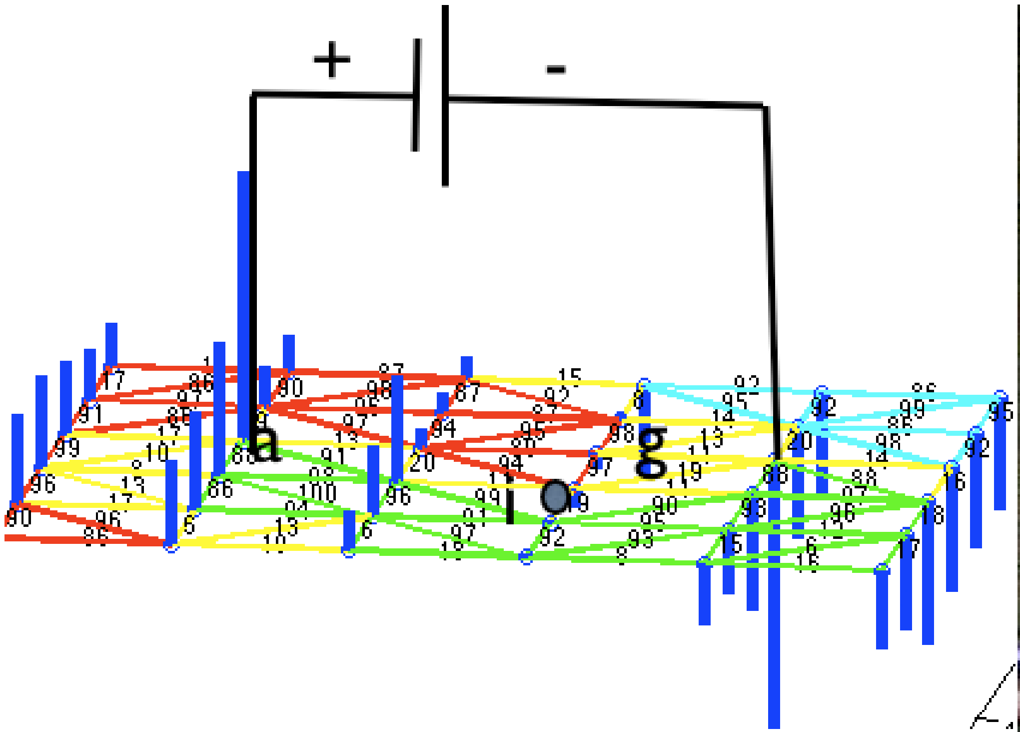
Statistical learning, complex systems, and pickles

Lecture 2, CSSS09

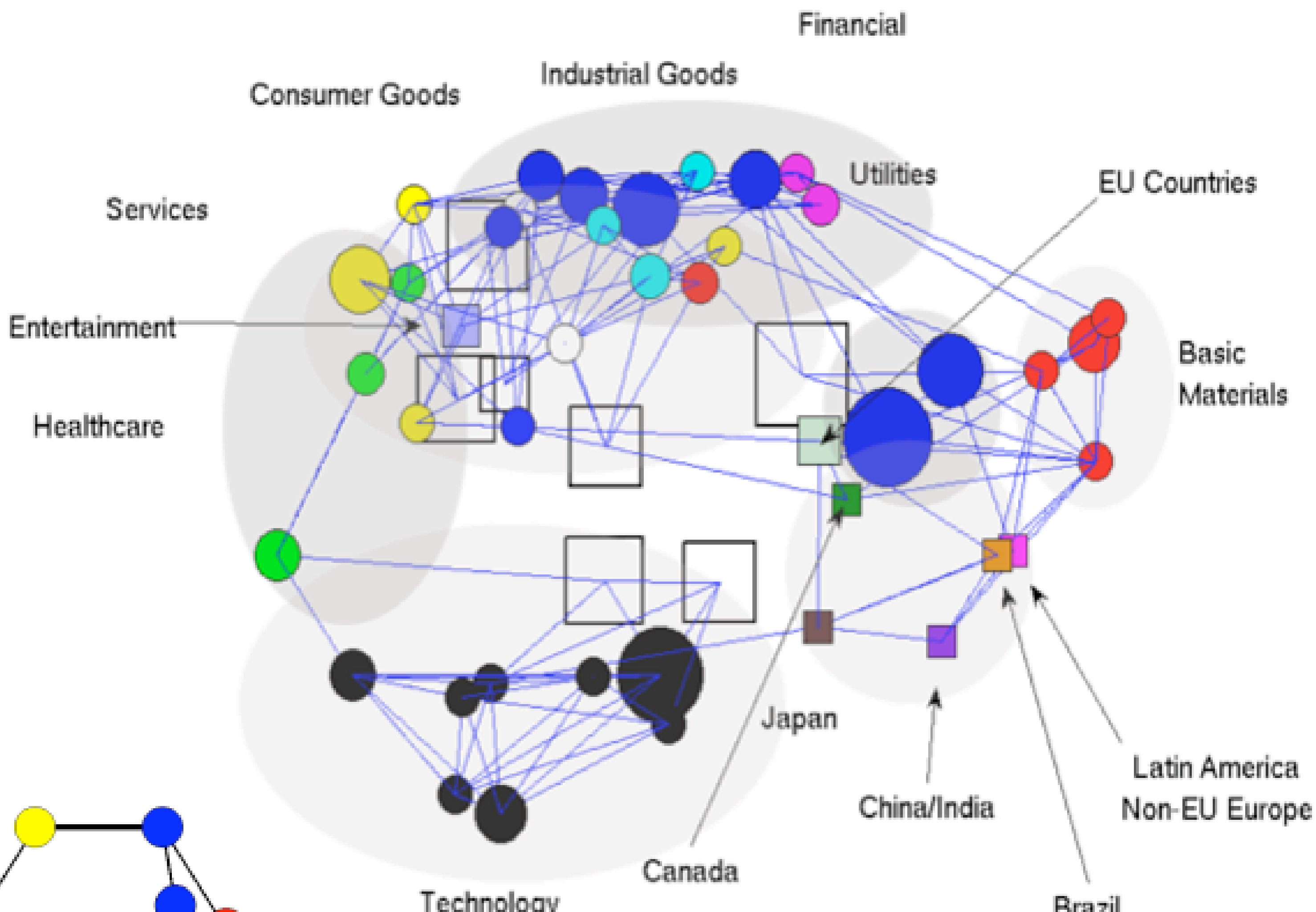


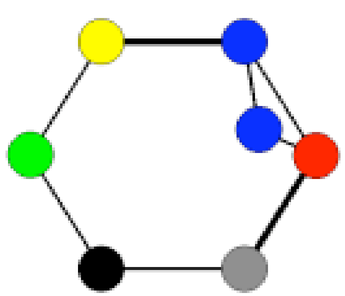
Greg Leibon
Memento, Inc
Dartmouth College

The Plan

- Part 1: *Spectral Clustering*
- Part 2: *Picklology*...and beyond!
- Appendix: Proof of the mighty *Commute Theorem*

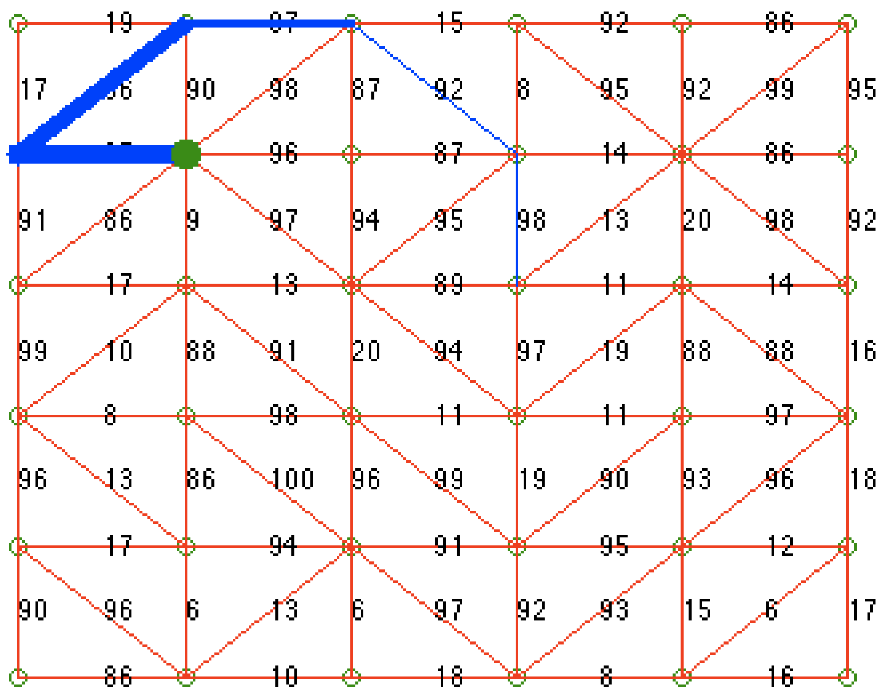
Part 1: Spectral Clustering



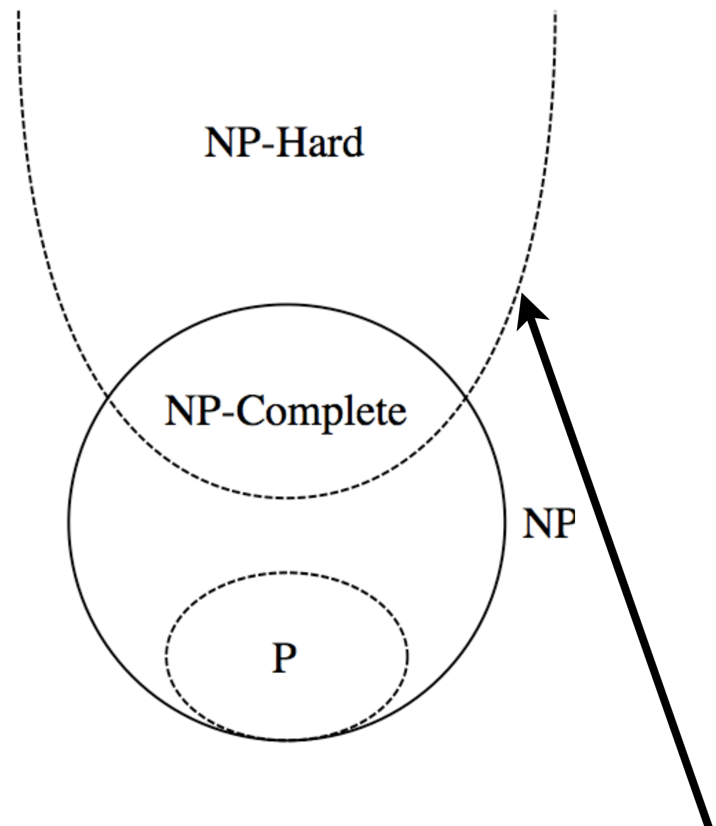


“Vagabond Clustering”: find a partition $\{A_k\}_{k=1}^{K}$ that minimizes

$$Vagabondliness = \sum P(X_{t+1} \in \overline{A}_k \mid X_t = A_k)$$

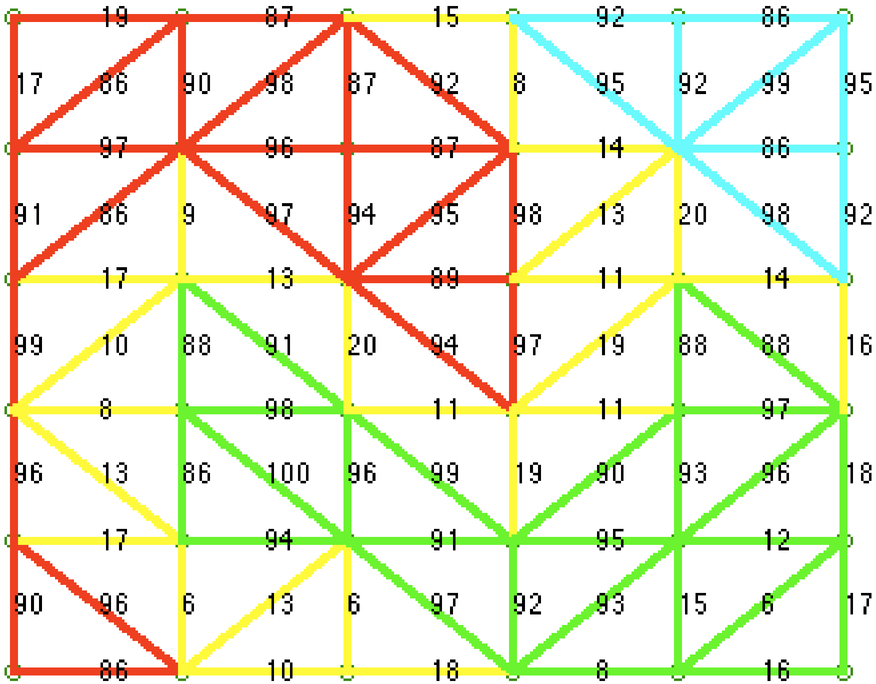


```
states=Snake(500,5,'No');
```

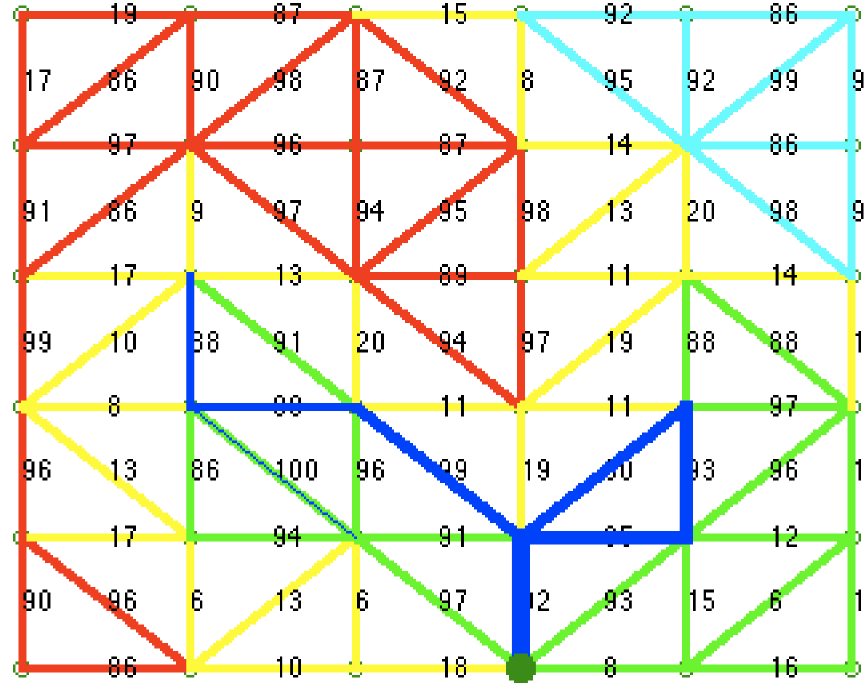


Finding this partition is usually called “NCut” and it is NP-hard.

Stoer, M. and Wagner, F. (1997). A simple min-cut algorithm. *J. ACM*, *44*(4), 585–591.



The Vagabond Partion



states=Snake(500,10,'Vg');

Since finding this partition is NP-hard, to find the solution we will need to "relax..."

