Introduction to Nonlinear Dynamics:
Maps, Representations and dirty hands.

Santa Fe Institute

Complex Systems Summer School

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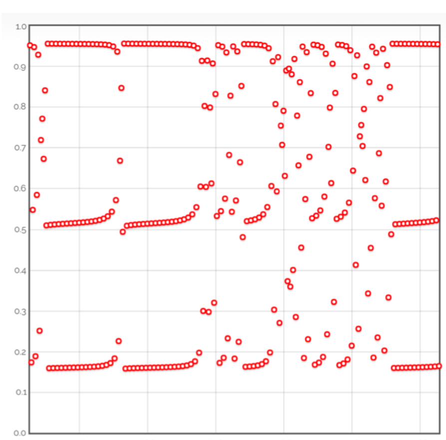
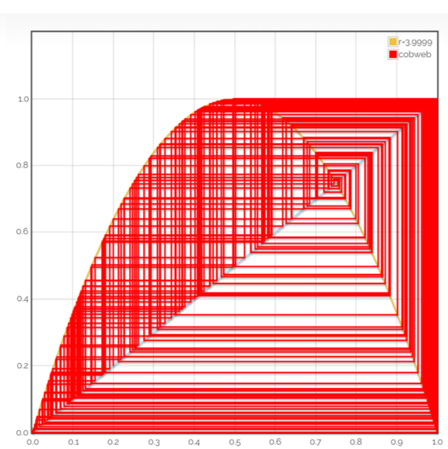
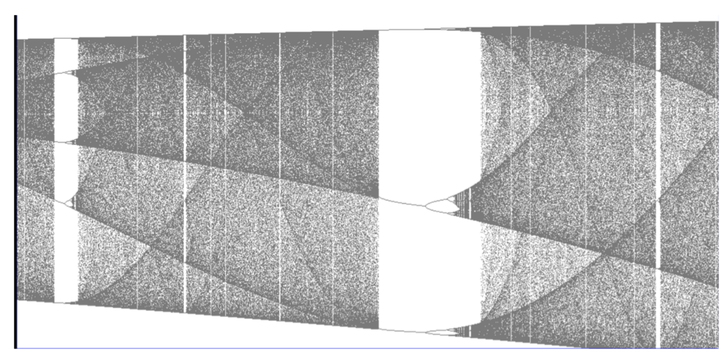






Two (almost) kinds of systems in the wild

- **discrete time systems:**
  - **"Operates in discrete time"**
  - **"maps"**
  - **modeling tool: difference equation**

- continuous time systems:
  - "Operates in continuous time"
  - "flows"
  - modeling tool: differential equations

What I mean by "operate in discrete time"

- Time proceeds in clicks
  - Dancing with a strobe light at a rave
  - A stock's ticker
  - Watching a movie with a frame rate of 60 fps

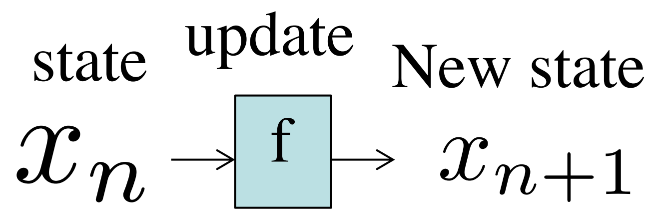
Important: Makes no sense to measure the state of the system between clicks. It simply does not exist

What do those beasts look like and how do we deal with them?

$$x_{n+1} = f(x_n)$$

Difference equations:

- e.g., $x_{n+1} = cos(x_n)$
- f is an "Update rule"



- given state x at time n, tells you state at time $n+1$
- solve by iterating

Lots of abstract stuff so let's make it concrete

A canonical difference equation:
"Logistic Map"

$$x_{n+1} = r x_n (1 - x_n)$$

- Simple Population Model
  - r is the "growth rate" parameter
  - x is the population state
- Simply compute next population = f(current population)

Logistic Map Example

$$x_{n+1} = rx_n(1 - x_n)$$

If $r = 1$ and $x_0 = 0.5$: "initial condition"

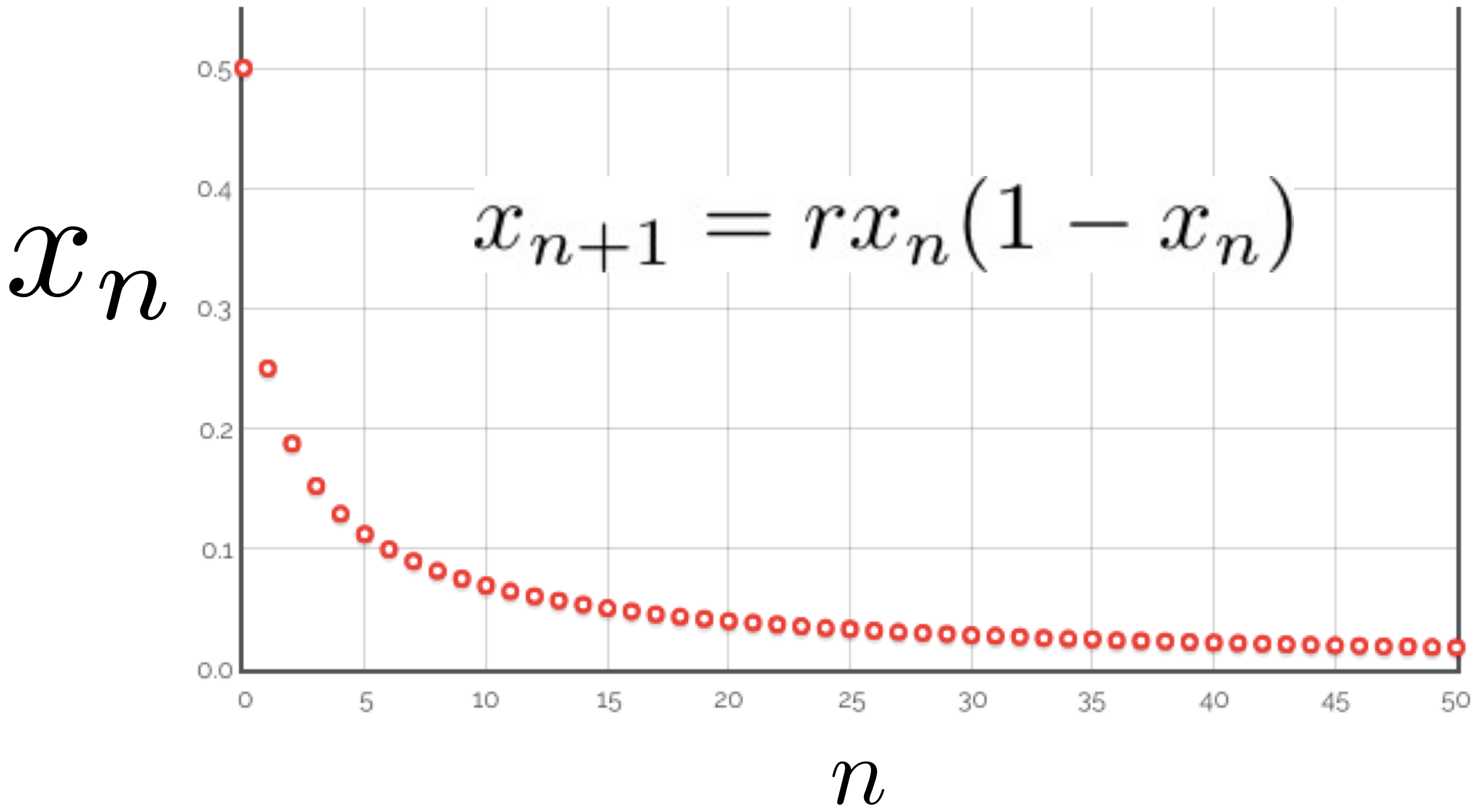
$$\begin{aligned} x_1 &= 1(0.5)(1-0.5) = 0.25 \\ x_2 &= 1(0.25)(1-0.25) = 0.1875 \\ &\dots \end{aligned}$$

Eventually, *x* settles down at *0*.

Population dies out… :(

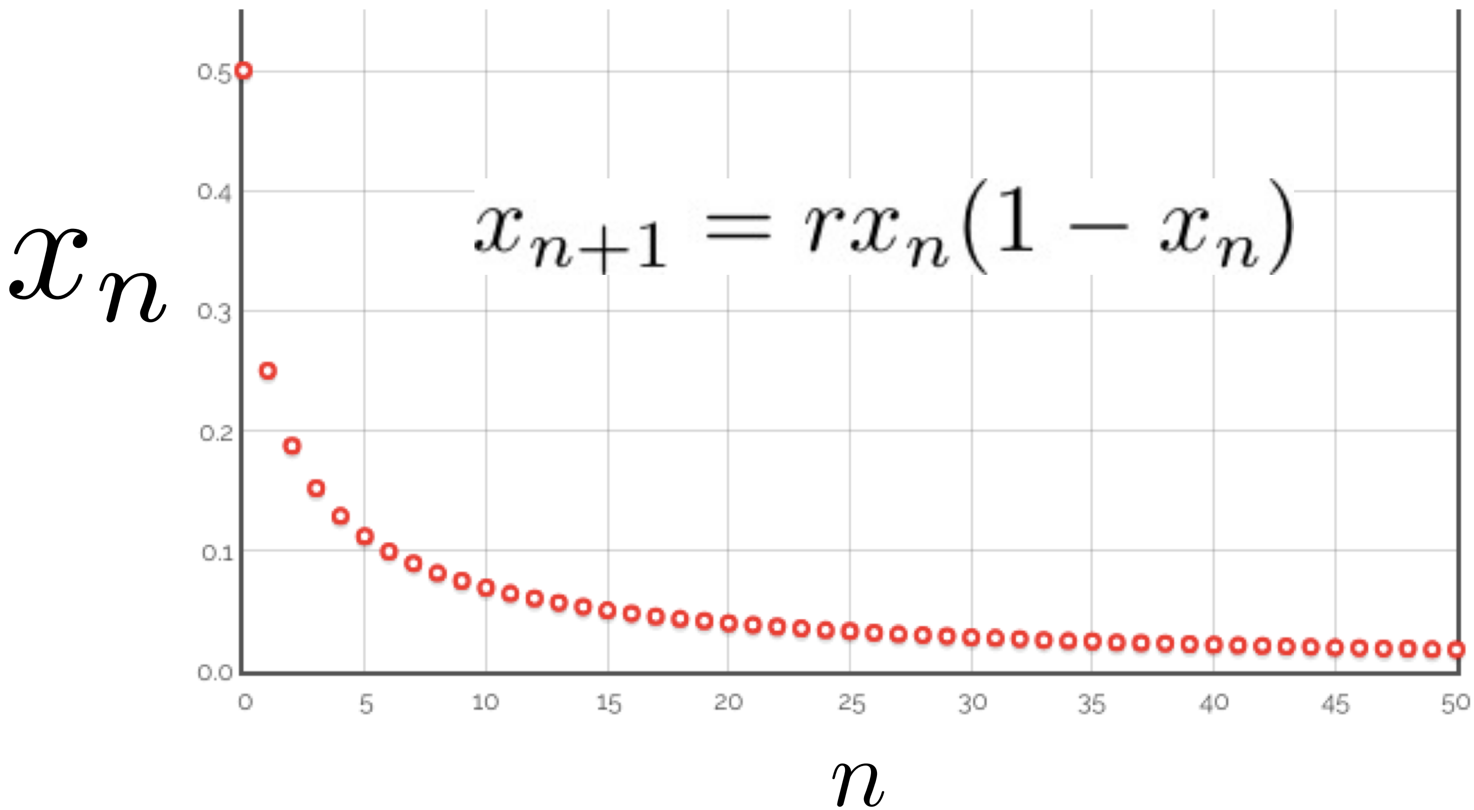
Brings us to our first graphical representation!

“Time Domain Plot” **or** “A Time Series”



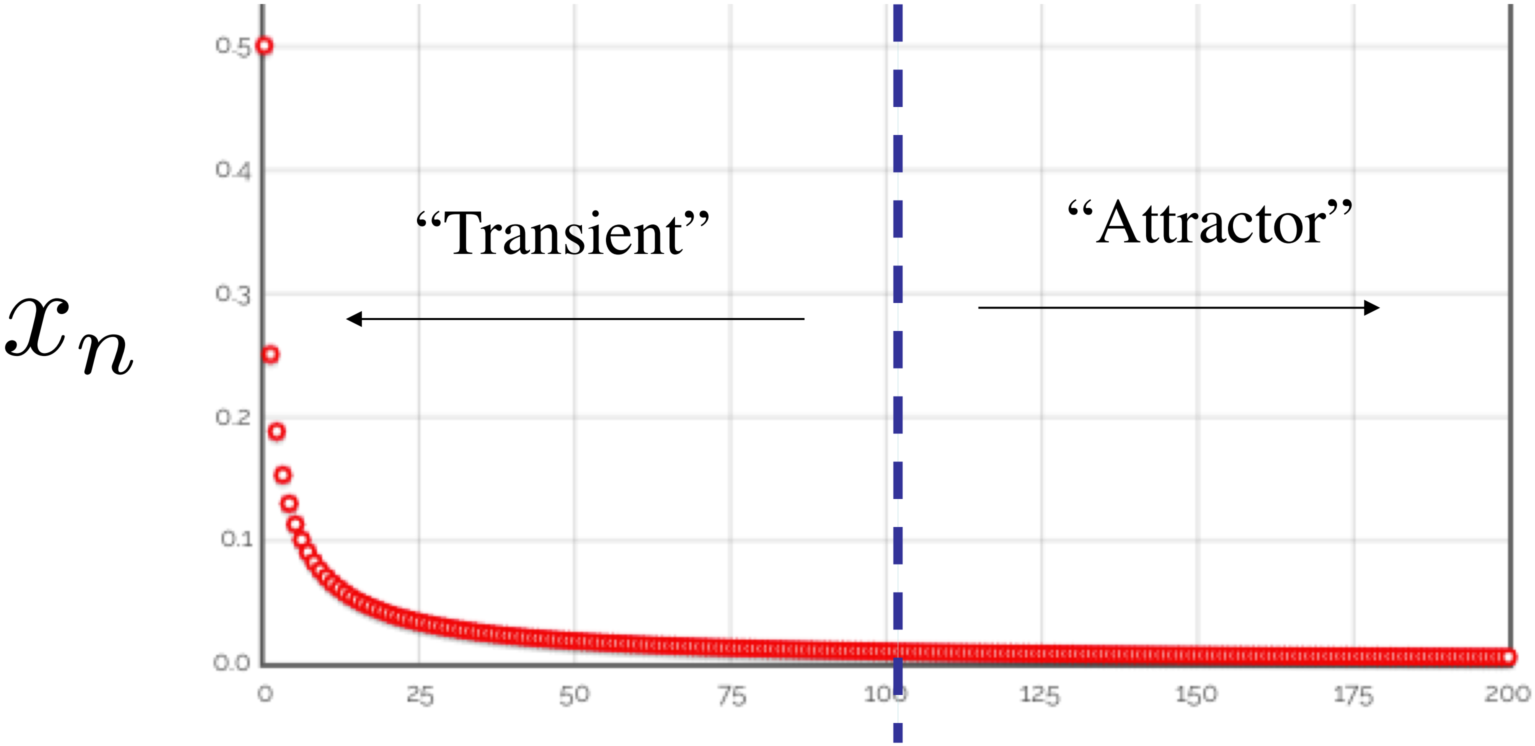
How do we describe this?

