Complexity

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St. Johns' College, Santa Fe, NM
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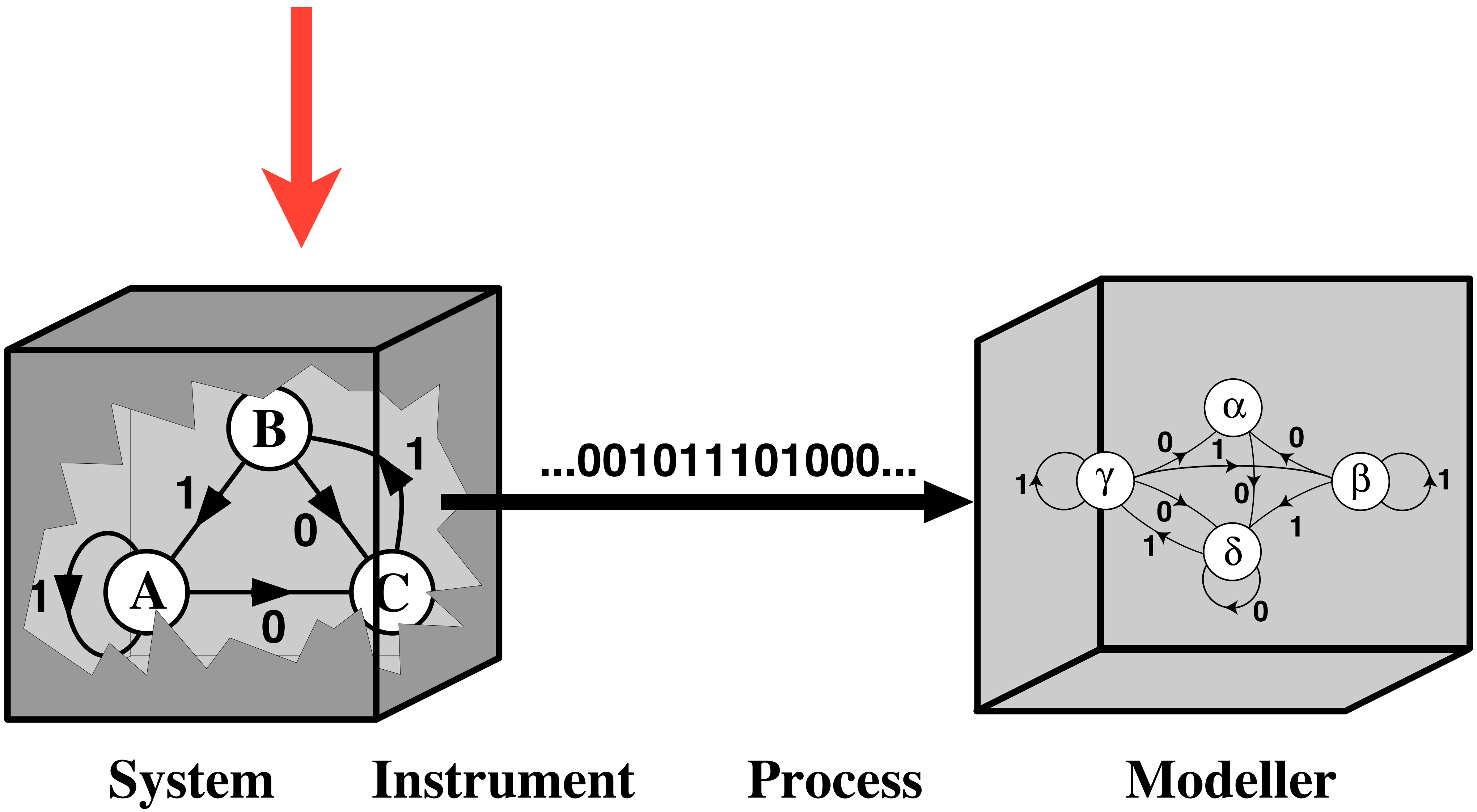
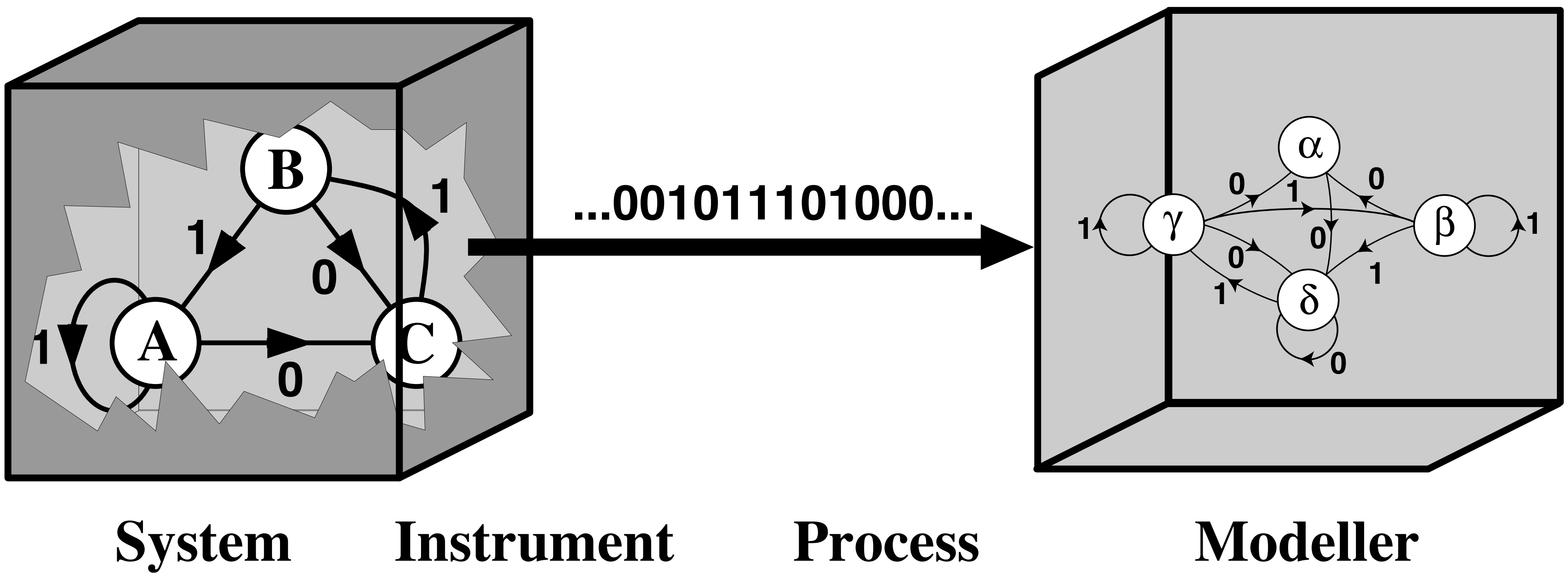
You Are Here

