# Are there quantitative mathematical laws underlying financial markets?

Talk given at Bariloche summer school
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# Do social systems obey quantitative laws?

- Psychology:  Individual human behavior?
- Anthropology/sociology:  Culture
- Economics: markets constrain and filter behavior
- Importance of institutions!
- Are there any laws?

# Statistical mechanics of human systems?

- Many human systems exhibit emergent phenomena generated by low level interactions of many individuals.
- In constrained settings do these exhibit consistent rules, like physical systems?
- Can we understand such phenomena with statistical mechanics-like theories?
- Two strategies:
  - ~ Find situations where institutional constraints dominate human choice.
  - ~ Find situations where we can use simple heuristics to characterize human reasoning.

# The Virtues and Vices of Equilibrium, and the Future of Financial Economics

J. Doyne Farmer* and John Geanakoplos†

January 11, 2008

**Abstract**

The use of equilibrium models in economics springs from the desire for parsimonious models of economic phenomena that take human reasoning into account. This approach has been the cornerstone of modern economic theory. We explain why this is so, extolling the virtues of equilibrium theory, then present a critique and describe why this approach is inherently limited, and why economics needs to move in new directions if it is to continue to make progress. We stress that this shouldn't be a question of dogma, but should be resolved empirically. There are situations where equilibrium models provide useful predictions and there are situations where they can never provide useful predictions. There are also many situations where the jury is still out, i.e., where so far they fail to provide a good description of the world, but where proper extensions might change this. Our goal is to convince the skeptics that equilibrium models can be useful, but also to make traditional economists more aware of the limitations of equilibrium models. We sketch some alternative approaches and discuss why they should play an important role in future research in economics.

# How markets slowly digest changes in supply and demand

Jean-Philippe Bouchaud, J. Doyne Farmer
Fabrizio Lillo

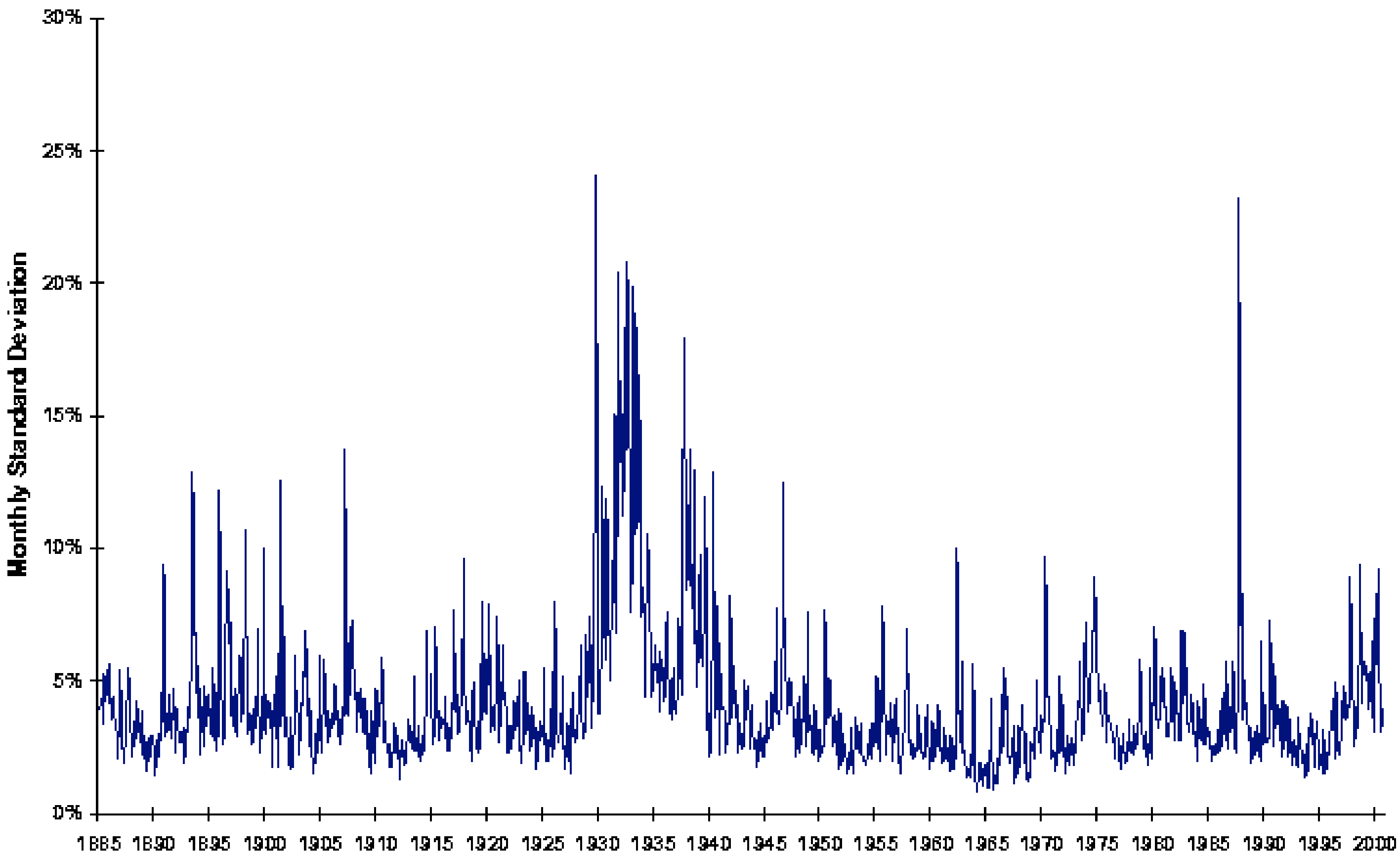
Abstract

In this article we revisit the classic problem of tatonnement, reviewing a recent body of theoretical and empirical work explaining how fluctuations in supply and demand are slowly incorporated into prices. For strategic reasons large orders to buy or sell are only traded incrementally, over periods of time that can be as long as months. Because of this fluctuations in supply and demand form a long-memory process, manifesting itself as highly persistent order flow. Liquidity dynamics plays a key role in determining volatility and in allowing the market to absorb large swings in supply and demand while remaining efficient. We review a body of theory that makes detailed quantitative predictions about the volume and time dependence of market impact, the bid ask spread, and order book dynamics, and show that the predictions of this body of theory compare well with empirical data. This approach suggests a novel interpretation of financial information, in which all agents are at best only weakly informed, price formation is extremely noisy, and most information comes from within rather than from outside the market. We review some preliminary studies of market ecology and argue that this should play a central role in the future.

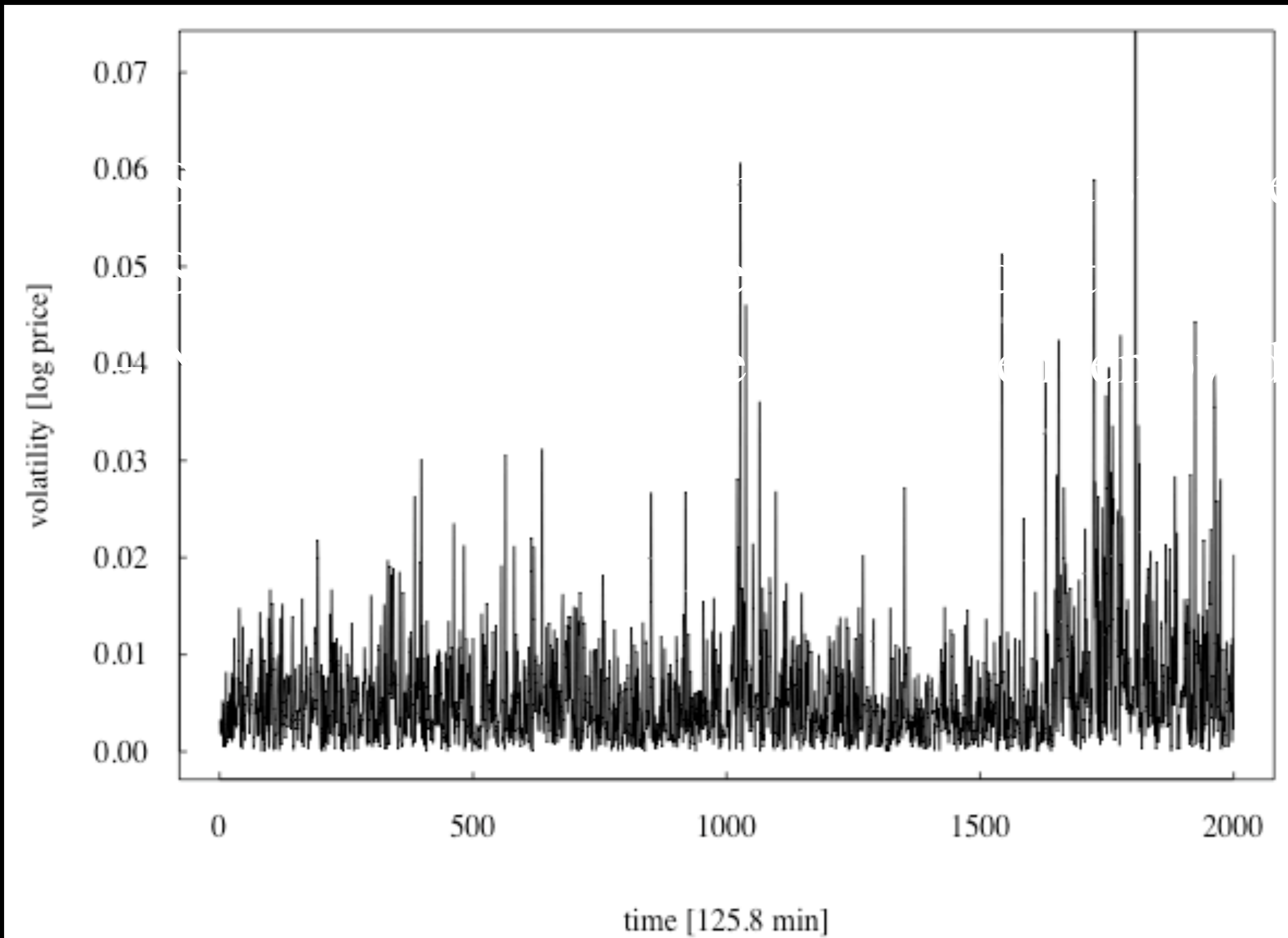
# Examples

- Long memory of volatility
- Long memory of supply and demand
- Law of market impact?
- Equation of state of price statistics and order flow
- Distribution of mutual fund sizes

**Standard Deviation of Monthly Stock Returns from Daily Returns in the Month, 1885-2000**



# Volatility at 2 hour timescale - AZN

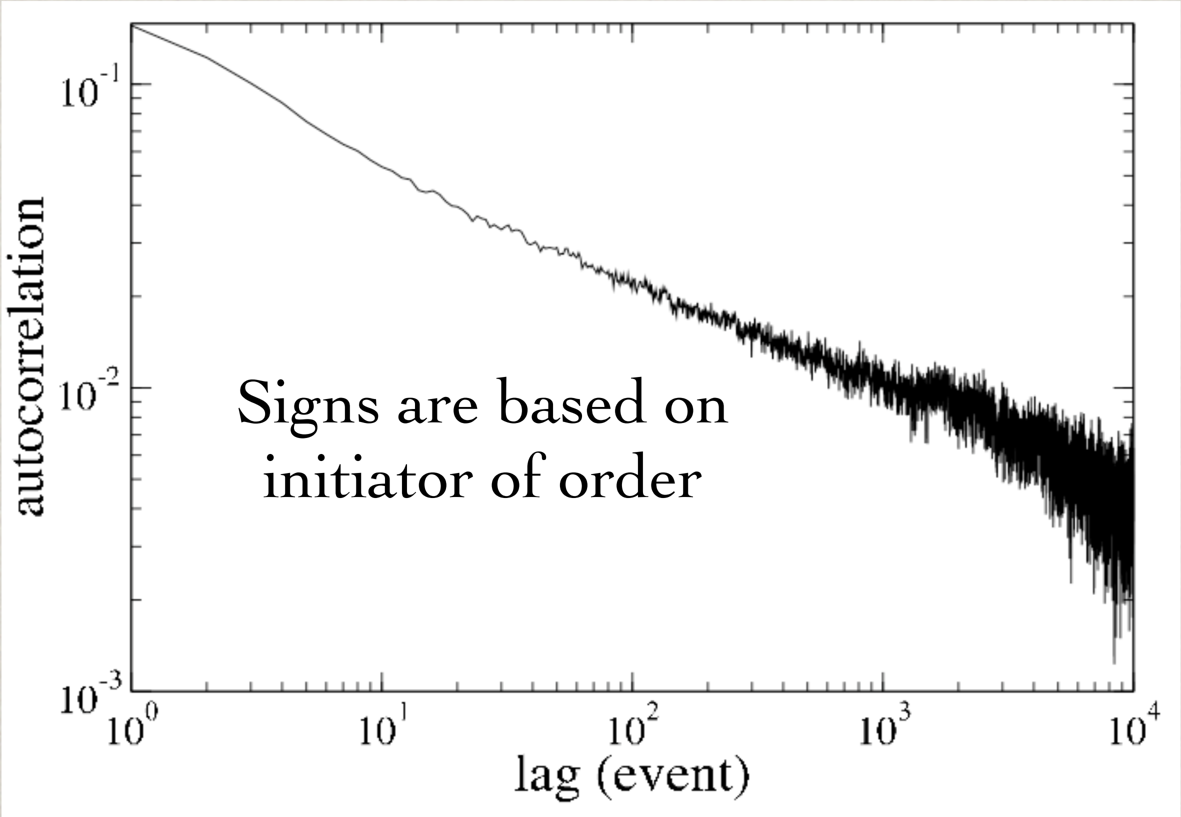


# Why does volatility have long-memory?

- At short time scales the dominant effect is fluctuations in liquidity (i.e. changes in the response of prices to fluctuations in supply and demand rather than changes in fluctuations in supply and demand themselves).
- Theory: Fluctuations in popularity of investor styles.
- New idea: Fluctuations in use of leverage.
- Not understood!