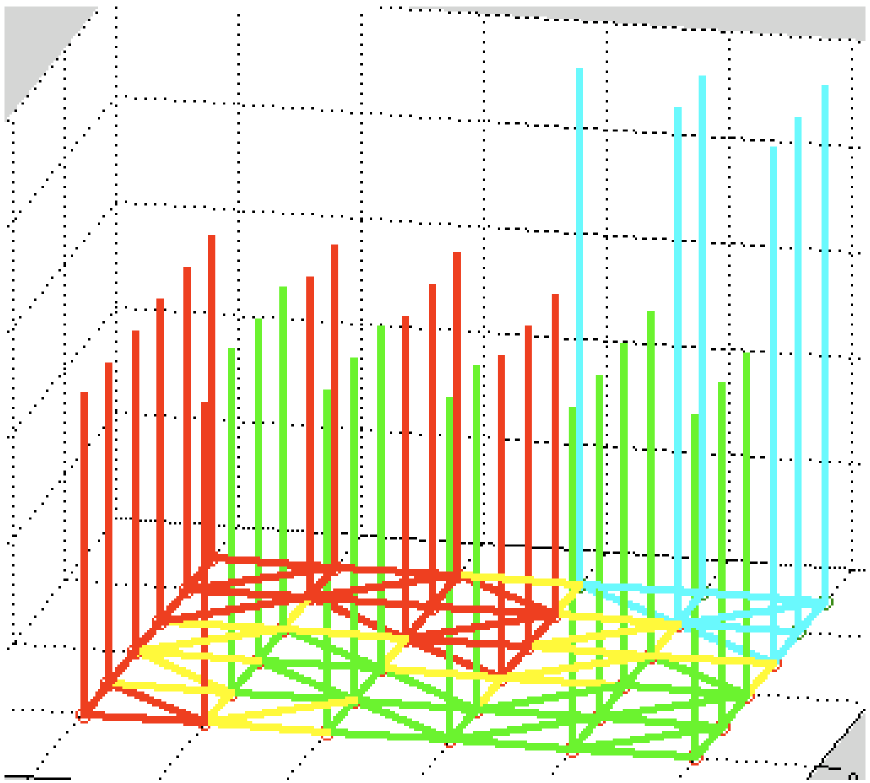
Key Theorem: $P(X_{t+1} \in \bar{A} \mid X_t \in A) = \langle \chi_A, \Delta \chi_A \rangle$

$$\chi_A(i) = \begin{cases} \frac{1}{\sqrt{w(A)}} & i \in A \\ 0 & i \notin A \end{cases}$$

for A and B disjoint

$$\langle \chi_A, \chi_B \rangle = 0$$

$$\langle \chi_A, \chi_A \rangle = 1$$



Key Theorem: $P(X_{t+1} \in \bar{A} \mid X_t \in A) = \langle \chi_A, \Delta \chi_A \rangle$

Approximate the Vagabond Clustering

$$\min_{partitions\{A_k\}} \sum_k \langle \chi_{A_k}, \Delta \chi_{A_k} \rangle$$

by "relaxing" and solving

$$\min_{\{v_i \mid \langle v_i, v_j \rangle = \delta_j^i\}} \sum_i \langle v_i, \Delta v_i \rangle$$

which from the *spectral theorem* is the first k Laplace eigen functions

Do not imagine that mathematics is hard and crabbed, and repulsive to common sense. It is merely the etherialisation of common sense."

~Lord Kelvin

Proof:
$$P(X_{t+1} \in \bar{A} \mid X_t \in A) = \frac{P(X_{t+1} \in \bar{A}, X_t \in A)}{P(X_t \in A)}$$

$$= \frac{\sum_{j \in \bar{A}, i \in A} P(X_{t+1} = j, X_t = i)}{P(X_t \in A)}$$

definition conditional probability

$$= \frac{\sum_{j \in \bar{A}, i \in A} P(X_{t+1} = j \mid X_t = i) P(X_t = i)}{P(X_t \in A)}$$

$$= \frac{\sum_{j \in \bar{A}, i \in A} P_i^j w^i}{\mu(A)}$$

THE Swanky formula Green-Kelvin Indtty

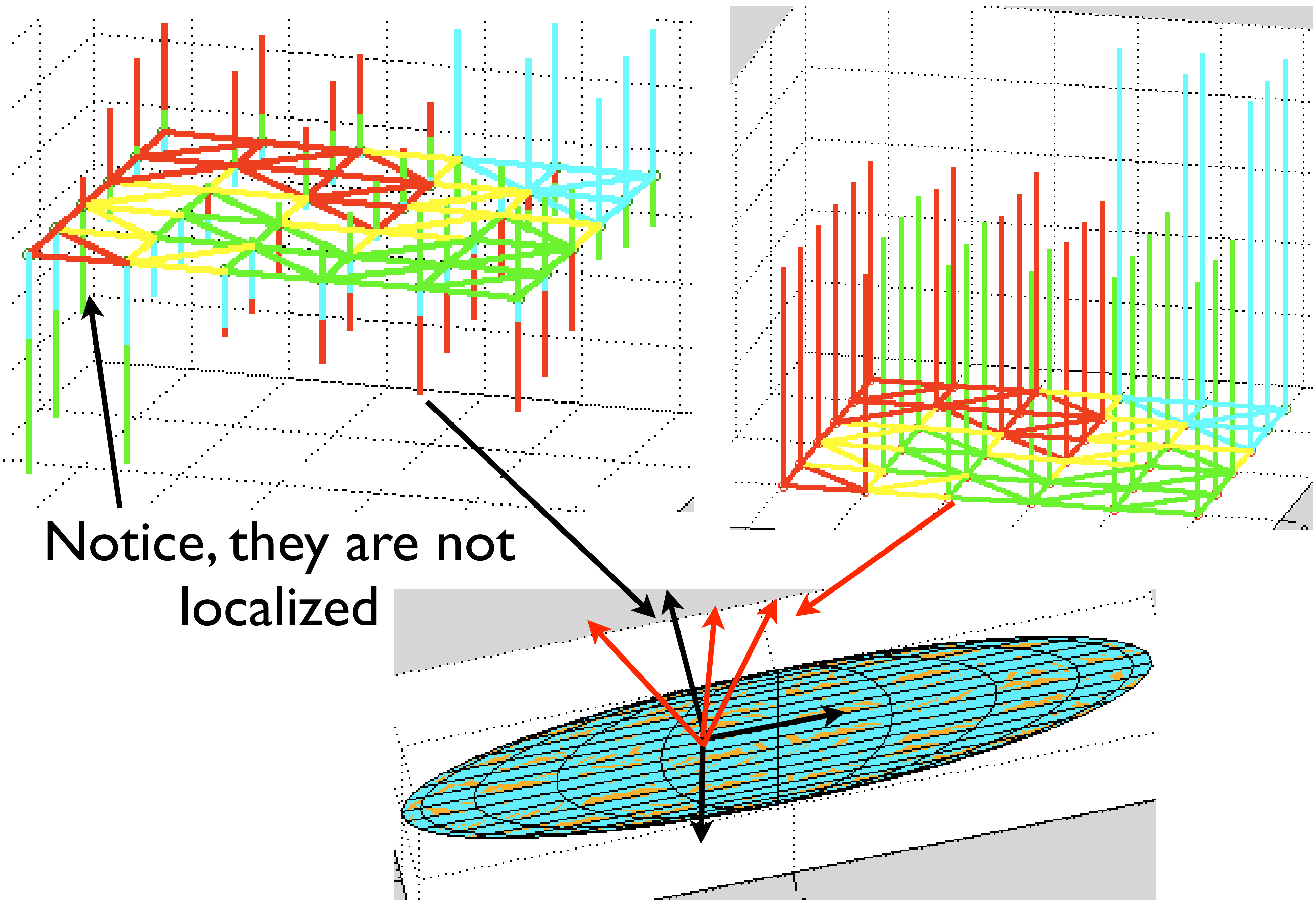
$$= \sum_{j \in \bar{A}, i \in A} P_i^j w^i \left(\chi_A(i) - \chi_A(j)\right)^2$$

$$= \frac{1}{2} \sum_{i,j} P_i^j w^i \left(\chi_A - \chi_A\right)^2 = \langle \chi_A, \Delta \chi_A \rangle$$

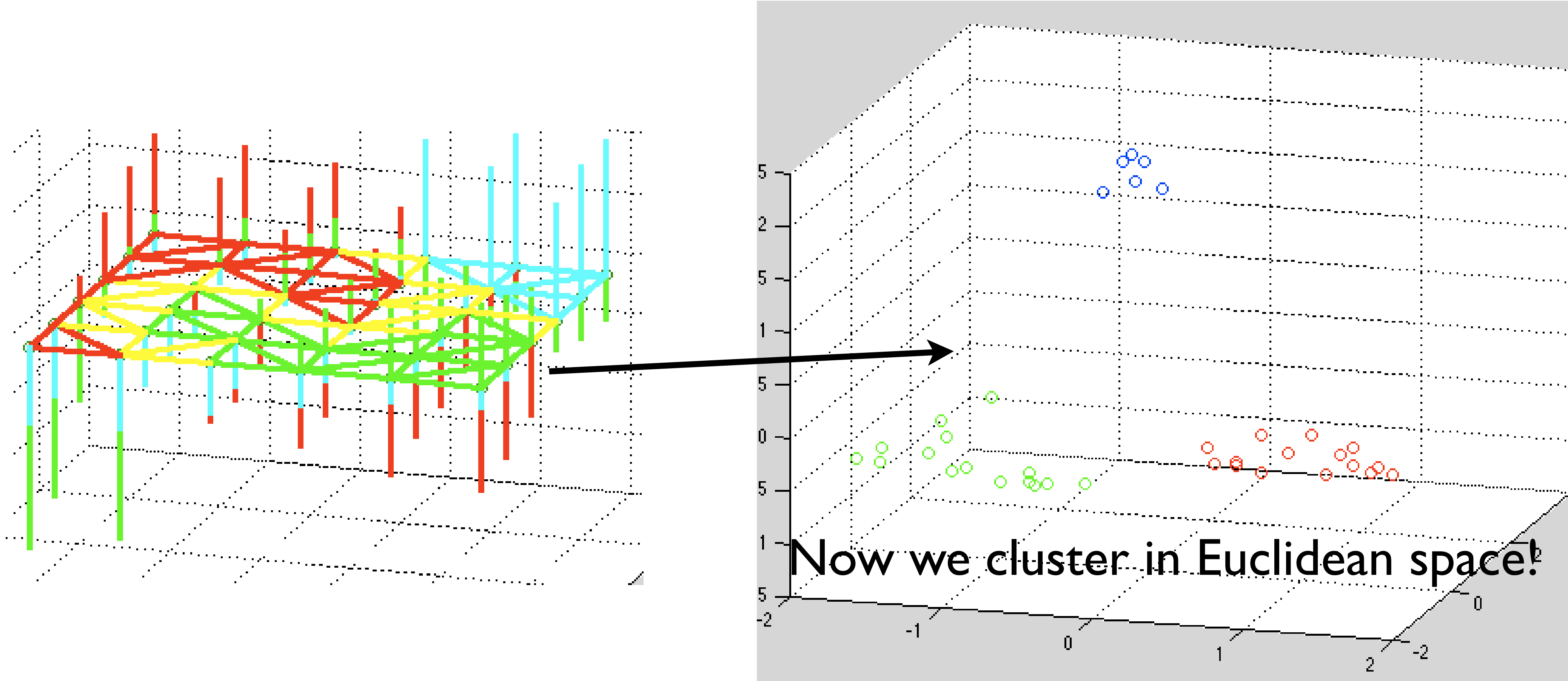
Q.E.D.

First three (non-trivial) Eigenfunctions

The Vagabond functions



Can use our eigenfunctions to embed our states in Euclidean space



See: On spectral clustering: Analysis and an algorithm. A. Y. Ng, M. I. Jordan, and Y. Weiss. In T. Dietterich, S. Becker and Z. Ghahramani (Eds.), *Advances in Neural Information Processing Systems (NIPS) 14*, 2002.

Shi, J. and Malik, J. (2000). Normalized cuts and image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(8), 888-905.

von Luxburg, U.: A Tutorial on Spectral Clustering. Statistics and Computing **17(4)**, 395-416 (12 2007)

Spectral Clustering Algorithm

1. Find the M orthonormal eigenvectors corresponding to the M smallest eigenvalues.

2. Using these eigenfunctions, embed our states into Euclidean space and then apply K-means.

Many Variations...

See: On spectral clustering: Analysis and an algorithm. A. Y. Ng, M. I. Jordan, and Y. Weiss. In T. Dietterich, S. Becker and Z. Ghahramani (Eds.), *Advances in Neural Information Processing Systems (NIPS) 14*, 2002.

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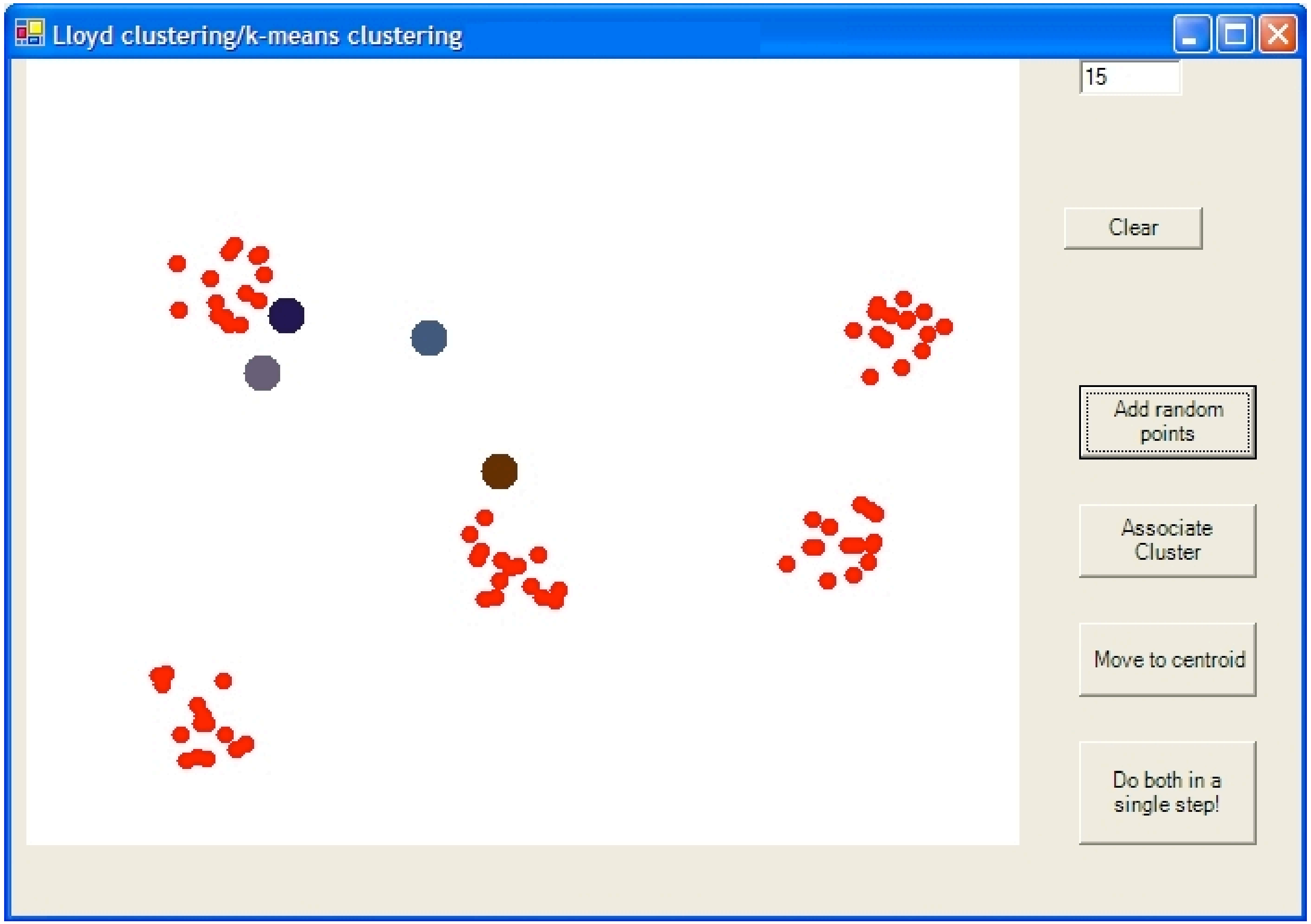
K-means