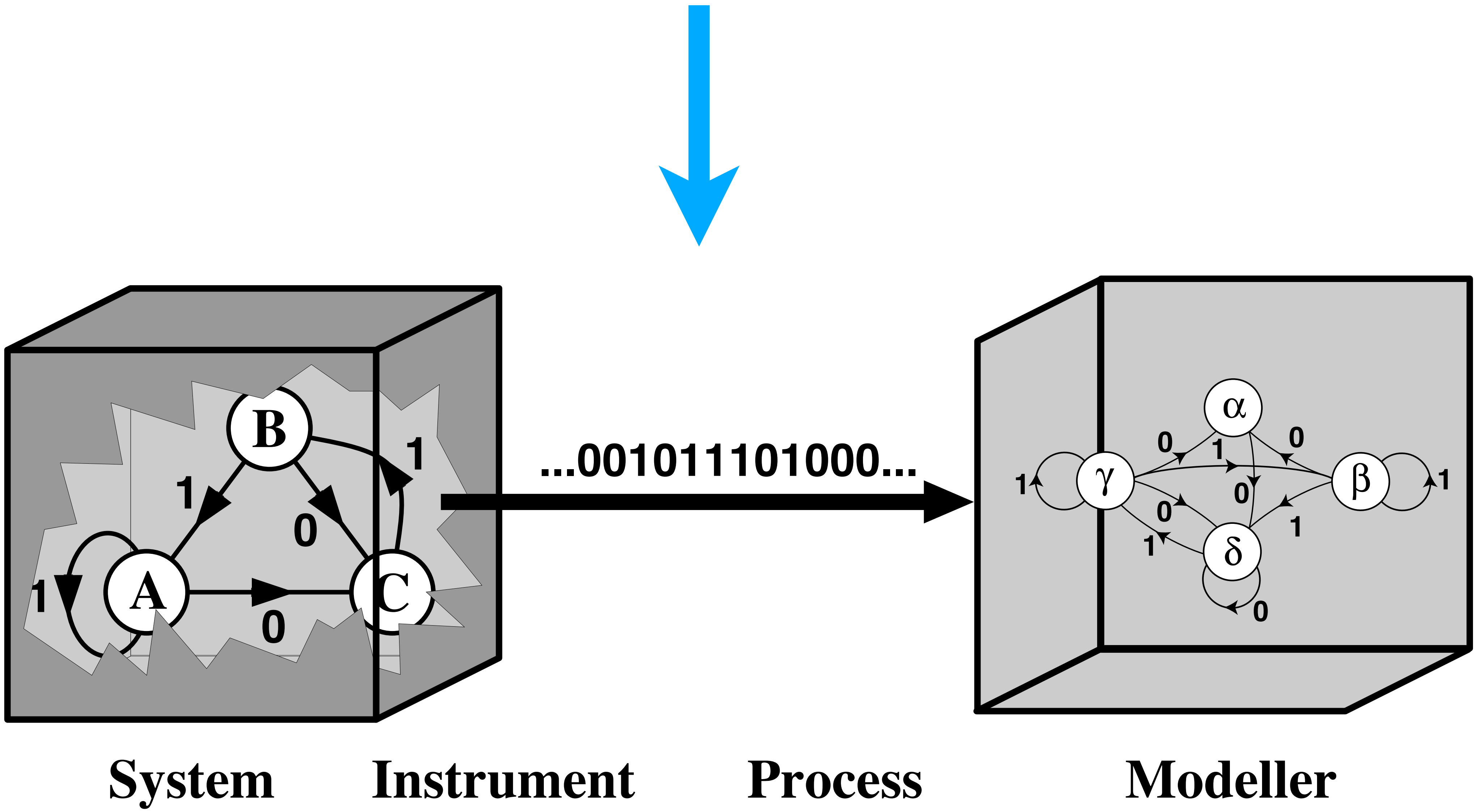


The Learning Channel

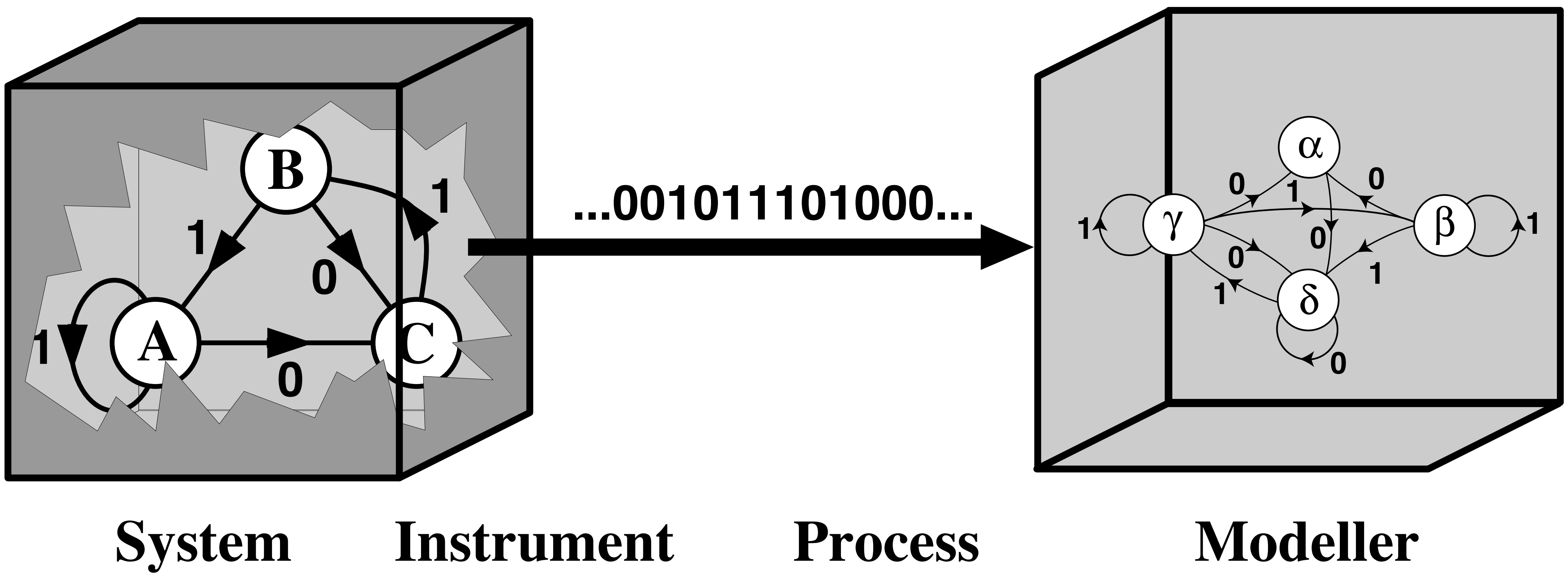


The Learning Channel

You Will Soon Be Here

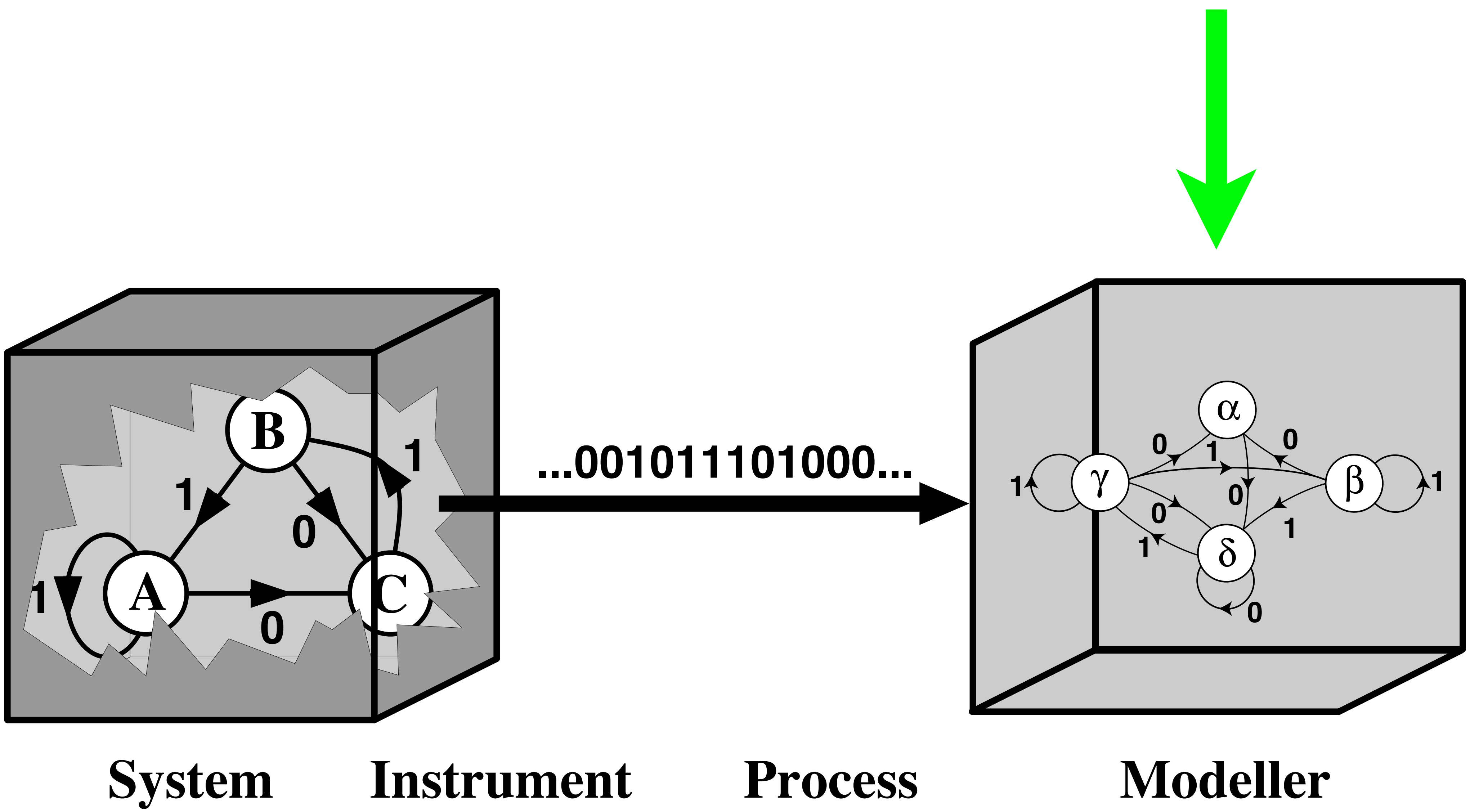


The Learning Channel



The Learning Channel

In Two Weeks



The Learning Channel

Complexity

Outline:

Lecture 1 (Thursday):
- Processes
- Information Theory: Basic and Processes

Lecture 2 (Friday):
- Structure
- Measures of Complexity
- Applications

Complexity

References? Many, for example:

Stanislaw Lem, *Chance and Order*, New Yorker **59** (1984) 88-98.

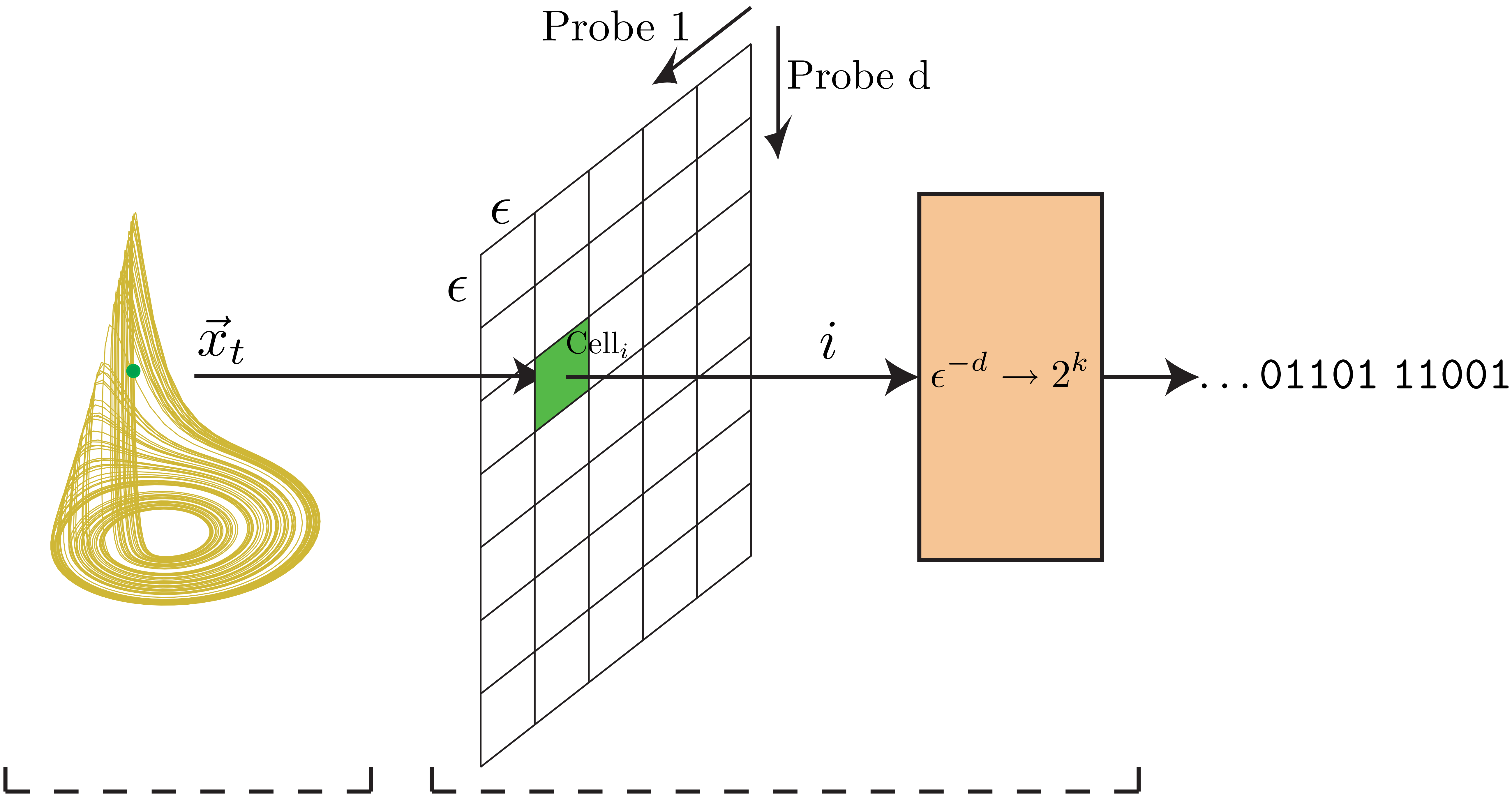
T. Cover and J. Thomas, *Elements of Information Theory*,
Wiley, Second Edition (2006) Chapters 1 - 7.

M. Li and P.M.B. Vitanyi, *An Introduction to Kolmogorov Complexity and its Applications*,
Springer, New York (1993).

J. P. Crutchfield and D. P. Feldman,
"Regularities Unseen, Randomness Observed: Levels of Entropy Convergence",CHAOS
13:1 (2003) 25-54.

J. P. Crutchfield, C. J. Ellison, and J. R. Mahoney,
"Time's Barbed Arrow: Irreversibility, Crypticity, and Stored Information",
Physical Review Letters **103**:9 (2009) 094101.

Processes and Their Models ...



System of Interest

Instrument

Measurement Channel

Complexity:
Processes and their Models

Previous Dynamics Lectures:

When measurements are faithful
study infinite discrete sequences
to learn about continuous-state dynamical system.

Now:

1. Processes
2. Information
3. Complexity

Processes and Their Models ...

Measurement Theory ...

Main questions now:

- How do we characterize the resulting process?
  - Measure degrees of unpredictability & randomness.
  - What correlational structure is there?
- How do we build a model from the process itself?
- How much can we reconstruct about the
  hidden internal dynamics?

Processes and Their Models ...

Stochastic Processes:

Chain of random variables:

$$\overleftrightarrow{S} \equiv \ldots S_{-2} S_{-1} S_0 S_1 S_2 \ldots$$

Random variable: S_t

Alphabet: $\mathcal{A}$

Realization:

$$\cdots s_{-2} s_{-1} s_0 s_1 s_2 \cdots \; ; \; s_t \in \mathcal{A}$$

Stochastic Processes:

Chain of random variables: $\overleftrightarrow{S} = \overleftarrow{S}_t \overrightarrow{S}_t$

Past: $\overleftarrow{S}_t = \ldots S_{t-3} S_{t-2} S_{t-1}$

Future: $\overrightarrow{S}_t = S_t S_{t+1} S_{t+2} \ldots$

L-Block: $S_t^L \equiv S_t S_{t+1} \ldots S_{t+L-1}$

Word: $s_t^L \equiv s_t s_{t+1} \ldots s_{t+L-1} \in \mathcal{A}^L$

Processes and Their Models ...