# Autocorrelation of order flow (long memory of supply and demand)

Autocorrelation of trade signs

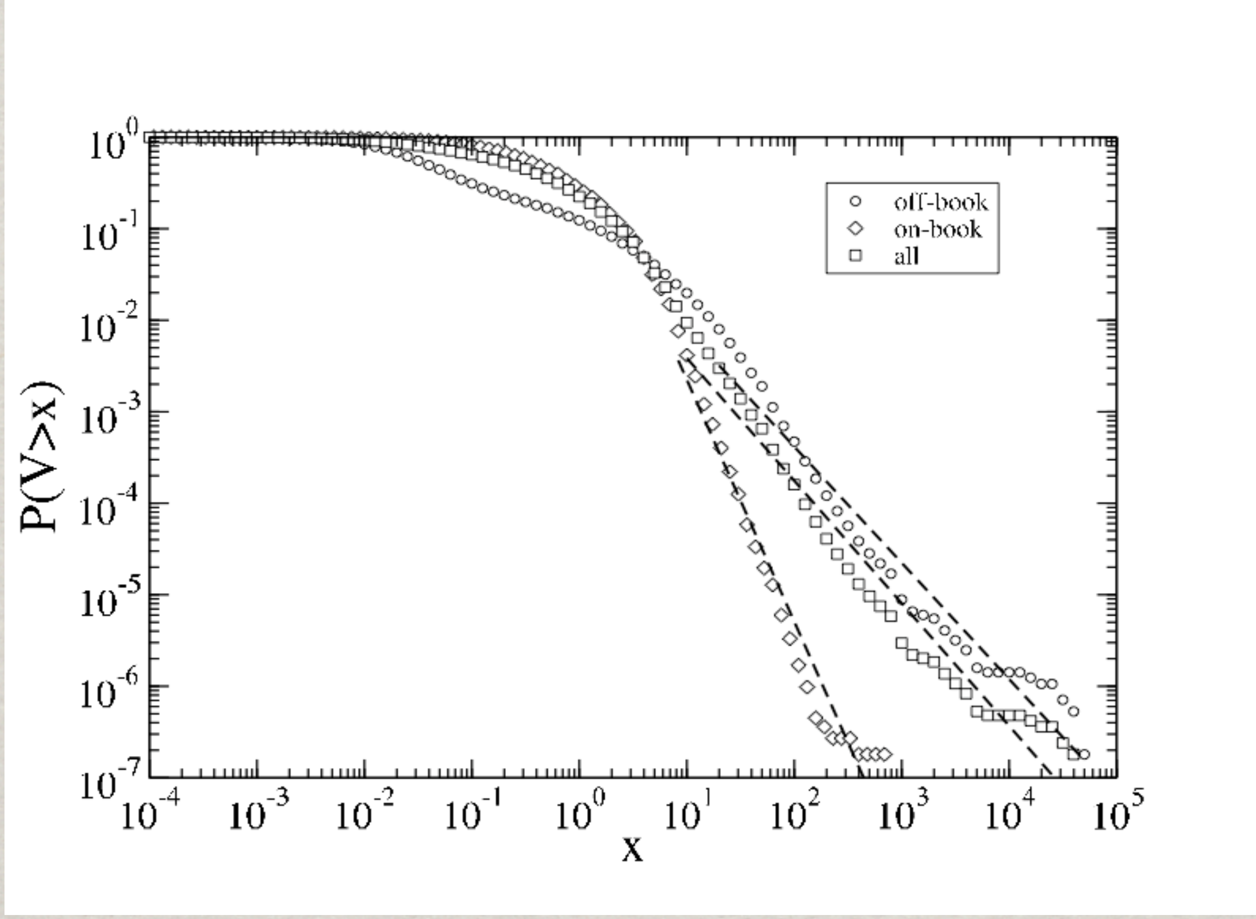



Bouchaud, Gefen, Potters, and Wyart (2004)
Lillo and Farmer (2004)

# What causes long-memory of order flow?

- Two theories:
  - Herding (Lebaron and Yeo)
  - Gradual revelation of information due to large hidden trading orders by large institutional investors, who execute their orders incrementally over periods as long as months (Lillo, Mike and Farmer)

# Cumulative distribution of trading volume for LSE Stock Astrazeneca




Gopikrishnan, Gabaix, Plerou and Stanley (2000)
Lillo, Mike, Farmer (2005)

# Order flow and hidden order sizes

- $C(\tau)$ = autocorrelation of transaction signs
- Assume all transactions come from hidden orders of size V drawn from P(V), split into uniform pieces and executed incrementally at a constant rate.
- The tail behavior of P(V) determines the persistence of order flow.
- E.g. $P(V > v) \sim v^{-\alpha}$ implies $C(\tau) \sim \tau^{-\gamma}$ with

$$\alpha = \gamma + 1$$

(Lillo, Mike and Farmer, 2005)

# Nonlinear order flow process

$M$ active hidden orders

Order $j$ is of size $N_j - n_j$ and sign $\epsilon_j = \pm 1$

At each time step random choose $j$ with probability $\pi_j$

Generate transaction of sign $\epsilon_j$

Increment $n_j$

If $n_j = N_j$ set $n_j = 0$;

draw new $N_j$ from $P(N_j > N) \sim N^{-\alpha}$

When P has heavy tails this generates a highly persistent and nonlinear order flow process

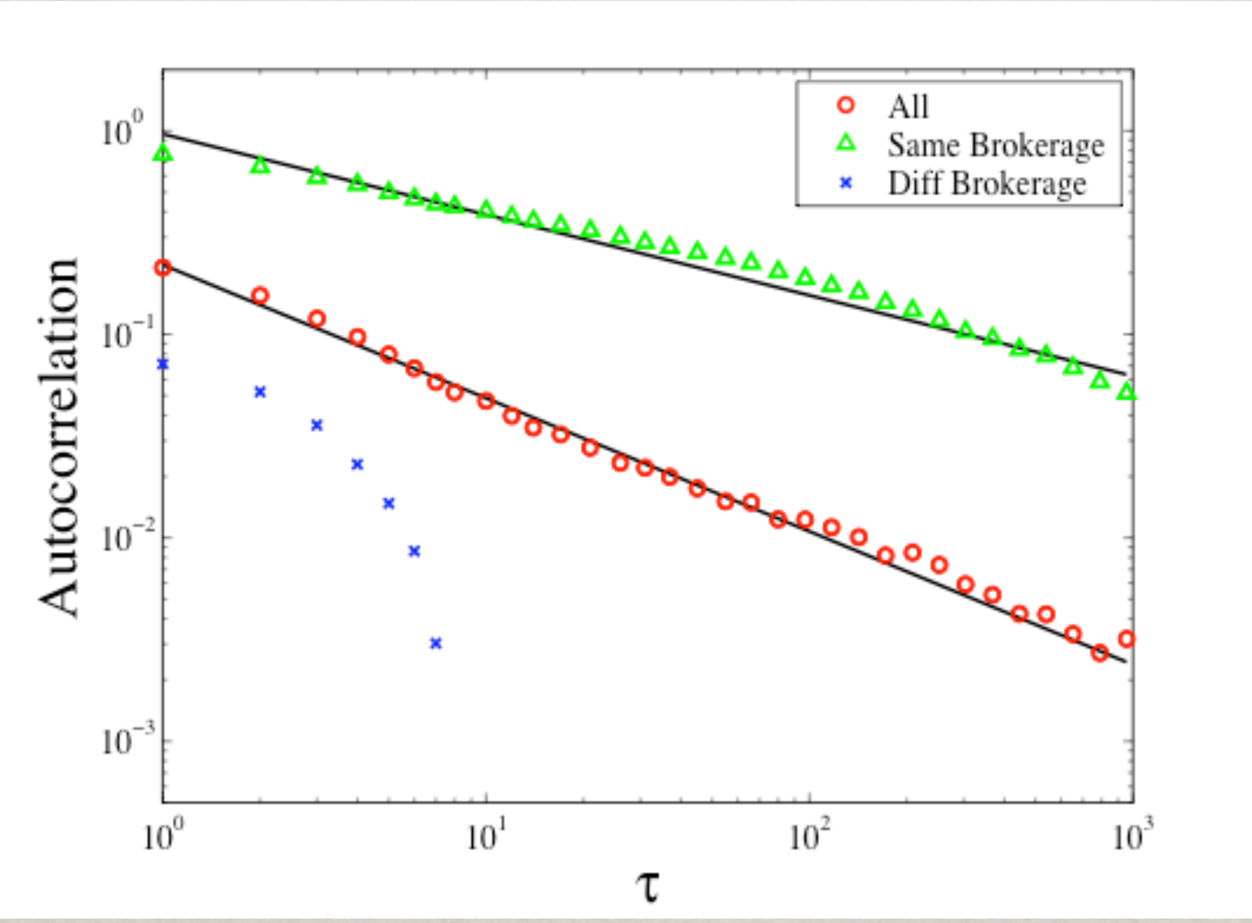
# AGREEMENT WITH DATA

$\alpha = \gamma + 1$ produces good agreement with the data

- Good fit to autocorrelation function of order flow with $M$ about 20.
- Long memory disappears for orders with different brokerage codes.

# Autocorrelation of trade signs (vs. time in #transactions)

Autocorrelation of trade signs
for same vs. different broker codes

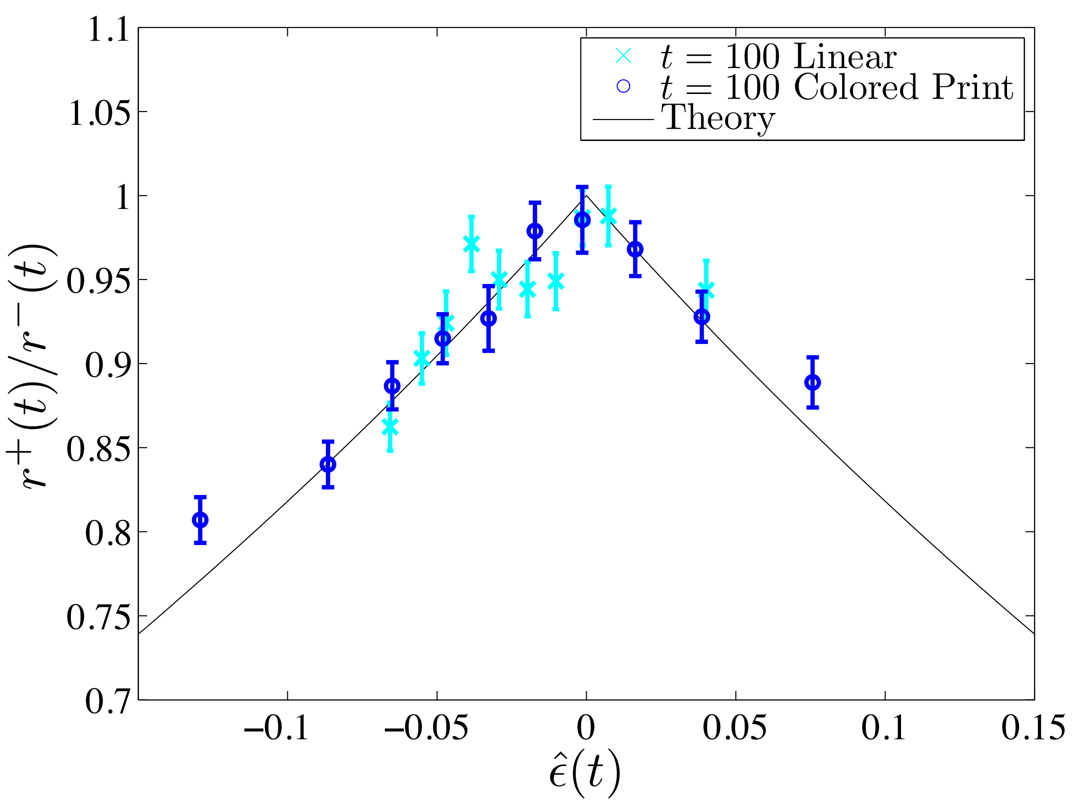


# Asymmetric liquidity

- Persistence of order flow requires market to adjust in order to enforce market efficiency.
- Implies asymmetric liquidity, i.e. price responses to buy orders are different from those to sell orders, depending on whether buy or sell hidden orders are dominant at a given time.

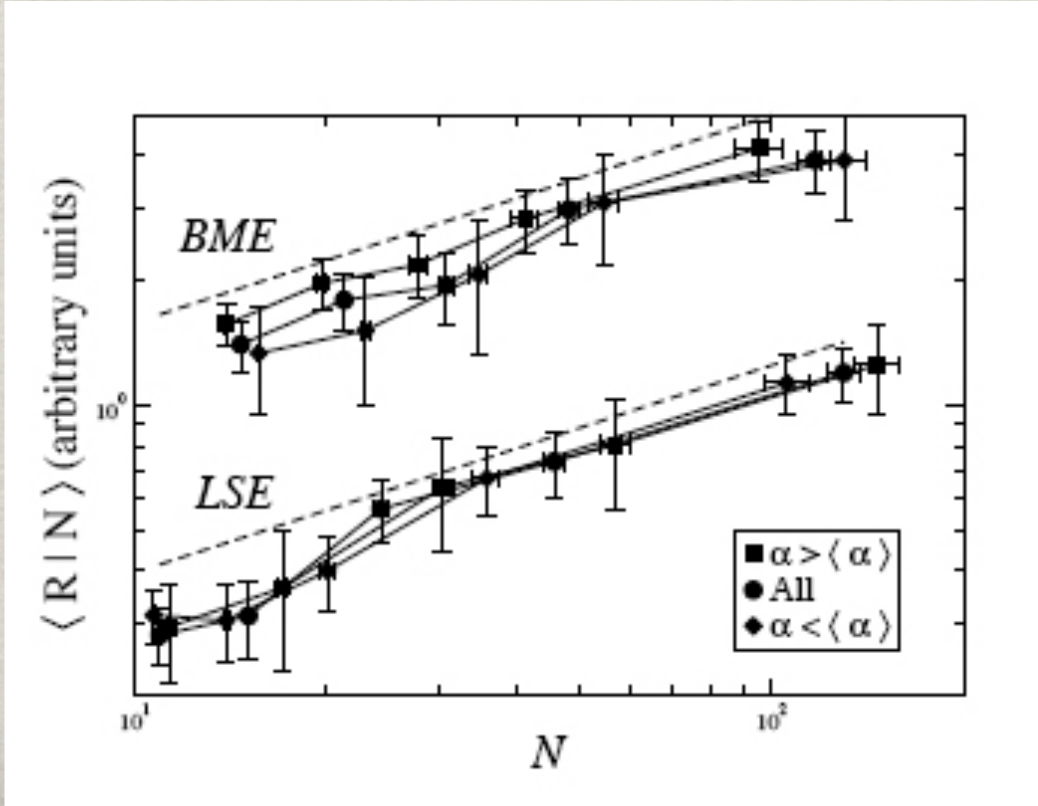
# Market efficiency implies asymmetric price response

To keep prices efficient (a martingale) when buy orders are really likely, responses to buy orders are smaller than responses to sell orders.