- Simplest clustering algorithm is k-means
- To run requires fixing K=#(Clusters)
- Requires an Euclidean type embedding
- We are attempting to minimizing a loss function:

$$L = \sum_{k=1}^{K} \sum_{x_i \in C_k} \left(x_i - \mu_k \right)^2$$

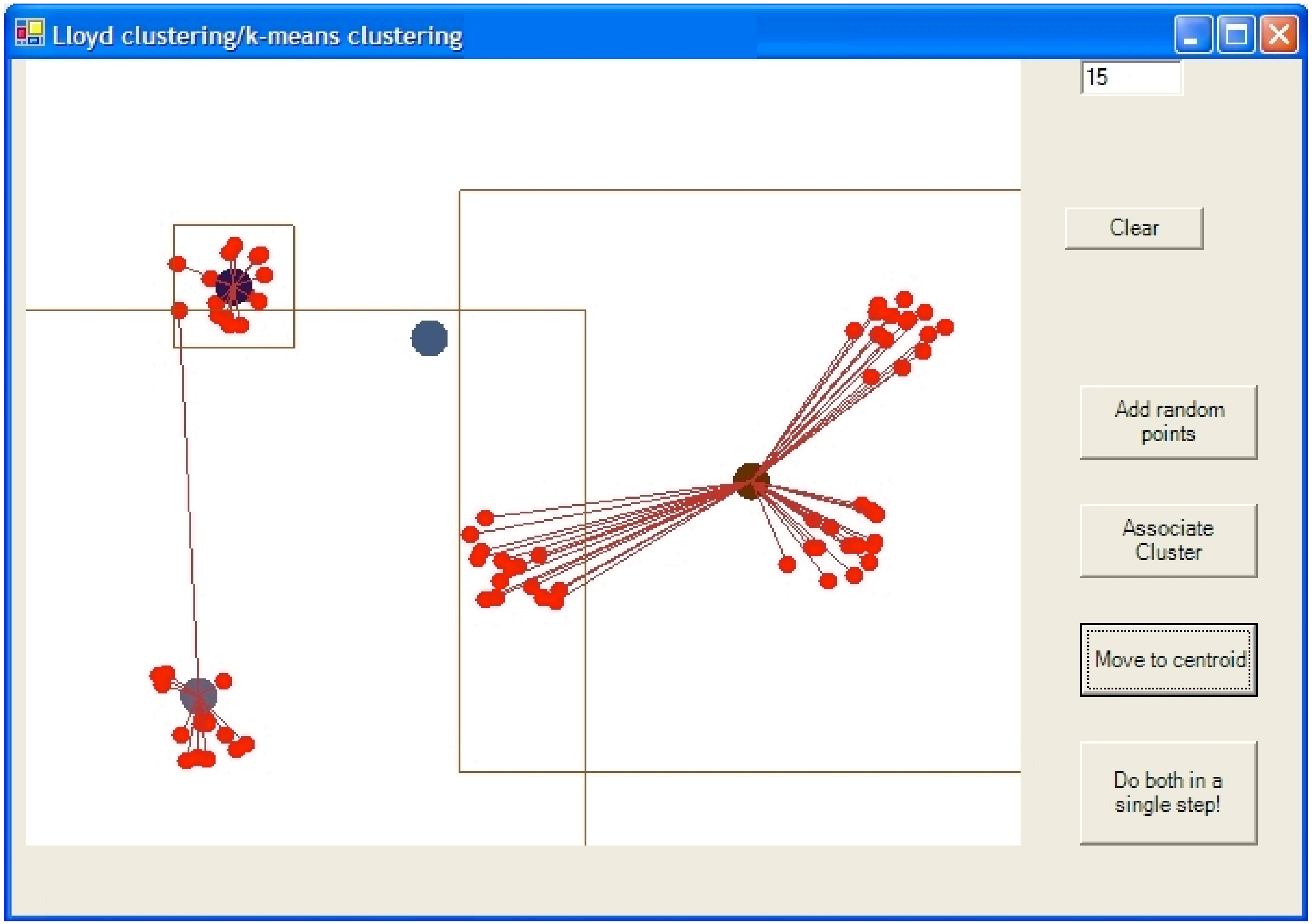
K-means algorithm:

1. Randomly choose points in each cluster and compute centroids.

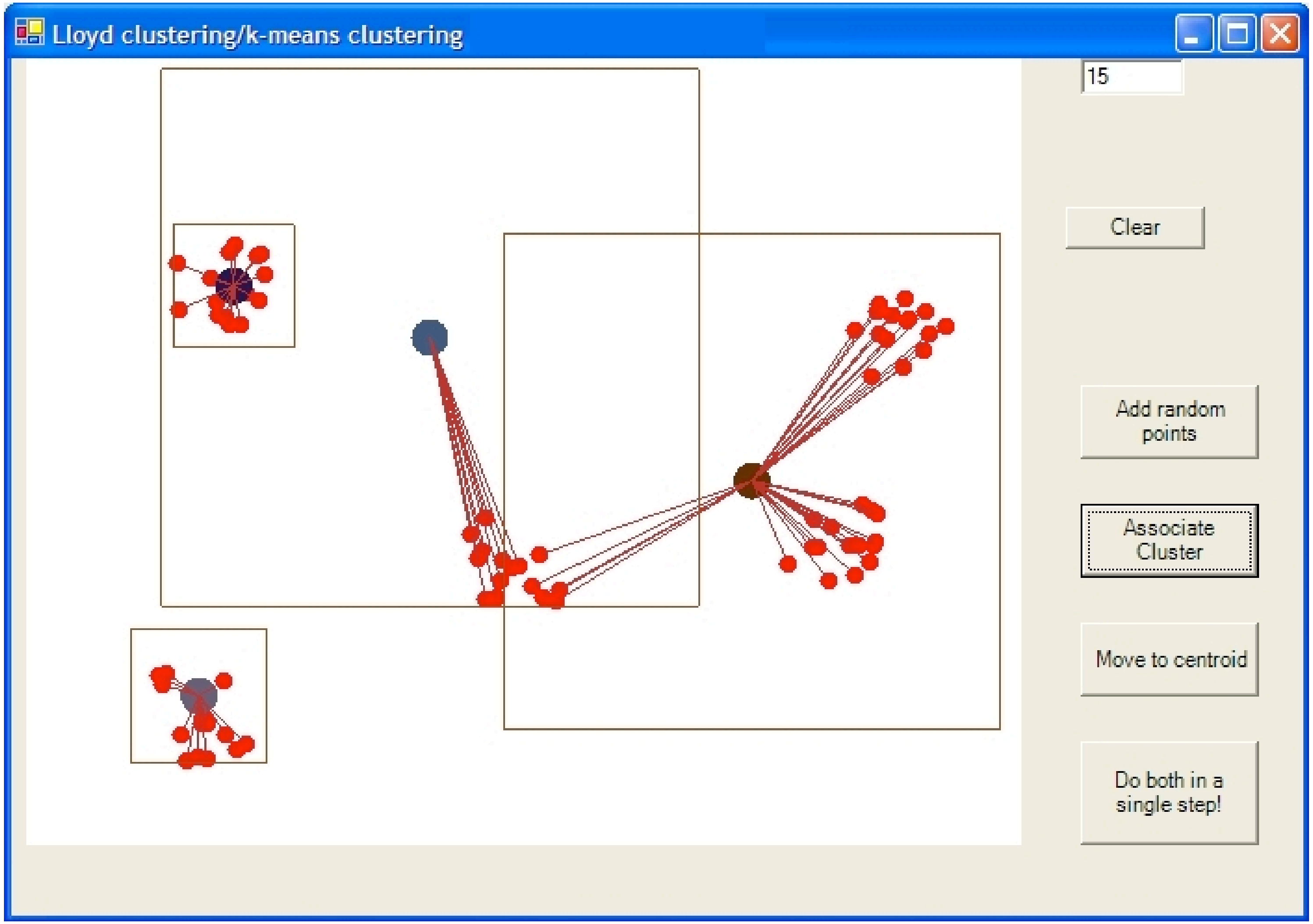


Example from: http://en.wikipedia.org/wiki/K-means_algorithm

2. Organize points by distance to the centroids.
3. Update centroids

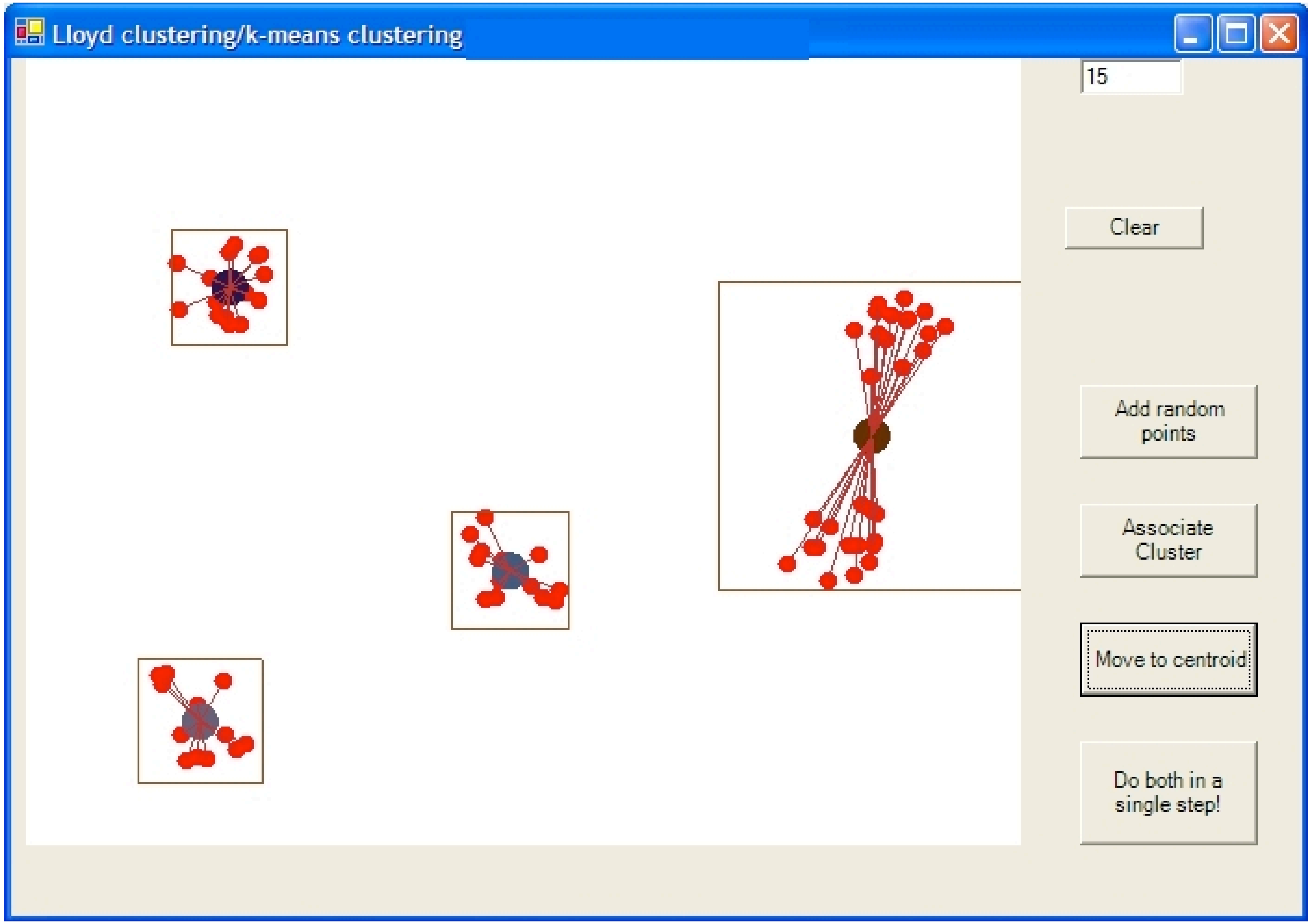


4. Repeat...



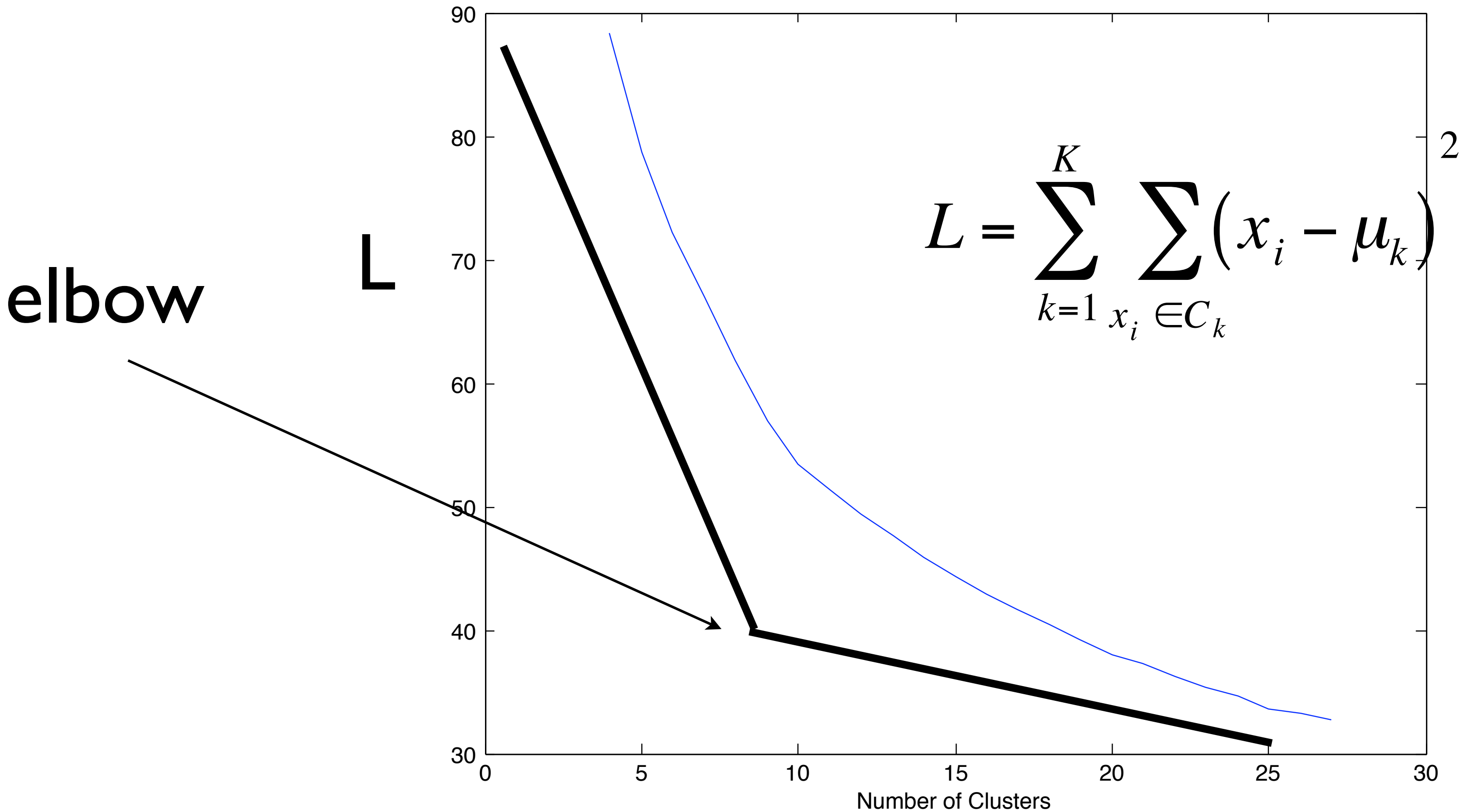
Lloyd clustering/k-means clustering
15
Clear
Add random points
Associate Cluster
Move to centroid
Do both in a single step!