Stochastic Processes ...

Process:

$$\Pr(\overleftrightarrow{S}) = \Pr(\ldots S_{-2} S_{-1} S_0 S_1 S_2 \ldots)$$

Sequence (or word) distributions:

$$\{\Pr(S_t^L) = \Pr(S_t S_{t+1} \ldots S_{t+L-1}) : S_t \in \mathcal{A}\}$$

Process:

$$\{\Pr(S_t^L) : \forall t, L\}$$

Consistency conditions:

$$\Pr(S_t^{L-1}) = \sum_{S_{t+L-1}} \Pr(S_t^L) \qquad \Pr(S_{t+1}^{L-1}) = \sum_{S_t} \Pr(S_t^L)$$

Types of Stochastic Process:

Stationary process:

$$\Pr(S_t S_{t+1} \ldots S_{t+L-1}) = \Pr(S_0 S_1 \ldots S_{L-1})$$

Assume stationarity, unless otherwise noted.

Notation: Drop time indices.

Types of Stochastic Process ...

Uniform Process:

Equal-length sequences occur with same probability

$$U^L : \ \ \Pr(s^L) \ = \ 1/|\mathcal{A}|^L$$

Example: Fair coin

$$\mathcal{A} = \{H, T\}$$

$$\Pr(H) = \Pr(T) = 1/2$$

$$\Pr(s^L) = 2^{-L}$$

Types of Stochastic Process ...

Independent, Identically Distributed (IID) Process:

$$\Pr(\overleftrightarrow{S}) = \ldots \Pr(S_t)\Pr(S_{t+1})\Pr(S_{t+2})\ldots$$

$$\Pr(S_t) = \Pr(S_\tau), \ \forall \ t, \tau$$

Example: Biased coin

$$\Pr(H) = p$$

$$\Pr(T) = 1 - p = q$$

$$\Pr(s^L) = p^n q^{L-n}$$

Number of heads in sequence: n

Types of Stochastic Process ...

Markov Process:

$$\Pr(\overleftrightarrow{S}) = \ldots \Pr(S_{t+1}|S_t)\Pr(S_{t+2}|S_{t+1})\Pr(S_{t+3}|S_{t+2})\ldots$$

Example: No Consecutive 0s (Golden Mean Process)

$$\mathcal{A} = \{0, 1\}$$

$$\Pr(0|0) = 0$$

$$\Pr(1|0) = 1$$

$$\Pr(0|1) = 1/2$$

$$\Pr(1|1) = 1/2$$

Not Noisy Period-2 Process: GMP @ L = 4 has 0110.

Types of Stochastic Process ...

Hidden Markov Process:

Internal Order-R Markov Process: $\Pr(\overleftrightarrow{S})$

$$\Pr(S_t | \ldots S_{t-2} S_{t-1}) = \Pr(S_t | S_{t-R} \ldots S_{t-1})$$

$$s_t \in \mathcal{A}$$

Observed via a function of the internal sequences

$$\overleftrightarrow{Y} = f(\overleftrightarrow{S})$$

Measurement alphabet: $y_t \in \mathcal{B}$

Measurement random variables: $\overleftrightarrow{Y} = \ldots Y_{-2} Y_{-1} Y_0 Y_1 \ldots$

Observation process: $\Pr(\overleftrightarrow{Y} \mid \overleftrightarrow{S})$

Observed process: $\Pr(\overleftrightarrow{Y})$

Block Distribution: $\Pr(Y^L)$

Example: The Even Process

Internal Process: Golden Mean

$$s_t \in \{0, 1\}$$

Observation Process: $y_t \in \{a, b\}$

$$Y_t = f(S_{t-1} S_t)$$

$$y_t = \begin{cases} a, & s_{t-1} s_t = 11 \\ b, & s_{t-1} s_t = 01 \text{ or } 10 \end{cases}$$

$$\overleftrightarrow{s} = \mathtt{110111011110101111011} \ldots$$

$$\overleftrightarrow{y} = \mathtt{.\ abbaabbaaabbbbaaaabba} \ldots$$

Models of Stochastic Processes:

Markov chain model of a Markov process:

States: $v \in \mathcal{A} = \{1, \ldots, k\}$

$$\overleftrightarrow{V} = \ldots V_{-2} V_{-1} V_0 V_1 \ldots$$

Transition matrix: $T_{ij} = \Pr(v_{t+1} | v_t) \equiv p_{vv'}$

$$T = \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix}$$

Stochastic matrix: $\sum_{i=1}^{k} T_{ij} = 1$

Processes and Their Models ...

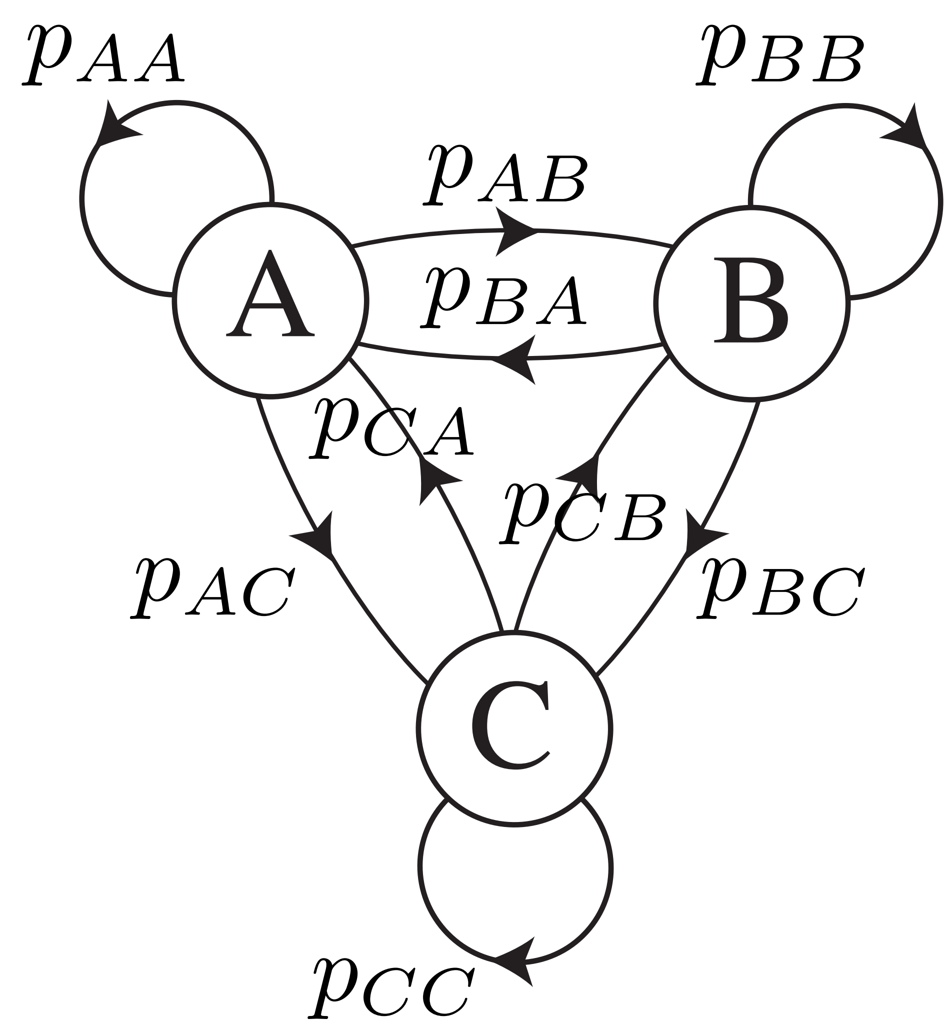
Models of Stochastic Processes ...

Markov chain ...

Example: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}$$

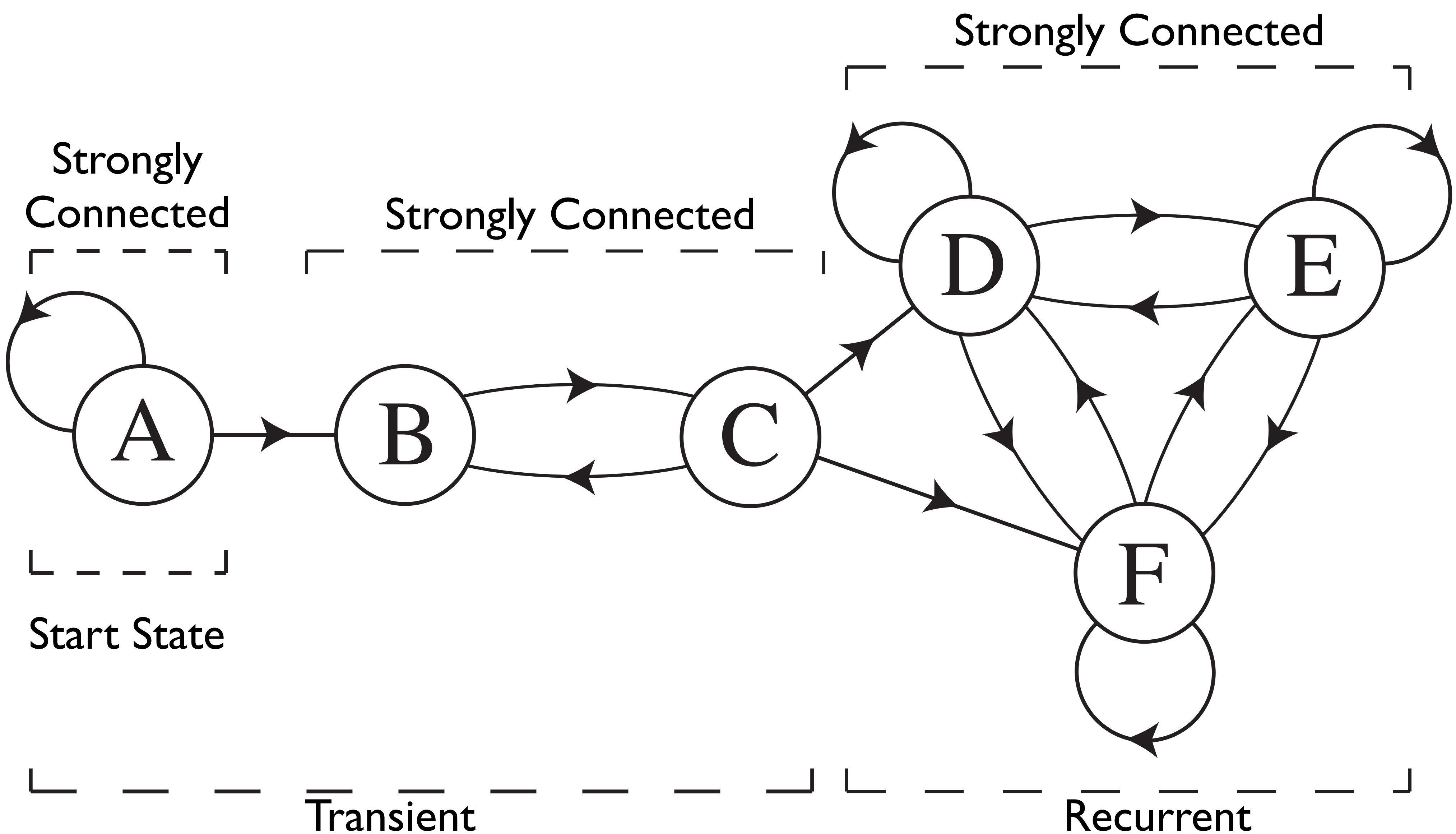
$$p_{AA} + p_{AB} + p_{AC} = 1$$
$$p_{BA} + p_{BB} + p_{BC} = 1$$
$$p_{CA} + p_{CB} + p_{CC} = 1$$



Processes and Their Models ...

Models of Stochastic Processes ...