# Market impact

- Market impact is the response in price to a trade (a change in demand - supply).
- Are there universal laws for market impact?



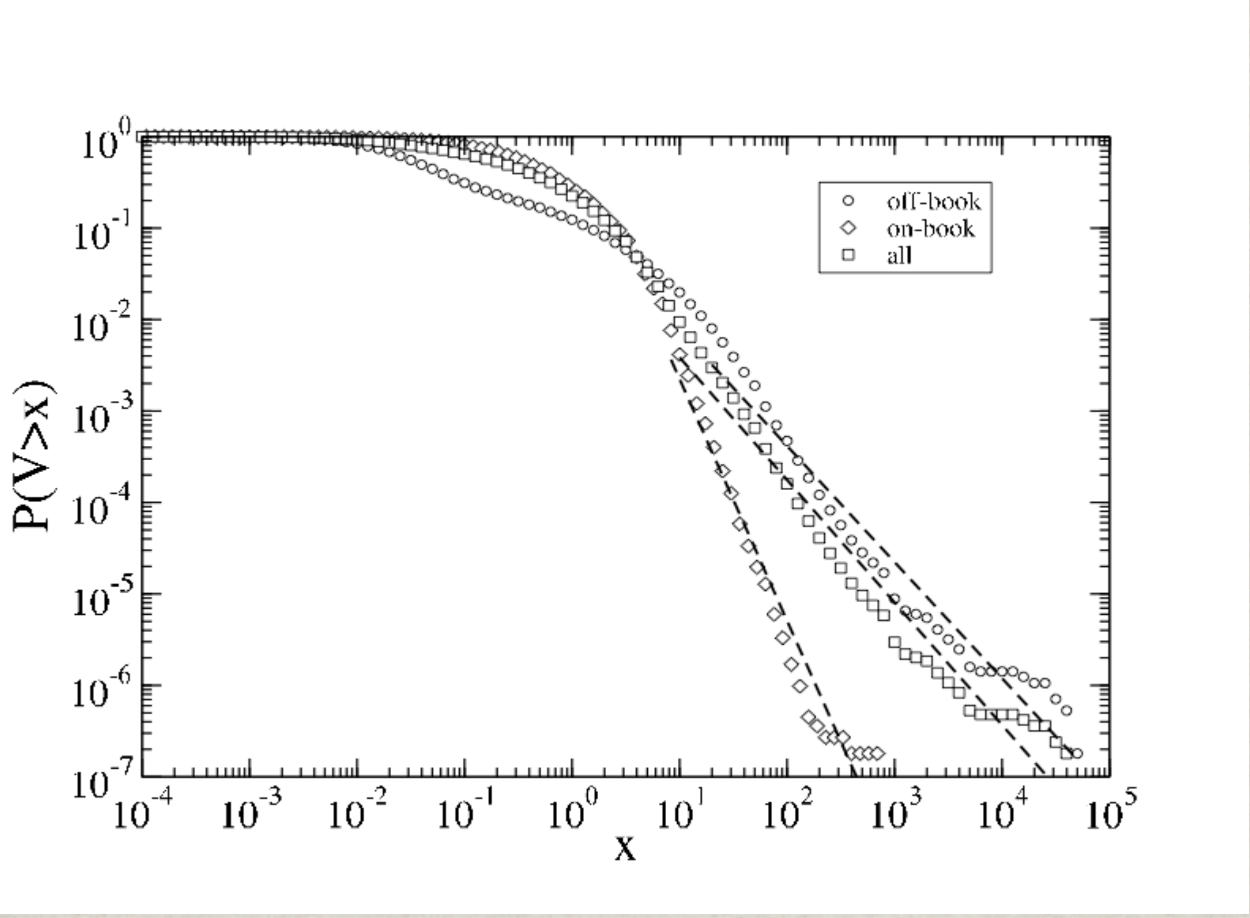
# Theory for average market impact

- Efficiency plus long-memory of order flow places strong constraints on market impact.
- Market impact:  price change corresponding to buyer or seller initiated trade of a given size.
  - Like poor-man's supply and demand functions.
  - Fundamental interaction rule of markets.
- With another key assumption can derive functional form -- work in progress.
  - (see also Gabaix, Gopikrishnan, Plerou, Stanley)

# Theory for market impact
## (Farmer, Gerig, Lillo, Waelbroeck)

- Assume large trades are split uniformly and executed at a constant rate, independent of size
- Market must remain efficient, i.e. price changes should be unpredictable.  Implies asymmetry of price response.
- Market is efficient for participants in every size category.
- Volume distribution for large trades is a power law with exponent alpha.
- Participant information:  There exists a time scale on which participants can detect that a large trade is taking place (and when it has stopped).
- Predicts impact increases as alpha - 1 power of volume
- Predicts permanent impact = 2/3  temporary impact at max.

# Cumulative distribution of trading volume for LSE Stock Astrazeneca

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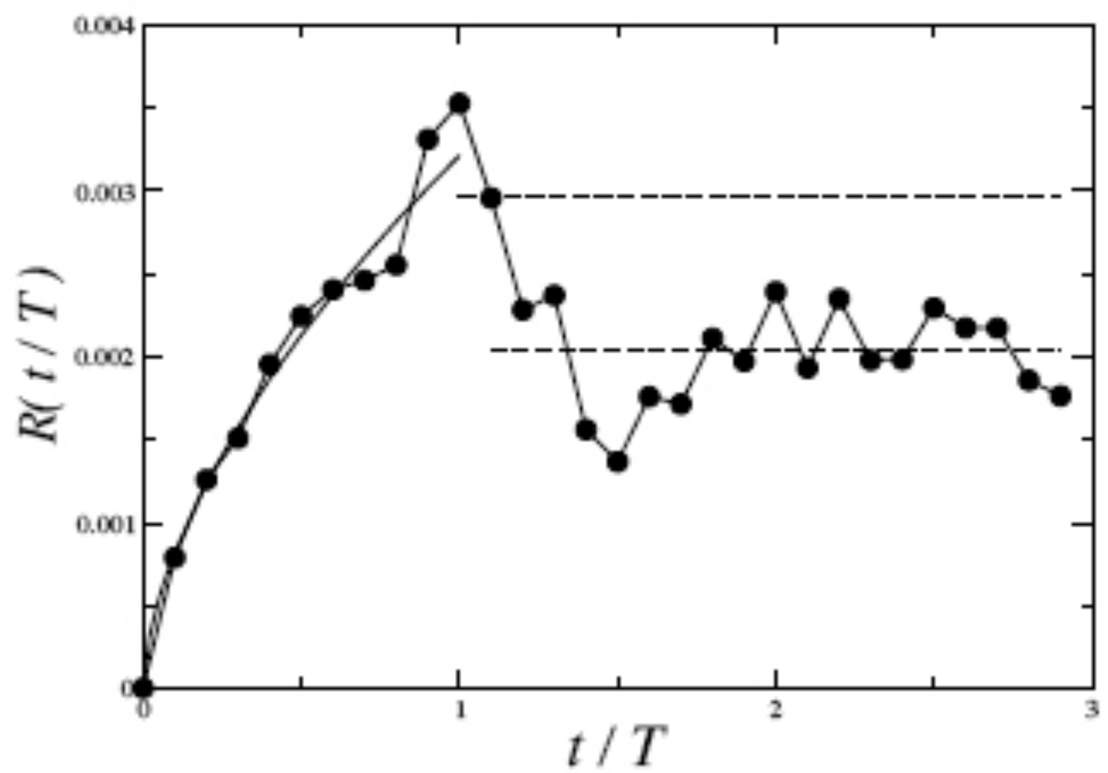
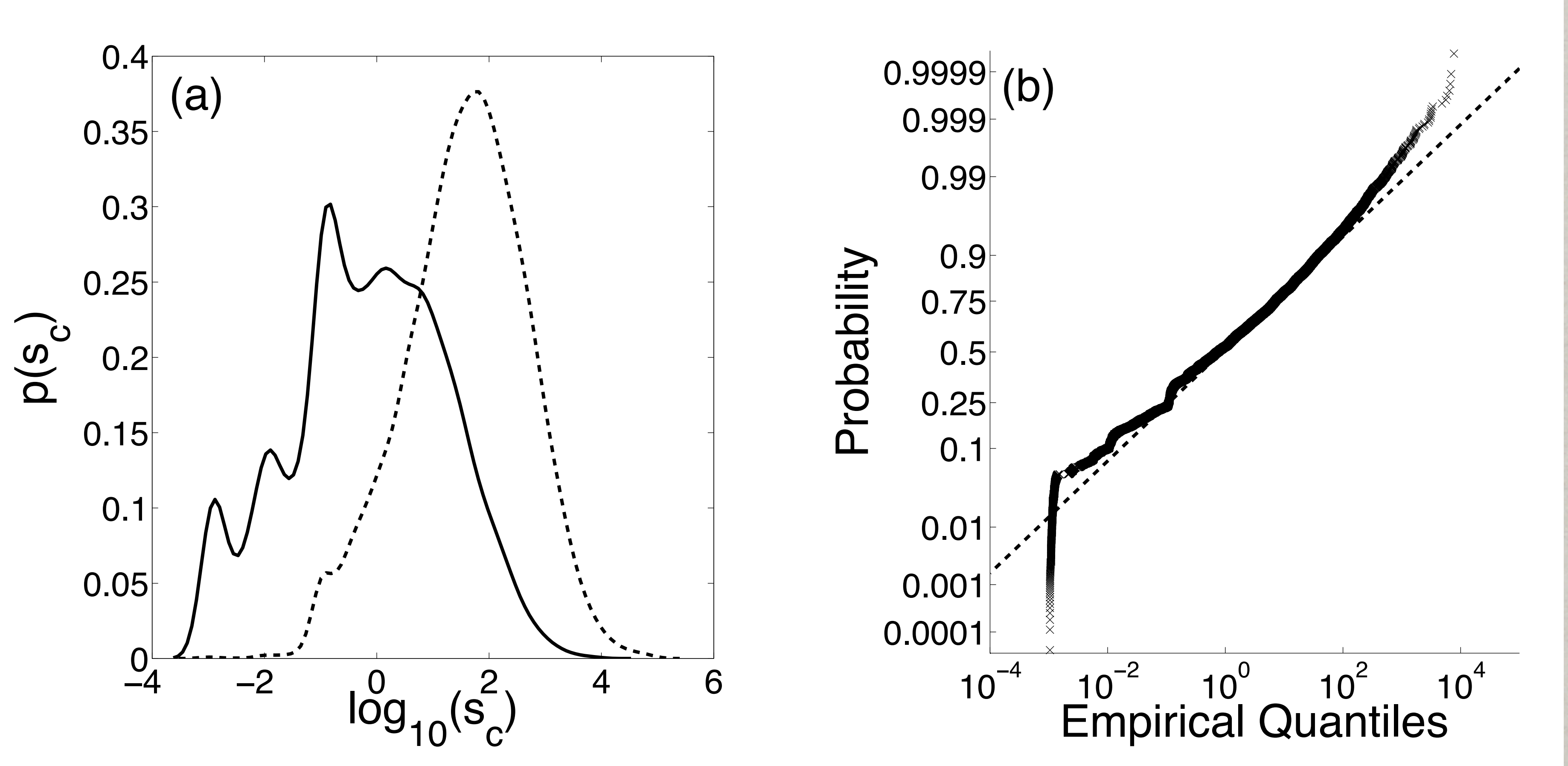


0.004
0.003
0.002
0.001
0
R( t / T )
0
1
2
3
t / T

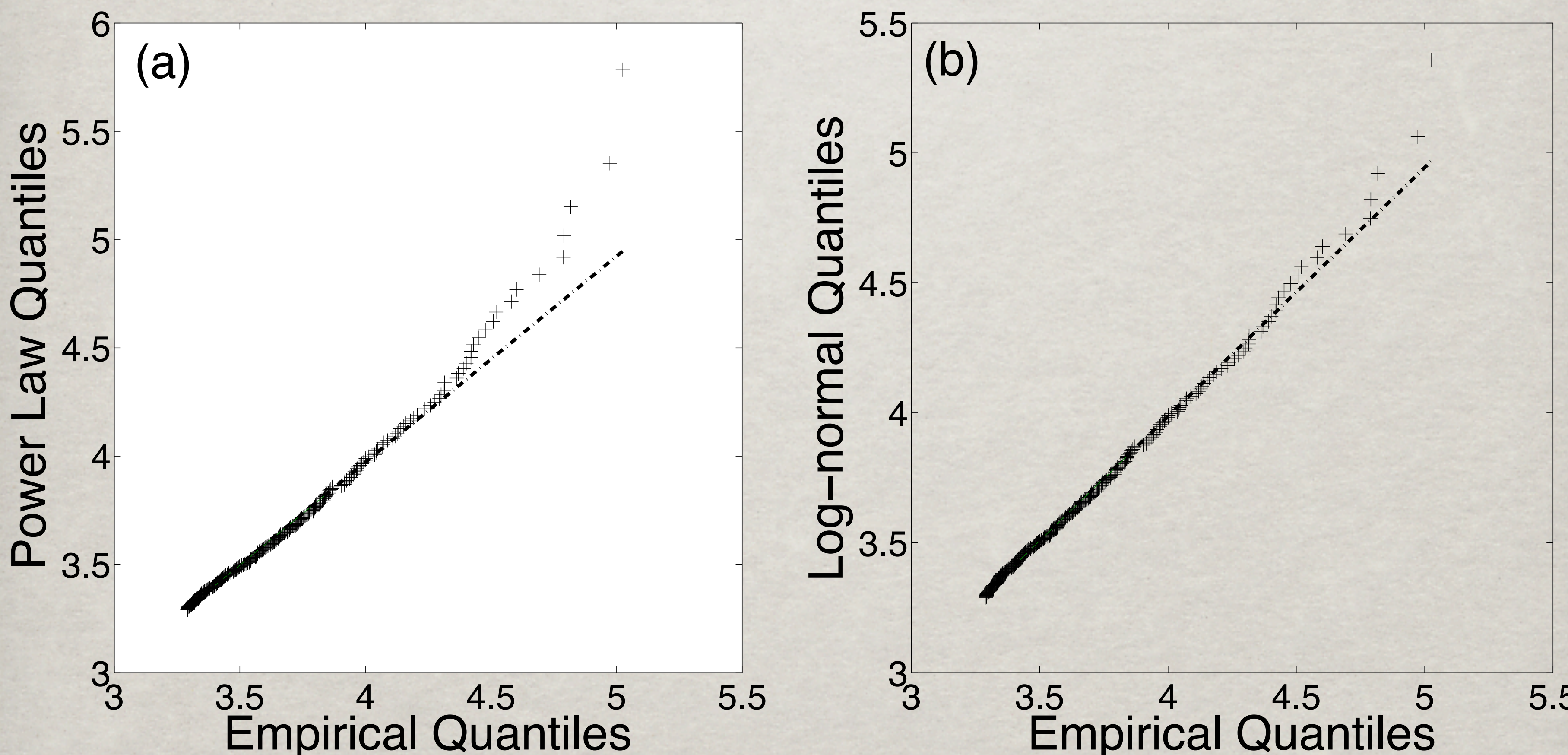
# Mutual fund size distribution

- Mutual funds have 23% of taxable assets, 4.4T$, 24% of US corporate equity holdings.
- Is there a characteristic size distribution for large funds?
- Will influence many properties of economy, such as distribution of trading volume.