“***dynamics*** are ***converging*** to ***the stable fixed point ‘attractor’*** at 0”



“**Converging** to the stable fixed point”

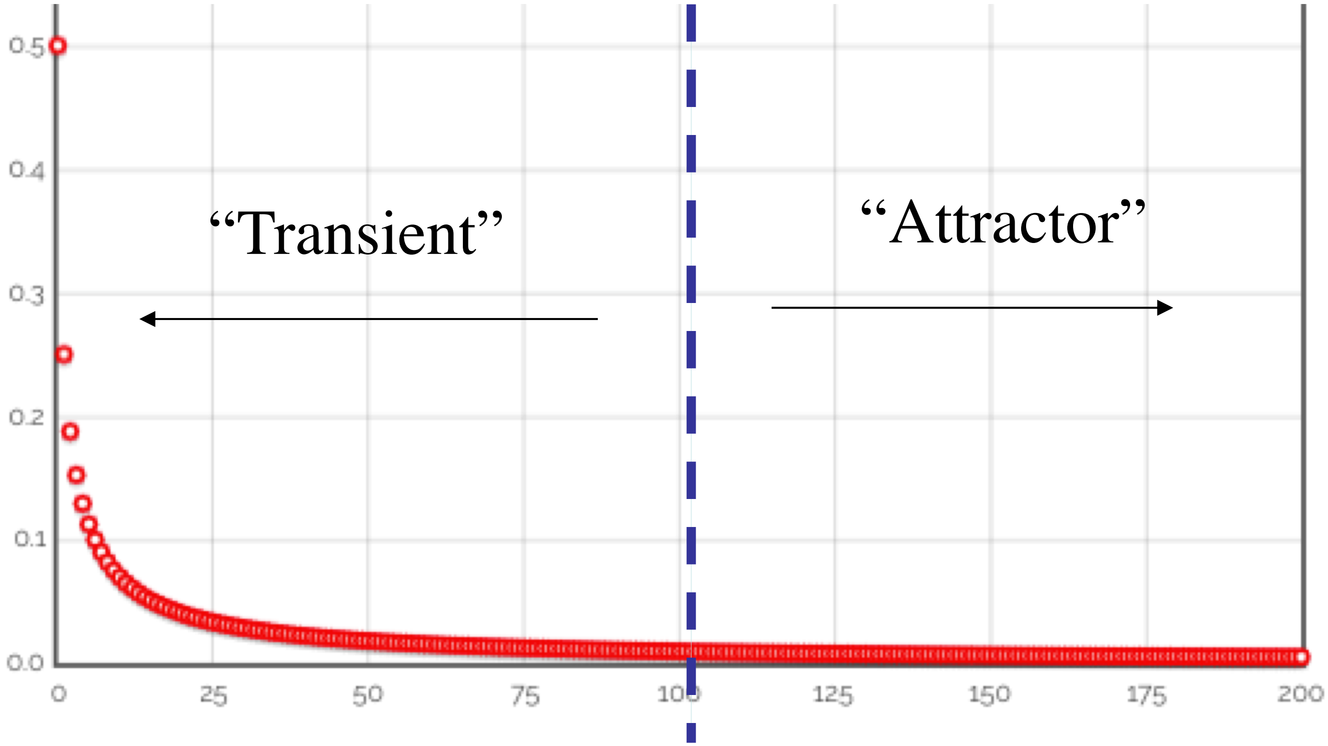
$$x_{n+1} = rx_n(1 - x_n)$$



“Converging to the stable **fixed point**”

$$x_{n+1} = r x_n (1 - x_n)$$

x_n

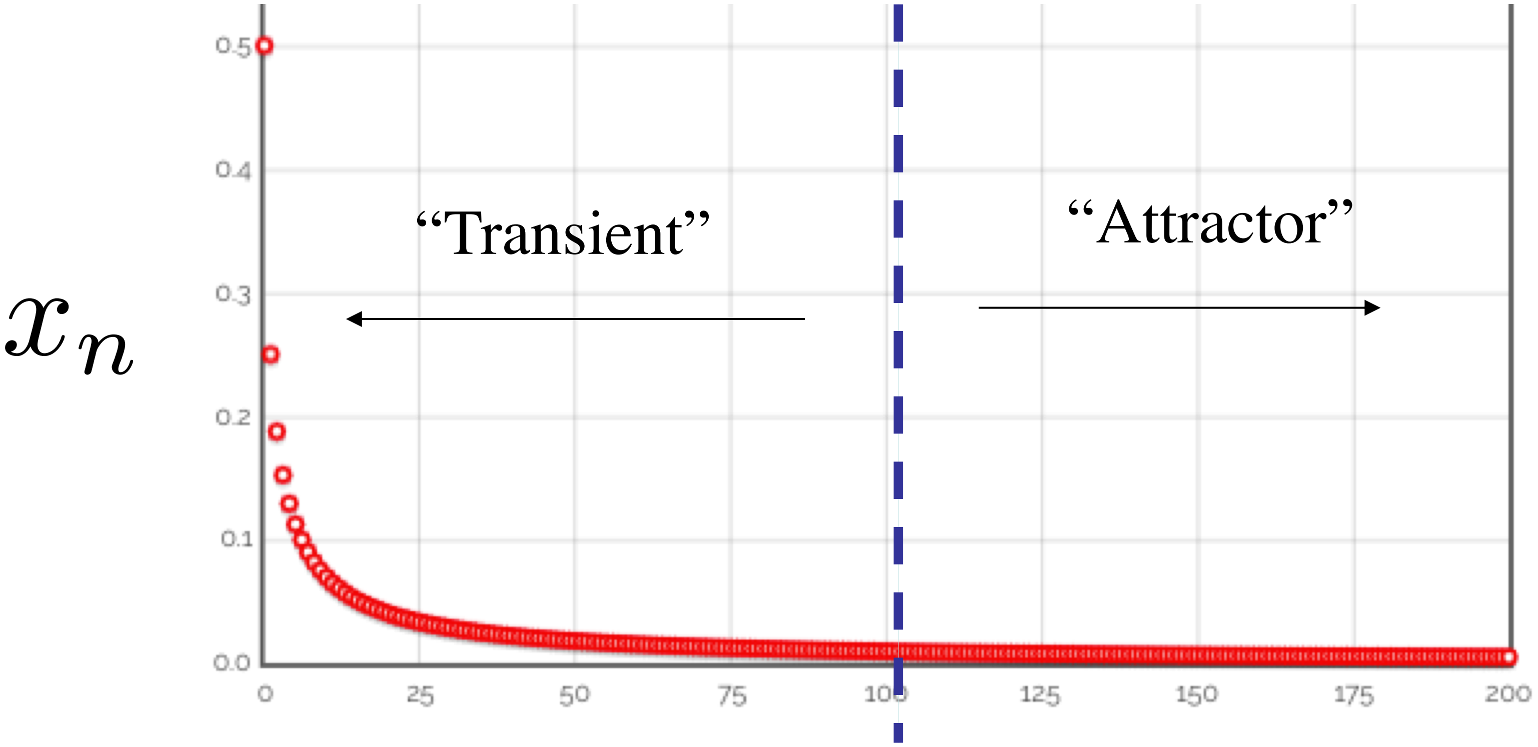


Formally, x^* is a fixed point if $x^* = f(x^*)$.

$$0 = f(0)$$

“Converging to the stable **fixed point**”

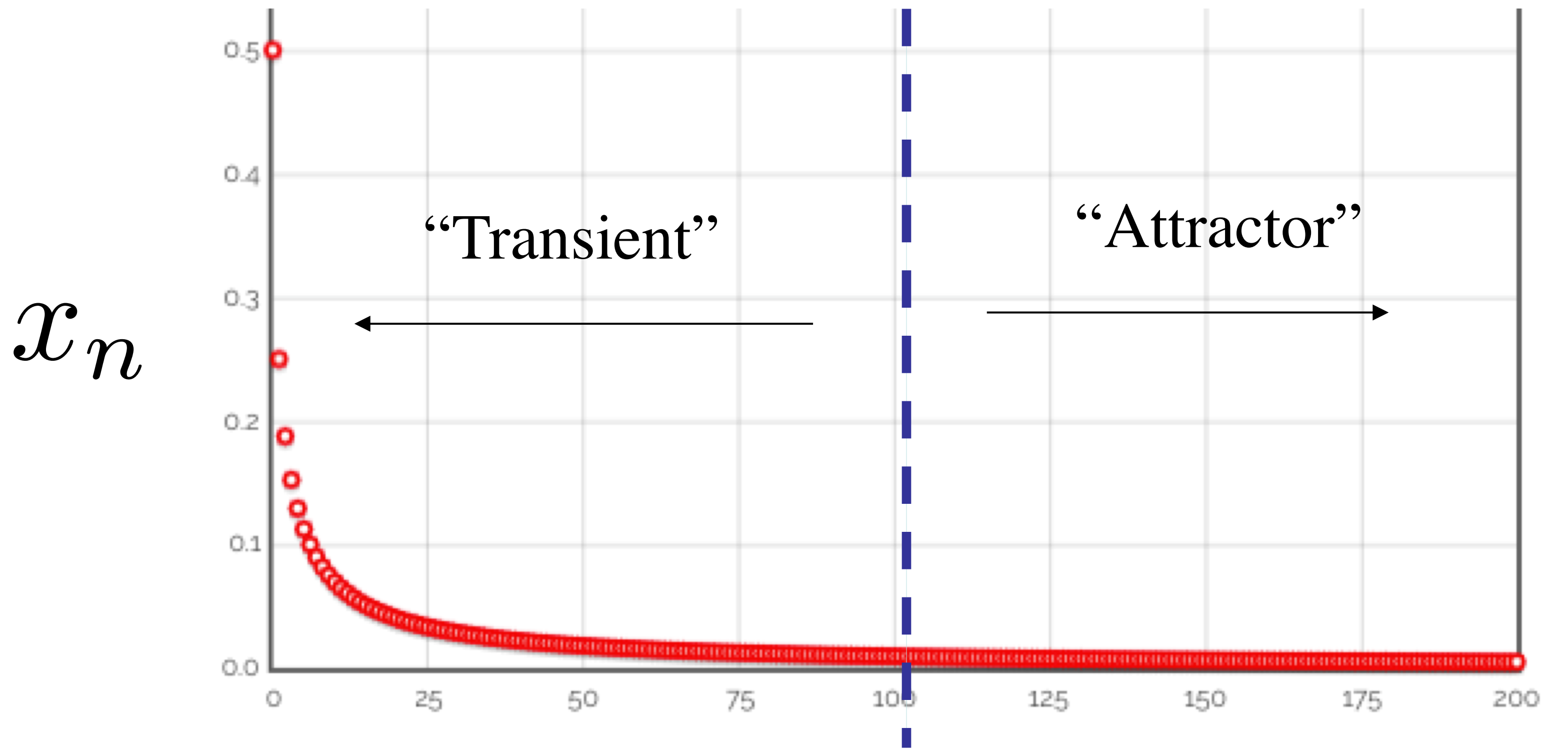
$$x_{n+1} = rx_n(1 - x_n)$$



Thought Experiment: What was the fixed point in the double pendulum?

"Converging to **the** stable fixed point"

$$x_{n+1} = r x_n (1 - x_n)$$



Logistic map only has one **attractor** at a time.

Stable/Unstable

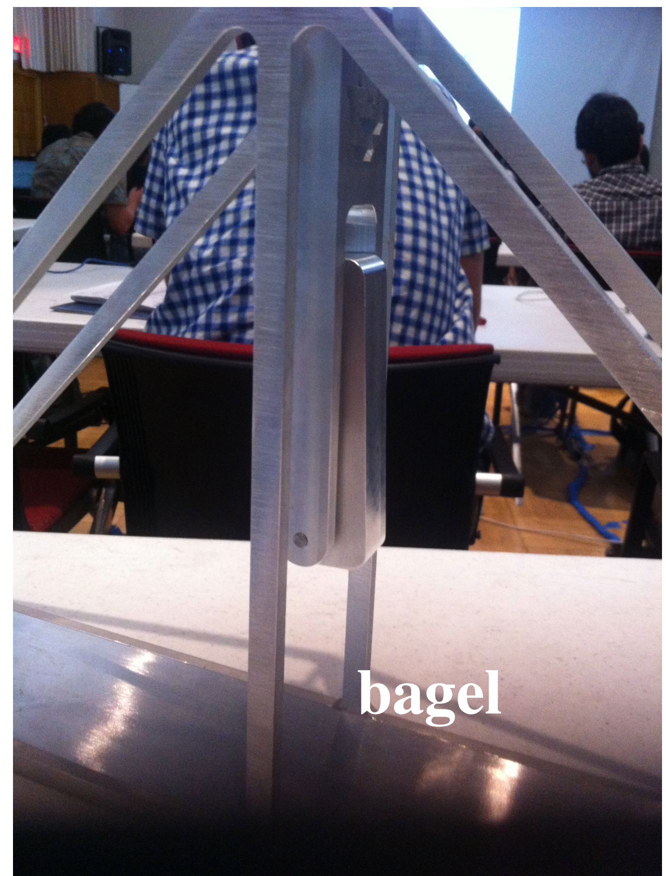
<u>Stable</u> or <u>attracting</u> fixed point:

local perturbations around x* shrink

<u>Unstable</u> or <u>repelling</u> fixed point

local perturbations around x* grow

teacup



bowl

Other Fixed Points?

$$x_{n+1} = rx_n(1 - x_n)$$

How could we find them?