...until stable.



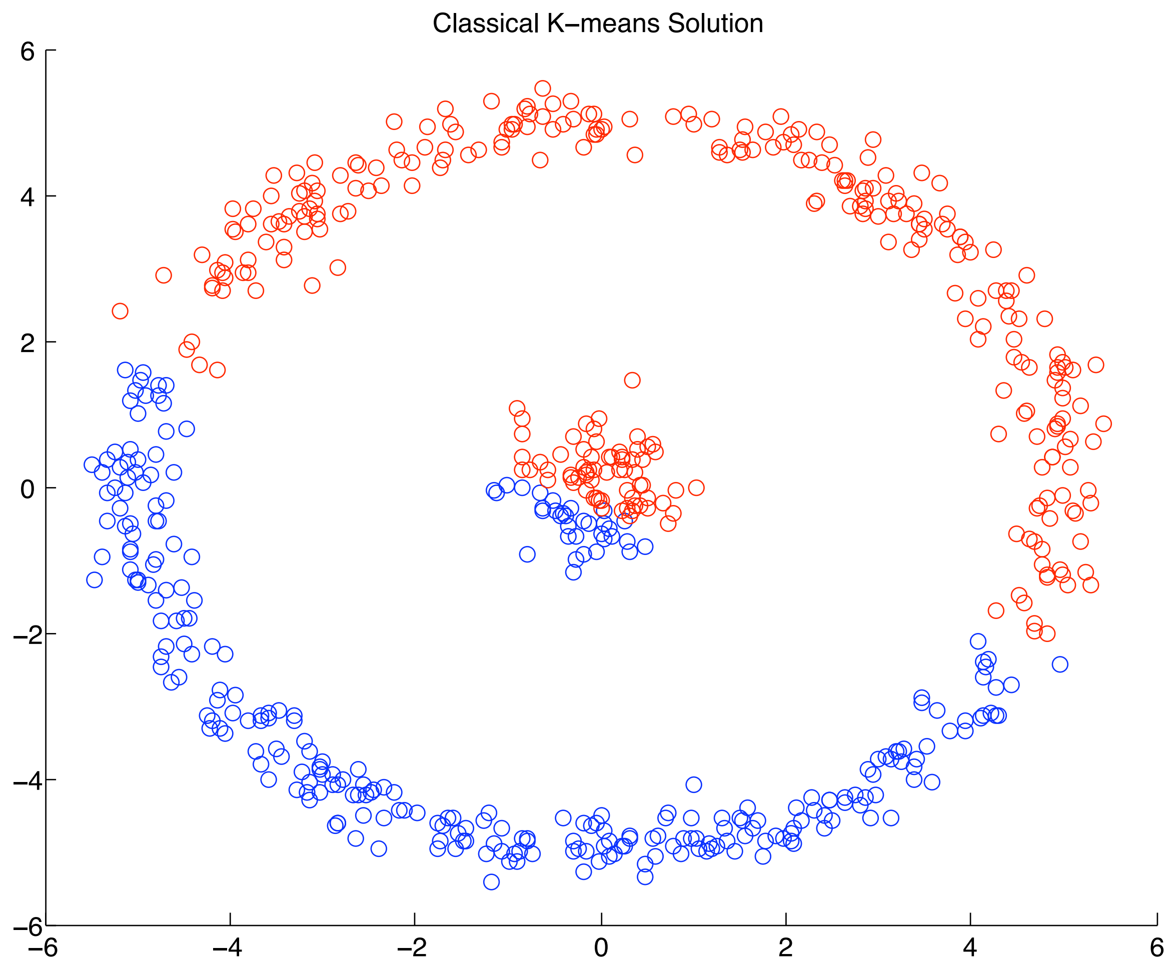
A hard part is choosing K=#(Clusters)

Elbowlogy:

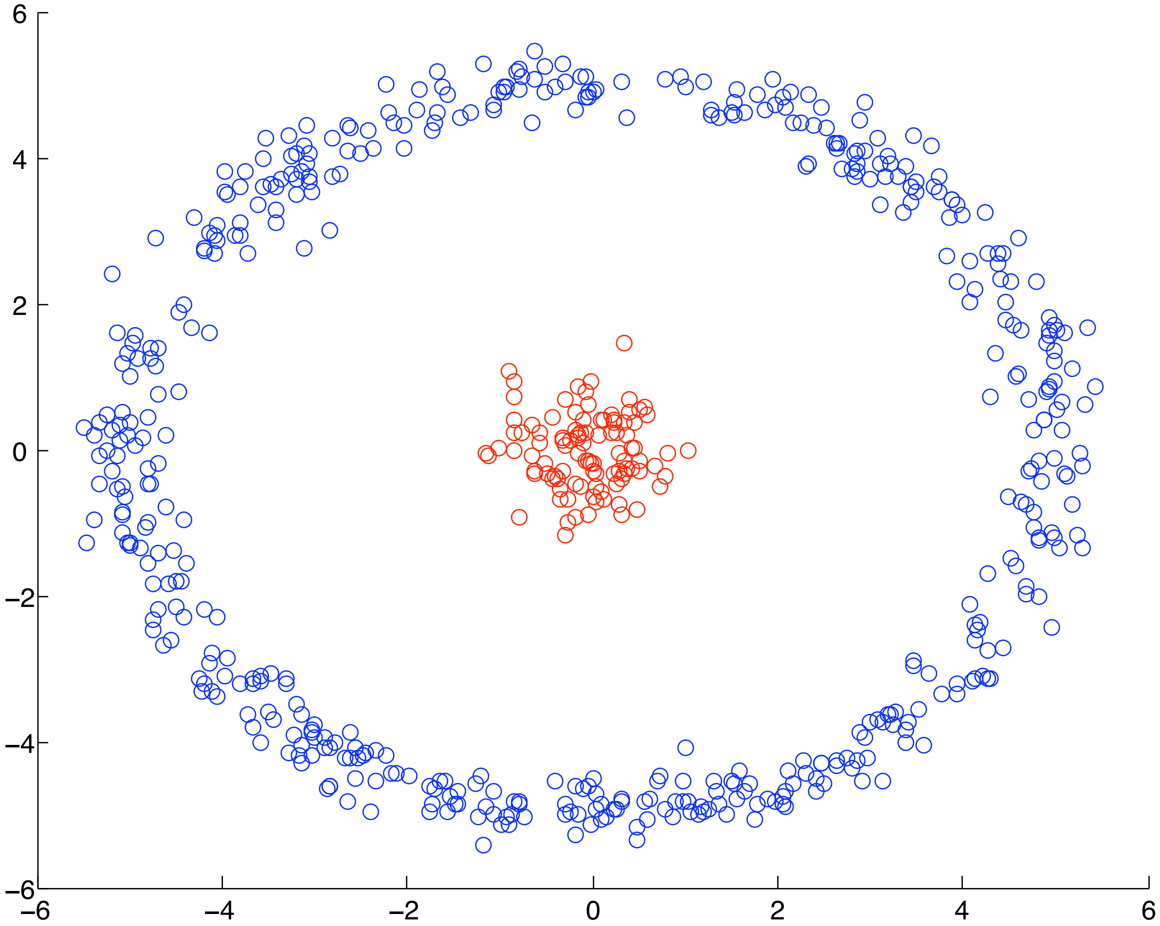


Though is practice this rarely works in a complex multi-scaar sysem system

Also, need to account for "non-ballish" geometry Classical K-means



Spectral Clustering



Manifold Learning

$$L = \sum_{k=1}^{N} \sum_{x_i \in C_k} d(x_i, \mu_k) \approx 488$$

Spectral clustering in Mat Lab.

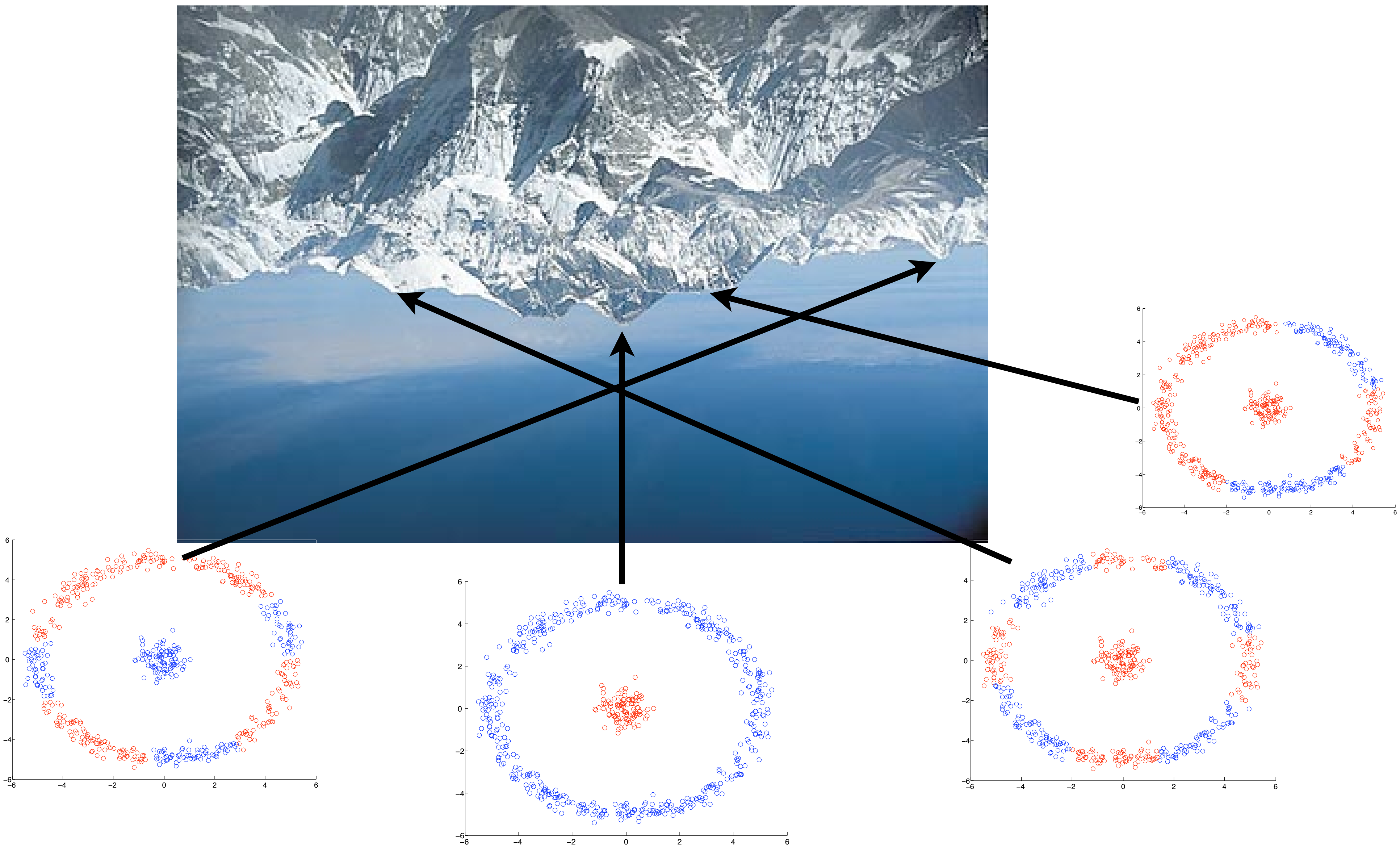
```
   %Sort by eigenvalue the eigenbasis [Eig,O]
Eig=diag(Eig);
[Eig Srt]=sort(Eig);
O=O(:,Srt);
    %Now apply K-means 'Rep' times
Emb=O(:,2:N);
[IDX,C,sumd,D] = kmeans(Emb,K,'emptyaction','drop');
for i=1:Rep
    [IDX0,C,sumd0,D] = kmeans(Emb,K,'emptyaction','drop');
    if (sum(sumd0)<sum(sumd))
        IDX=IDX0;
        sumd=sumd0;
    end;
end;
```

why repeat?

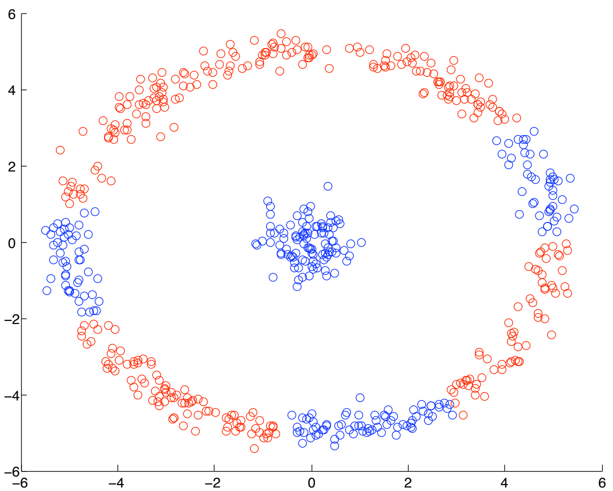
Yet another tricky part....