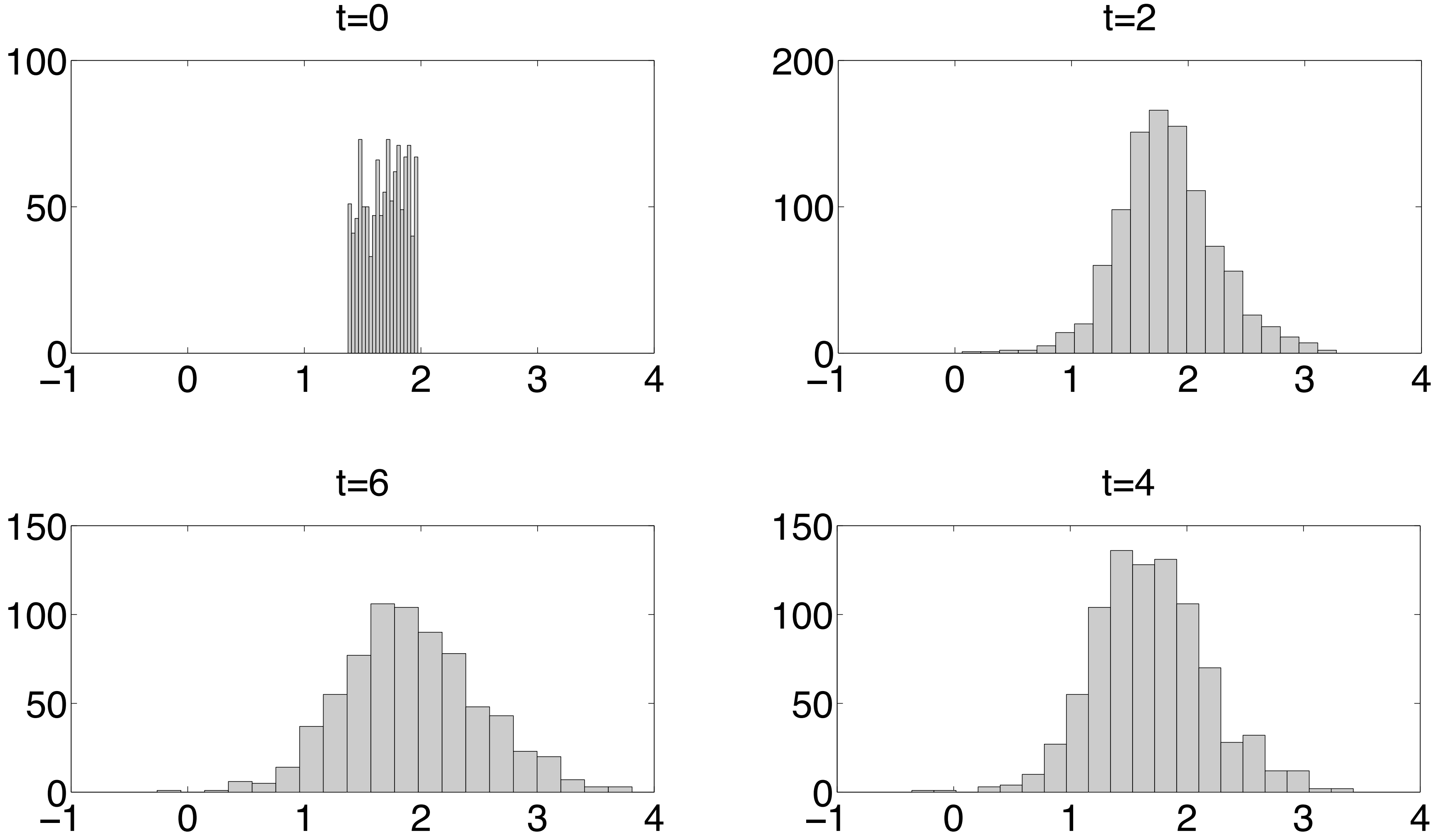


(a)
p(s_c)
log_10(s_c)
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−4
−2
0
2
4
6
(b)
Probability
Empirical Quantiles
0.9999
0.999
0.99
0.9
0.75
0.5
0.25
0.1
0.01
0.001
0.0001
10^−4
10^−2
10^0
10^2
10^4

# Empirical distribution is log-normal (not power law)



# Mutual fund growth

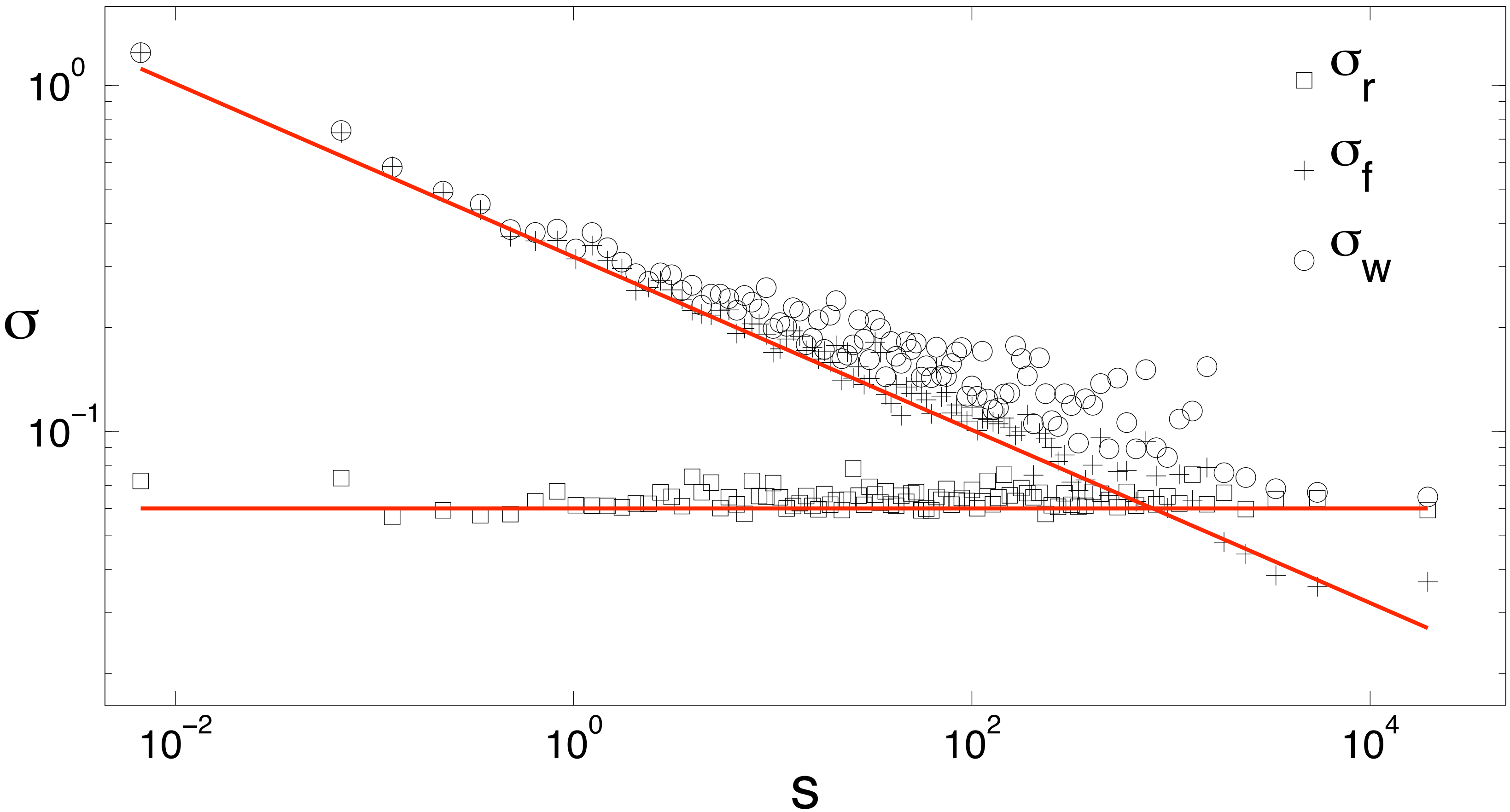


# Model

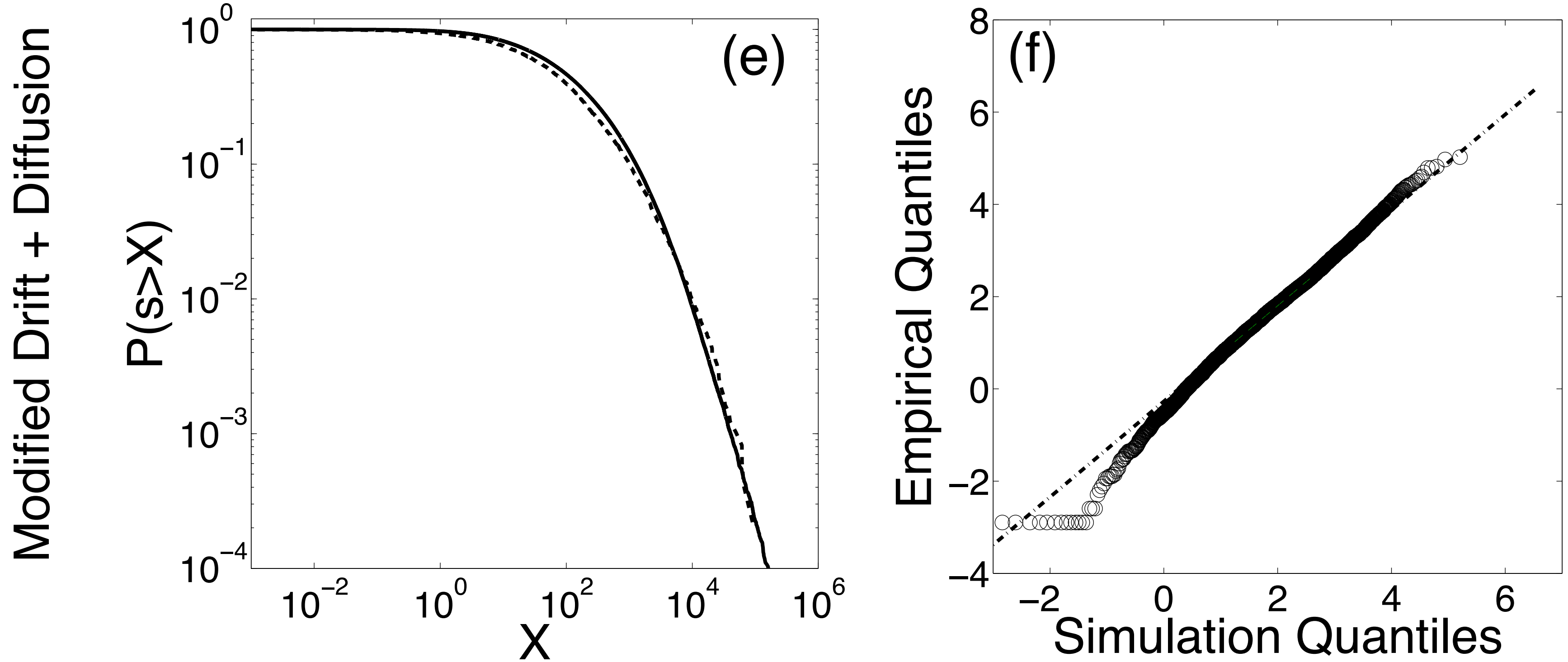
- Model as diffusion process with growth and death.  Initially assume size independent.
- To improve model, add size dependence of diffusion process (Amaral et al.)
- Solutions are asymptotically power laws.
- Resolution:  Asymptotic solution takes a very long time, transients are close to log-normal.

# Size dependence

- investor money flux vs. intrinsic returns



# Theory vs. empirical



# MUTUAL FUND DISTRIBUTION CONCLUSIONS

- Mutual fund distribution can be explained very well by stochastic process model with random birth, death, and size-dependent diffusion.
- Reason for size dependent diffusion remains unclear.
- Surprising that transaction cost does not seem to play a major role.
- Analogy to animal size (Clauset and Erwin).