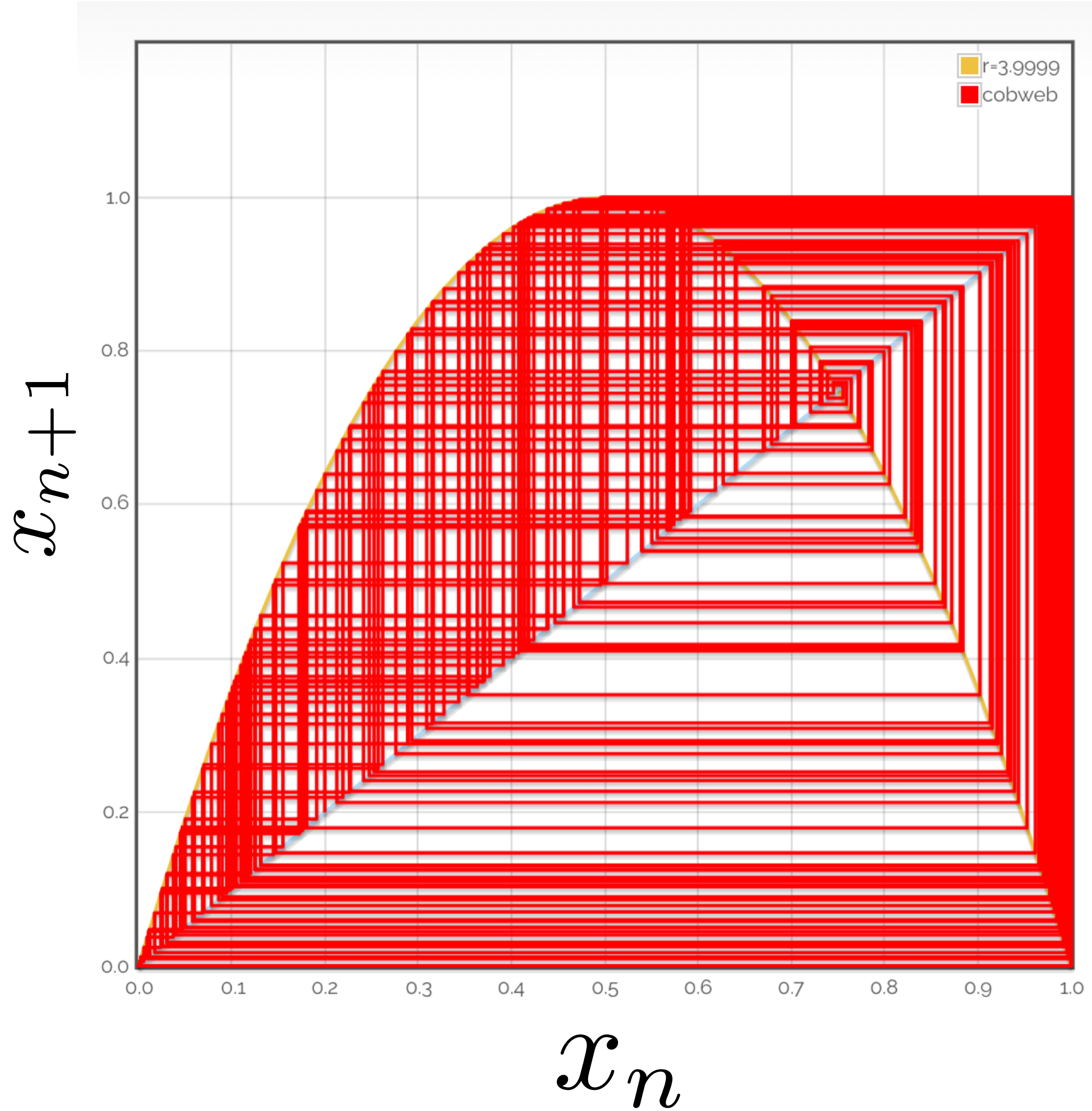
Other Fixed Points?

$$x_{n+1} = r x_n (1 - x_n)$$

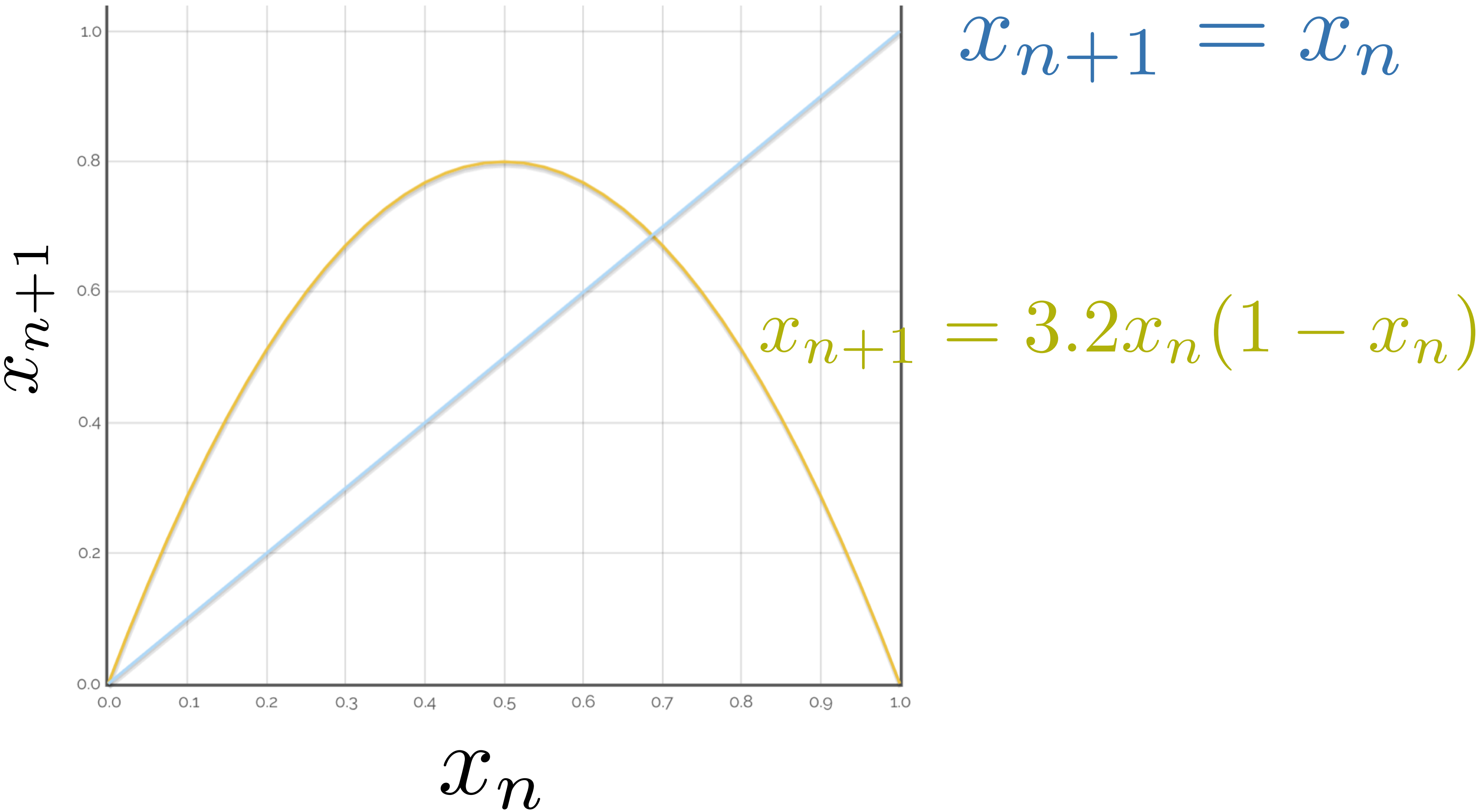
Do they depend on *r*?

A useful graphical solution technique

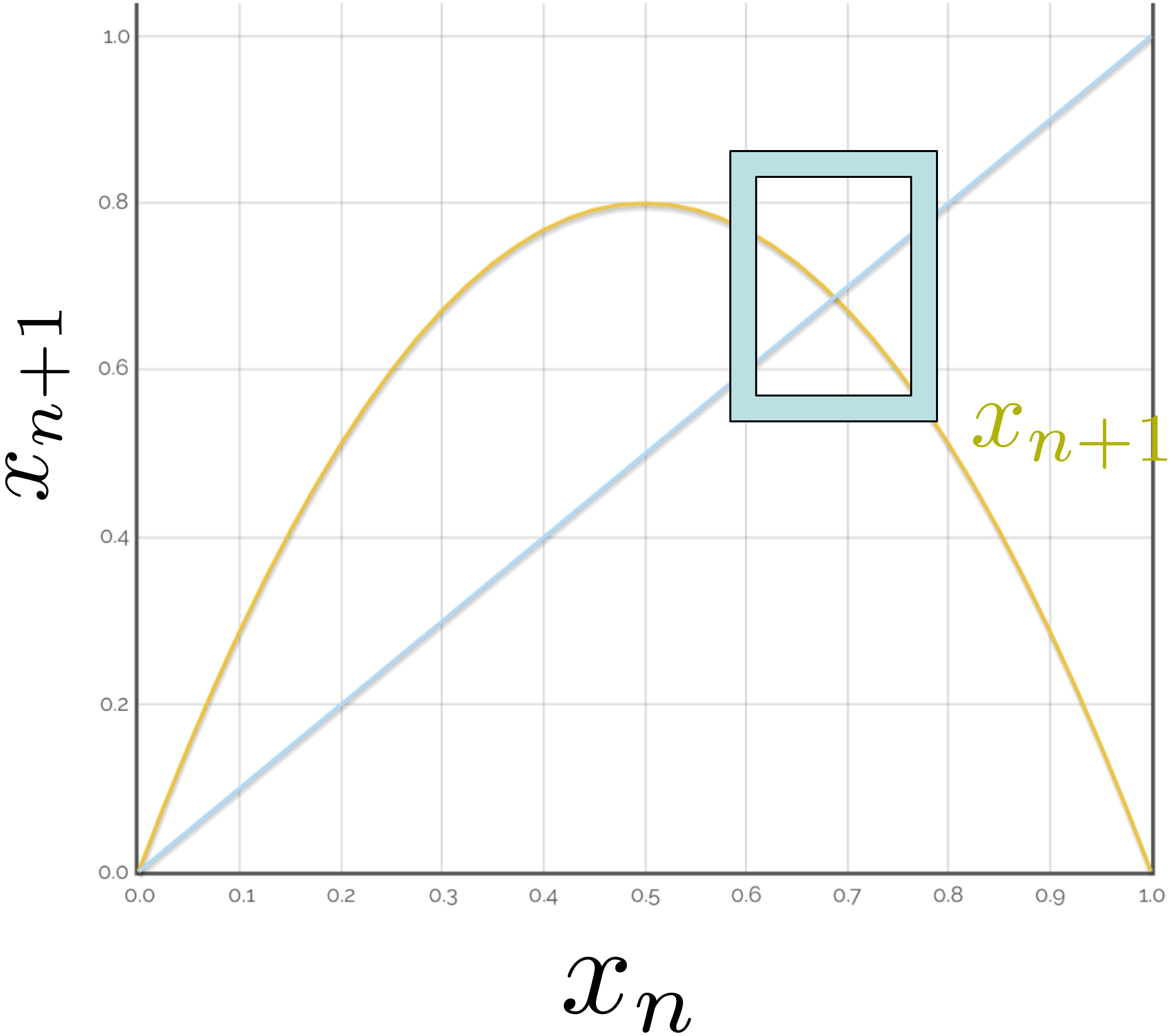
- "cobweb" diagram
- *aka* return map
- *aka* correlation plot



Cobweb Diagrams



Cobweb Diagrams

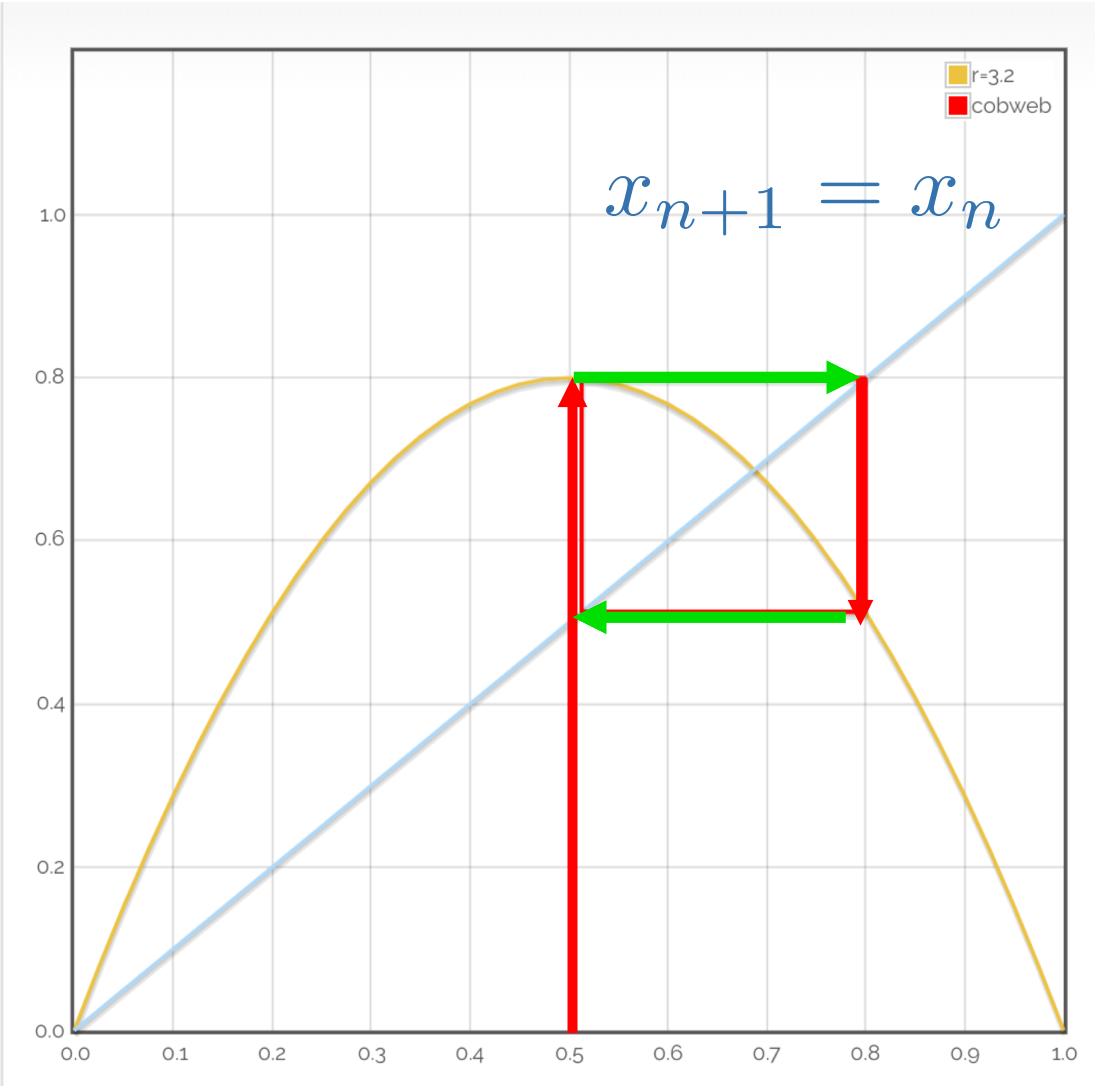


$$x_{n+1} = x_n$$

$$x_{n+1} = 3.2x_n(1 - x_n)$$

Significance?