Himalayas

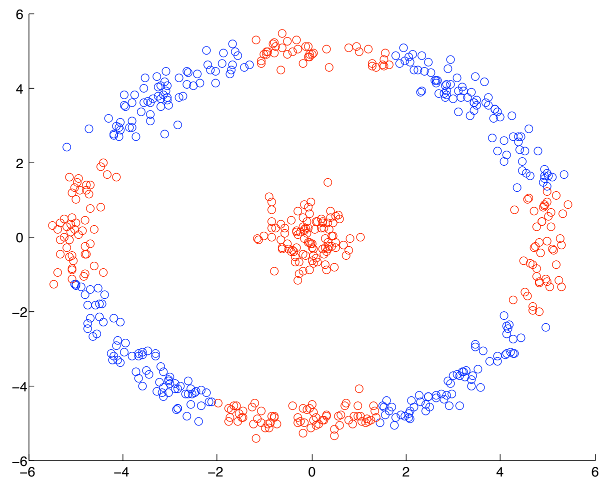
$$L = \sum_{k=1}^{K} \sum_{x_i \in C_k} \left(x_i - \mu_k \right)^2$$



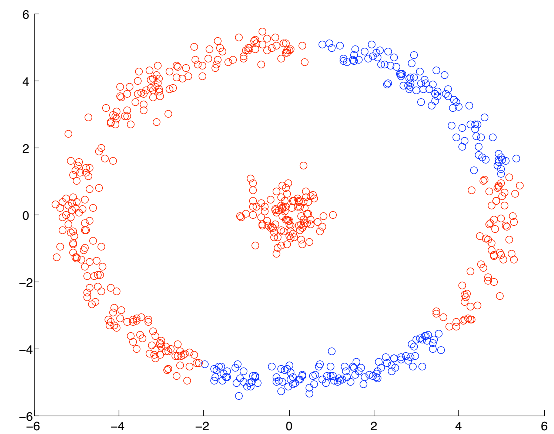
Watch Out! 100 convergent runs of the spectral k-means were performed. $\sigma = 1$



$$L = \sum_{k=1}^{N} \sum_{x_i \in C_k} d(x_i, \mu_k) \approx 536$$

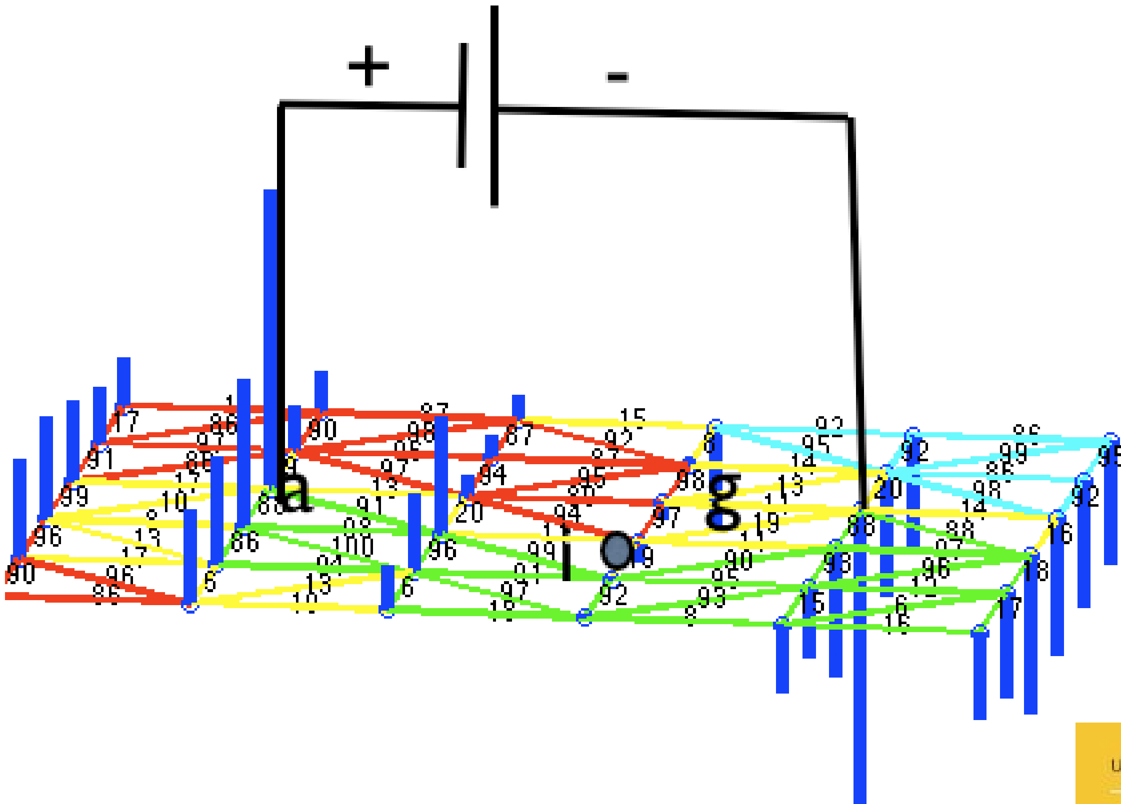


$$L = \sum_{k=1}^{N} \sum_{x_i \in C_k} d(x_i, \mu_k) \approx 538$$



$$L = \sum_{k=1}^{N} \sum_{x_i \in C_k} d(x_i, \mu_k) \approx 534$$

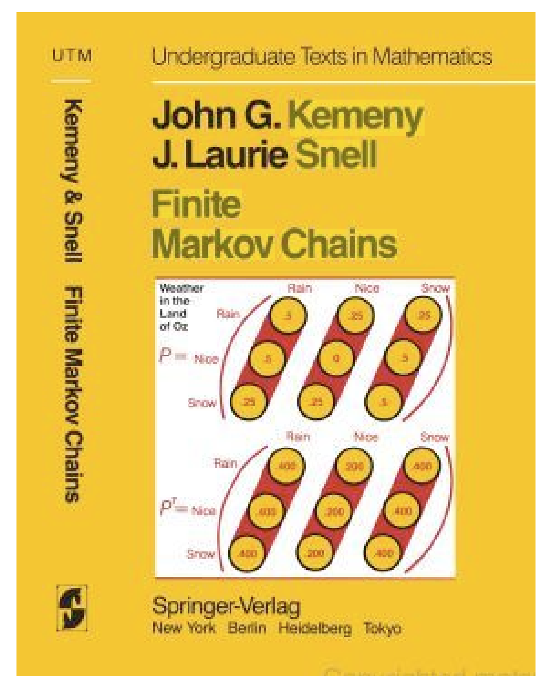
Part 2: Picklology...and beyond



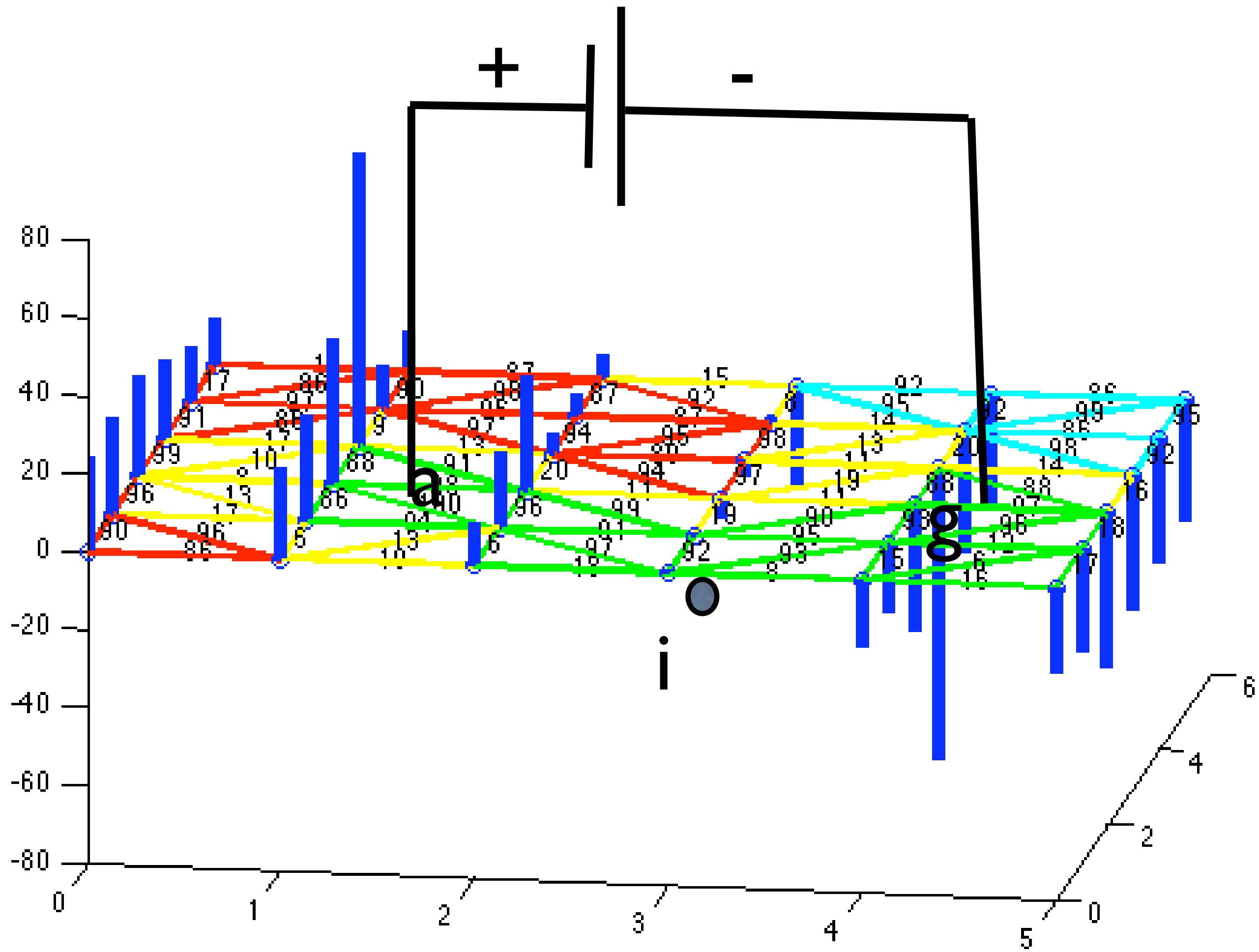
We thus see that α and A have nearly the same properties in the ergodic case as they did for regular chains; only, (a) had to be weakened to summability in place of convergence. We will now show that ergodic chains have a fundamental matrix which behaves just like the fundamental matrix of regular chains.

5.1.3 THEOREM. *If P is an ergodic transition matrix, then the inverse matrix $Z = (I - (P - A))^{-1}$ exists, and*

(a) $PZ = ZP$
(b) $Z\xi = \xi$
(c) $\alpha Z = \alpha$
(d) $(I - P)Z = I - A$.



Define the pickle embeddings of a chain: $Z_a^g(i)$



$$\Delta Z_a^g = \frac{e_a}{w^a} - \frac{e_g}{w^g}$$

$$\sum_i Z_a^g(i) = 0$$

What is Z...

MATHEMATICAL PAPERS

OF THE LATE

GEORGE GREEN,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

AN ESSAY

ON THE APPLICATION OF MATHEMATICAL ANALYSIS

TO THE THEORIES

OF ELECTRICITY AND MAGNETISM.*

Circa 1828

Green's Function

$$G(x,y)$$

Potential at x for point source at y.

$$\Delta G(x,y) = \delta y$$

(1.) THE function which represents the sum of all the electric particles acting on a given point divided by their respective distances from this point, has the property of giving, in a very simple form, the forces by which it is solicited, arising from the whole electrified mass.—We shall, in what follows, endeavour to discover some relations between this function, and the density of the electricity in the mass or masses producing it, and apply the relations thus obtained, to the theory of electricity.

Firstly, let us consider a body of any form whatever, through which the electricity is distributed according to any given law, and fixed there, and let x', y', z', be the rectangular co-ordinates of a particle of this body, ρ' the density of the electricity in this particle, so that $dx'dy'dz'$ being the volume of the particle, $\rho' dx'dy'dz'$ shall be the quantity of electricity it contains: moreover, let r' be the distance between this particle and a point p exterior to the body, and V represent the sum of all the particles of electricity divided by their respective distances from this point, whose co-ordinates are supposed to be x, y, z, then shall we have

$$r' = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2},$$

and

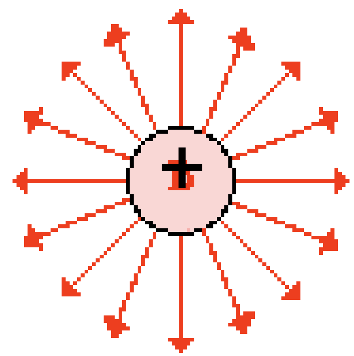
$$V = \int \frac{\rho' dx'dy'dz'}{r'};$$

the integral comprehending every particle in the electrified mass under consideration.

2—2

Example (R^3):

$$G(x,y) = \frac{1}{4\pi |x-y|}$$



....with a second source at infinity

LAPLACE has shown, in his Méc. Céleste, that the function V has the property of satisfying the equation

$$0=\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2},$$

and as this equation will be incessantly recurring in what follows, we shall write it in the abridged form $0=\delta V$; the symbol δ being used in no other sense throughout the whole of this Essay.

In order to prove that $0=\delta V$, we have only to remark, that by differentiation we immediately obtain $0=\delta\frac{1}{r'}$, and consequently each element of V substituted for V in the above equation satisfies it; hence the whole integral (being considered as the sum of all these elements) will also satisfy it. This reasoning ceases to hold good when the point p is within the body, for then, the coefficients of some of the elements which enter into V becoming infinite, it does not therefore necessarily follow that V satisfies the equation

$$0=\delta V,$$

although each of its elements, considered separately, may do so.

...

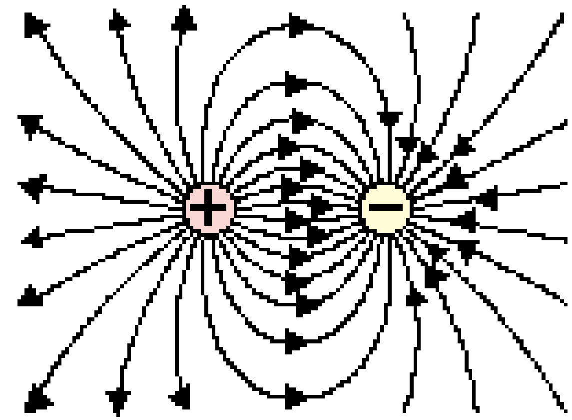
Hence, throughout the interior of the mass

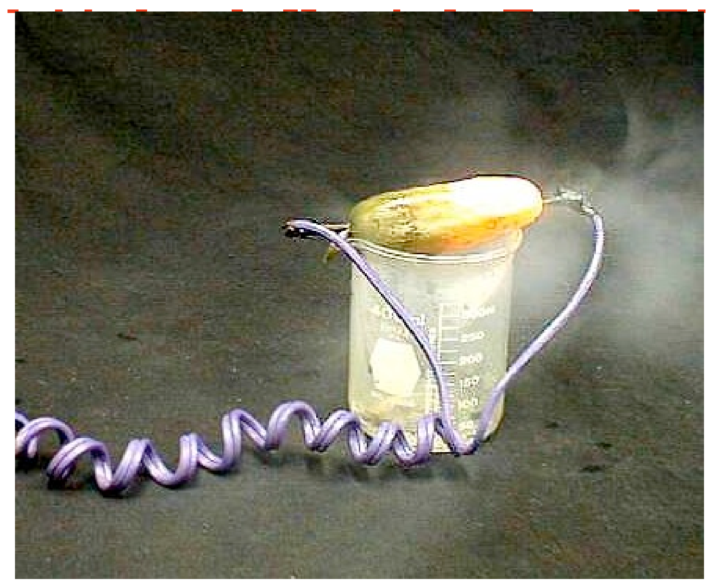
$$0=\delta V+4\pi\rho;$$

of which, the equation $0=\delta V$ for any point exterior to the body is a particular case, seeing that, here $\rho=0$.