Kinds of state:



Complexity Lecture 1: Processes and information (CSSS 2011) Jim Crutchfield

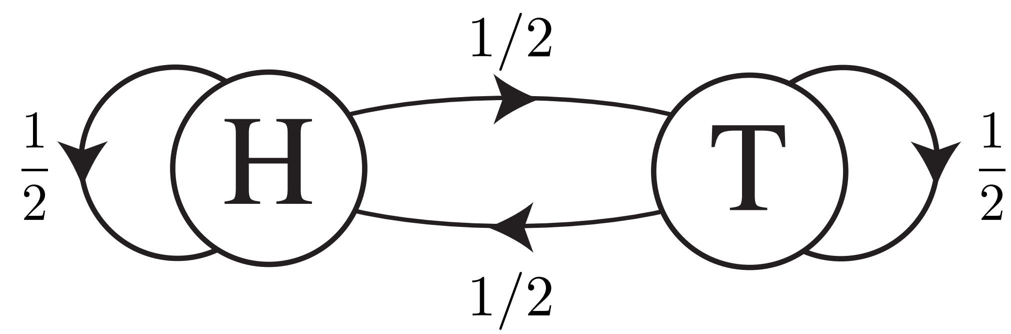
Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Fair Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



$$\pi = (1/2, 1/2)$$

$$\Pr(H) = \Pr(T) = 1/2$$

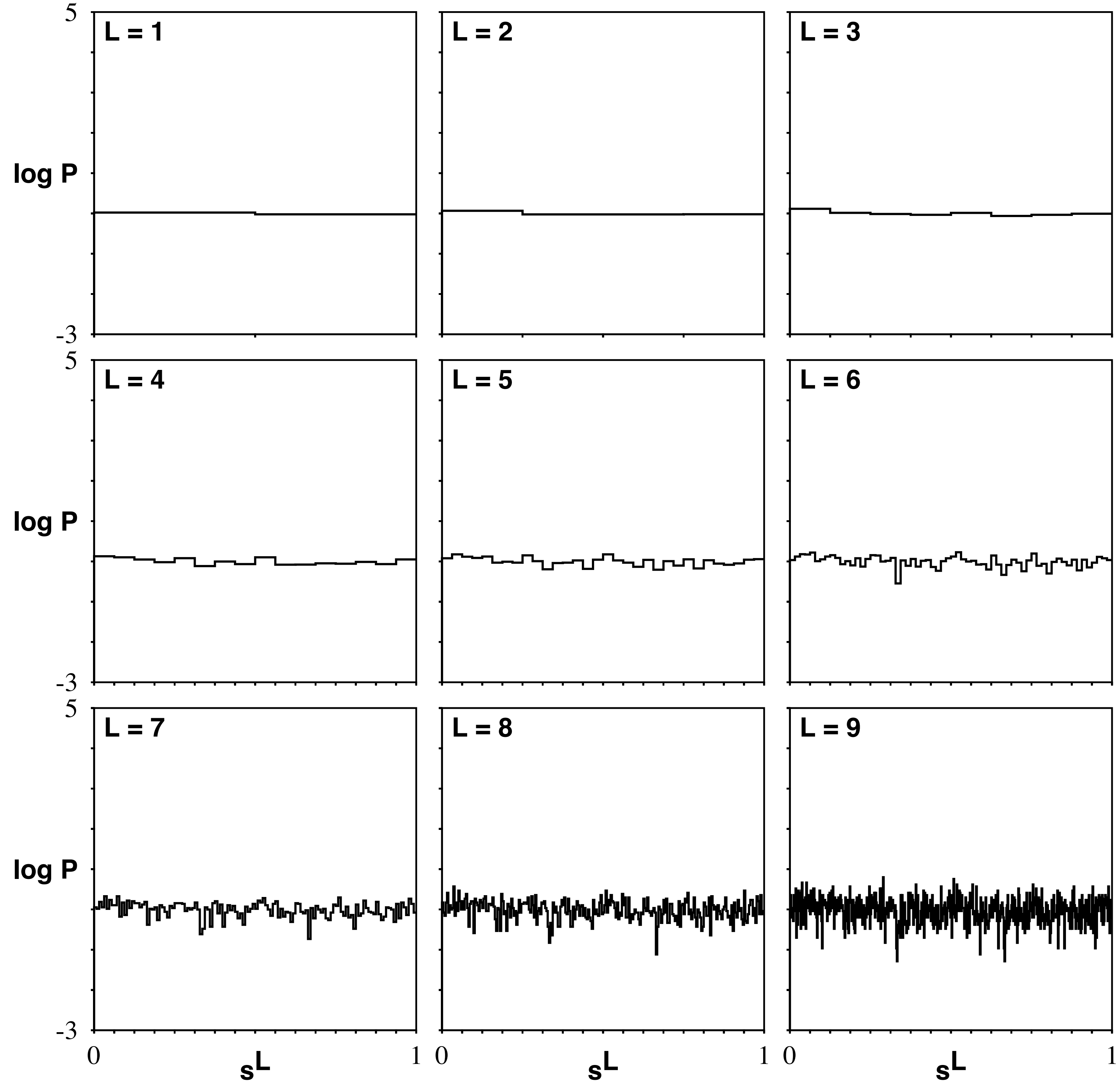
Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Fair Coin ...

Sequence Distribution: $\Pr(v^L) = 2^{-L}$



Word as binary fraction:

$$s^L = s_1 s_2 \ldots s_L$$

$$\text{“}s^L\text{”} = \sum_{i=1}^{L} \frac{s_i}{2^i}$$

$$s^L \in [0, 1]$$

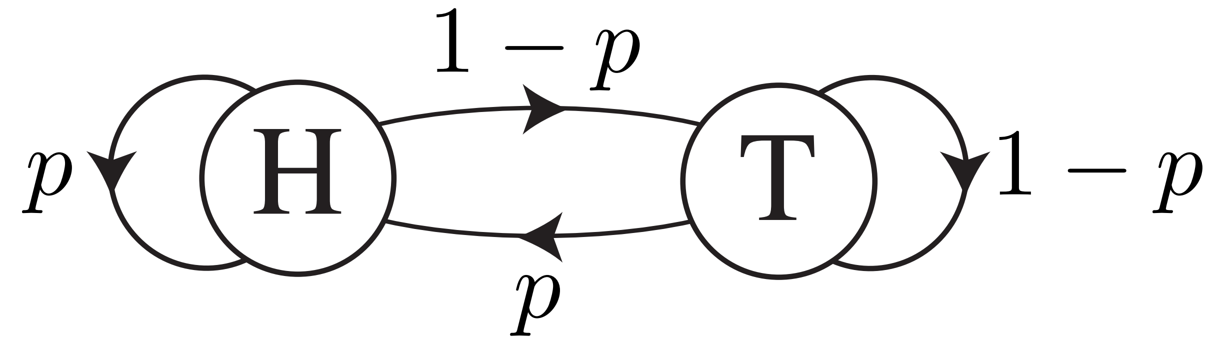
Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Biased Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$



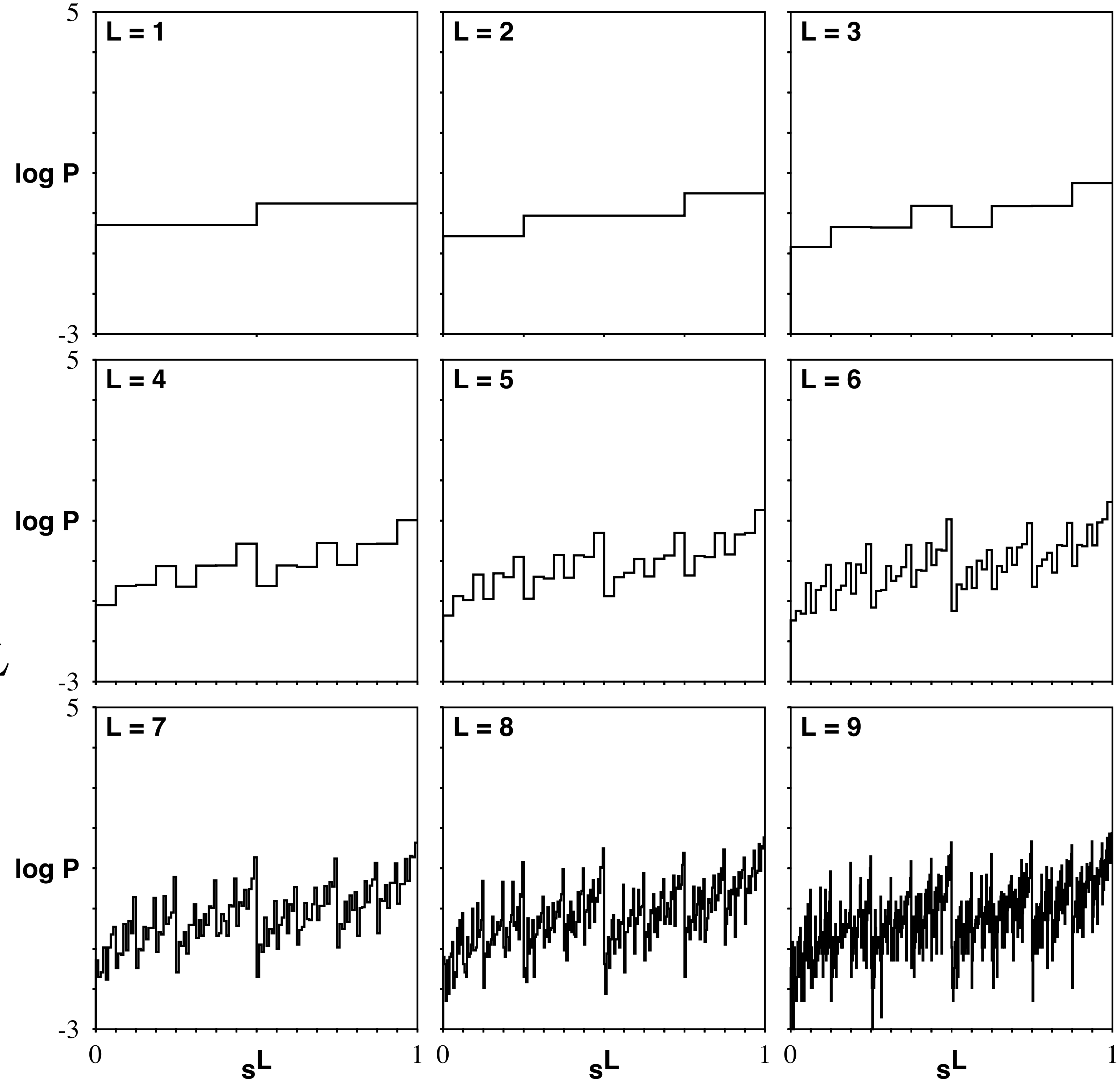
$$\pi = (p, 1-p)$$

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Biased Coin ...



Sequence Distribution:

$$\Pr(s^L) = p^n (1-p)^{L-n},$$

$$n = \text{Number } H\text{s in } s^L$$

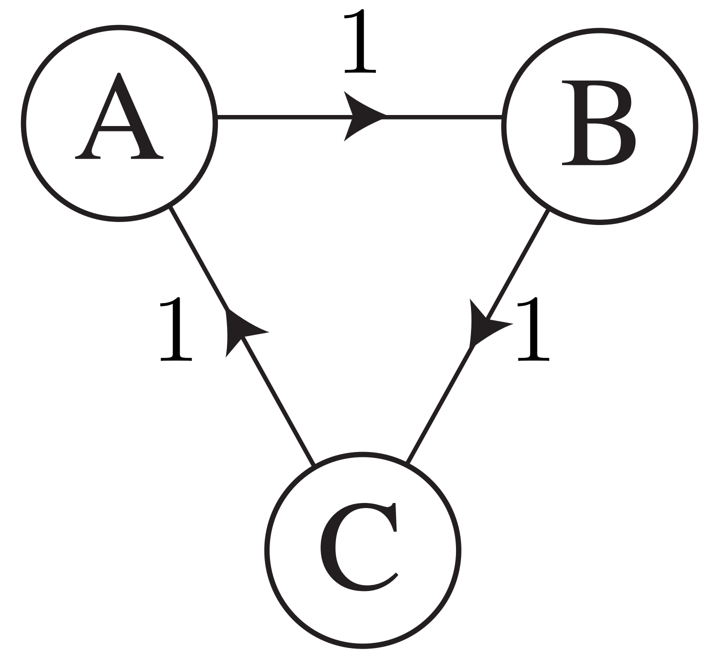
Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Periodic: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



$$\pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$ Careful!