Cobweb Diagrams

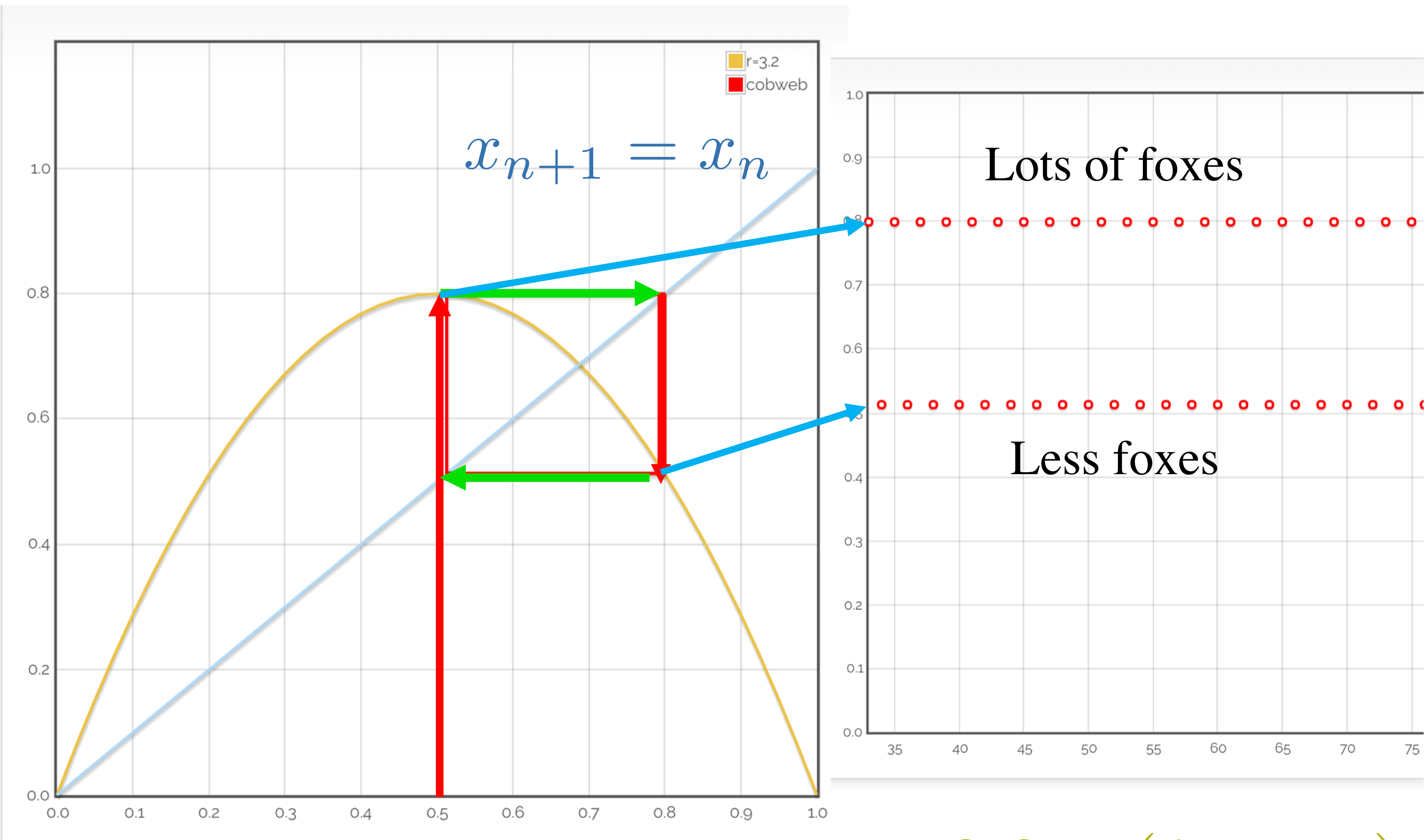


Evaluate f

set $x_n = x_{n+1}$

$$x_{n+1} = 3.2x_n(1 - x_n)$$

Back to the time domain



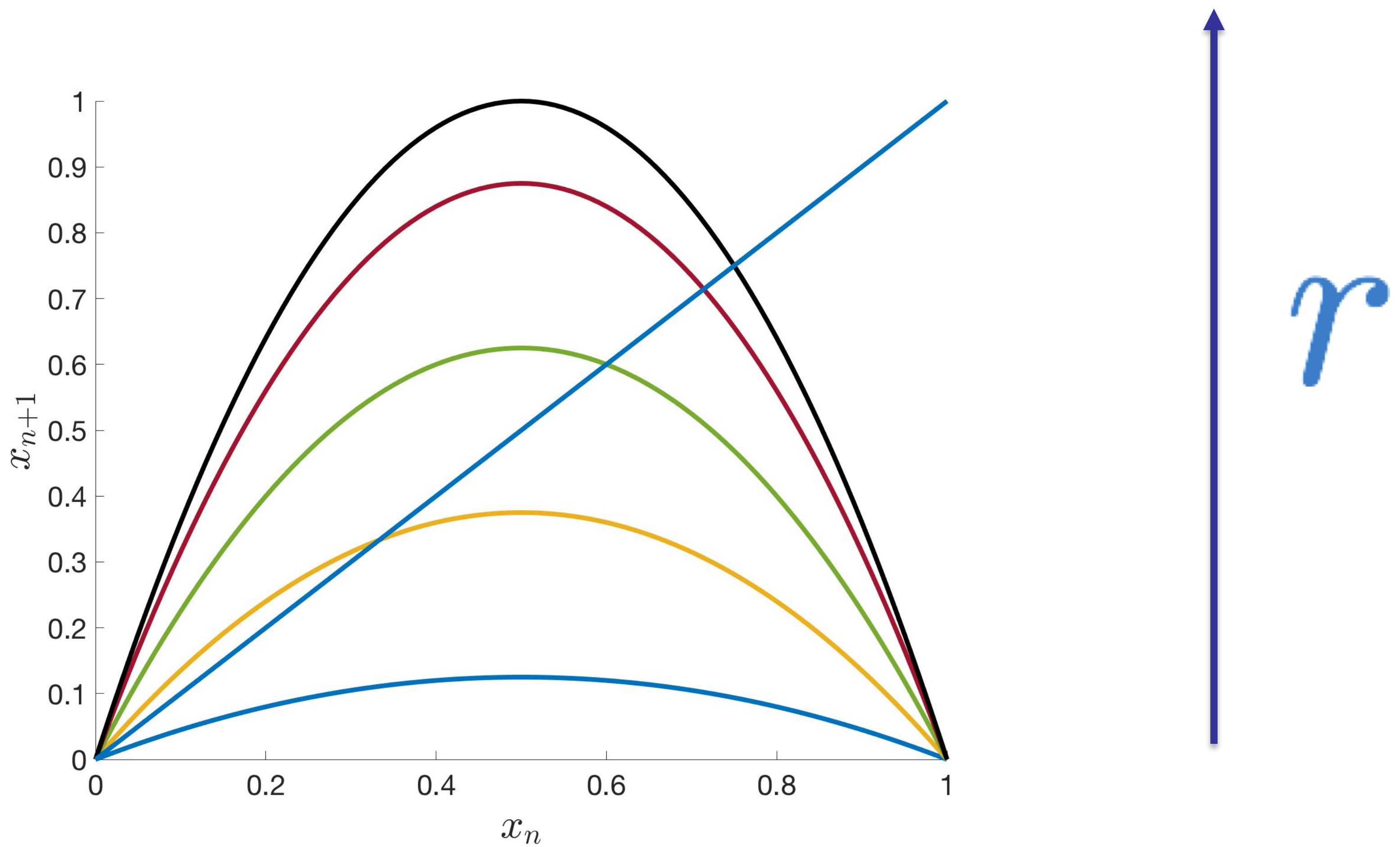
$$x_{n+1} = 3.2x_n(1 - x_n)$$

What happens if we crank up r?

"CRANK UP ZE POWER!"

What happens if we crank up r?

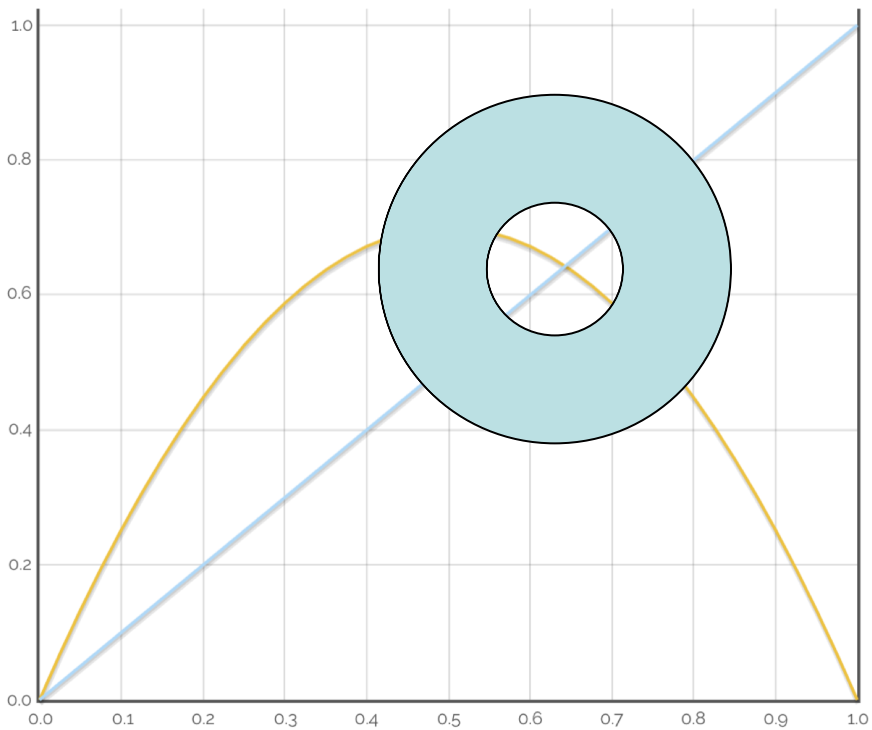
The logistic map: $x_{n+1} = rx_n(1 - x_n)$



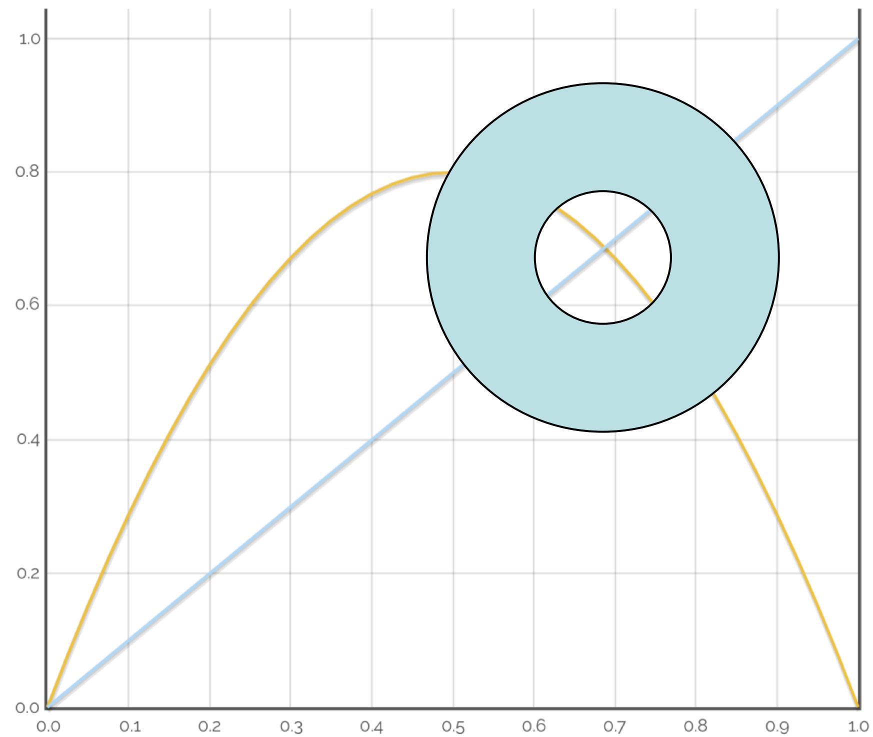
Still a fixed point?

The logistic map: $x_{n+1} = rx_n(1 - x_n)$

$r = 2.8$



$r = 3.2$

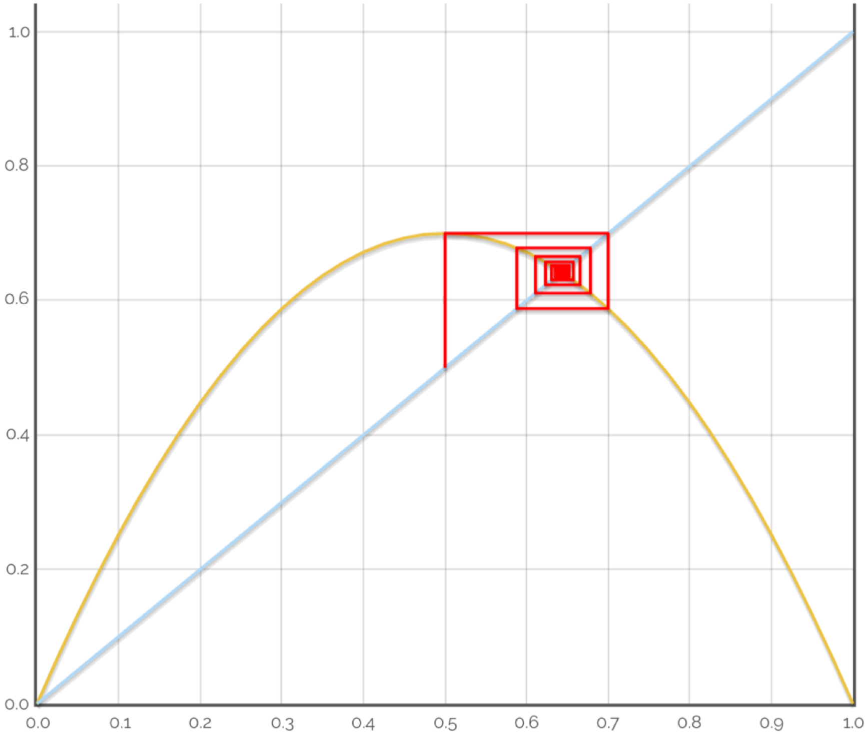


We can verify this with a cobweb!