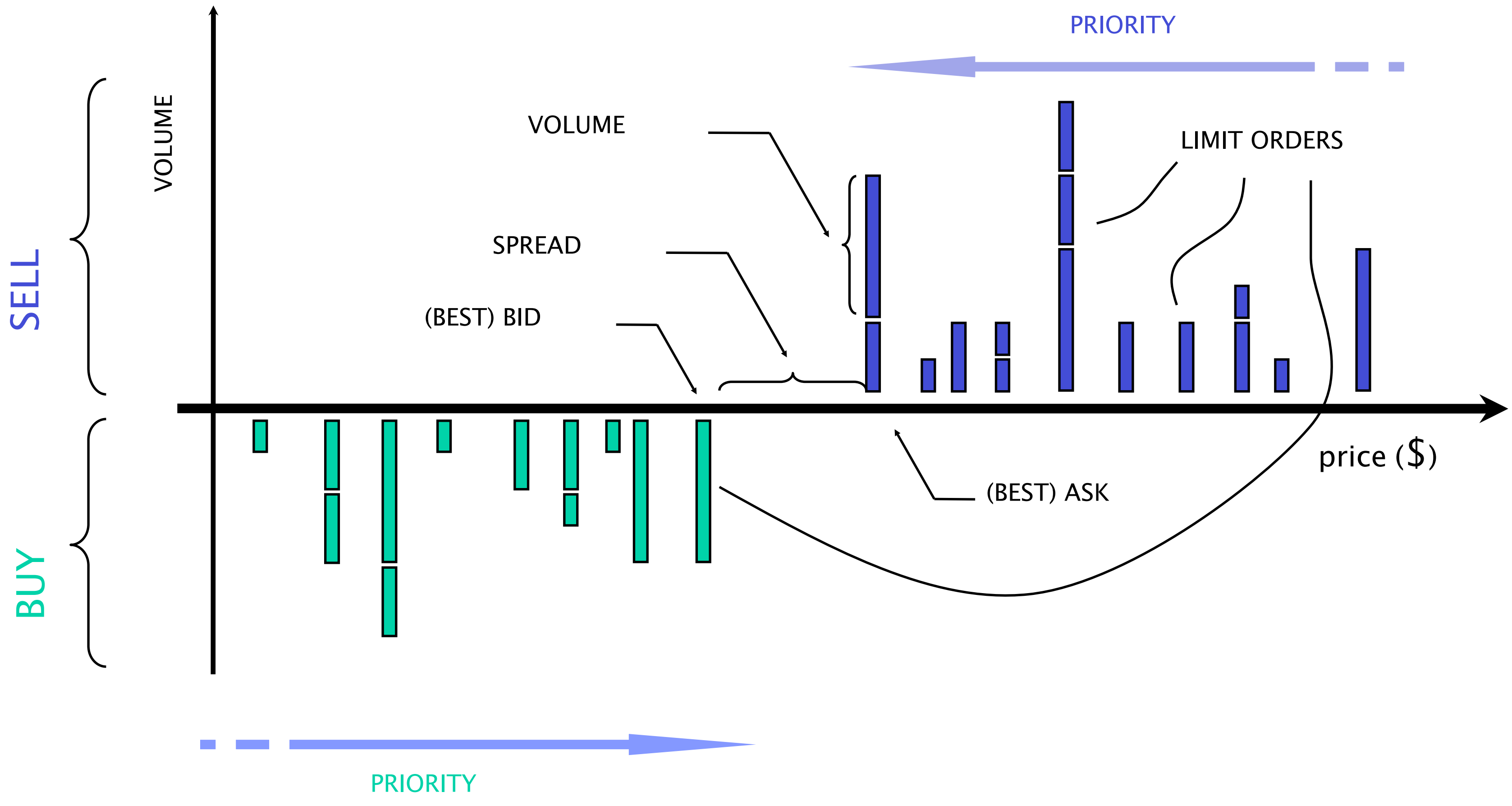
# Equation of state connecting order flow to statistics of prices

- Zero intelligence model (Daniels, Farmer, Iori, Smith, 2003, Smith, Farmer, Gillemot, Krishnamurthy, 2003)
- Assumes order placement and cancellation are Poisson processes.
- Predicts equation of state relating order flow to spread and volatility (approximate but not correct).
- More realistic non-Poisson simulation model by Mike and Farmer (2008) makes more realistic prediction and reproduces distribution of price changes, at least for a class of stocks.

# Poisson model

- Assume agents randomly place orders to buy or sell, at random prices at random times.
- Agents cancel existing orders according to a Poisson process.
- Prices change according to continuous double auction (which is just algorithm).
- Orders that cause transactions can change prices.
- This changes boundary conditions for order placement -- key is that distribution is referenced to prices of best quotes.

# Double continuous auction



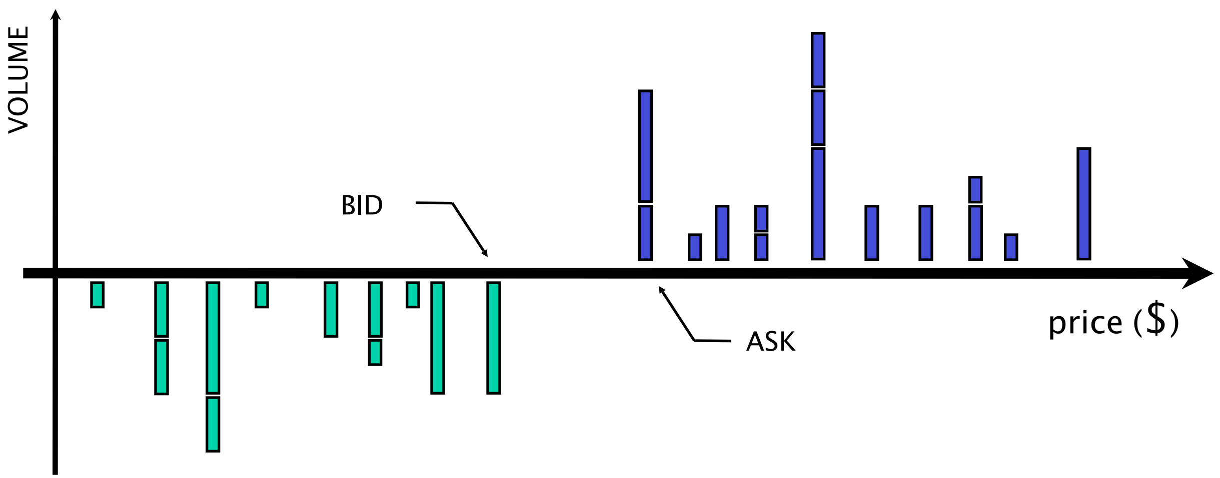
# Three event types

- Market order
  - Any component of an order resulting in an immediate transaction
- Limit order
  - Any component of an order that does not result in an immediate transaction
- Cancellation
  - Limit order removal for any reason

# Patient trading

- Patient traders place **non-marketable limit orders** that do not lead to an immediate transaction
- Non-marketable limit orders accumulate
- Limit order book is a storage device

| **Limit Order** |
|---|
| BUY / SELL |
| # OF SHARES |
| LIMIT PRICE |



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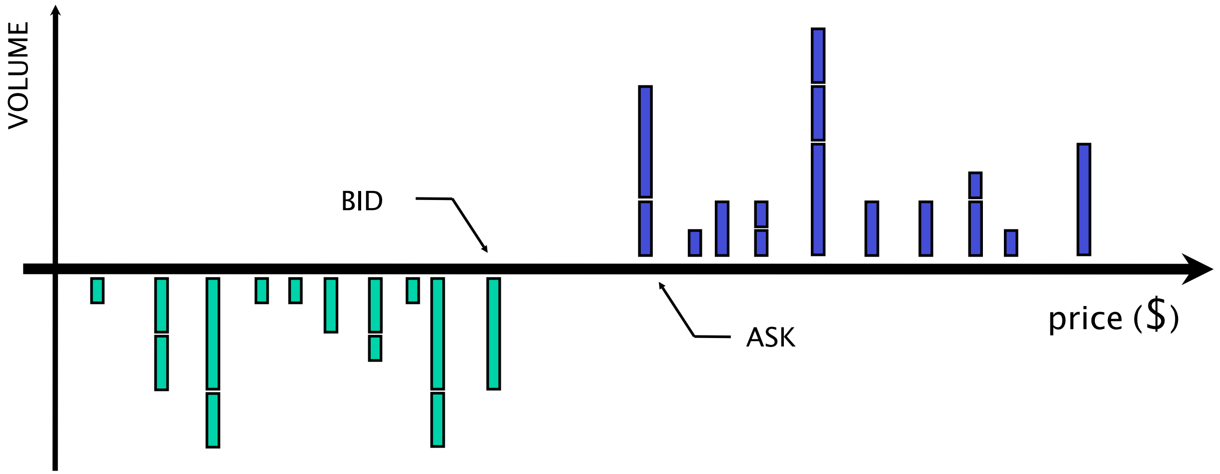

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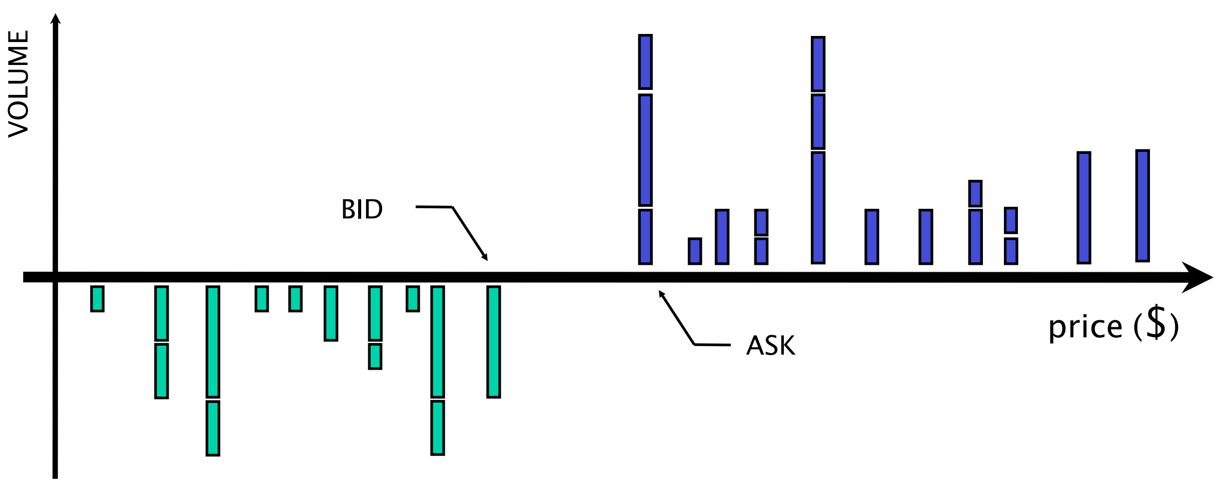

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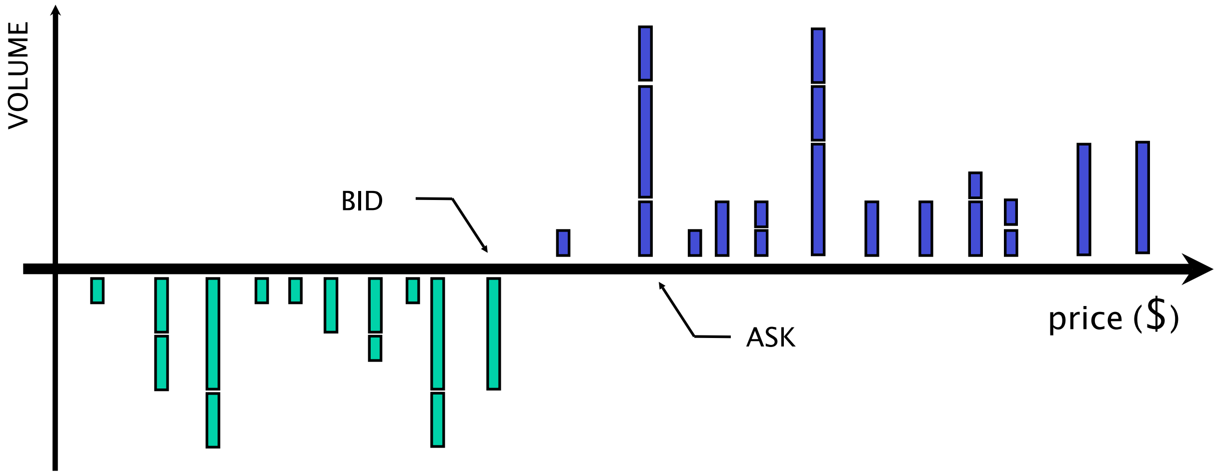

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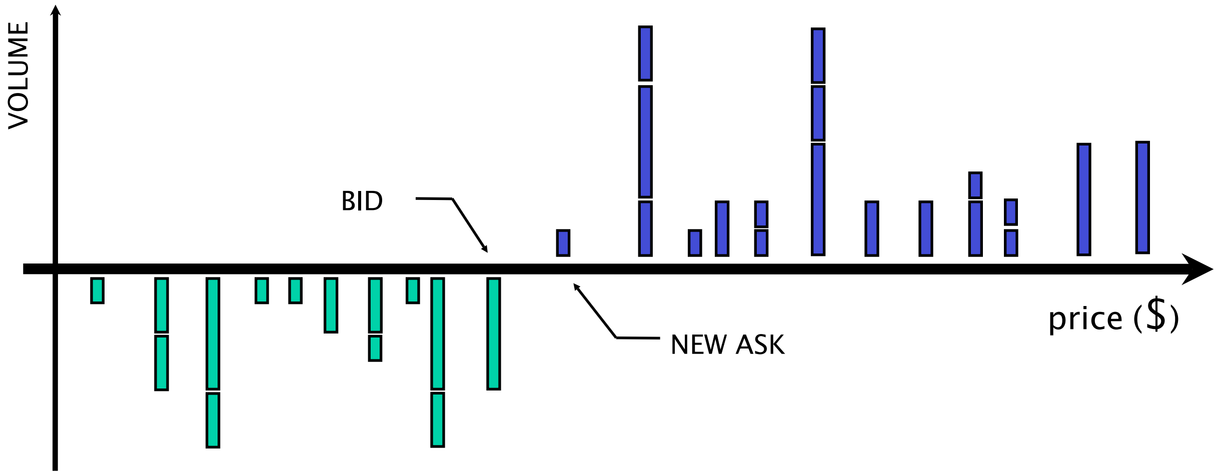

# Impatient trading

**Market order:**

- An order to buy or sell up to a given volume
- No limit price is defined
- Executed immediately
- Often causes unfavorable price impact