Sequence distribution:

$$\Pr(A) = \Pr(B) = \Pr(C) = \tfrac{1}{3}$$

$$\Pr(AB) = \Pr(BC) = \Pr(CA) = \tfrac{1}{3} \qquad \Pr(s^2) = 0$$ otherwise

$$\Pr(ABC) = \Pr(BCA) = \Pr(CAB) = \tfrac{1}{3} \qquad \Pr(s^3) = 0$$ otherwise

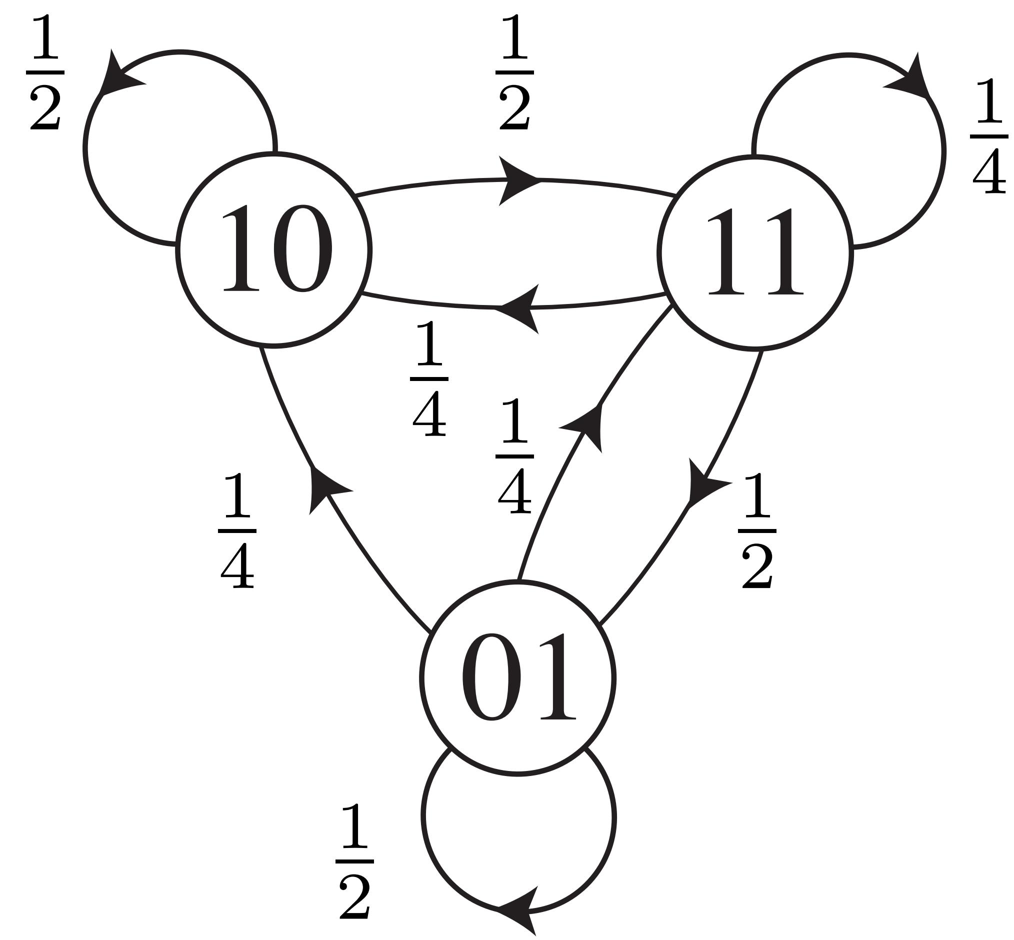
Models of Stochastic Processes ...

Example:

Golden Mean over 2-Blocks: $\mathcal{A} = \{10, 01, 11\}$

$$
\begin{array}{cc}
 & \begin{array}{ccc} 10 & 01 & 11 \end{array} \\
T = & \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}
\end{array}
$$

$$\pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$



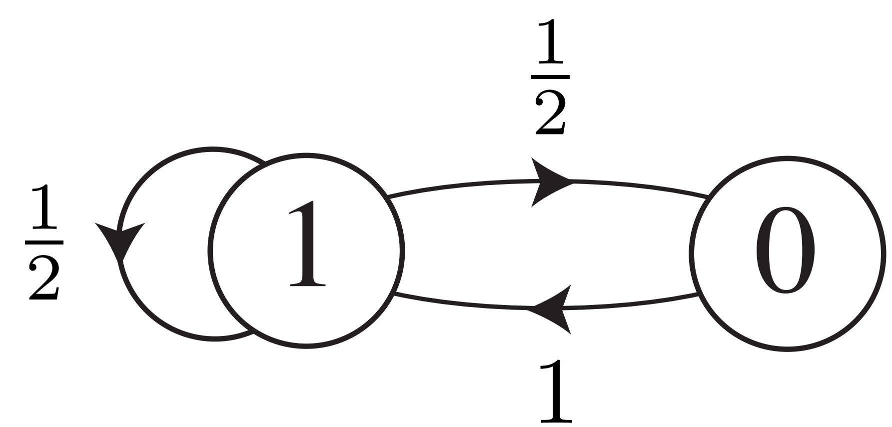
Models of Stochastic Processes ...

Example ...

Golden Mean over 1-Blocks: $\mathcal{A} = \{0, 1\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\pi = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$



Also an order-1 Markov chain. Minimal order.

Previous model and this:

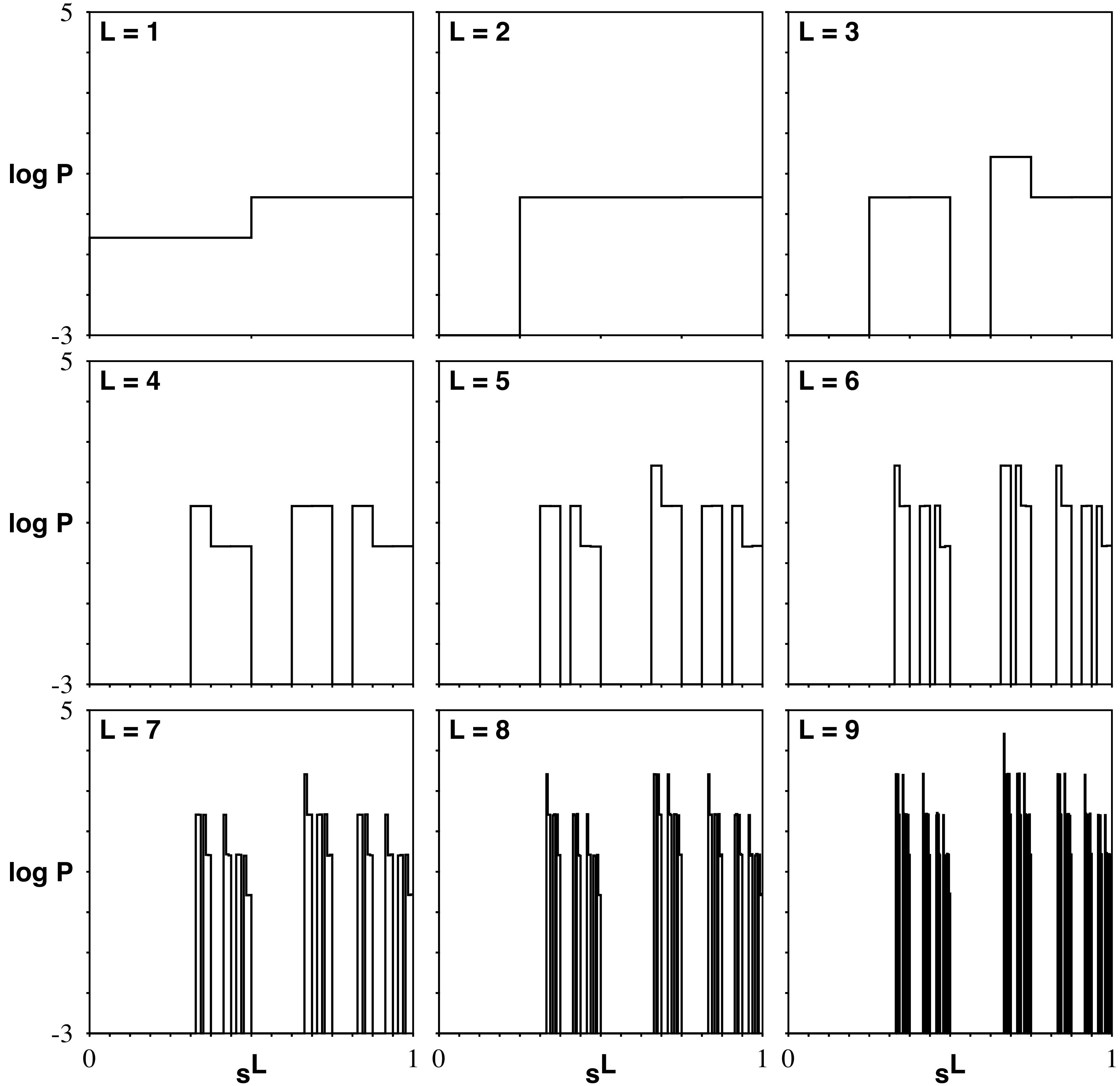
Different presentations of the Golden Mean Process

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Golden mean:



Models of Stochastic Processes ...

Two Lessons:

Structure in the behavior: $\mathrm{supp}\ \Pr(s^L)$

Structure in the distribution of behaviors: $\Pr(s^L)$

Models of Stochastic Processes ...

Hidden Markov Models of Processes:

Internal states: $v \in \mathcal{A}$

Transition matrix: $T = \Pr(v'|v), \; v, v' \in \mathcal{A}$

Observation: Symbol-labeled transition matrices

$$T^{(s)} = \Pr(v', s|v), \; s \in \mathcal{B}$$

$$T = \sum_{s \in \mathcal{B}} T^{(s)}$$

Stochastic matrices:

$$\sum_j T_{ij} = \sum_j \sum_s T_{ij}^{(s)} = 1$$

Internal state distribution: $\vec{p}_V = (p_1, p_2, \ldots, p_k)$

Evolve internal distribution: $\vec{p}_n = \vec{p}_0 T^n$

State sequence distribution: $v^L = v_0 v_1 v_2 \ldots v_{L-1}$

$$\Pr(v^L) = \pi(v_0) p(v_1|v_0) p(v_2|v_1) \cdots p(v_{L-1}|v_{L-2})$$

Observed sequence distribution: $s^L = s_0 s_1 s_2 \ldots s_{L-1}$

$$\Pr(s^L) = \sum_{v^L \in \mathcal{A}^L} \pi(v_0) p(v_1, s_1|v_0) p(v_2, s_2|v_1) \cdots p(v_{L-1}, s_{L-1}|v_{L-2})$$

No longer 1-1 map between internal & observed sequences:
Multiple state sequences can produce *same* observed sequence.

Processes and Their Models ...

Models of Stochastic Processes ...

Hidden Markov Models ...