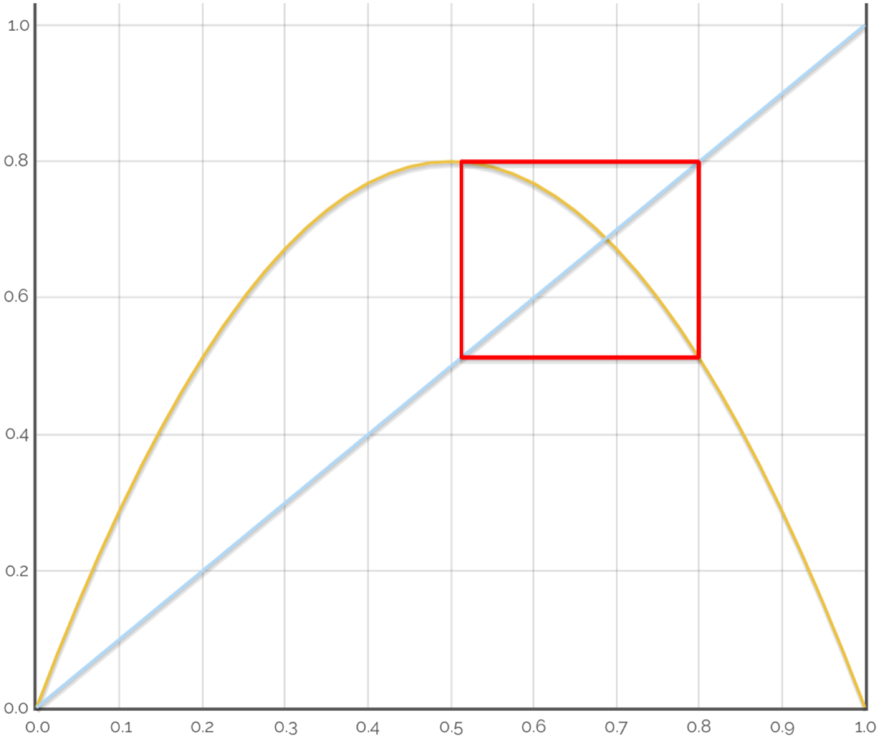
Still a fixed point? *What the hell…*

The logistic map: $$x_{n+1} = r x_n (1 - x_n)$$

$r = 2.8$



$r = 3.2$



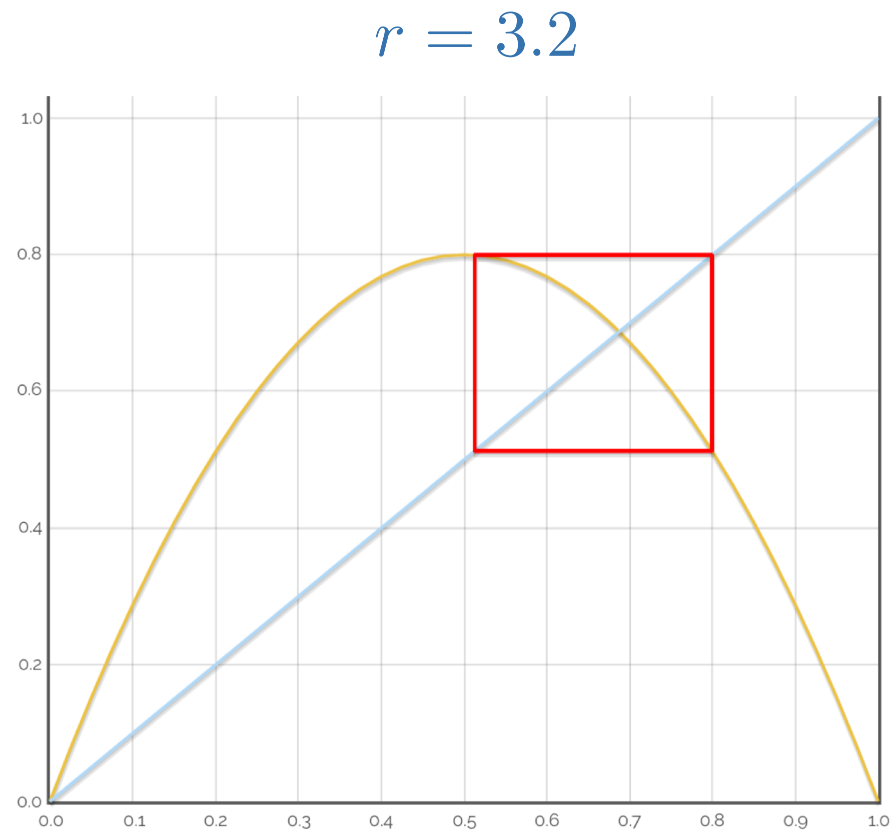
The fixed point has destabilized…

The logistic map: $x_{n+1} = rx_n(1 - x_n)$

Changing a **parameter** can **qualitatively** change the dynamics.

This is called a “Bifurcation”

As r increased the stable FP vanished and was replaced by something called a stable “2-cycle”
Or a “period 2 orbit”



Bifurcations

Qualitative changes in the dynamics caused by changes in *parameters:*

- Heart: Vfib…Parameters change and you die.
- Eddy in creek: water level
- Brain: coke makes you periodic
- Logistic map: R parameter…

Let’s play name those dynamics!

Go to App….

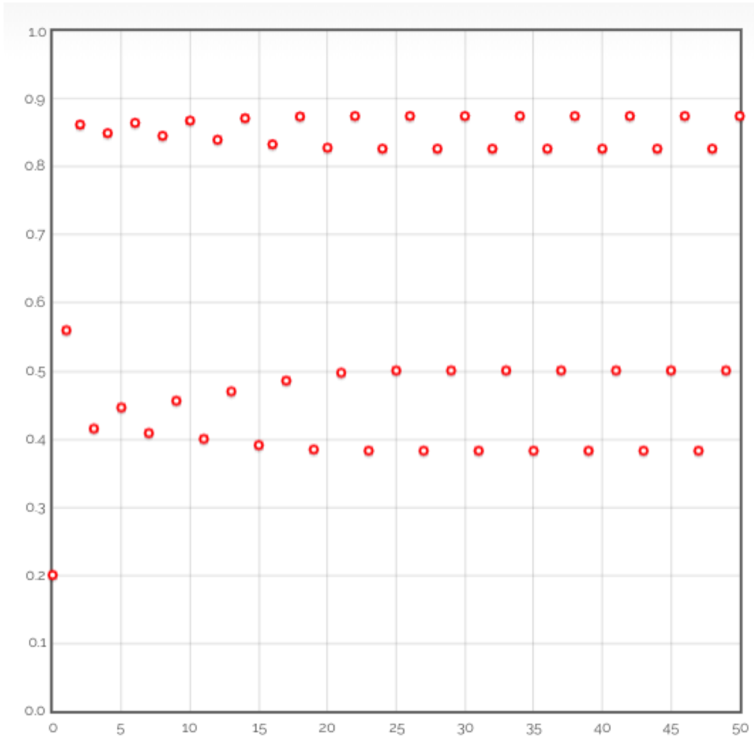
2.8 -> 3.1 -> 3.2 -> 3.5 -> 3.55 ->3.6->3.83->3.8281

How can we get our heads around all the available dynamics?