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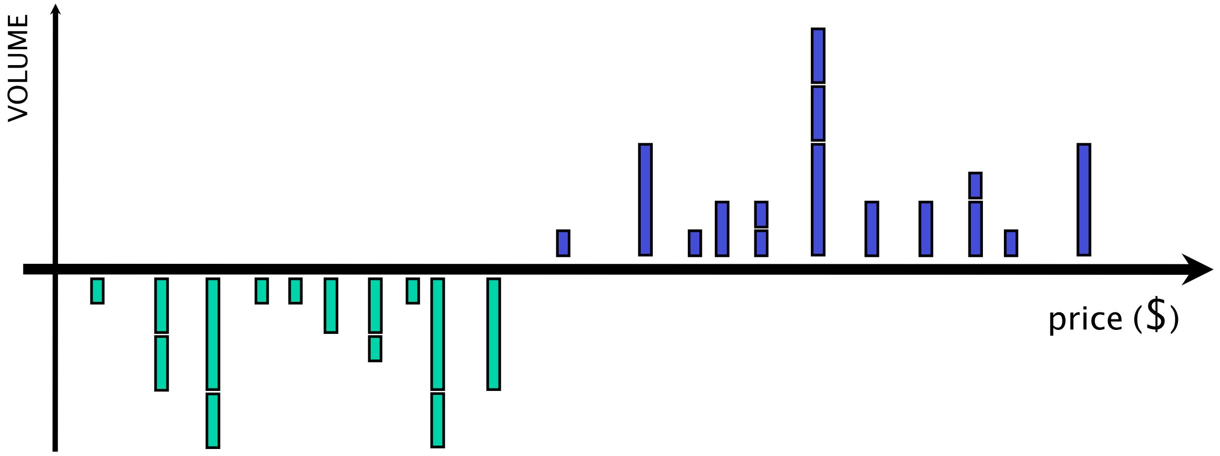

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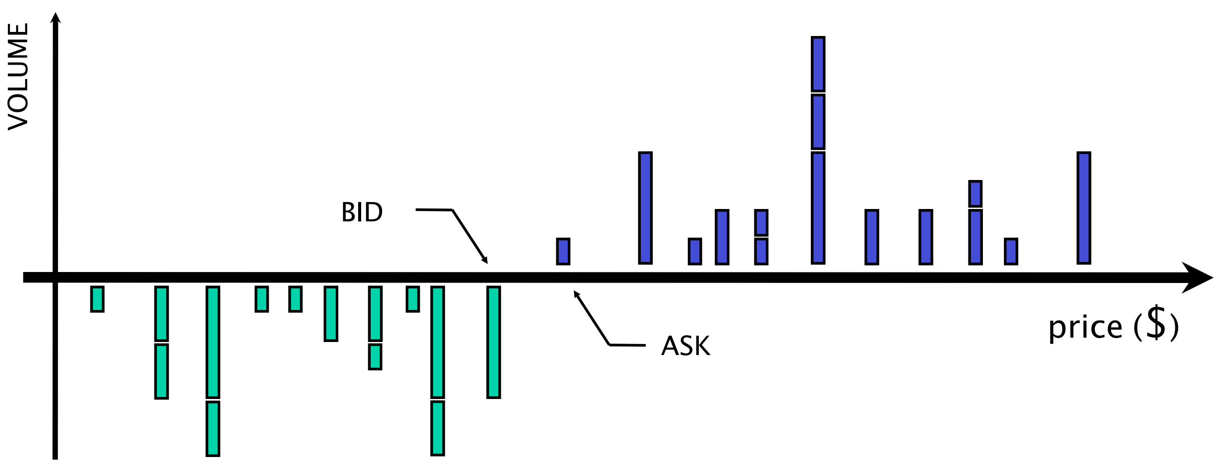

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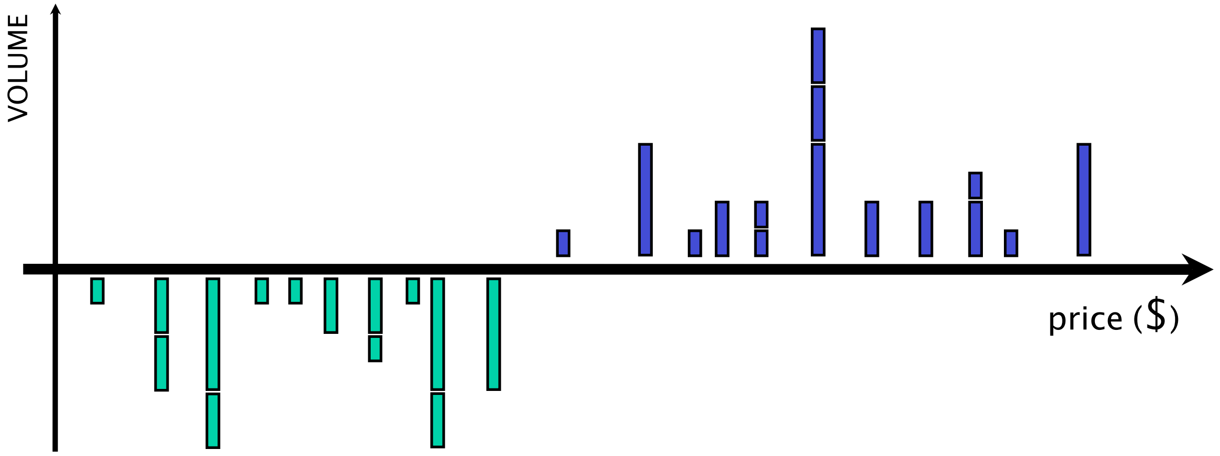

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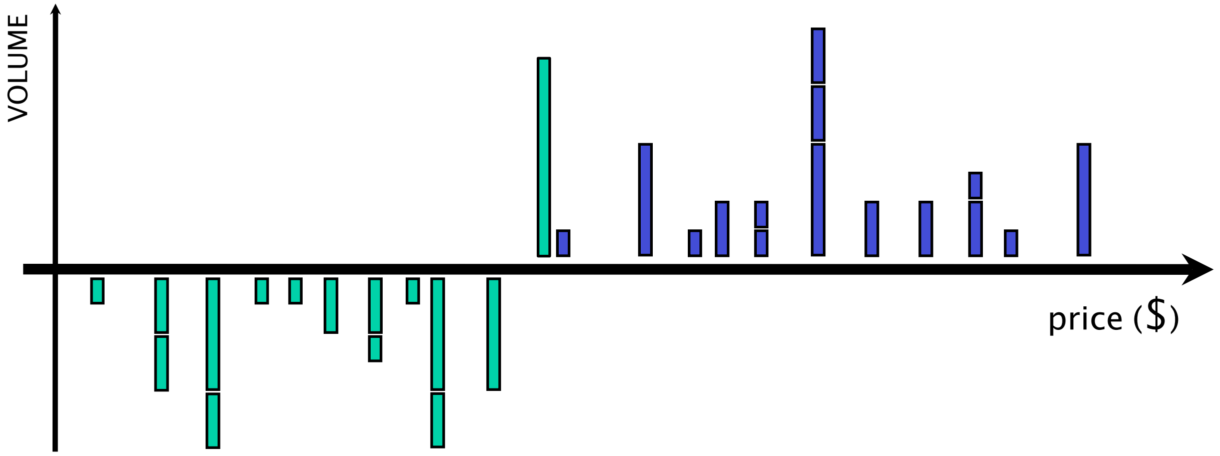

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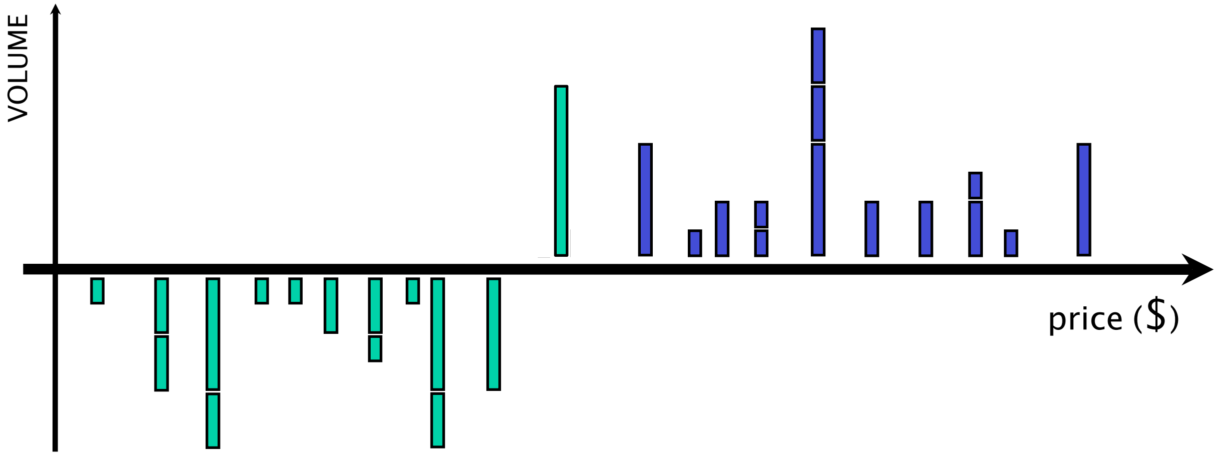

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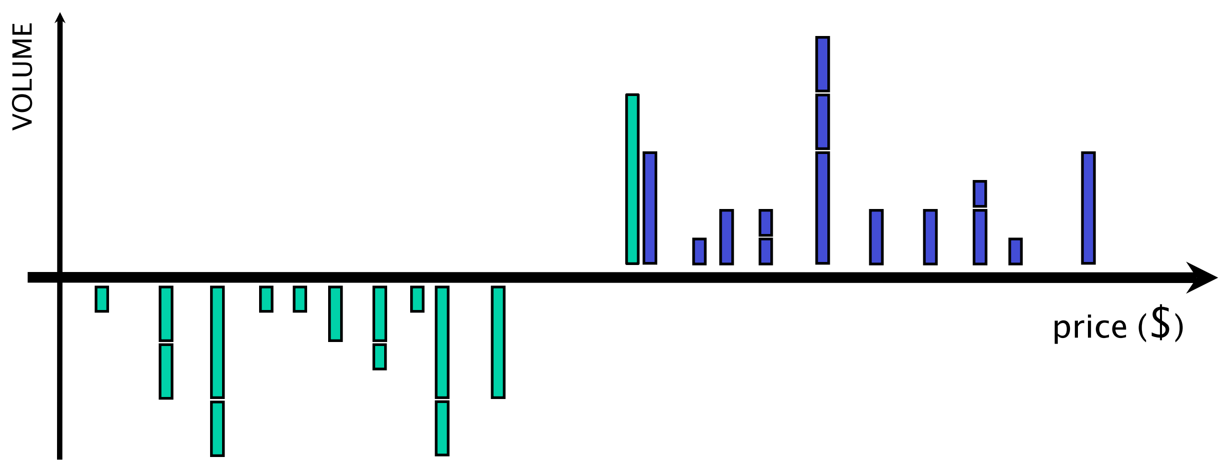

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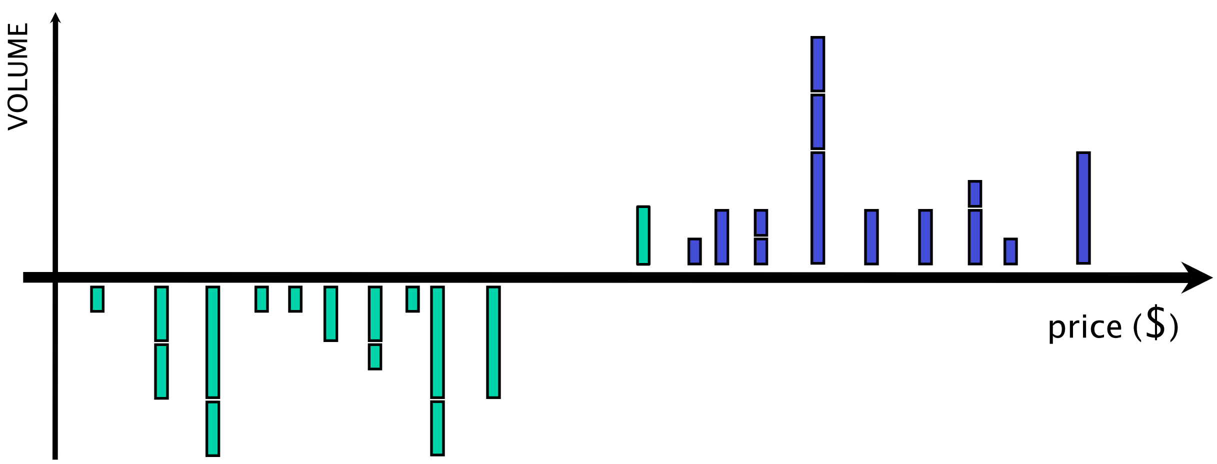

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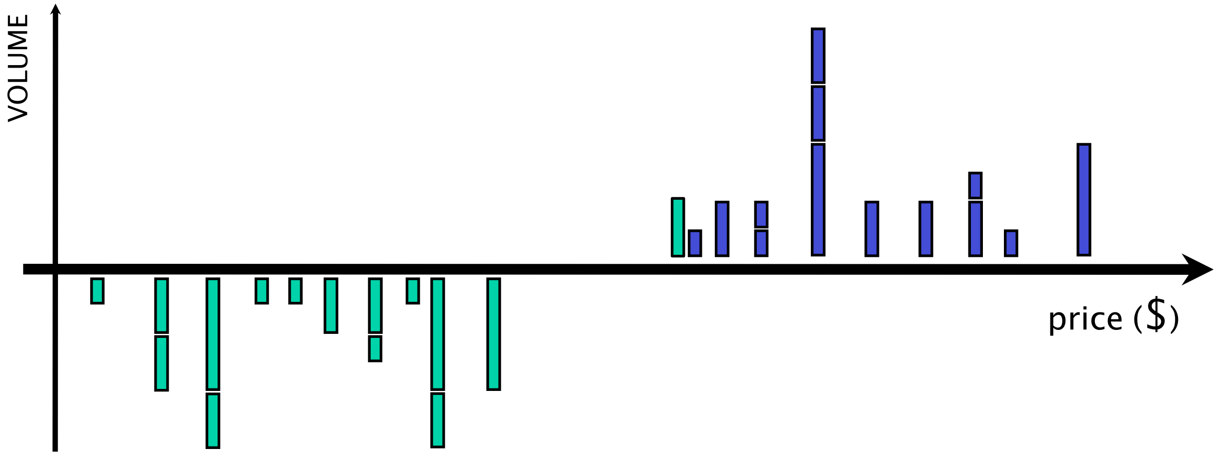