$$G(x,y_1,y_2)$$

$$G(x,y_1)-G(x,y_2)$$

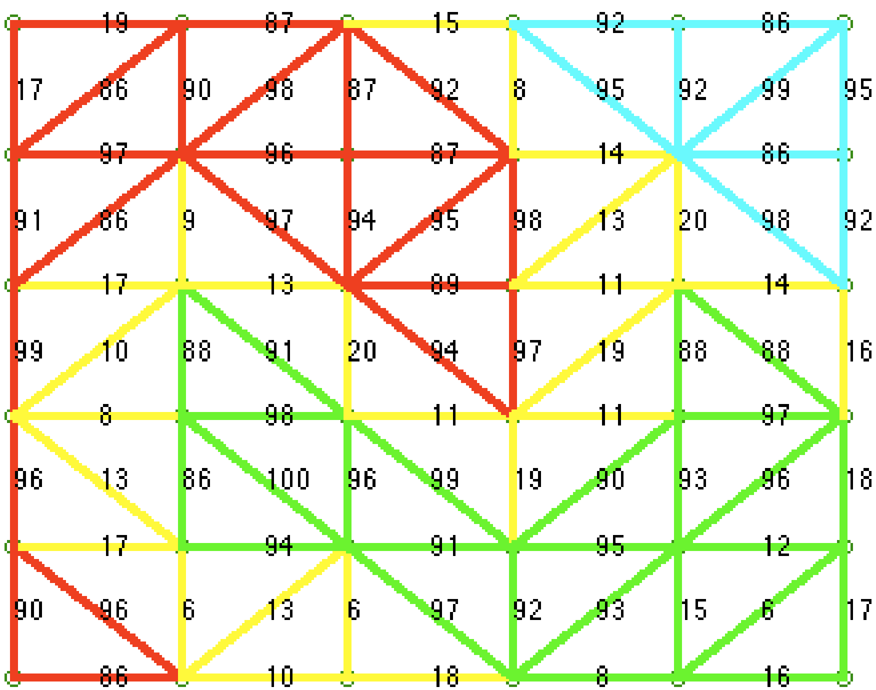


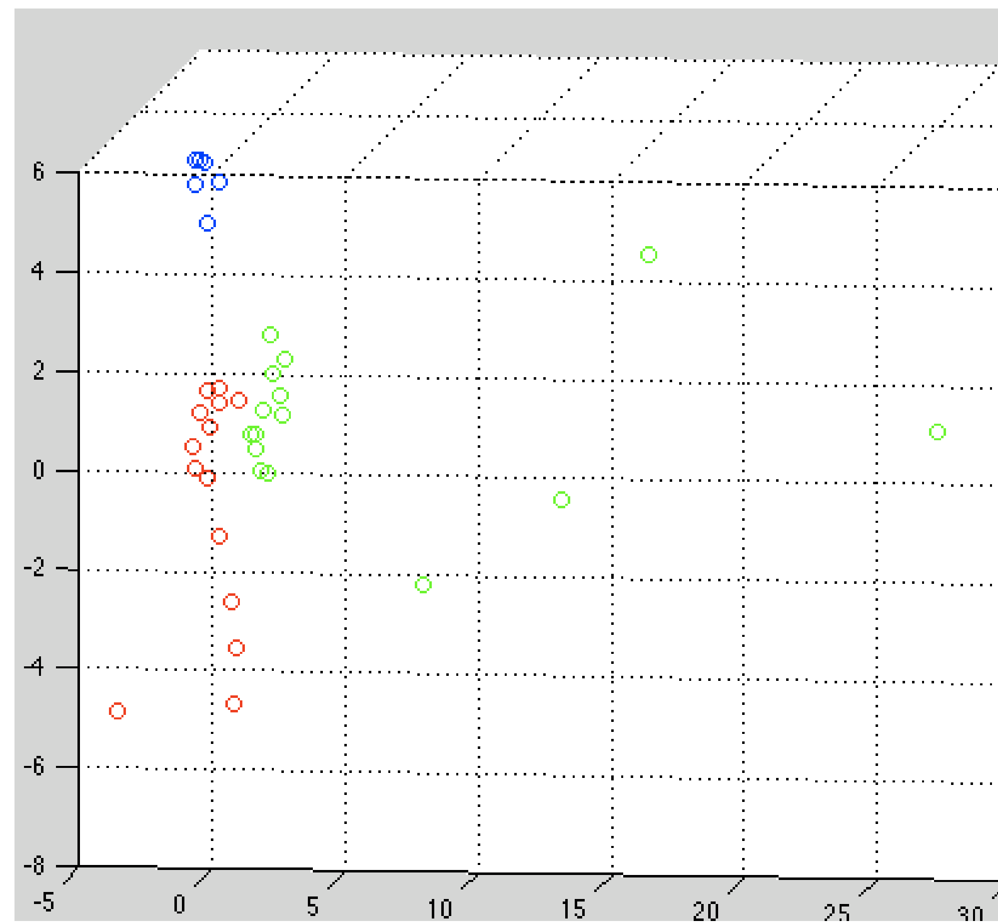


$$<\Delta f,c>=<f,\Delta c>=0$$

The Pickle Embedding
(reduced to 3 dimensions)

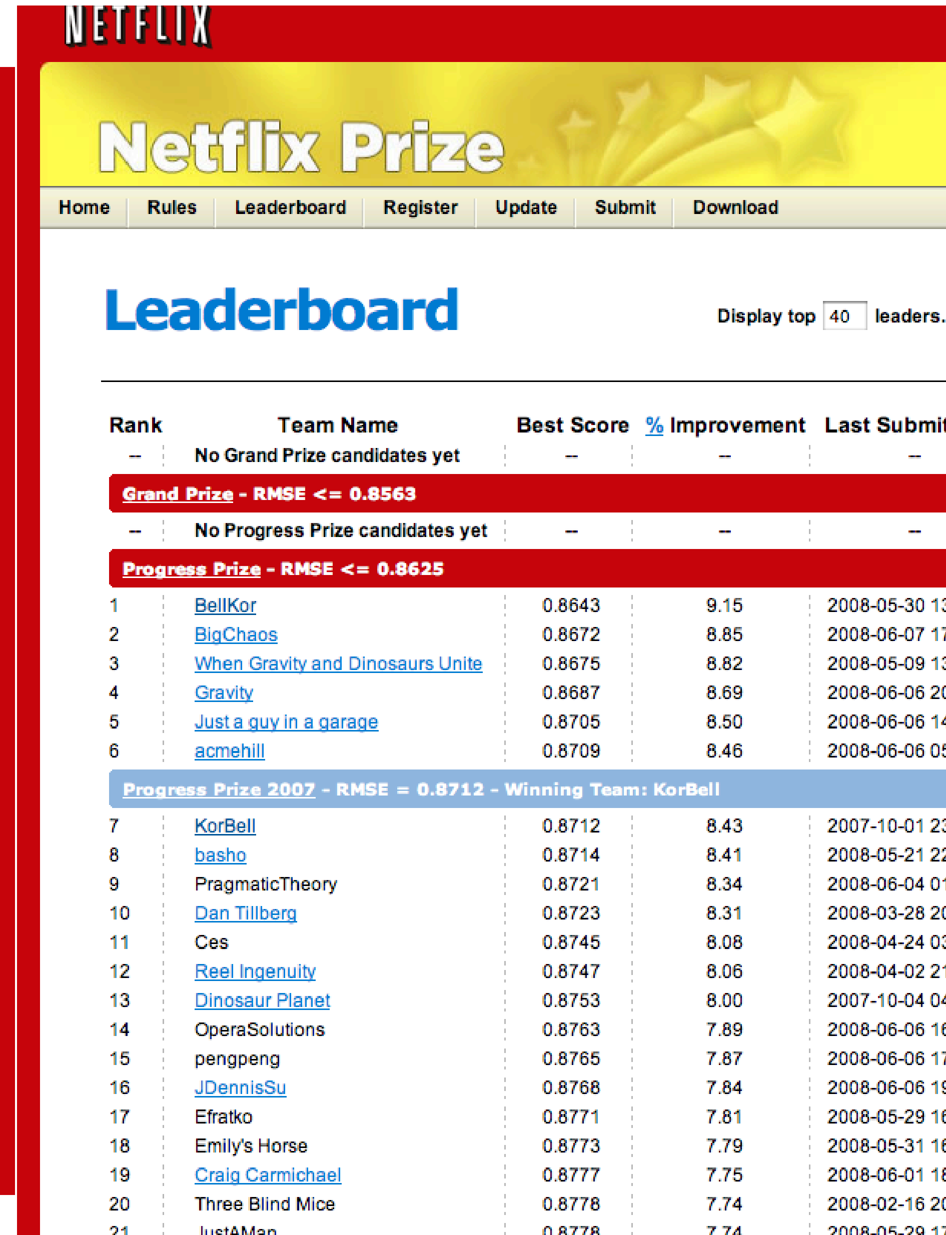
$$a \longrightarrow Z_a^g$$





Applications

The Netflix Challenge




MEET THE FEEBLES
THE ADULT PUPPET MOVIE
with something to offend everybody!
A PETER JACKSON FILM
18
SPECIAL COLLECTOR'S EDITION
ATTACK OF THE KILLER TOMATOES
More adventure than humanly possible.
Howard the Duck (1986).jpg
HOWARD THE DUCK
COMING AUGUST 1st
BEST PICTURE
KEVIN COSTNER
DANCES WITH WOLVES
Twilight of the Cockroaches
MOVIEGOODS
billy crystal
meg ryan
when harry met sally

IEEE TRANSACTIONS ON KNOWLEDGE AND DATA ENGINEERING, VOL. 19, NO. 3, MARCH 2007

Random-Walk Computation of Similarities between Nodes of a Graph with Application to Collaborative Recommendation

François Fouss, Alain Pirotte, *Member, IEEE*, Jean-Michel Renders, and Marco Saerens, *Member, IEEE*

Abstract—This work presents a new perspective on characterizing the similarity between elements of a database or, more generally, nodes of a weighted and undirected graph. It is based on a Markov-chain model of random walk through the database. More precisely, we compute quantities (the **average commute time**, the **pseudoinverse of the Laplacian matrix** of the graph, etc.) that provide similarities between any pair of nodes, having the nice property of increasing when the number of paths connecting those elements increases and when the "length" of paths decreases. It turns out that the square root of the average commute time is a Euclidean distance and that the pseudoinverse of the Laplacian matrix is a kernel matrix (its elements are inner products closely related to commute times). A principal component analysis (PCA) of the graph is introduced for computing the subspace projection of the node vectors in a manner that preserves as much variance as possible in terms of the Euclidean commute-time distance. This graph PCA provides a nice interpretation to the "**Fiedler vector**," widely used for graph partitioning. The model is evaluated on a collaborative-recommendation task where suggestions are made about which movies people should watch based upon what they watched in the past. Experimental results on the MovieLens database show that the Laplacian-based similarities perform well in comparison with other methods. The model, which nicely fits into the so-called "statistical relational learning" framework, could also be used to compute document or word similarities, and, more generally, it could be applied to machine-learning and pattern-recognition tasks involving a relational database.

6 EXPERIMENTS

6.1 Experimental Methodology

Remember that each element of the `people` and the `movie` sets corresponds to a node of the graph. Each node of the `people` set is connected by a link to each movie watched by the corresponding person. Notice that, in this special case, the graph is bipartite. The results shown here do not take into account the numerical value of the ratings provided by the persons but only the fact that a person has or has not watched a movie (i.e., entries in the person-movie matrix are 0s and 1s). Moreover, our experiments do not take the `movie_category` set into account so that comparisons between the various scoring algorithms remain fair. Indeed, three standard scoring algorithms (i.e., maximum frequency, cosine, and nearest-neighbor algorithms) cannot naturally use the `movie_category` set to rank the movies.

6.1.1 Data Set

Our experiments were performed on a real movie database from the Web-based recommender system MovieLens (http://www.movielens.umn.edu). We used a sample of their database as suggested in [59]: Enough people (i.e., 943 people) were randomly selected to obtain 100,000 ratings (considering only persons that had rated 20 or more movies on a total of 1,682 movies).

A Wiki Adventure!

Table 1: Output of **GREEN** on the articles used for evaluation.

Clique (graph theory)	*Germany*	*Hungarian language*	*Pierre de Fermat*	*Star Wars*	*Theory of relativity*	*1989*
1. Clique (graph theory)	1. Germany	1. Hungarian language	1. Pierre de Fermat	1. Star Wars	1. Theory of relativity	1. 1989
2. Graph (mathematics)	2. Berlin	2. Slovakia	2. Toulouse	2. Dates in Star Wars	2. Special relativity	2. Cold War
3. Graph theory	3. German language	3. Romania	3. Fermat's Last Theorem	3. Palpatine	3. General relativity	3. 1912
4. Category:Graph theory	4. Christian Democratic Union (Germany)	4. Slovenia	4. Diophantine equation	4. Jedi	4. Spacetime	4. Tiananmen Square protests of 1989
5. NP-complete	5. Austria	5. Hungarian alphabet	5. Fermat's little theorem	5. Expanded Universe (Star Wars)	5. Lorentz covariance	5. Soviet Union
6. Complement graph	6. Hamburg	6. Hungary	6. Fermat number	6. Star Wars Episode I: The Phantom Menace	6. Albert Einstein	6. German Democratic Republic
7. Clique problem	7. German reunification	7. Croatia	7. Grandes écoles	7. Star Wars Episode IV: A New Hope	7. Principle of relativity	7. George H. W. Bush
8. Complete graph	8. Social Democratic Party of Germany	8. Category:Hungarian language	8. Blaise Pascal	8. Obi-Wan Kenobi	8. Electromagnetism	8. 1903
9. Independent set	9. German Empire	9. Turkic languages	9. France	9. Star Wars Episode III: Revenge of the Sith	9. Lorentz transformation	9. Communism
10. Maximum common subgraph isomorphism problem	10. German Democratic Republic	10. Finno-Ugric languages	10. Pseudoprime	10. Coruscant	10. Inertial frame of reference	10. 1908
11. Planar graph	11. Bavaria	11. Austria	11. Lagrange's four-square theorem	11. Anakin Skywalker	11. Speed of light	11. 1929
12. Glossary of graph theory	12. Stuttgart	12. Serbia	12. Number theory	12. Lando Calrissian	12. Galilean transformation	12. Ruhollah Khomeini
13. Mathematics	13. States of Germany	13. Uralic languages	13. Fermat polygonal number theorem	13. Luke Skywalker	13. Local symmetry	13. March 1
14. Connectivity (graph theory)	14. Munich	14. Ukraine	14. Holographic will	14. Star Wars: Clone Wars	14. Category:Relativity	14. Czechoslovakia
15. Computer science	15. European Union	15. Hungarian grammar (verbs)	15. Diophantus	15. List of Star Wars books	15. Galilean invariance	15. June 4
16. David S. Johnson	16. National Socialist German Workers Party	16. German language	16. Euler's theorem	16. George Lucas	16. Gravitation	16. The Satanic Verses (novel)
17. Independent set problem	17. World War II	17. Hungarian grammar	17. Pell's equation	17. Star Wars Episode II: Attack of the Clones	17. Global symmetry	17. 1902
18. Computational complexity theory	18. Jean Edward Smith	18. Khanty language	18. Fermat's theorem on sums of two squares	18. Splinter of the Mind's Eye	18. Tensor	18. November 7
19. Set	19. Soviet Union	19. Hungarian phonology	19. Fermat's spiral	19. List of Star Wars comic books	19. Maxwell's equations	19. October 9
20. Michael Garey	20. Rhine	20. Finnish language	20. Fermat's factorization method	20. The Force (Star Wars)	20. Introduction to general relativity	20. March 14
Mark: 7.6/10	Mark: 7.0/10	Mark: 6.2/10	Mark: 7.3/10	Mark: 7.4/10	Mark: 8.1/10	Mark: 5.4/10