Bifurcation Diagram

$r = 3.5$

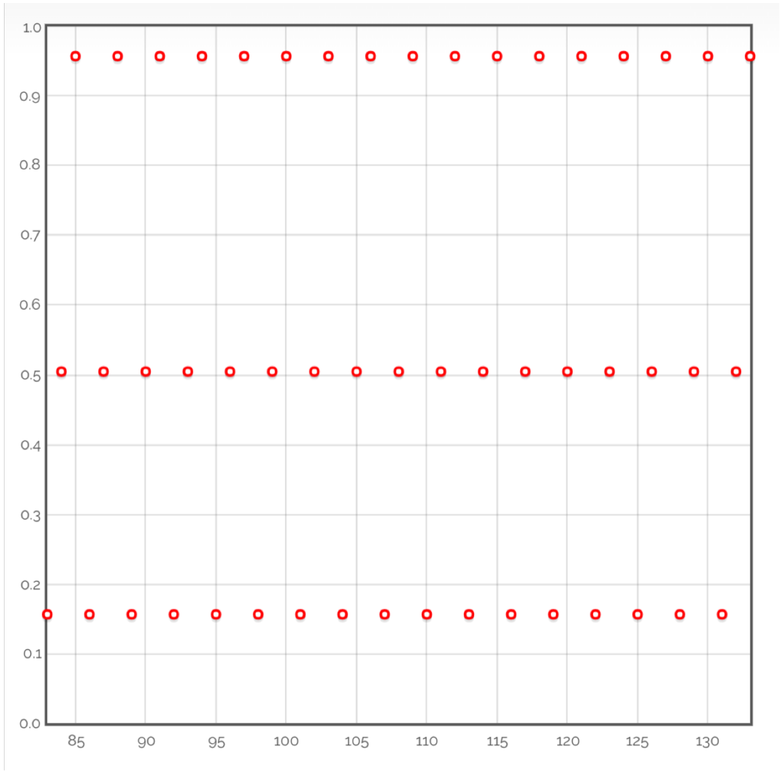
x_n



n

Bifurcation Diagram

x_n

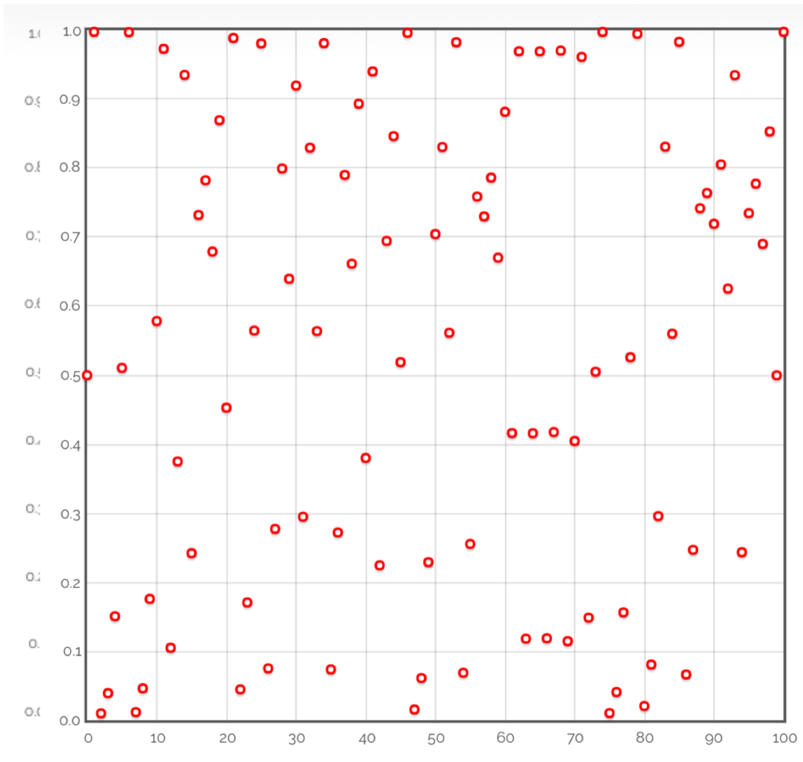


n

$r = 3.83$

Bifurcation Diagram

x_n



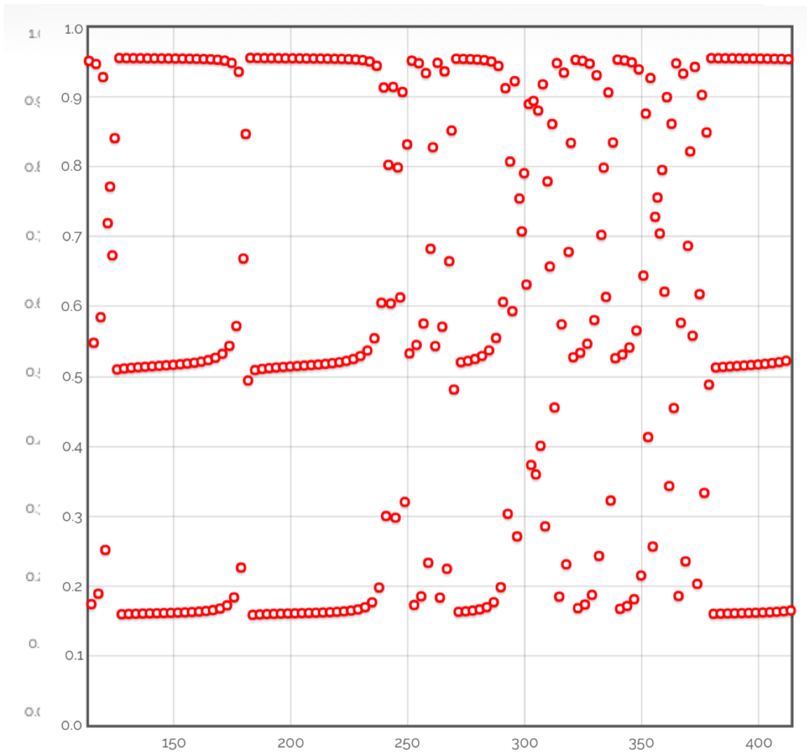
n

$$r = 3.99$$

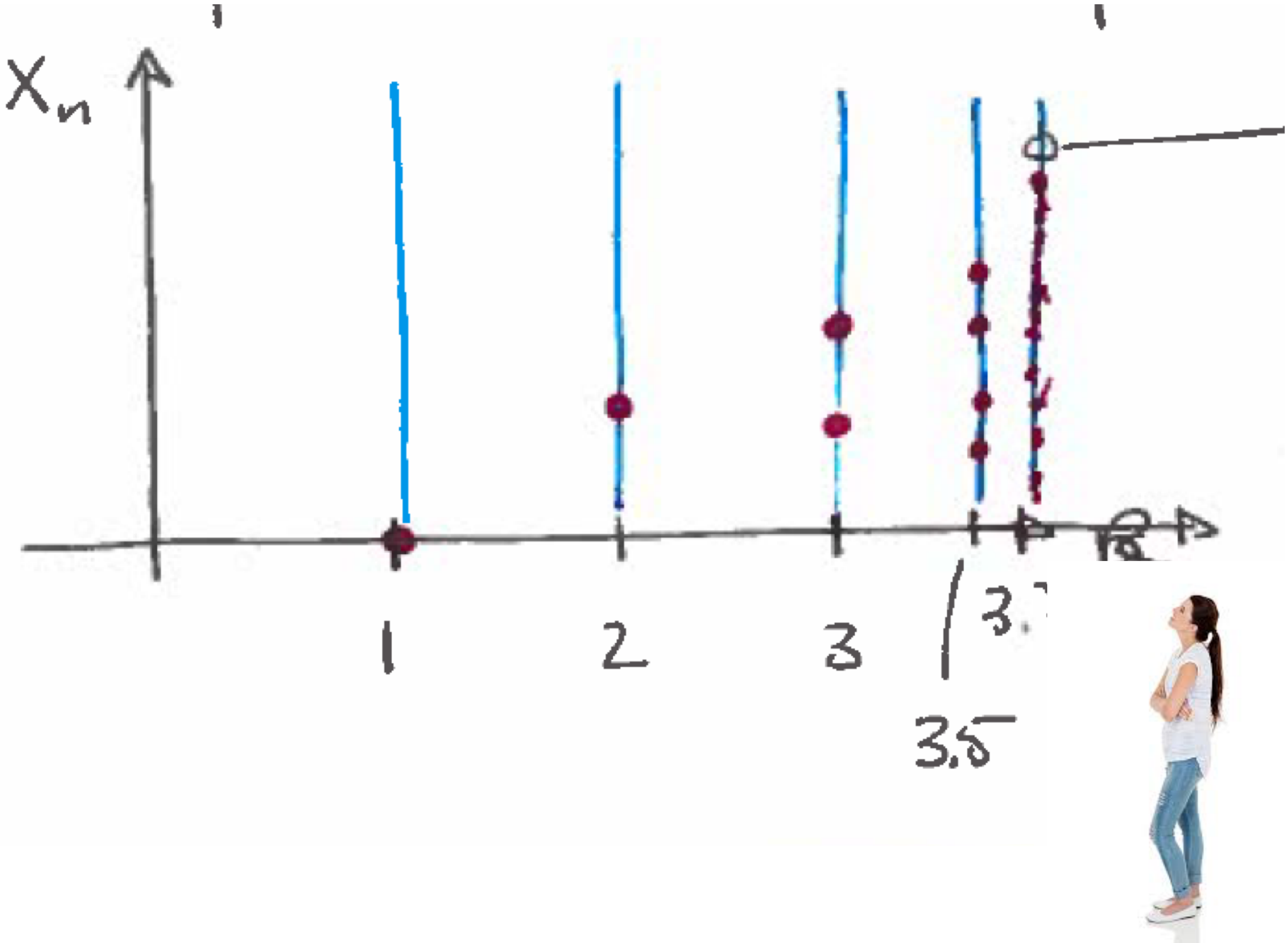
Bifurcation Diagram

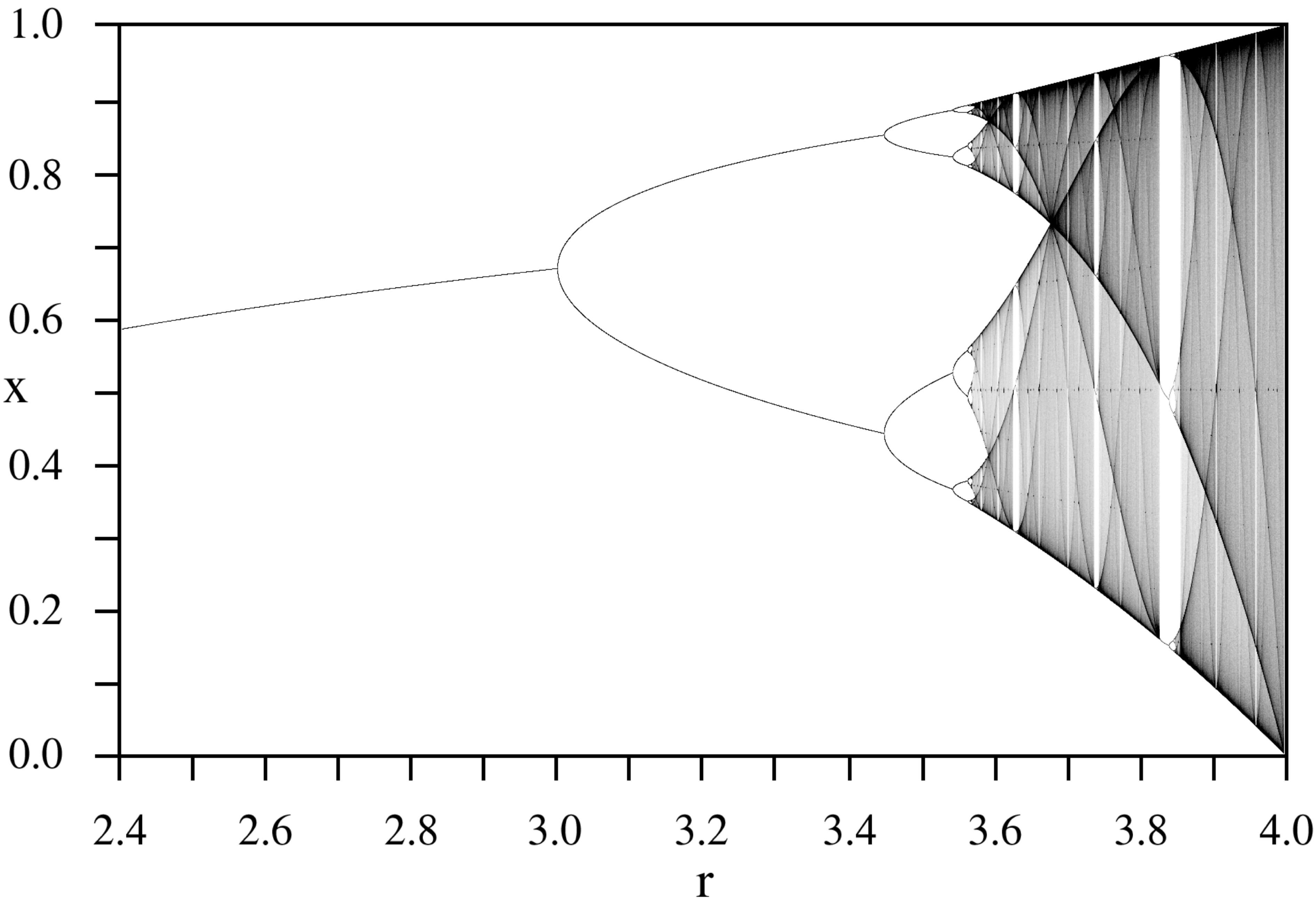
$$r = 3.8281$$

x_n



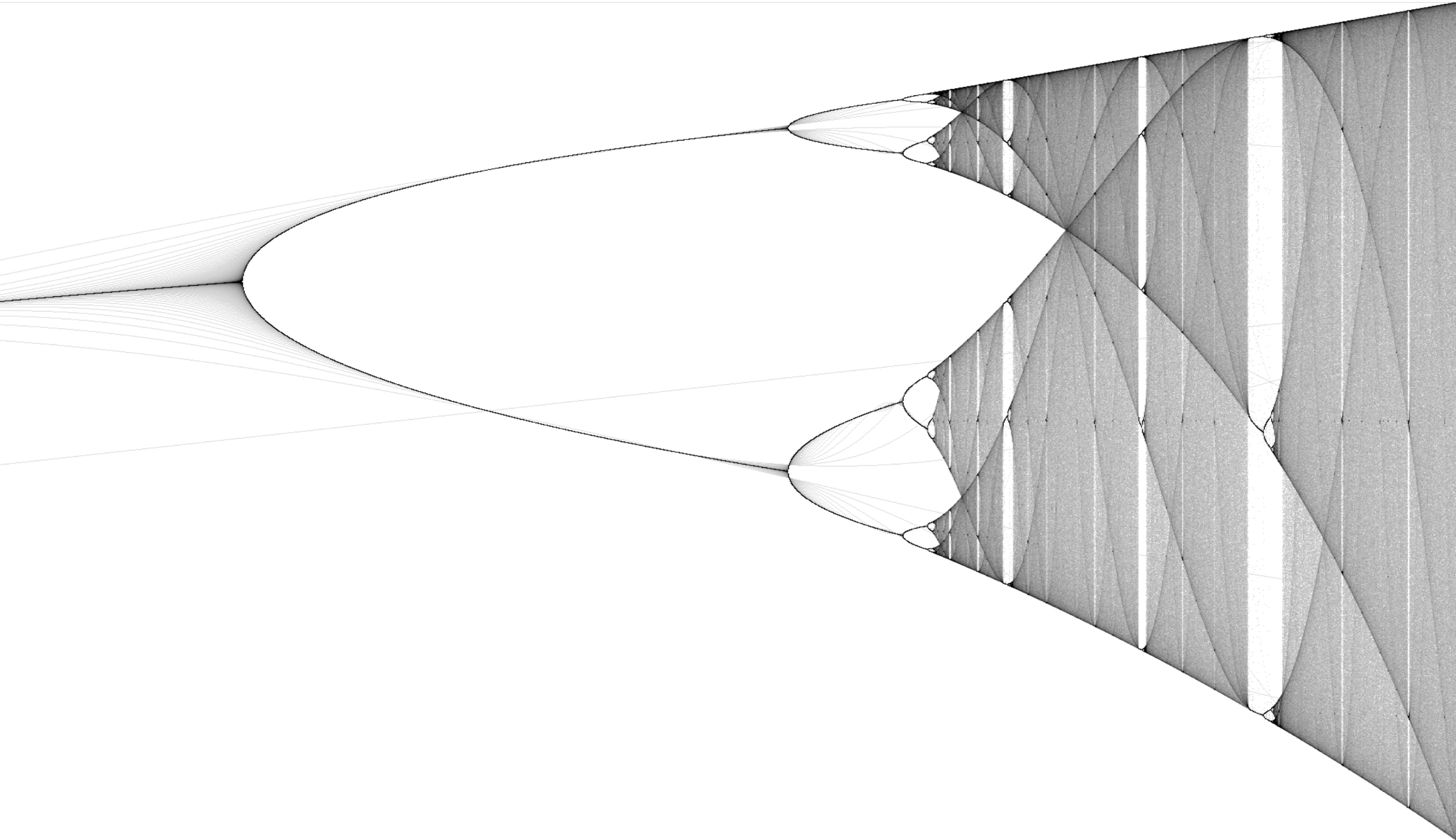
n

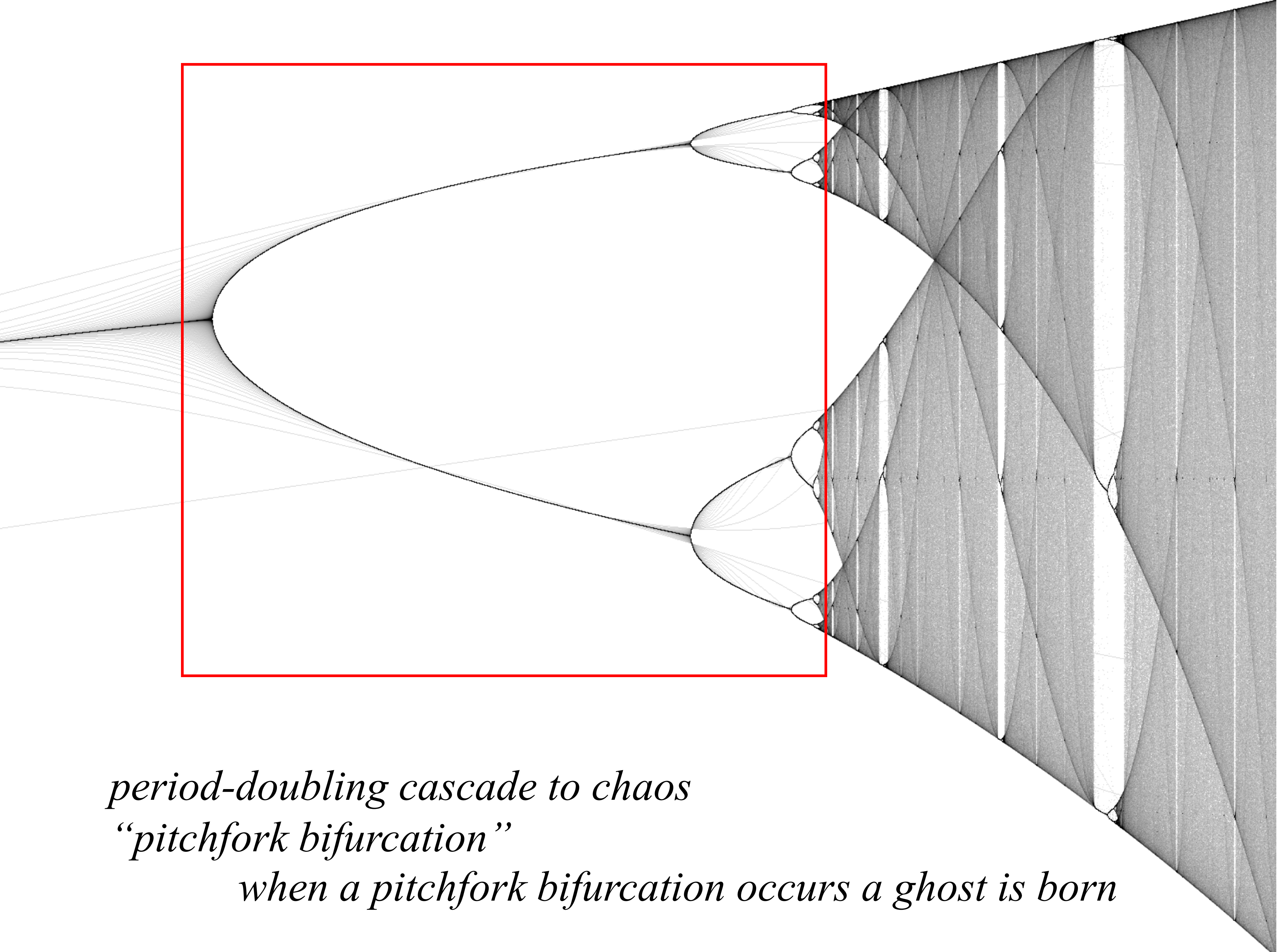

X_n
1
2
3
3.5
R


1.0
0.8
0.6
x
0.4
0.2
0.0
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
r

Things to note

- Chaotic bands



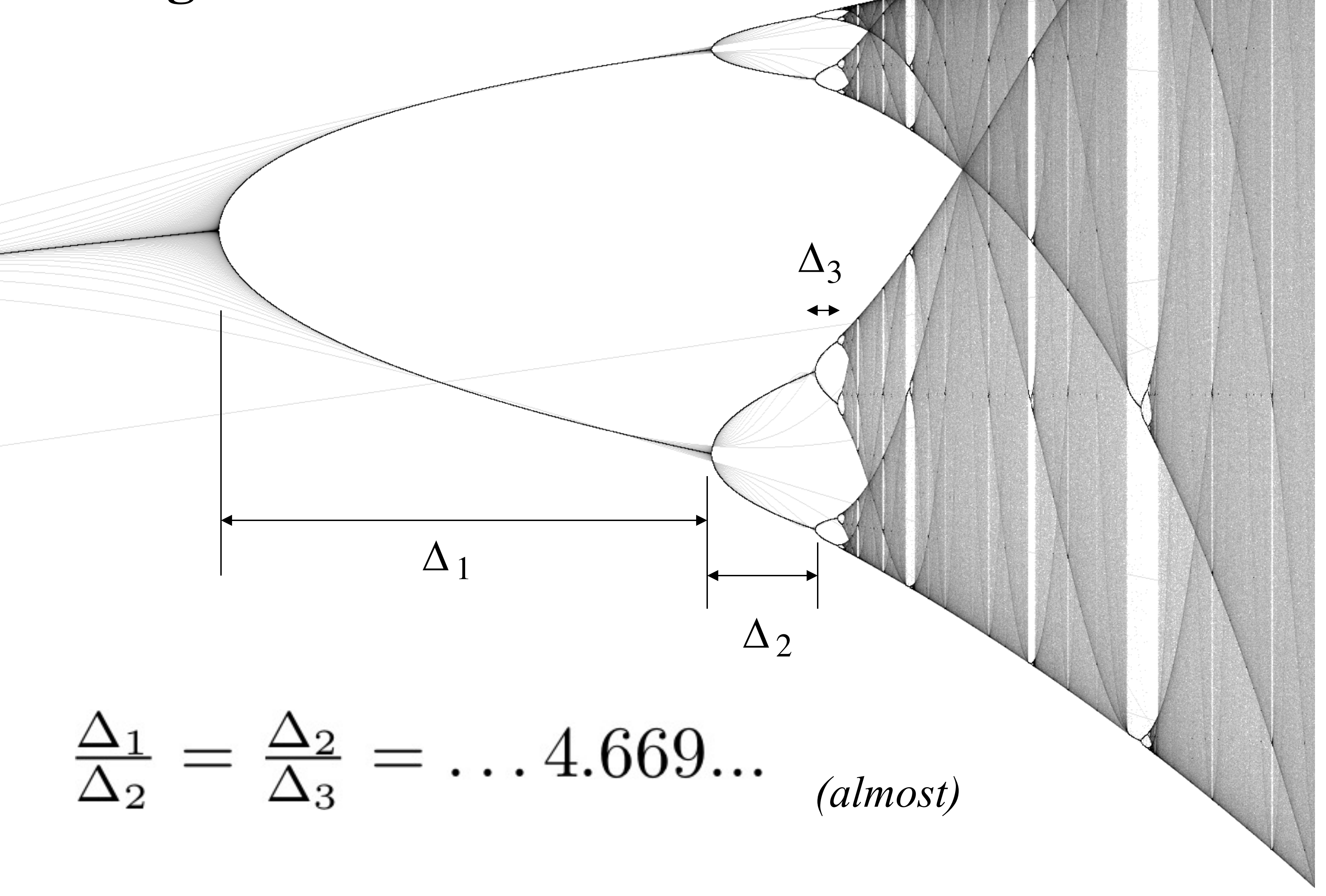


period-doubling cascade to chaos

“pitchfork bifurcation”