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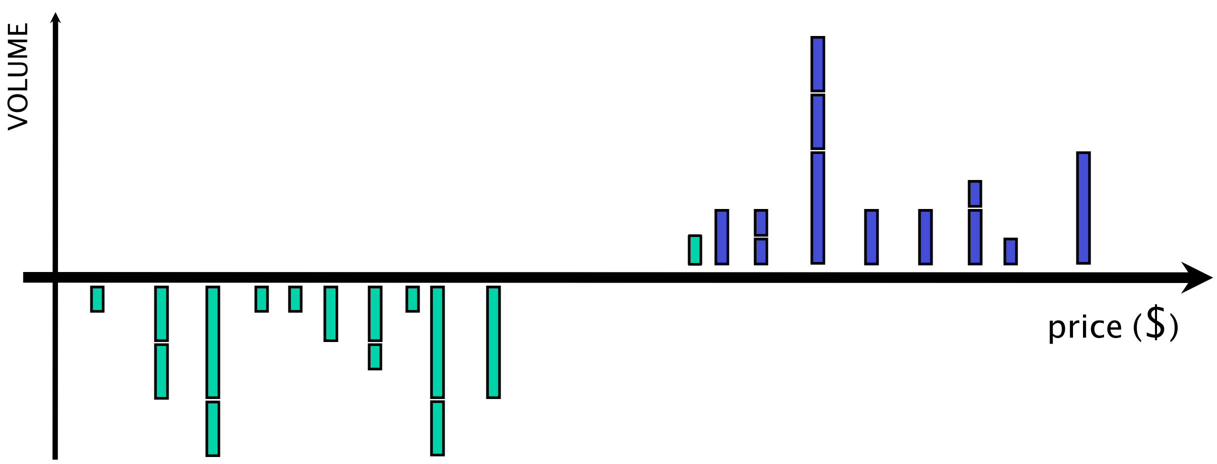

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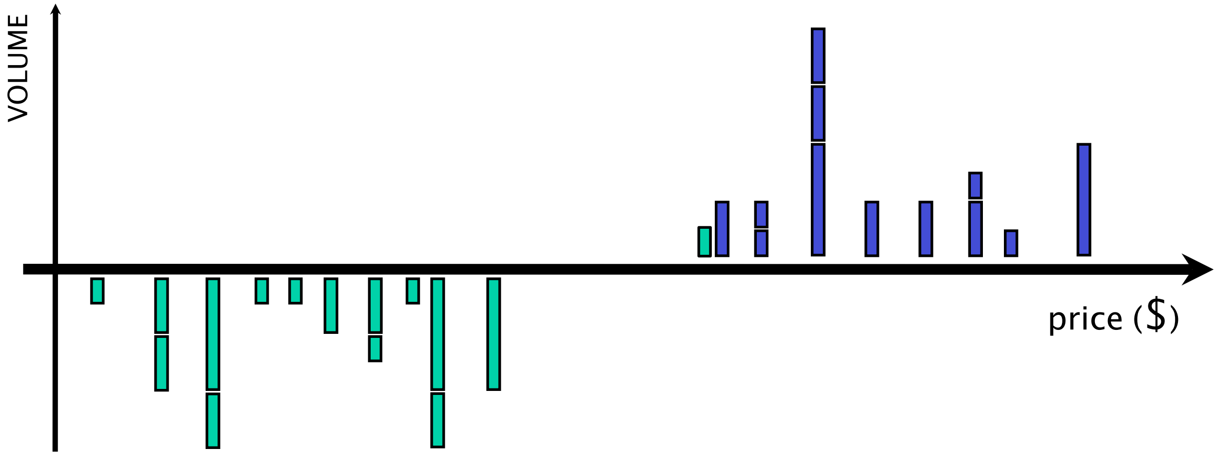

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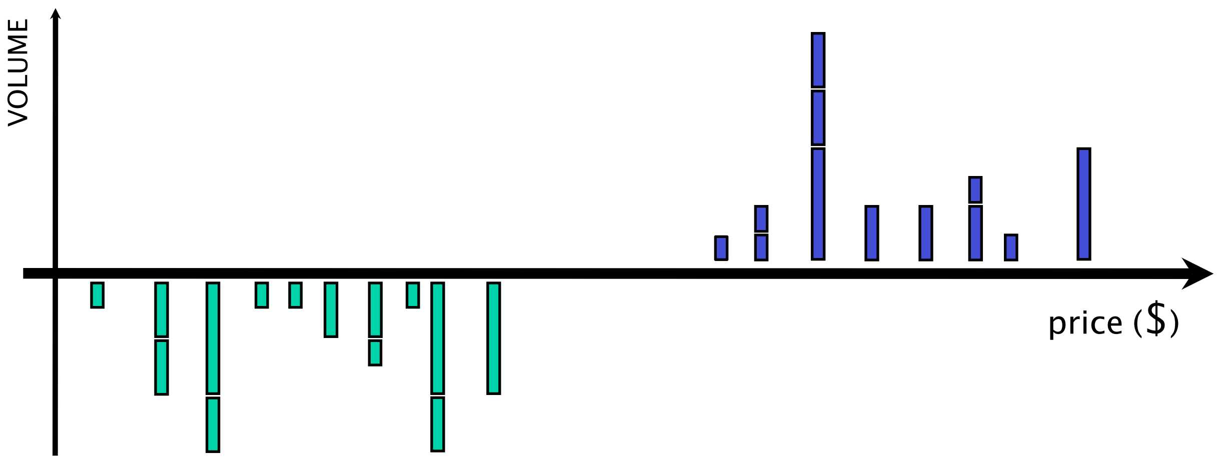

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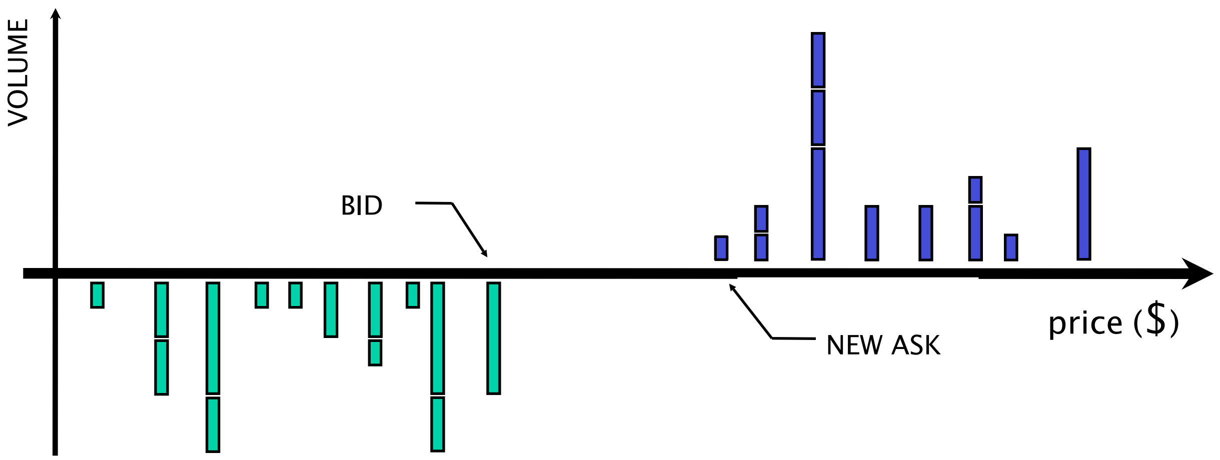
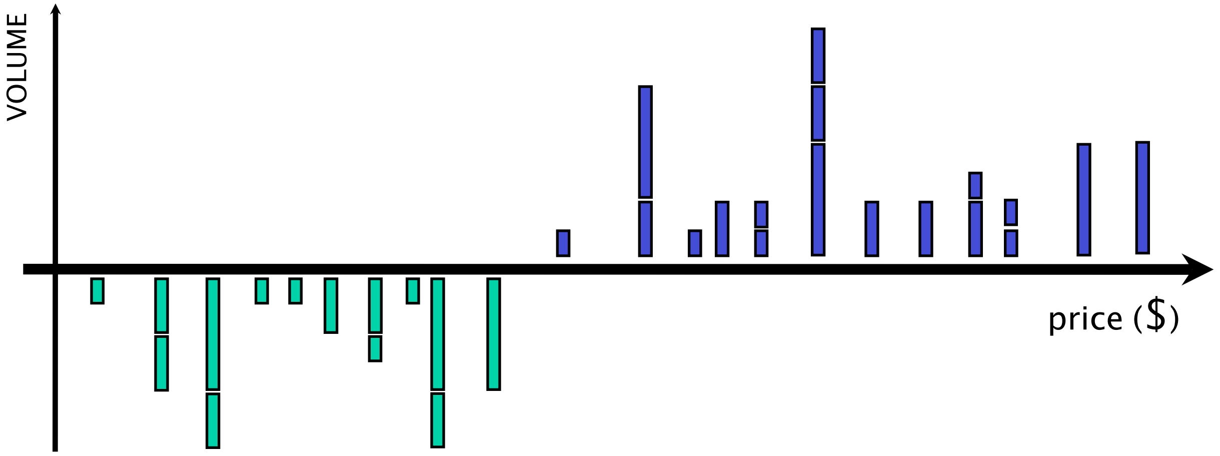


# Order cancellation

**Limit order cancellations:**

- Limit orders can be cancelled by the owner
- Market defined expiration



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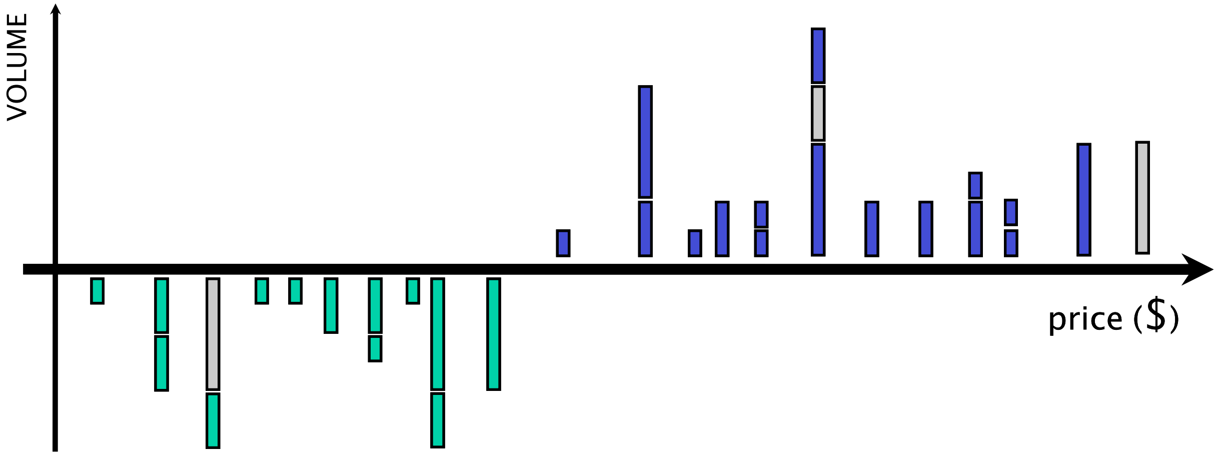

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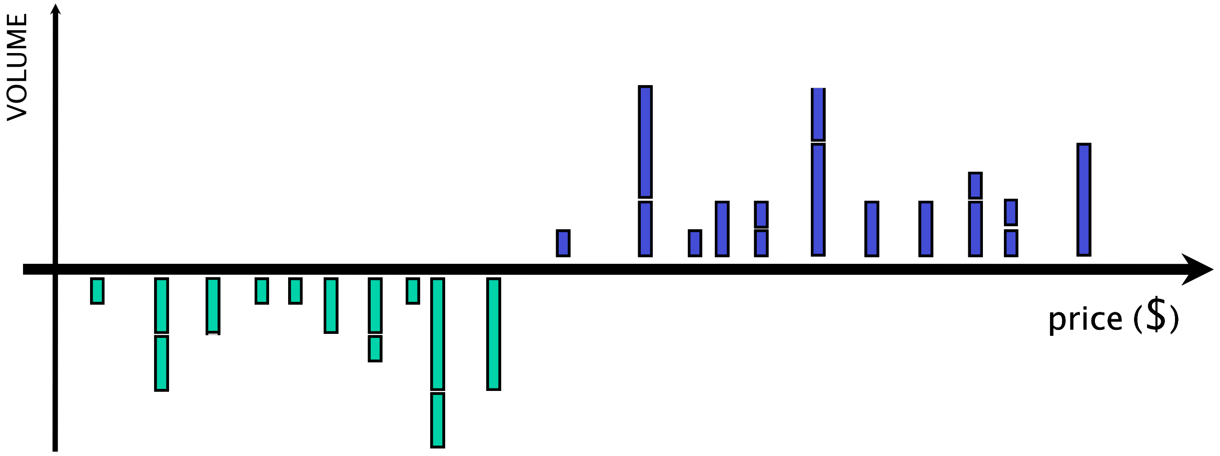


# Basic model

Giulia Iori

Eric Smith

Marcus Daniels

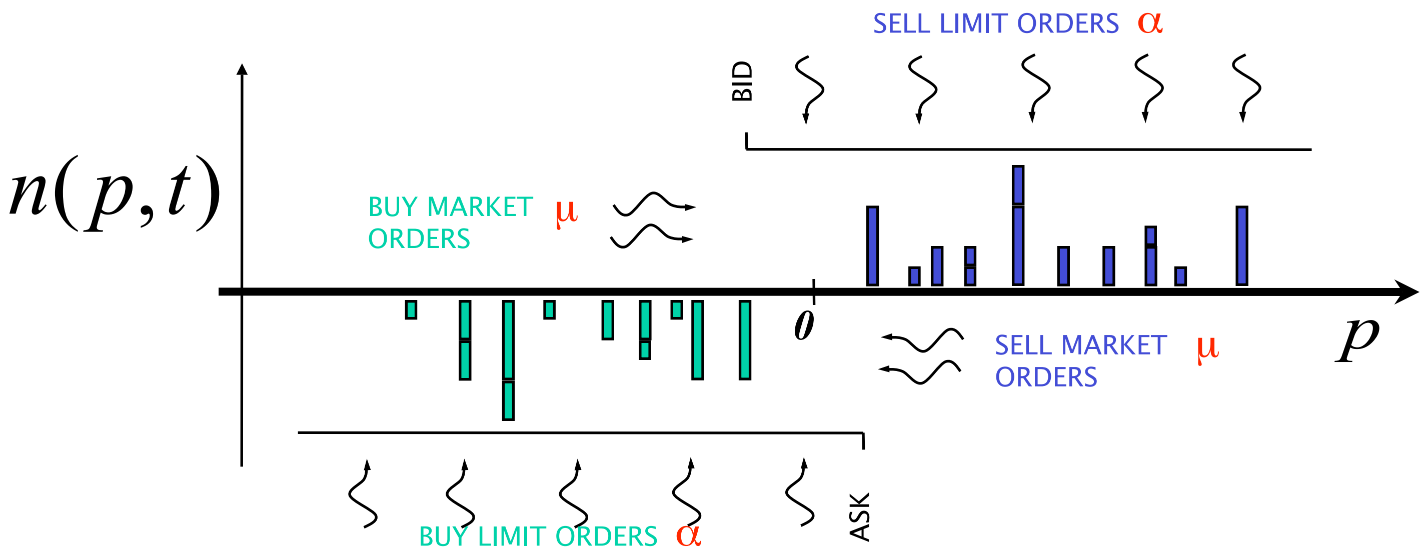
Laszlo Gillemot

Supriya Krishnamurthy

# The Simplest Poisson Model

- Limit order arrival: Poisson process in time & price; $\alpha$
  - Market order arrival: Poisson process in time; $\mu$
  - Cancellation: random in time (like radioactive decay); $\delta$
  - Separate processes for buying and selling, with same parameters.