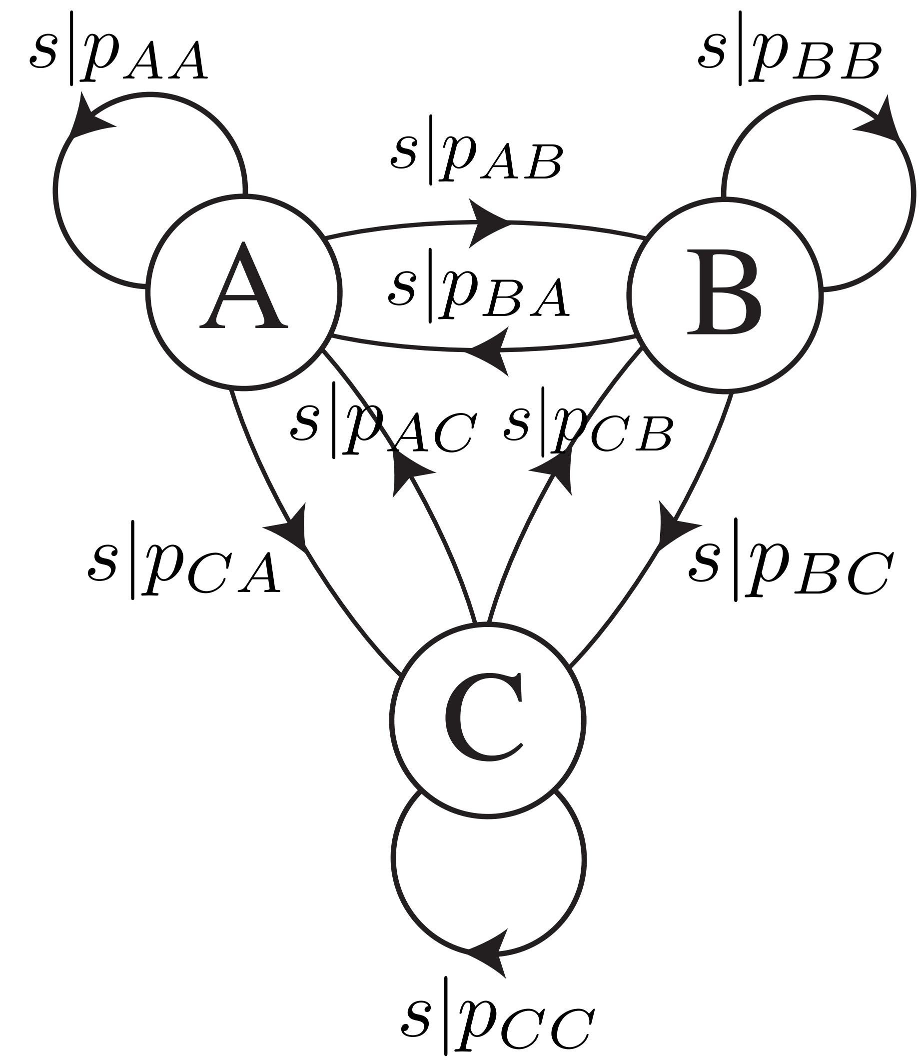
Internal: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(s)} = \begin{pmatrix} p_{AA;s} & p_{AB;s} & p_{AC;s} \\ p_{BA;s} & p_{BB;s} & p_{BC;s} \\ p_{CA;s} & p_{CB;s} & p_{CC;s} \end{pmatrix}$$

$$p_{AA} = \sum_{s \in \mathcal{B}} p_{AA;s}$$



symbol | transition probability

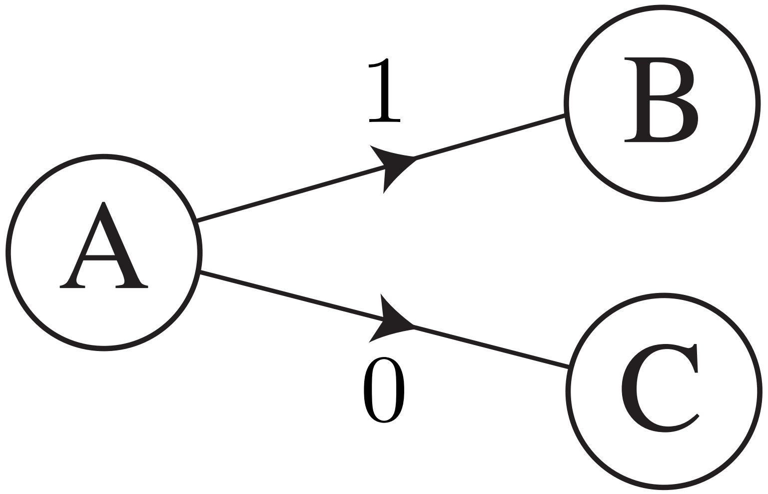
Processes and Their Models ...

Models of Stochastic Processes ...

Types of Hidden Markov Model:

"Unifilar": current state + symbol "determine" next state

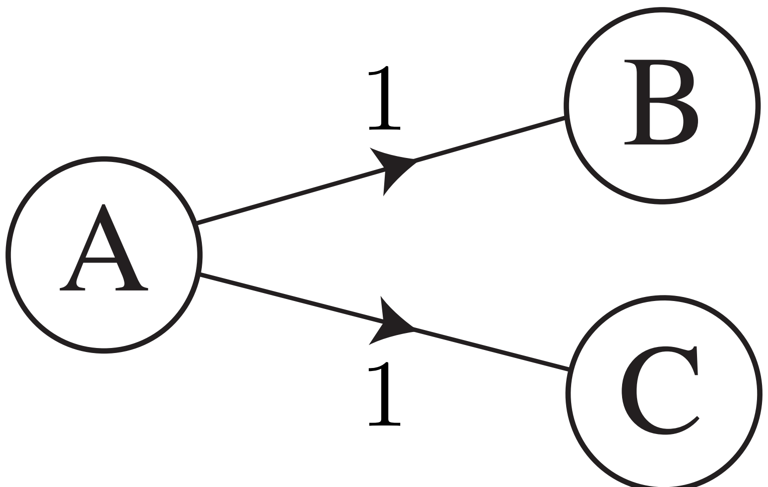
$$\Pr(v'|v,s) = \begin{cases} 1 \\ 0 \end{cases}$$



$$\Pr(v',s|v) = p(s|v)$$

$$\Pr(v'|v) = \sum_{s \in \mathcal{A}} p(s|v)$$

"Nonunifilar": no restriction



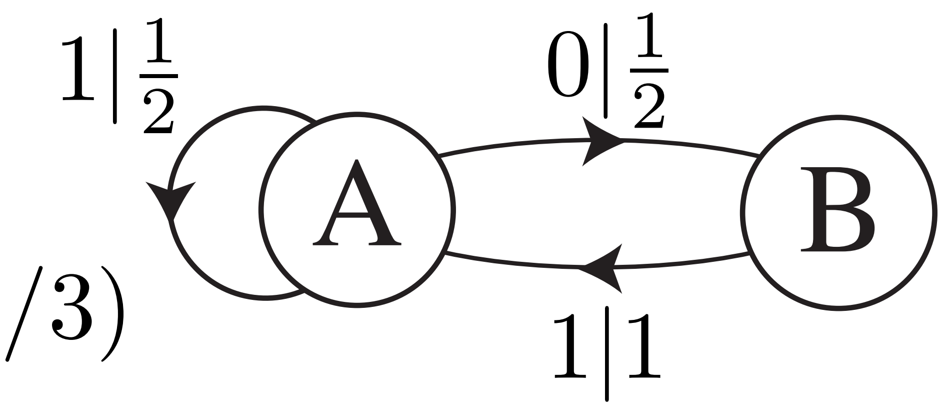
Multiple internal edge paths can generate same observed sequence.

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Golden Mean Process as a unifilar HMM:



Internal: $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

Initial ambiguity only: At most 2-to-1 mapping

$$BA^n = 1^n$$

$$AA^n = 1^n$$

Sync'd: $s = 0 \Rightarrow v = B$
$s = 1 \Rightarrow v = A$

Irreducible forbidden words: $\mathcal{F} = \{00\}$

Processes and Their Models ...

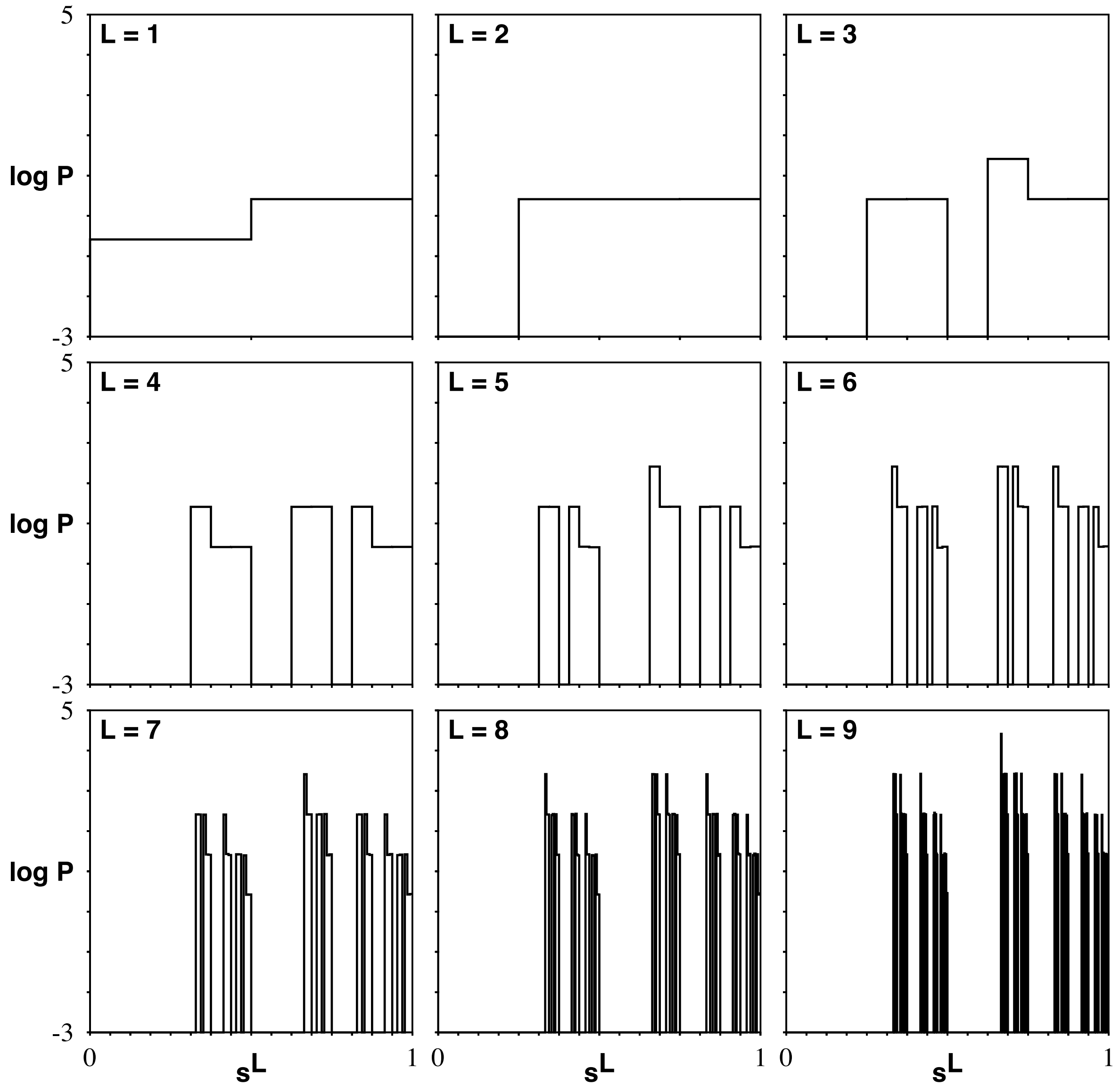
Models of Stochastic Processes ...