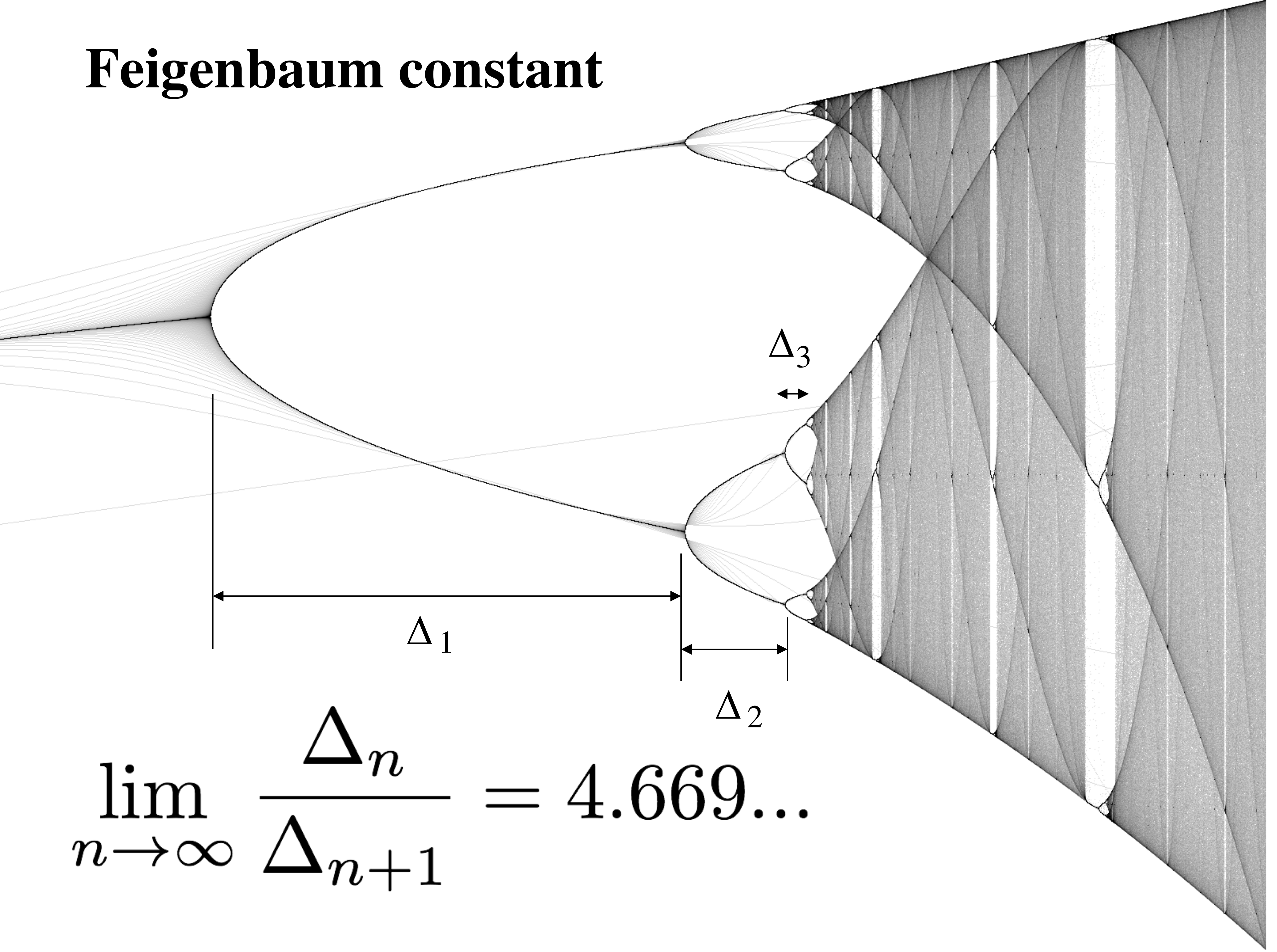
when a pitchfork bifurcation occurs a ghost is born

Feigenbaum constant



$$\frac{\Delta_1}{\Delta_2} = \frac{\Delta_2}{\Delta_3} = \ldots 4.669\ldots$$ *(almost)*

Feigenbaum constant



$$\lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+1}} = 4.669...$$

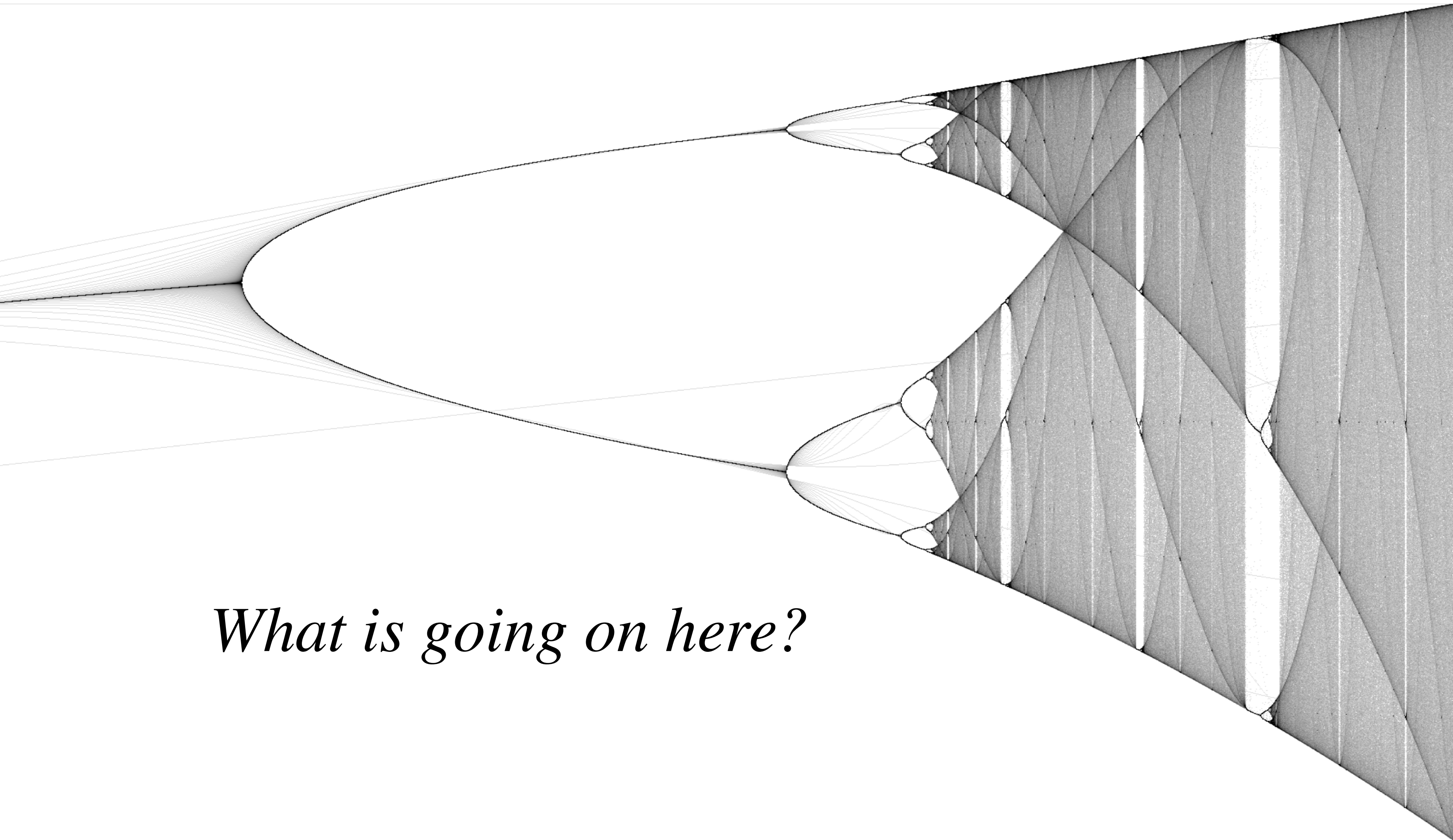
Universality!

Feigenbaum constant (and many other interesting chaotic/dynamical properties) hold *for any 1D map with a quadratic maximum*.

The ***Feigenbaum constant*** is a new physical constant as fundamental to 1D maps as π is to circles---Strogatz

Things to note

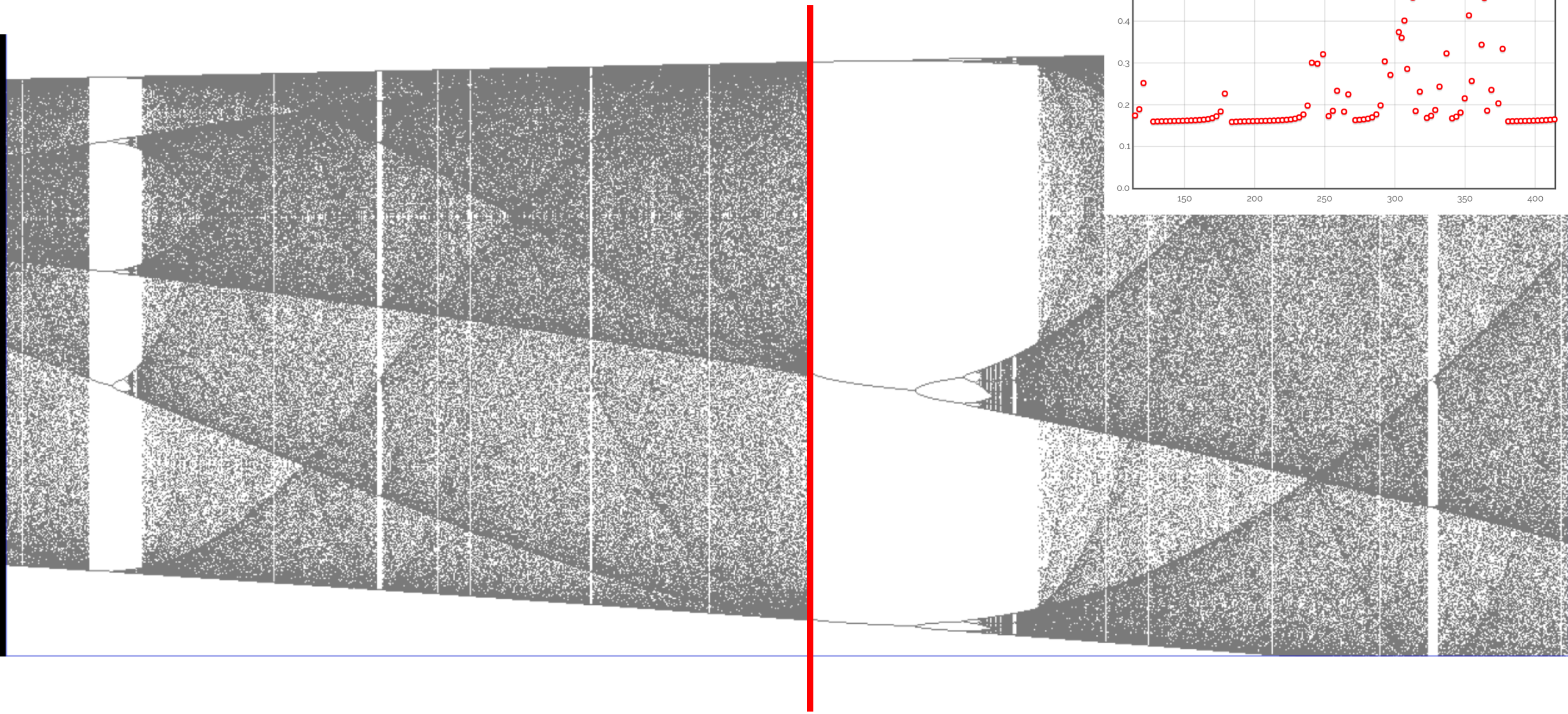
• veils

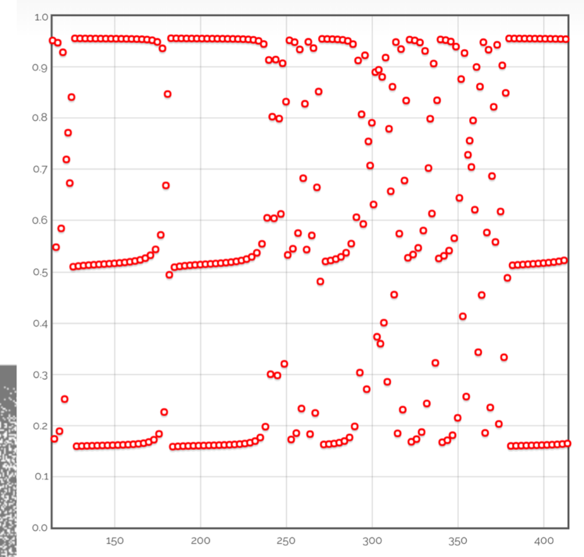


What is going on here?