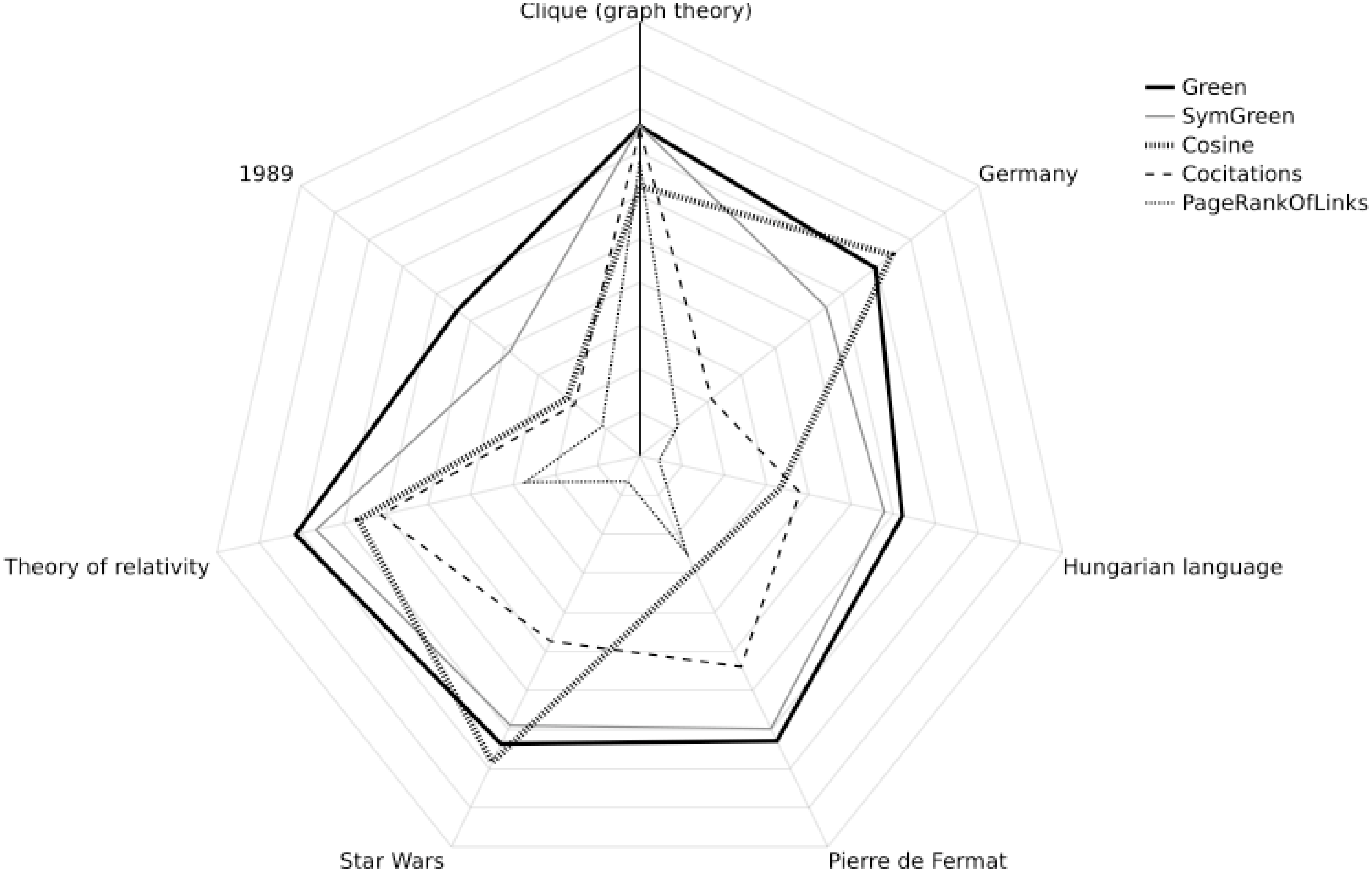
Y. Ollivier and P. Senellart, Finding Related Pages Using Green Measures: An Illustration with Wikipedia. In *Proc. AAAI*, pp. 1427–
Conference article ver, Canada, July 2007. (pdf | slides | slides, short | website | bib)

Abstract

We introduce a new method for finding nodes semantically related to a given node in a hyperlinked graph: the Green method, based on a classical Markov chain tool. It is generic, adjustment-free and easy to implement. We test it in the case of the hyperlink structure of the English version of Wikipedia, the on-line encyclopedia. We present an extensive comparative study of the performance of our method versus several other classical methods in the case of Wikipedia. The Green method is found to have both the best average results and the best robustness.

Evaluation Methodology. We carried out a blind evaluation of the methods on 7 different articles, chosen for their diversity: (i) *Clique (graph theory)*: a very short, technical article. (ii) *Germany*: a very large article. (iii) *Hungarian language*: a medium-sized, quite technical article. (iv) *Pierre de Fermat*: a short biographical article. (v) *Star Wars*: a large article, with an important number of links. (vi) *Theory of relativity*: a short introductory article pointing to more specialized articles. (vii) *1989*: a very large article, containing all the important events of year 1989. It was unreasonable to expect our testers to evaluate more articles. In order to avoid any bias, we did not run the methods on these 7 articles before the evaluation procedure was launched.

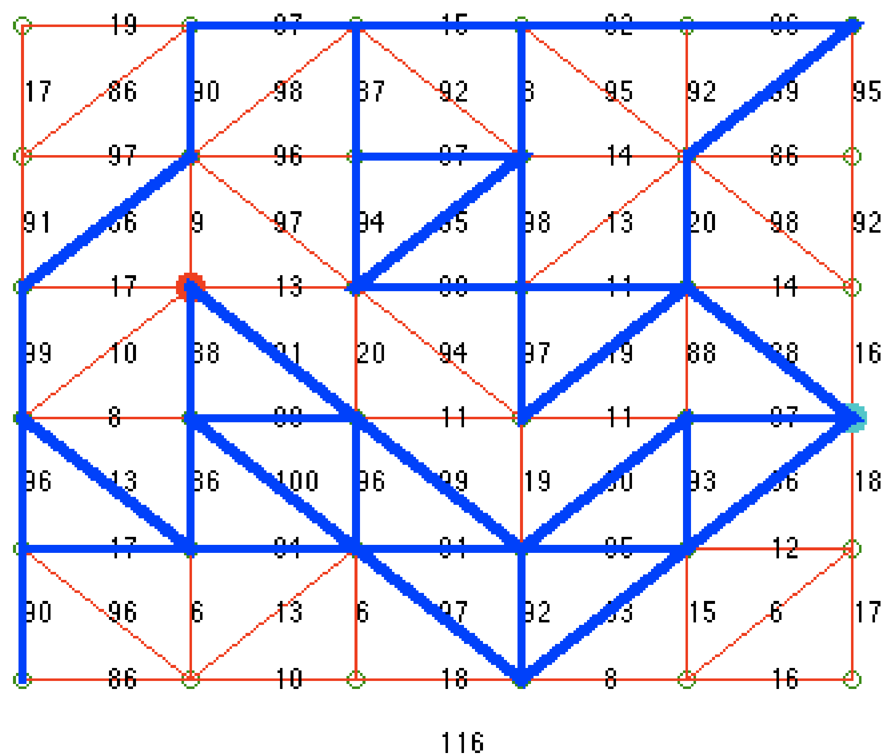
People were asked to assign a mark between 0 and 10 (10 being the best) to the list of the first 20 results returned by



Clique (graph theory)
Germany
Hungarian language
Pierre de Fermat
Star Wars
Theory of relativity
1989
Green
SymGreen
Cosine
Cocitations
PageRankOfLinks

Why do we like the Pickle Embedding?

$||Z_a^g - Z_b^g||$ has a natural interpretation via the Commute Time from i to j (denoted T_{ab}) is the expected number up step needed to get from i to j and back to i.




```
[states Time]=Commute(15,25,1000);
```

The Commute Theorem

$$||Z^g_a - Z^g_b||^2_{Dir} = ||Z^b_a||^2_{Dir} = T_{ab}$$

Note: linear combinations simply translate the image, and we can form

$$Z = \sum_{n=0}^{\infty}(P^n - P^{\infty})$$

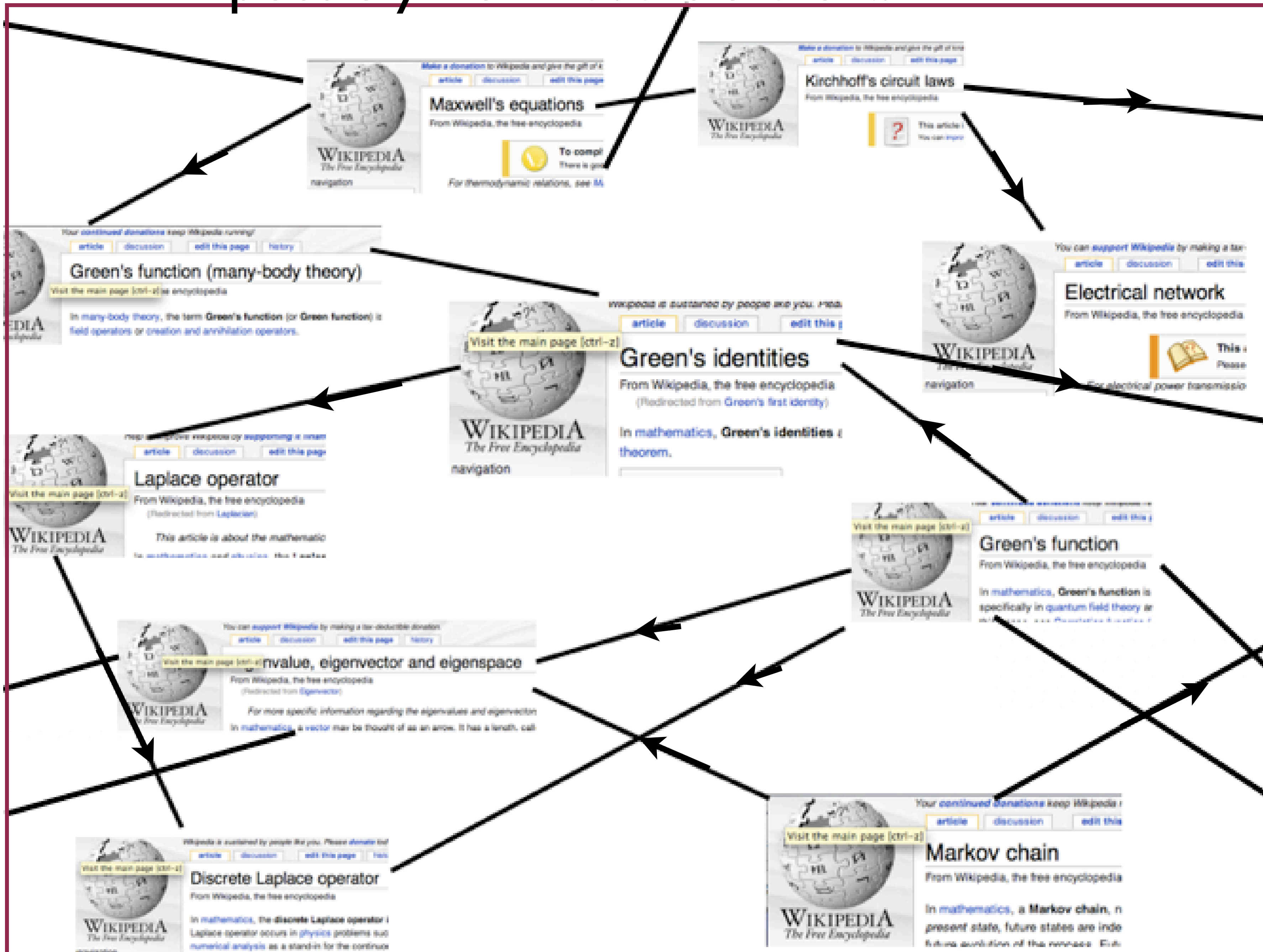
For a reversible, ergodic Markov chain

A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari. The electrical resistance of a graph captures its commute and co
Symposium on Theory of Computing, pages 574--586, Seattle, WA, May 1989. http://citeseer.ist.psu.edu/chandra89electrical.html More

Klein, D. and Randic, M. (1993). Resistance distance. *Journal of Mathematical Chemistry*, *12*, 81-95.

Francois Fouss, Alain Pirotte, Jean-Michel Renders, Marco Saerens: Random-Walk Computation of Similarities between Nodes of a Graph with Application to Collaborative Recommendation. IEEE Trans. Knowl. Data Eng. 19(3): 355-369 (2007)

A Wiki Adventure, done correctly! i.e. a possibly non-reversible chain



Yes!!

The Commute Theorem

$$||Z_a^g - Z_b^g||_{Dir}^2 = ||Z_a^b||_{Dir}^2 = T_{ab}$$

and we can use

$$Z = \sum_{n=0}^{\infty}(P^n - P^\infty)$$

is true for any ergodic Markov chain!