Example:

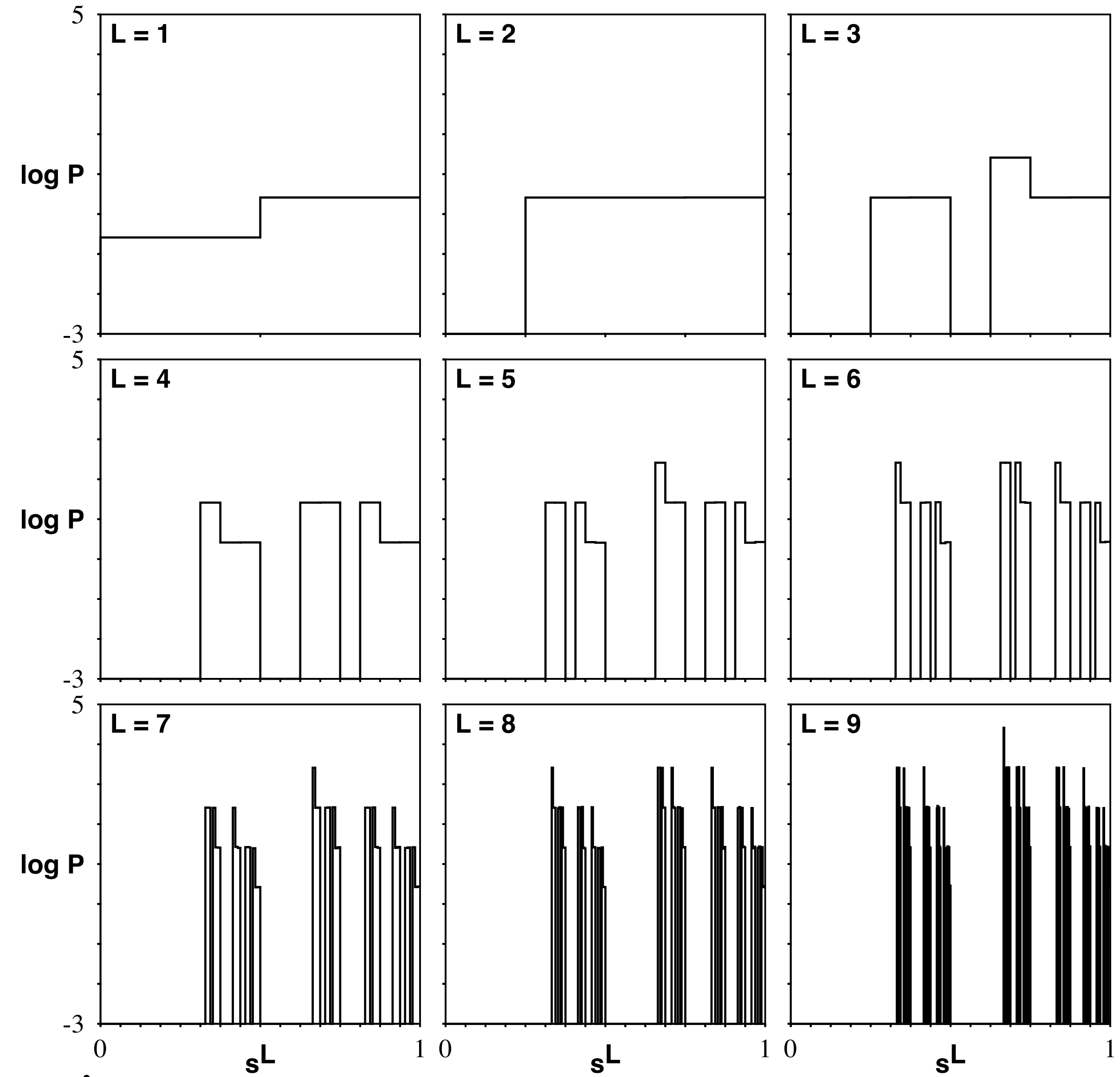
Golden Mean Process ... Sequence distributions:

Internal state sequences

(A = 1; B = 0)



Observed sequences



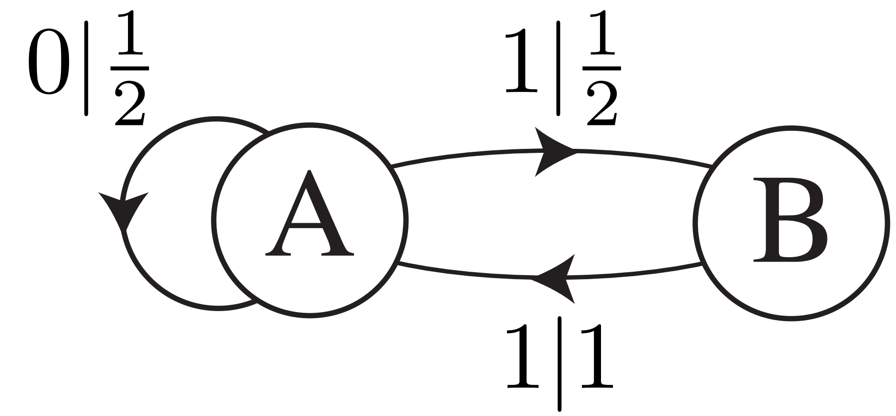
Same!

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Even Process as a unifilar HMM:



Internal (= GMP): $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$v^L = \ldots AABAABABAA \ldots$$

$$s^L = \ldots .\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0 \ldots \qquad s^L = \{\ldots 01^{2n}0 \ldots\}$$

Irreducible forbidden words: $\mathcal{F} = \{010, 01110, 0111110, \ldots\}$

No finite-order Markov process can model the Even process!

Lesson: Finite Markov Chains are a subset of HMMs.

Processes and Their Models ...

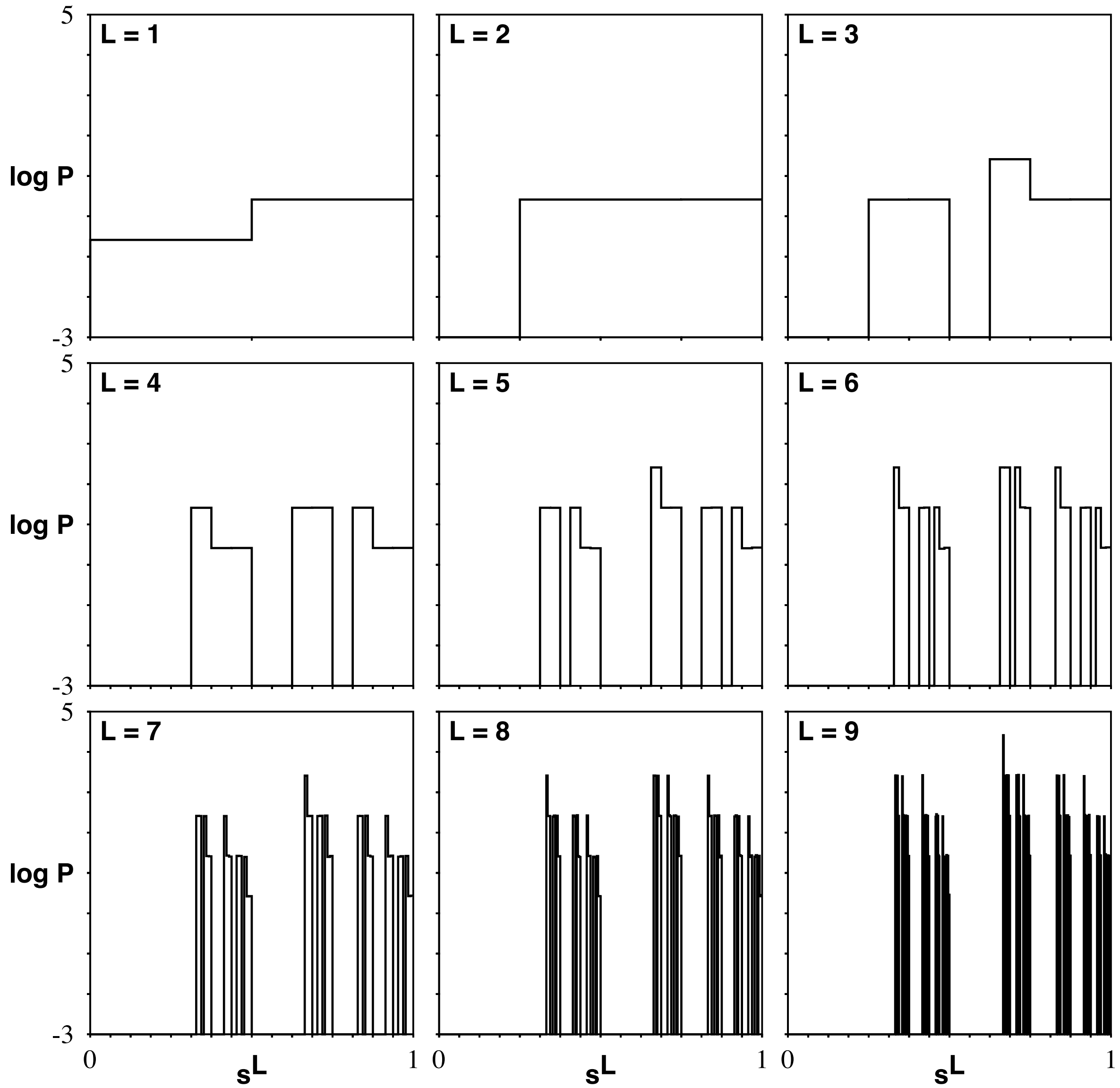
Models of Stochastic Processes ...

Example:

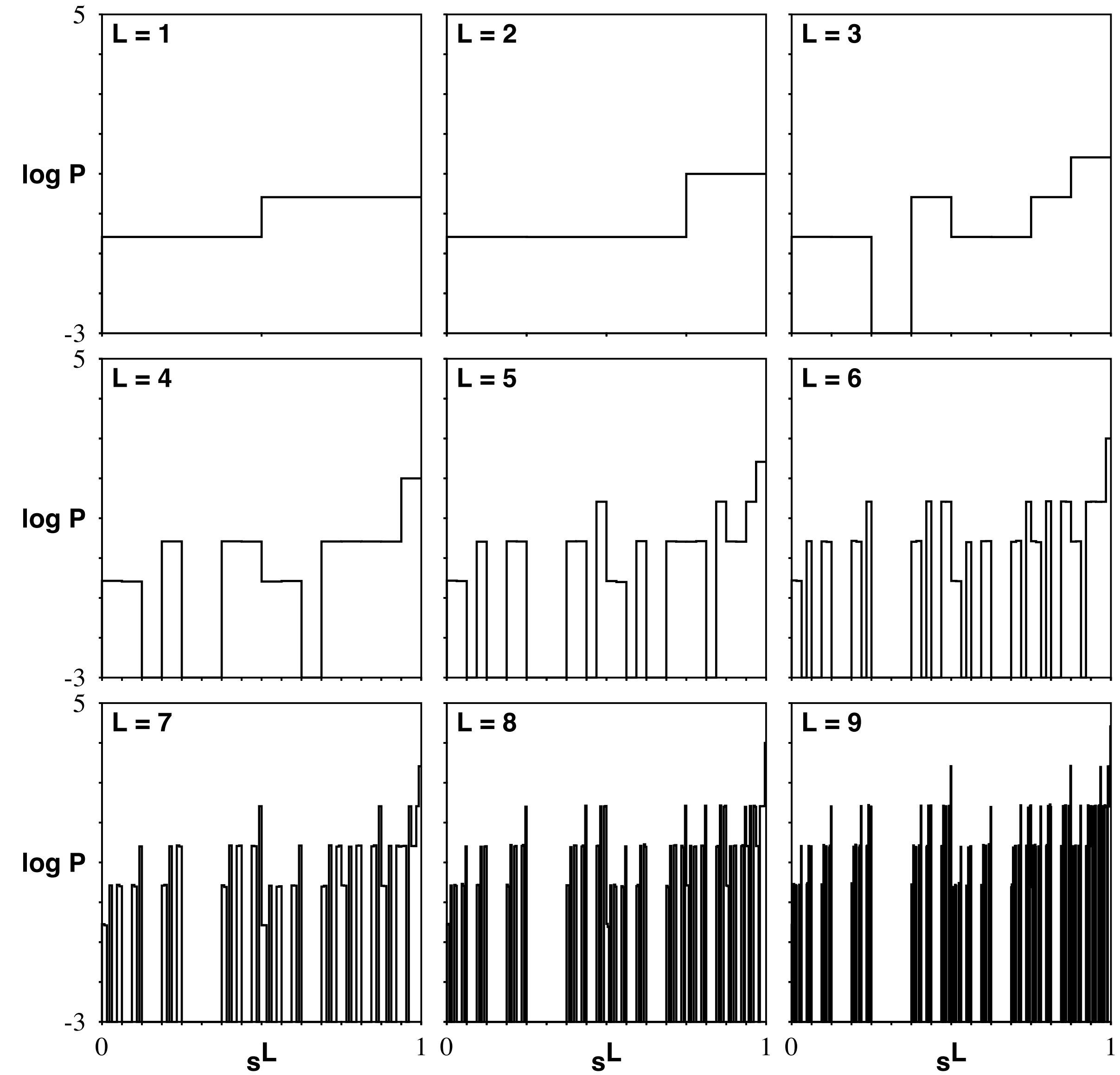
Even Process ... Sequence distributions:

Internal states (= GMP)

(A = 1; B = 0)



Observed sequences



Rather different!