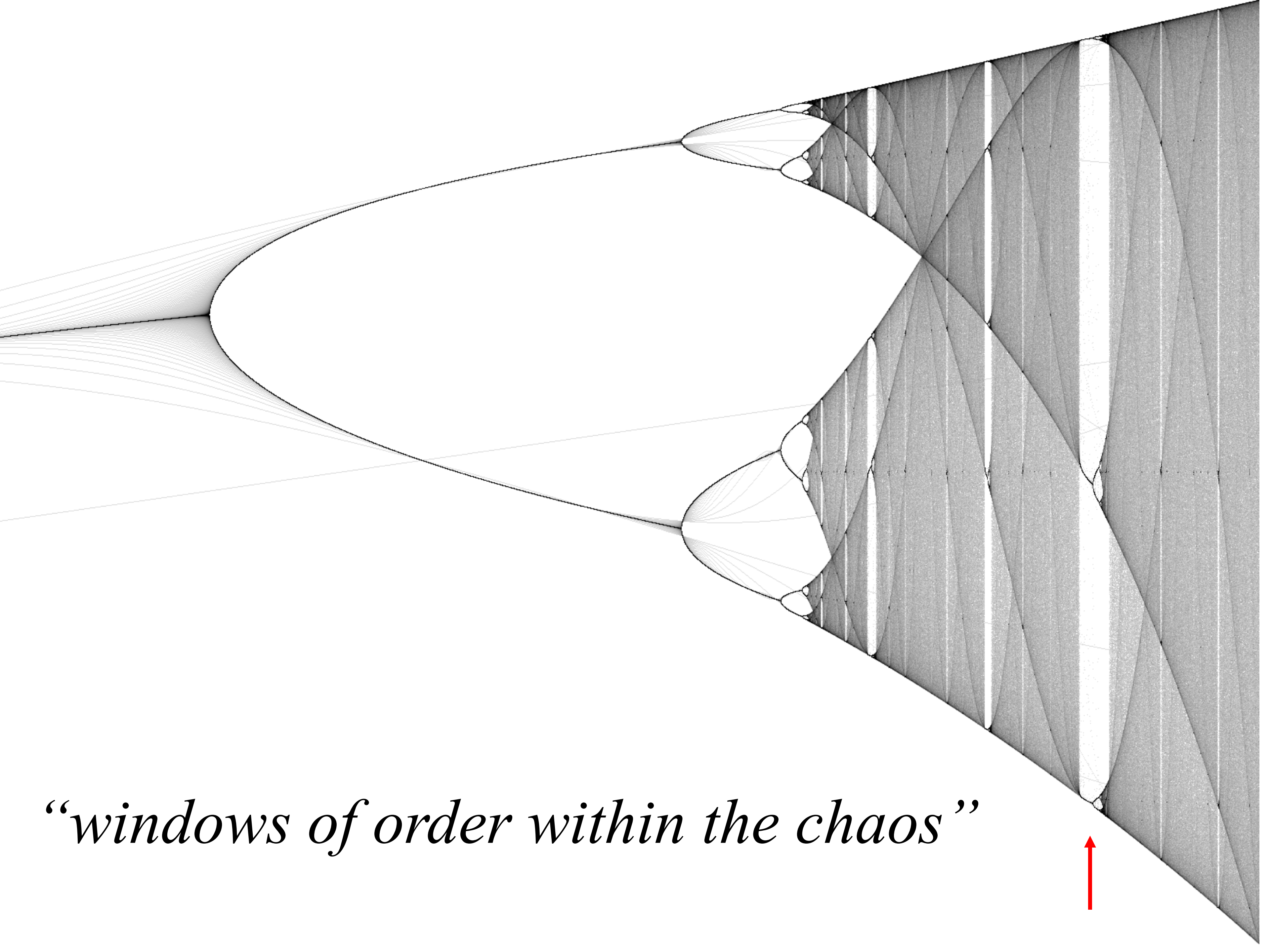
Things to note

- veils: teacups, doughnuts and loneliness





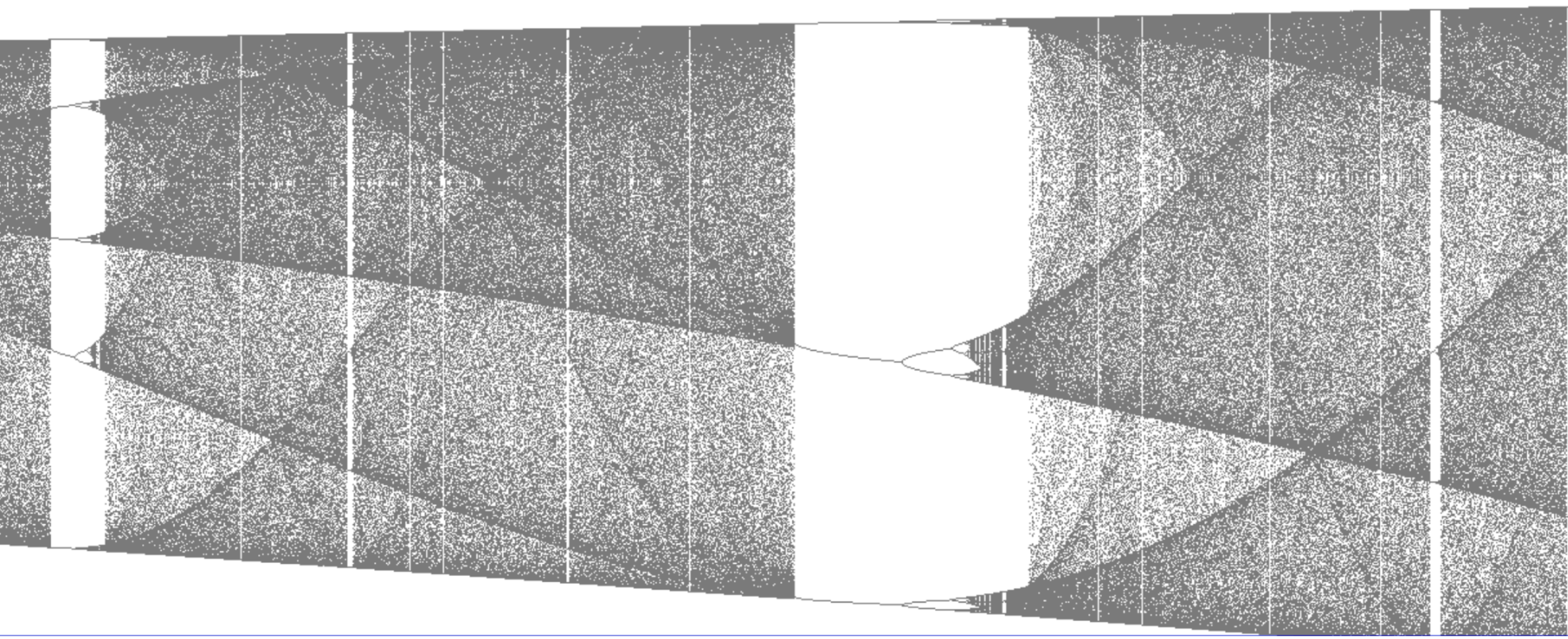
Ghosts and the loneliness of chaos



"windows of order within the chaos"

Things to note

•Regions of order w/in chaotic region, this massive 3 cycle

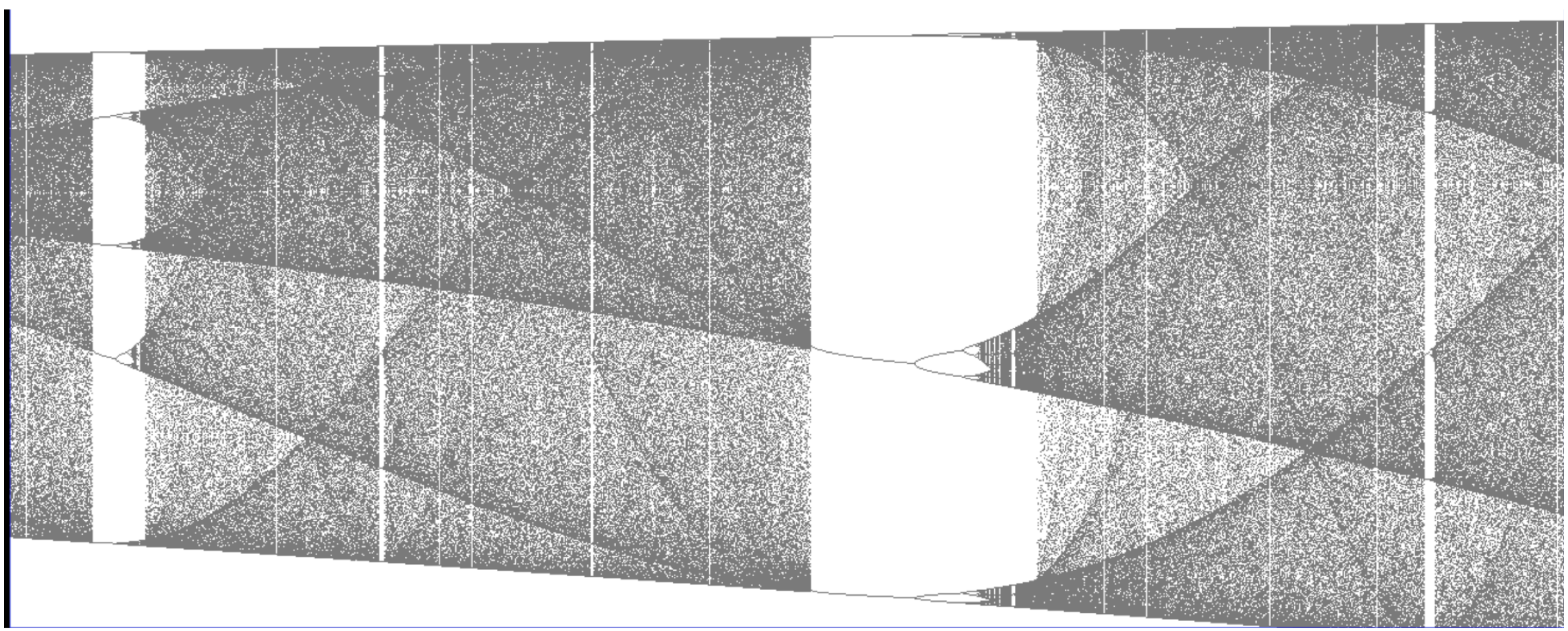


There's something very special about 3…

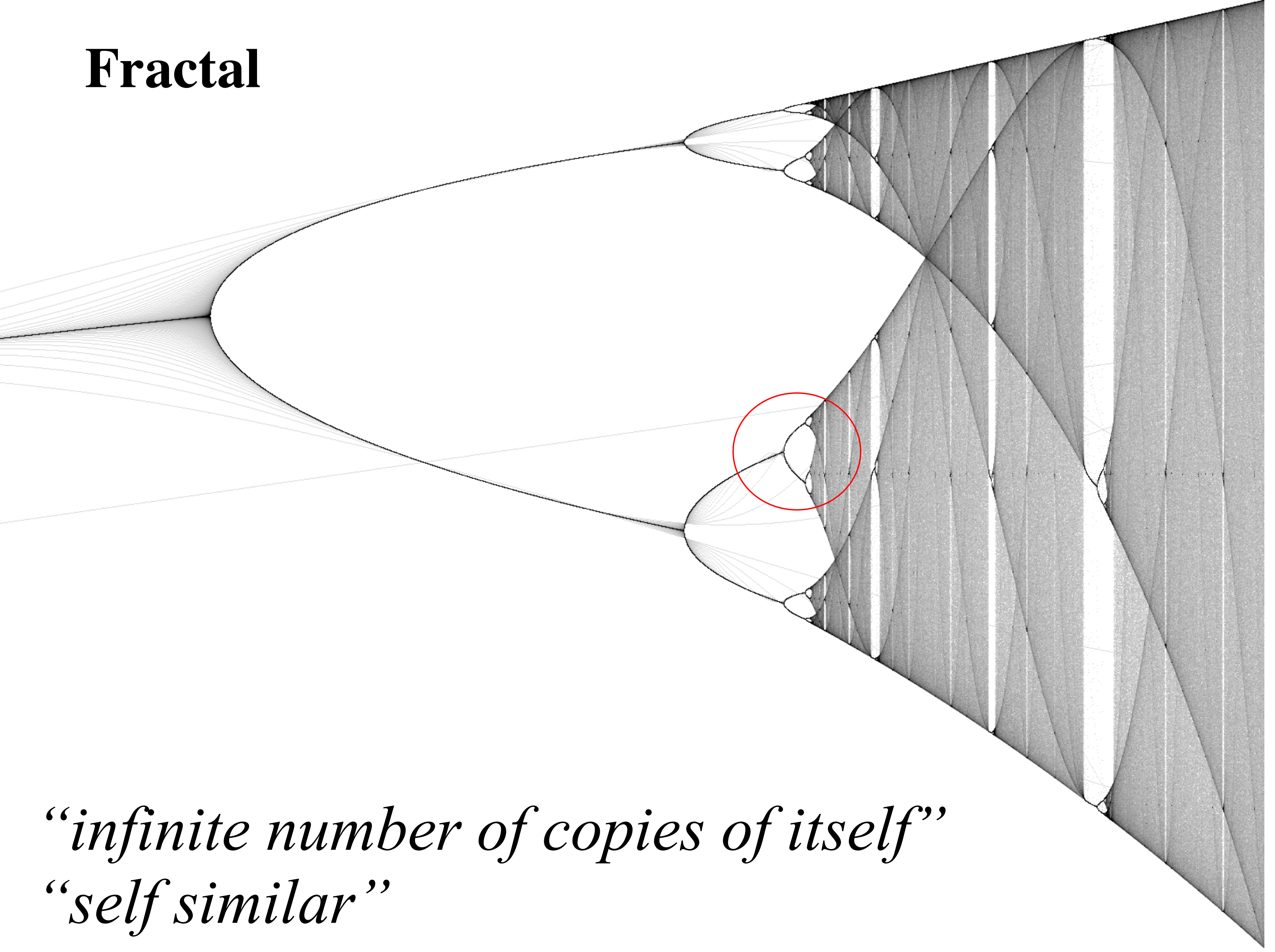
- Sarkovskii (1964)

  3, 5, 7, …3x2, 5x2, …$3x2^2$, $5x2^2$, … 2^2, 2, 1

- Yorke (1975) "Period 3 implies chaos" (in 1D maps)
- Metropolis *et al.* (1973)

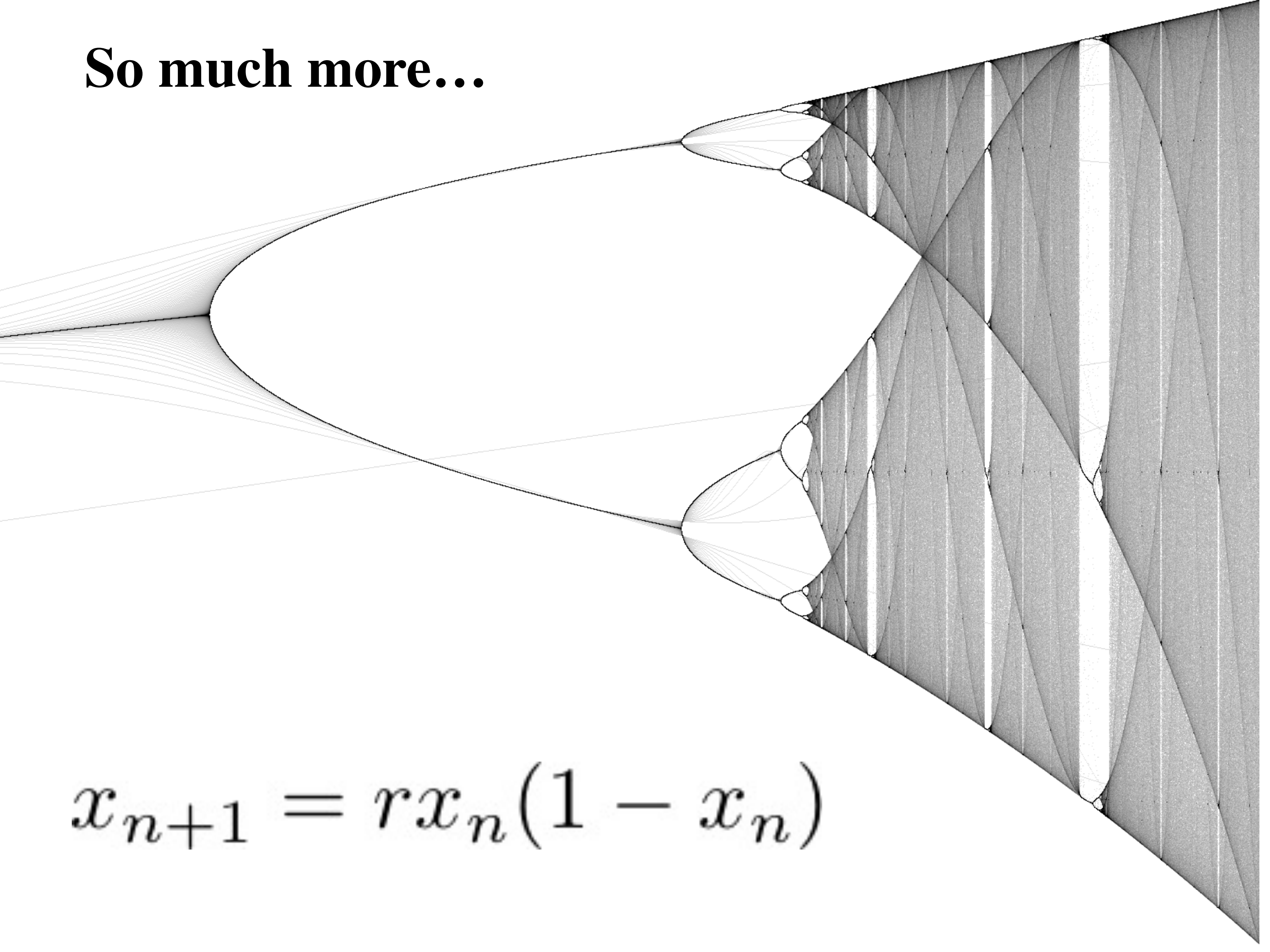


Fractal



"infinite number of copies of itself"
"self similar"

So much more…



$$x_{n+1} = rx_n(1 - x_n)$$

Get your hands dirty!

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