

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• and these look a lot like real, physical dynamics…
• source: algorithms, arithmetic system, timestep, etc.
• Q: what could you do to diagnose whether your results included spurious numerical dynamics?
  • change the timestep
  • change the method
  • change the arithmetic
So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….
Shadowing lemma:

Every noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

Important: this is for *state* noise, not *parameter* noise.
Solving PDEs
Projection:
Section:
Section of a UPO:
Aside: finding UPOs

- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers
Back to sections... *time-slice* ones now.
Time-slice sections of periodic orbits: some thought experiments

• pendulum rotating @ 1 Hz and strobe @ 1 Hz?
• pendulum rotating @ 1 Hz and strobe @ 2 Hz?
• pendulum rotating @ 1 Hz and strobe @ 3 Hz?
• pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
• pendulum rotating @ 1 Hz and strobe @ $\pi$ Hz? (or some other irrational)
When this becomes really useful:
Poincare section:
What bifurcations look like on a Poincare section:
Computing sections:

- Space-slice
- Time-slice
Stability, $\lambda$, and the un/stable manifolds
Lyapunov exponents:

- nonlinear analogs of eigenvalues: one $\lambda$ for each dimension
- $\Sigma \lambda < 0$ for dissipative systems
- $\lambda$ are same for all ICs in one basin
- negative $\lambda$ compress state space along stable manifolds
- positive $\lambda$ stretch it along unstable manifolds
- biggest one $\lambda_1$ dominates as $t \to$ infinity
- *positive $\lambda_1$ is a signature of chaos*
These $\lambda$ & manifolds play a role in control of chaos...
We’ve been assuming that we can measure all the state variables:
But often you can’t:
How to reconstruct the other state vars?

derivatives magnify noise!
What this looks like in the state space:

This is not useful for computation.
What we want here is to undo a projection:
Delay-coordinate embedding:

“reinflate” that squashed data to get a topologically identical copy of the original thing.
Reconstruction space:
Mechanics:

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<th>t</th>
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\[ x(t+\tau) \]

\[ x(t+2\tau) \]

\[ \tau = 0.2 \]

\[ m = 3 \]
\[
x(t + \tau) = m = 3
\]
What this looks like:

Data:

Reconstruction space:
Take\n
Theorem: 

For the right $\tau$ and enough dimensions, the dynamics in this reconstruction space are diffeomorphic to the original state-space dynamics.

* Whitney, Mane, …
Diffeomorphisms and topology:

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

• qualitatively the same shape
• have same dynamical invariants (e.g., $\lambda$)
Picking $\tau$: 
Picking $m$:

$m > 2d$: **sufficient** to ensure no crossings in reconstruction space:

…may be overkill.

“Embedology” paper: $m > 2 \, dc$

(box-counting dimension)
If $\Delta t$ is not uniform:

Theorem (Takens): for $\tau > 0$ and $m > 2d$, the reconstructed trajectory is diffeomorphic to the true trajectory.

Conditions: evenly sampled in time