Processes and Their Models ...

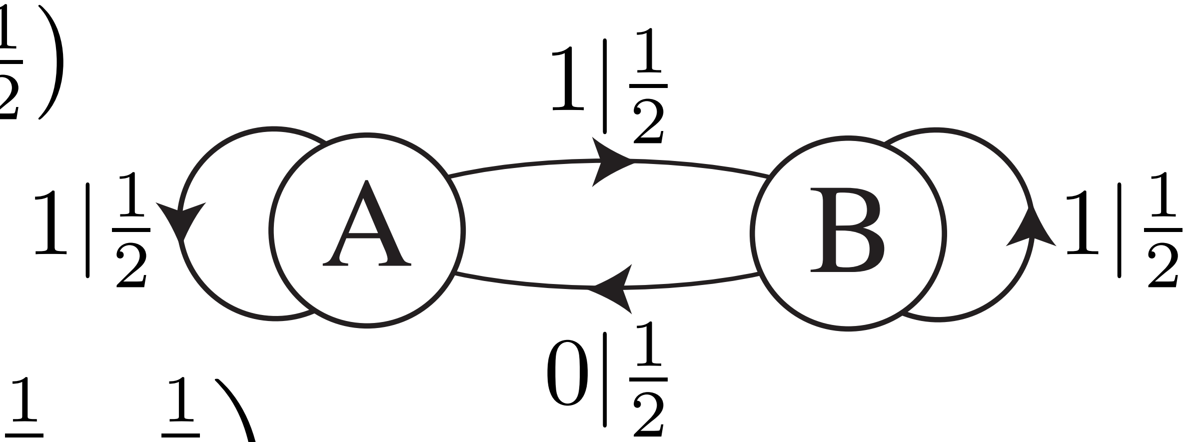
Models of Stochastic Processes ...

Example:

Simple Nonunifilar Source:

Internal (= Fair Coin): $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \pi_V = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$



Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Many to one: $1111111 \Leftarrow \begin{cases} AAAAAAAA \ldots \\ ABBBBBBB \ldots \\ AABBBBBB \ldots \\ AAABBBBB \ldots \\ \ldots \\ BBBBBBBB \ldots \end{cases}$

Is there a unifilar HMM presentation of the observed process?

Processes and Their Models ...

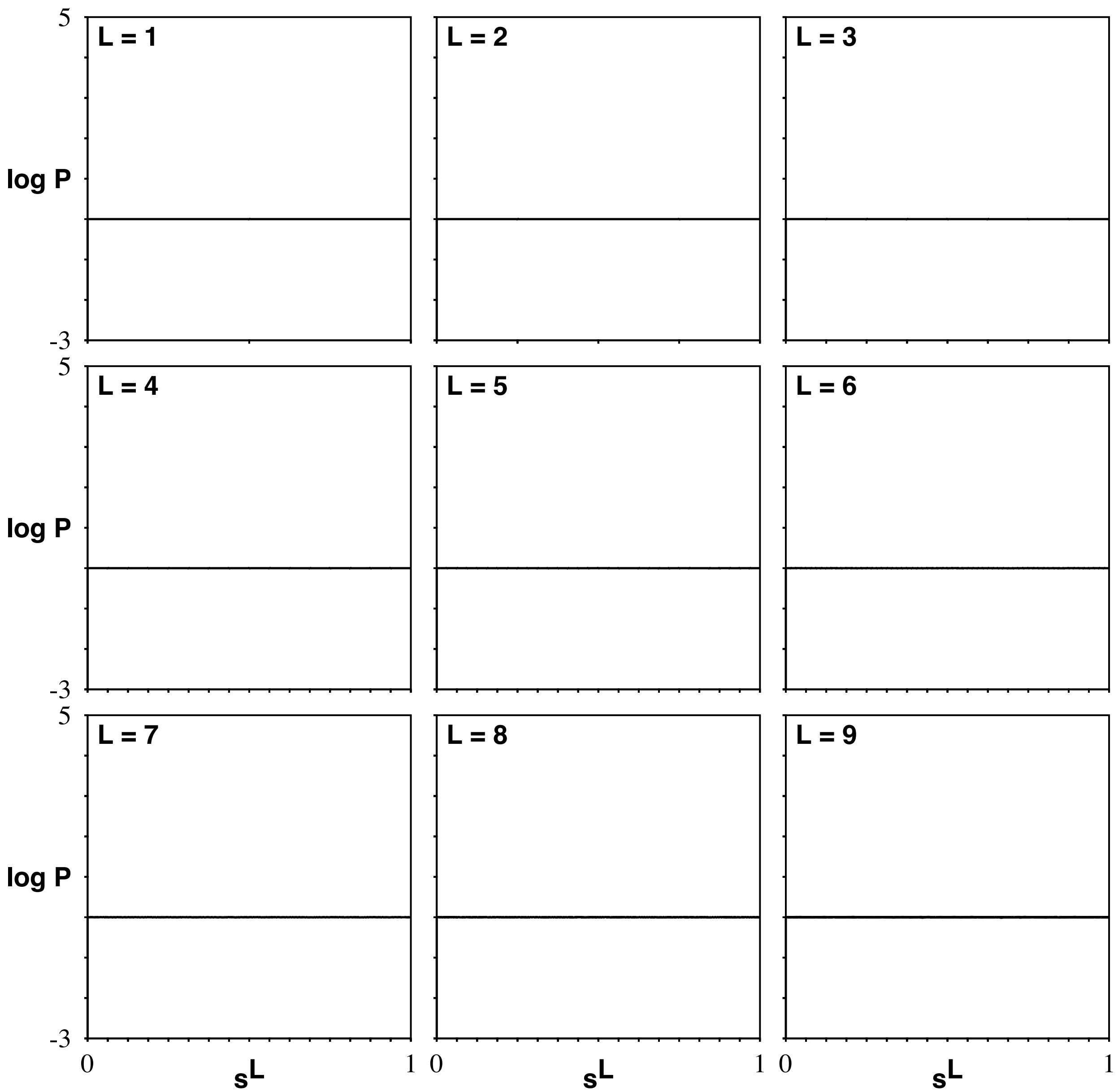
Models of Stochastic Processes ...

Example:

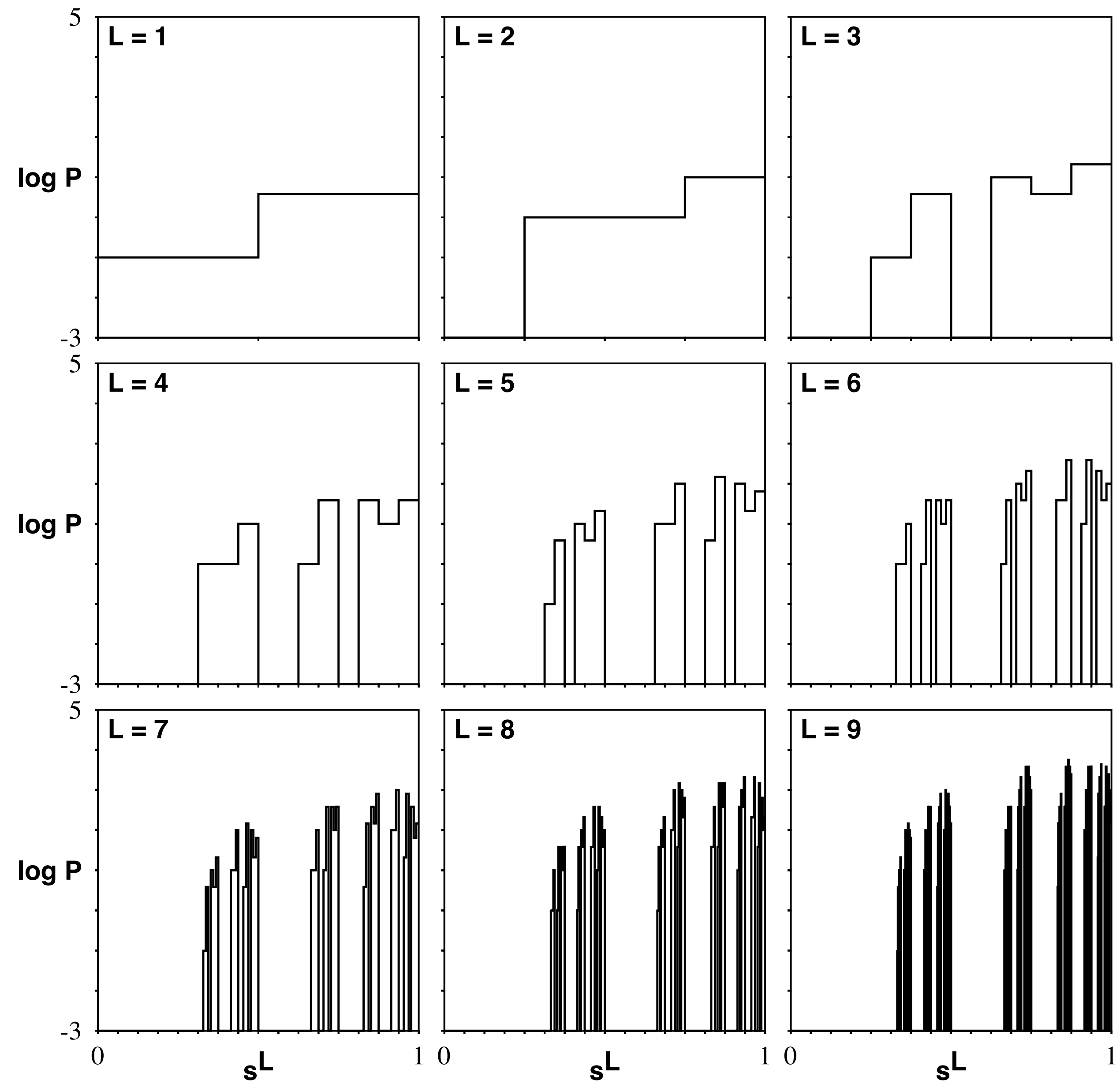
Simple Nonunifilar Process ...

Internal states (= Fair coin)

(A = 1; B = 0)



Observed sequences



Complexity Lecture 1: Processes and information (CSSS 2011) Jim Crutchfield

Processes and Their Models ...

Next lectures: "Complexity Module"

1. Information theory for general stochastic processes

2. Measures of complexity

3. Optimal models and how to build them

4. Applications

Labs:

Track these topics closely.

Ryan will describe.

Work through them on your own.