Commuting time geometry of ergodic Markov chains

Peter G. Doyle Jean Steiner

Version 1A7 dated 26 January 2009
GNU FDL*

$$(z_1; z_2; z_3; z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

$$N_{z_1;z_2;z_3;z_4} = \frac{-1}{2\pi} log(|(z_1; z_2; z_3; z_4)|)$$

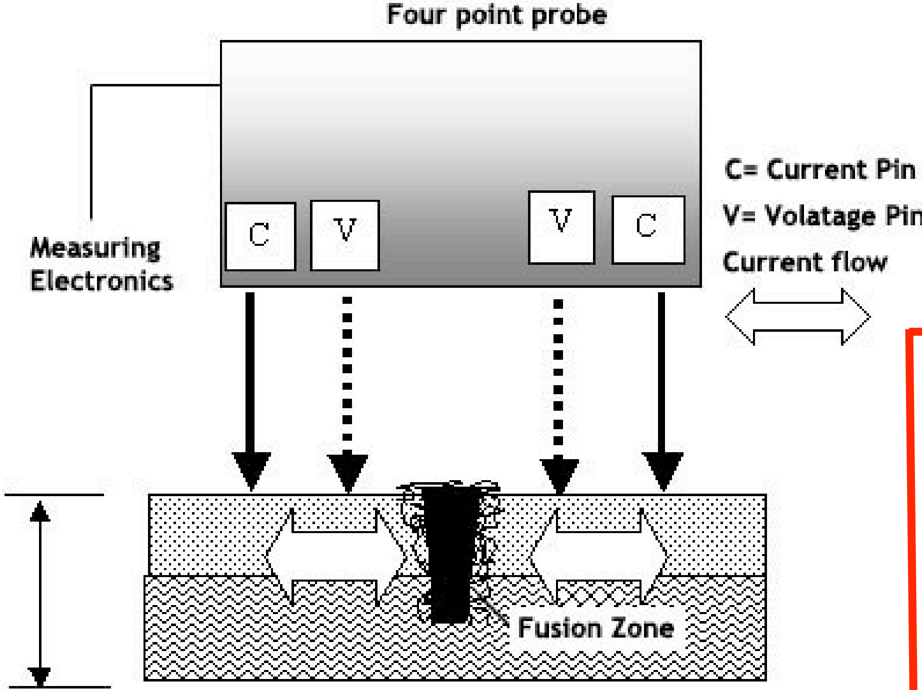
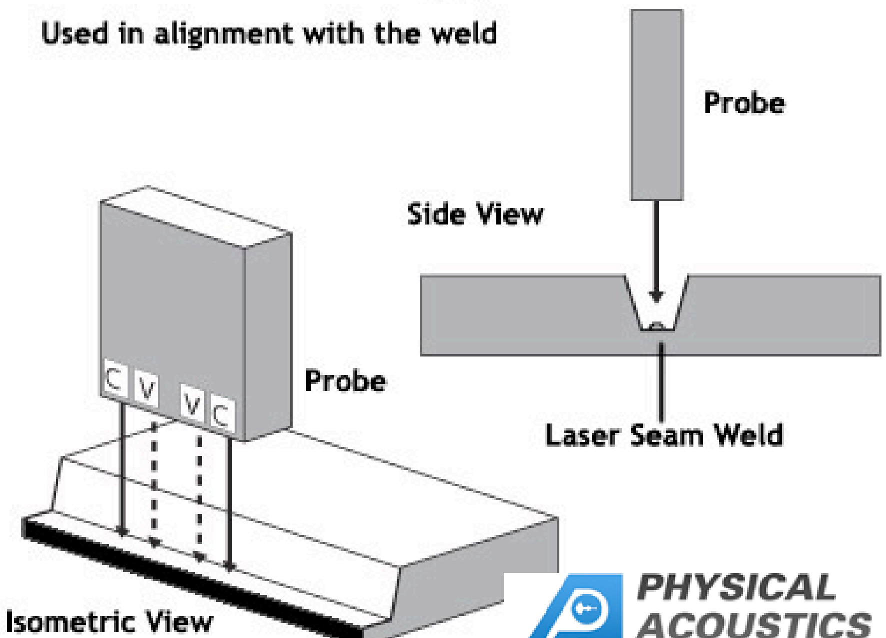


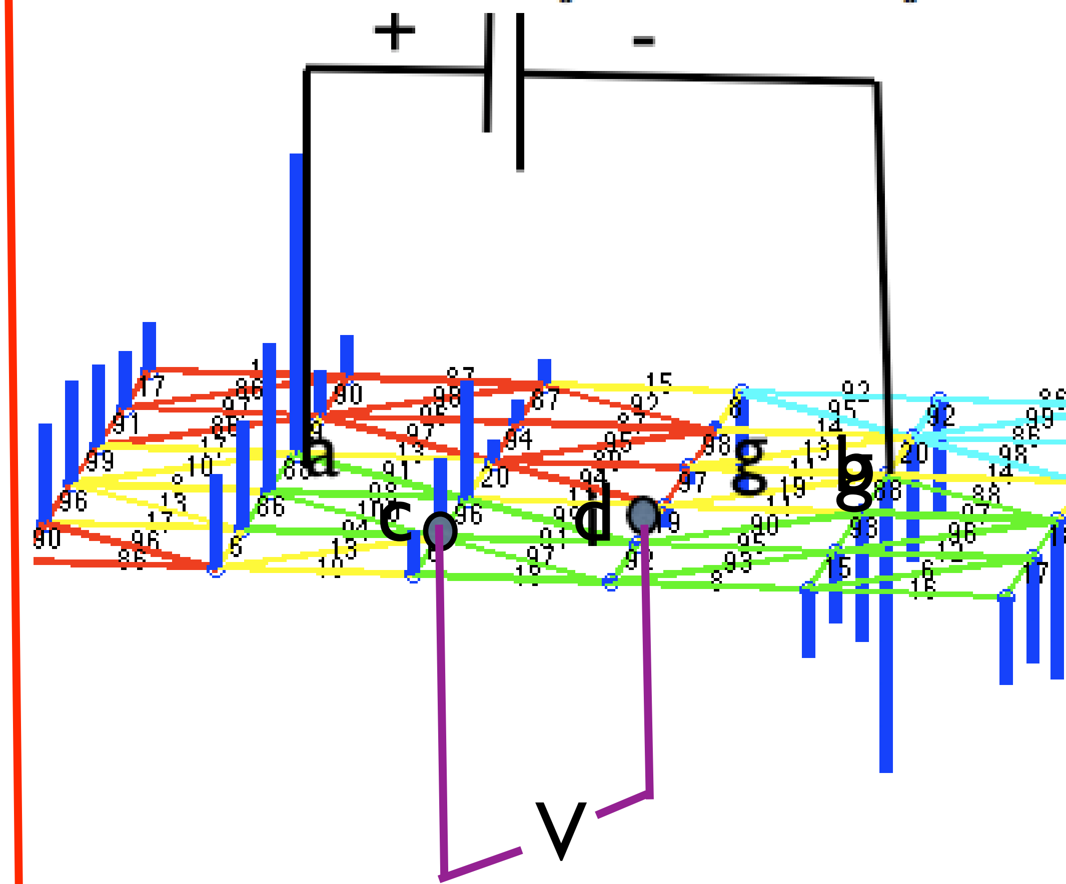
Figure 1. Four-point probe contact Resistivity Measurement Technique(Used across the weld)

OR

Used in alignment with the weld



$$N_{a;b;c;d} = \frac{Z_{ac} - Z_{bc}}{w_c} - \frac{Z_{ad} + Z_{bd}}{w_c}$$



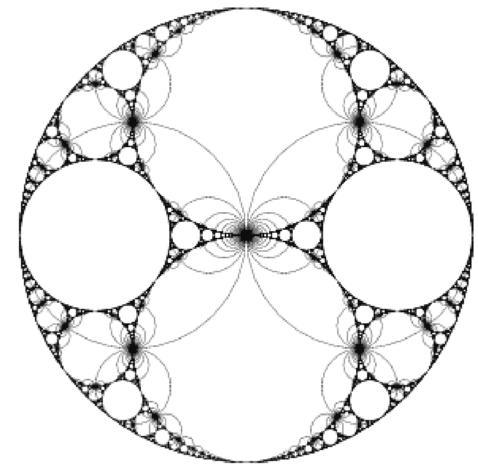
PHYSICAL ACOUSTICS CORPORATION

Geometry and conformal invariance

$$(z_1; z_2; z_3; z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

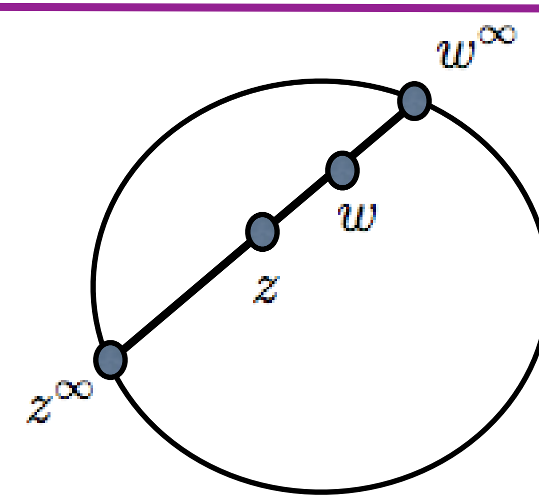
$$N_{z_1;z_2;z_3;z_4} = \frac{-1}{2\pi} log(|(z_1; z_2; z_3; z_4)|)$$

need 4th point to avoid infinity



This is the distance in the Hyperbolic plane (Klein Model)

$$d(z, w) = cN_{z,w,z^{\infty},w^{\infty}}$$



Why should we care...

Theorem (Doyle , Steiner 2009) N determines the conformally invariant part of a Markov chain and with w the whole chain.

Appendix: Proof of the Commute Theorem

$$||Z_a^b||^2 = T_{ab}$$

Why we are doing it.... to do some mathematics (yeah!)....

- Visit the almighty *maximum principle*
- Dwell on the *fundamental mysteries of probability theory*
- Directly interact with the single most important fact about infinity

Commute Theorem:

$$||Z_a^b||^2 = T_{ab}$$

Proof: Let p_e be the probability of reaching j before i when starting at i.

Proposition 1 (Renewal Theory) $T_{ab} = \frac{1}{p_{esc} w^a}$

Proposition 2 (Maximum Principle, Potential Theory)

$$||Z_a^b||^2 = \frac{1}{p_{esc} w^a}$$

First...

The second fundamental mystery of probability theory is that:

$$E(X) = E(E(X \mid Y)).$$

As an application let

$$X = \sum_{k=1}^{N} R_k$$

where N is a positive integer valued random variable and let the $\{R_i\}$ share the expected value $E(R)$. Then

2FMPT

$$E\left(\sum_{k=1}^{N} R_k\right) = E\left(E\left(\sum_{k=1}^{N} R_k \mid N = n\right)\right)$$

$$= \sum_{n=1}^{\infty} E(\sum_{k=1}^{n} R_k) P(N = n) = \sum_{n=1}^{\infty} n E(R) P(N = n)$$

1FMPT

$$= E(R) \sum_{n=1}^{\infty} n P(N = n) = E(R) E(N)$$

Wald's Theorem (Renewal Theory)

Proposition I

$$T_{ab} = \frac{1}{p_{esc} w^a}$$

$$R = \#\ \text{Steps taken going from a to a}$$

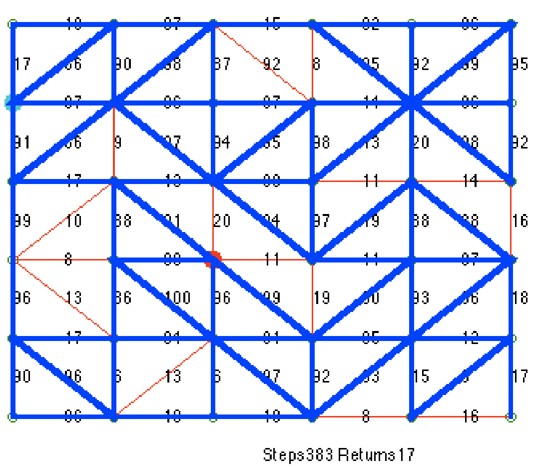
$$N = \#\ \text{Returns to i on commute}$$

$$E(T_{ab}) = E(\sum_{k=1}^{N} R_k)$$

Second Fundamental Mystery of Probability Theorey

(see the appendix)

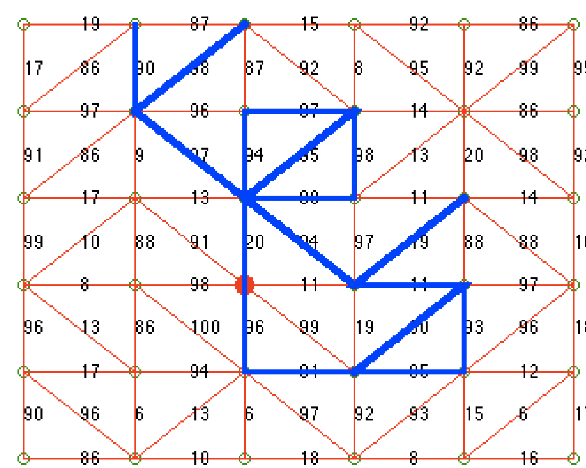
$$= E(N)E(R)$$



Commute(15,25,1000);

Very well known!

$$E(R) = \frac{1}{w^a}$$



Loop(15,1000);

$$E(N)$$

$$= 1(p_e) + 2(p_e(1 - p_{esc})) + \ldots$$

$$= p_e \sum_{k=0}^{\infty} k(1-p_e)^{k-1}$$

$$= p_e \frac{1}{(1-(1-p_e))^2} = \frac{1}{p_e}$$

Fundamental Theorem of Analysis

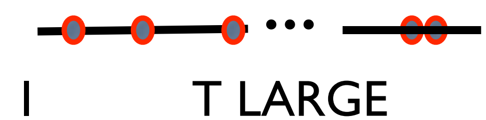
$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

$$\sum_{i=0}^{\infty} i r^{i-1} = \frac{d}{dr}\left(\sum_{i=1}^{\infty} r^i\right)$$

$$= \frac{1}{(1-r)^2}$$

Here's one argument....

N=#Loops=#(S=i)-1



$$E(R)$$

$$= \lim_{T\to\infty} \frac{1}{N} \sum_{k=1}^{N} R_k$$

only small stuff on ends missing

$$= \lim_{T\to\infty} \frac{T}{N}$$

Law of Large Numbers

$$= \frac{1}{\lim_{T\to\infty} \frac{N}{T}}$$

$$= \frac{1}{\lim_{T\to\infty} \frac{\sum_{k=1}^{T} 1_k(i)}{T}}$$

$$= \frac{1}{w^i}$$

$$T_{ab} = \frac{1}{p_{esc} w^a}$$

Proposition 2

$$||Z_a^b||^2 = \frac{1}{p_{esc} w^a}$$

$$\Delta \tilde{Z}_a^b = 0 \quad i \neq a, b$$

$$\tilde{Z}_a^b(a) = 0$$

$$\tilde{Z}_a^b(b) = 1$$

Dirichlet Problem

Unique solution up to constant from Maximum Principle

$$Z_a^b(i) = C\tilde{Z}_a^b(i) + D$$

$\Downarrow$

$\tilde{Z}_a^b(i)$ equals the probability starting a i of reaching b before a.

$\Downarrow$

$\tilde{Z}_a^b(a)$ is minus the probability of escaping from a to b staring at a

$\Downarrow$

$$\Delta \tilde{Z}_a^b(b) = \frac{w^a p_e}{w^b}$$

constant function in kernel

$$Z_a^b = \frac{1}{w^a p_e} \tilde{Z}_a^b$$

$$||\tilde{Z}_a^b||_{Dir}^2 = \left\langle \tilde{Z}_a^b, \Delta \tilde{Z}_a^b \right\rangle = w^b \frac{w^a p_e}{w^b} = w^a p_e$$

$$\Downarrow \qquad \Downarrow$$

$$||Z_a^b||_{Dir}^2 = \frac{1}{(w^a p_e)^2} ||\tilde{Z}_a^b||_{Dir}^2 = \frac{1}{w^a p_e}$$

Poyla's Theorem

On the 2-d Euclidean lattice a drunk Lord will always find his way home. In three dimensions he may not be so lucky!

"I have not had a moment's peace or happiness in respect to electromagnetic theory since November 28, 1846. All this time I have been liable to fits of ether dipsomania, kept away at intervals only by rigorous abstention from thought on the subject."
~Lord Kelvin~ (to FitzGerald 1896)

* **Dipsomania** is a term USUALLY related to an incontrollable craving for alcohol.... the obsession is so compulsive that the dipsomanic will ingest whatever intoxifying liquid is at hand, whether it is fit for consumption or not. Dipsomania differs from alcoholism in that it is an uncontrollable periodic lust for alcohol, with, in the interim, no desire for alcoholic beverages.

Ether Dipsomania is a term related to an incontrollable craving for a consistent and appealing theory of something in the form of an analogon to the theory of electromagnetism...the obsession is so compulsive that the ether dipsomaniac will....