***Revealed supply and demand*** $n(p,t)$: Number of shares in limit order book at log price $p$, time $t$.



# Analytic treatment

- Like evaporation/deposition problems in physics
- Two approaches
  - Master equation
  - Independent interval analysis (models gaps between orders)
- Approximate mean field theory solutions in steady state
- Agrees well with simulations in most cases.

# Comparison to real data


Paolo Patelli

Ilija Zovko

# London Stock Exchange data set

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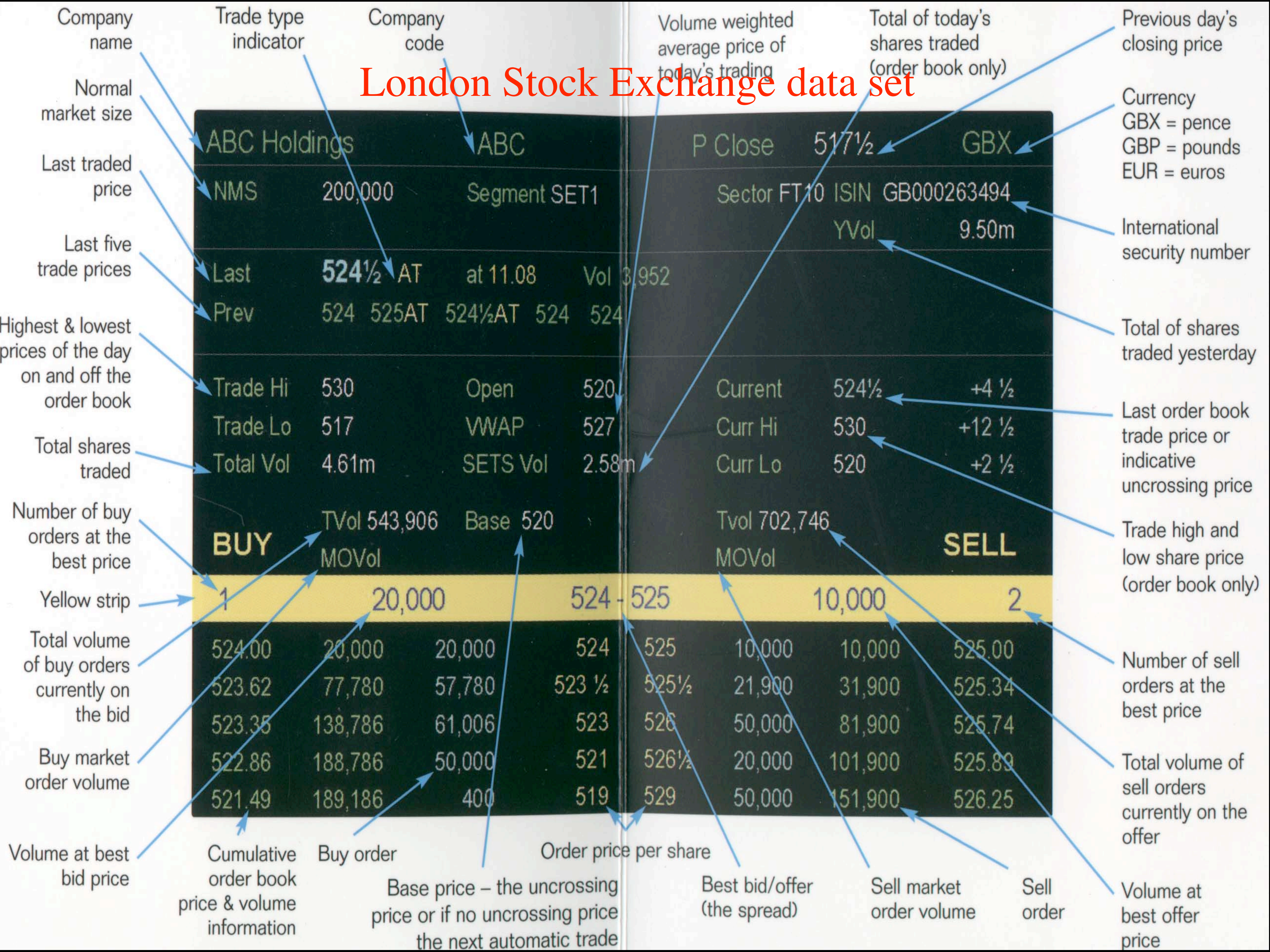


# Parameters of model

$\alpha$ = limit order rate

$\mu$ = market order rate

$\delta$ = order cancellation rate

Order flow rates

$\sigma$ = typical order size

$dp$ = tick size

Discreteness parameters

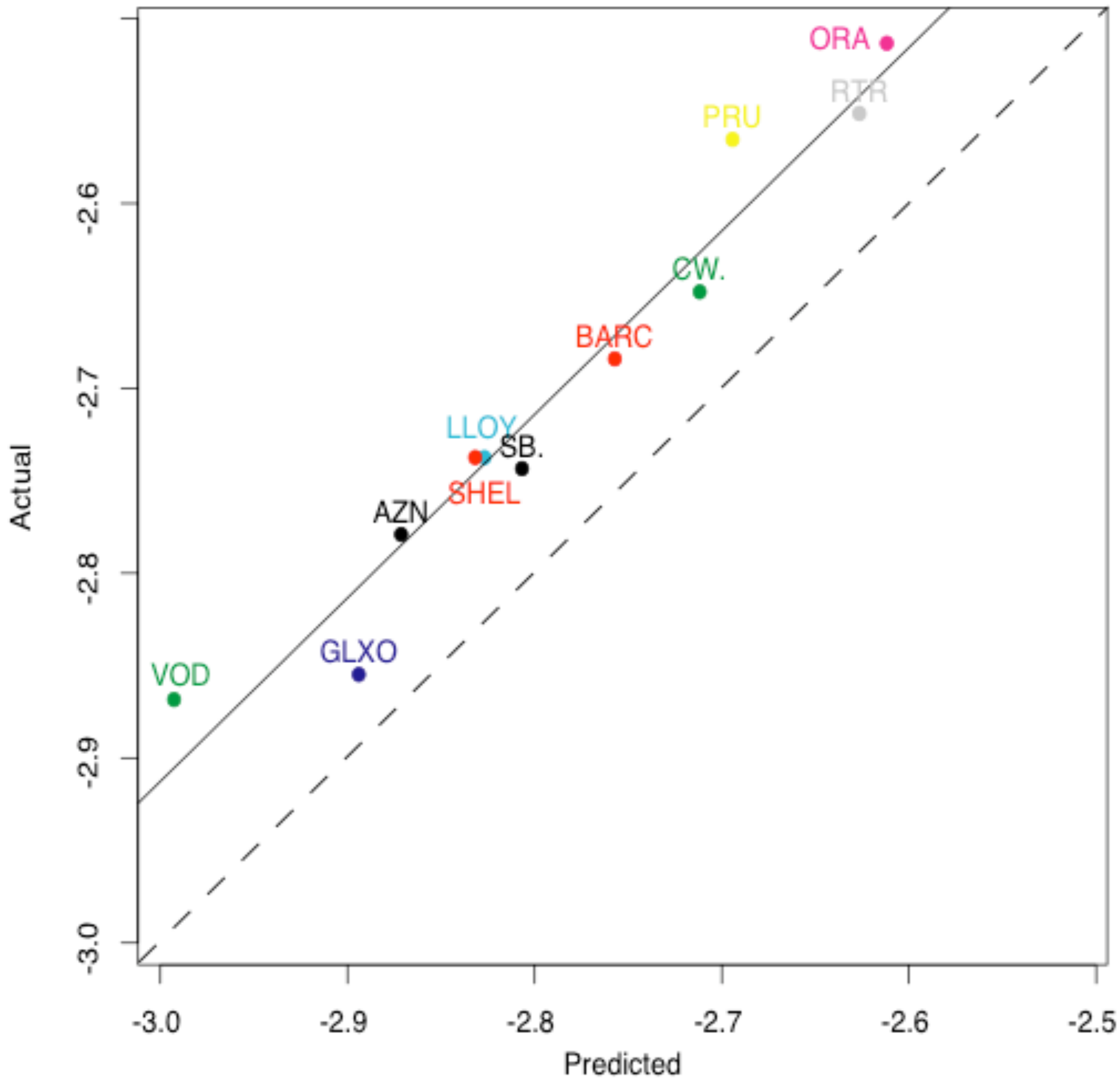
Three fundamental dimensional quantities:
*shares, price, time*

# Testing prediction of spread

- From theory: $\text{mean spread} = \dfrac{\mu}{\alpha} f(\varepsilon)$

$$\varepsilon = \frac{\sigma\delta}{\mu}$$

- From data for each of 11 stocks, measure average spread over 21 month period.
- Measure average parameters of model.
- Compare predictions to actual values.
  - Note: Only one free parameter.


ORA
RTR
PRU
CW.
BARC
LLOY
SB.
SHEL
AZN
GLXO
VOD
-2.6
-2.7
-2.8
-2.9
-3.0
Actual
-3.0
-2.9
-2.8
-2.7
-2.6
-2.5
Predicted

# Predicted price diffusion rate

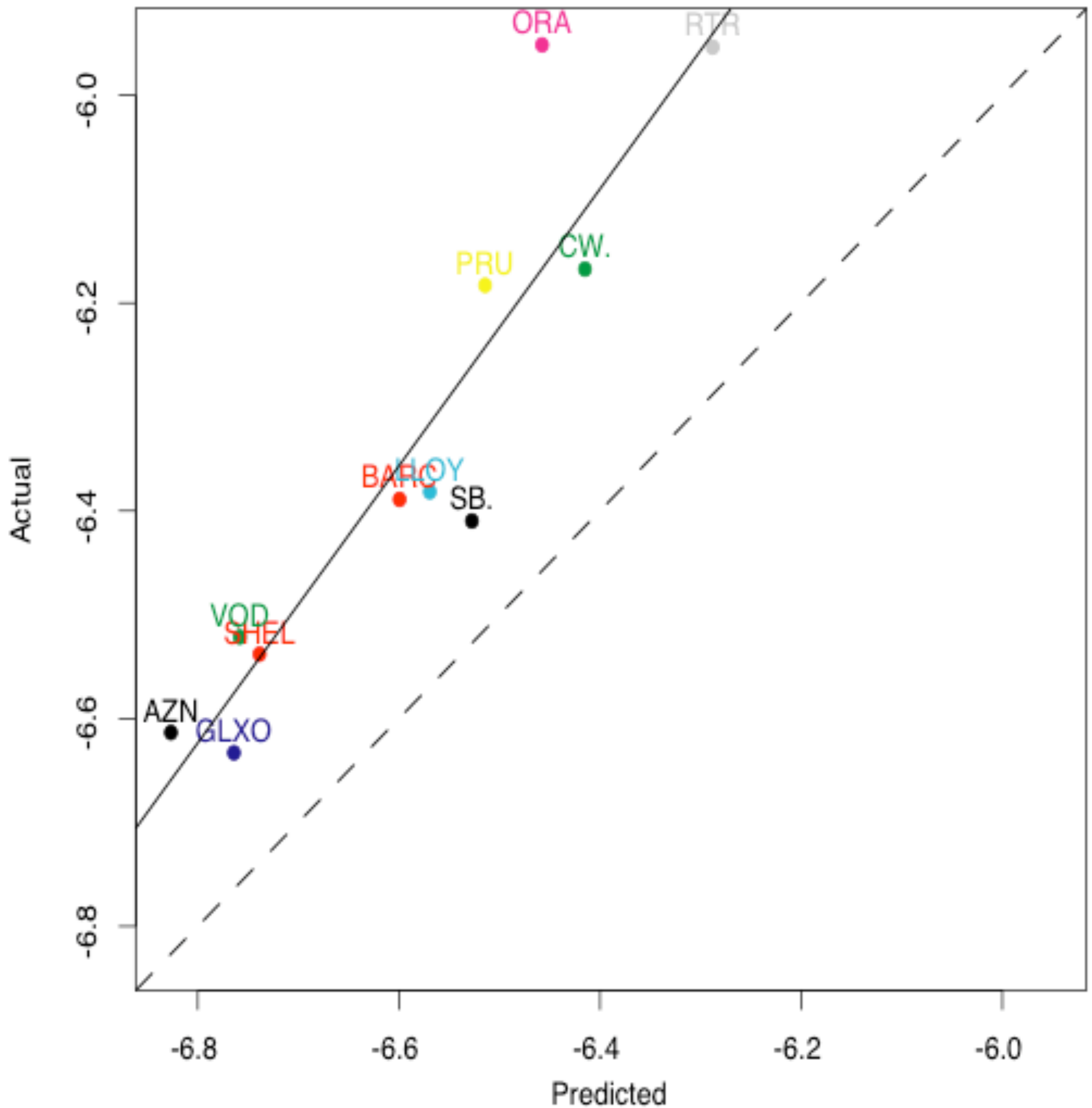
(from simulation)

$$= \frac{\mu^2 \delta}{\alpha^2} \varepsilon^{-1/2}$$

$$= \frac{\mu^{5/2} \delta^{1/2} \sigma^{-1/2}}{\alpha^2}$$

Price diffusion rate D: $Var(\text{random walk}) = Dt$

Price diffusion rate describes volatility

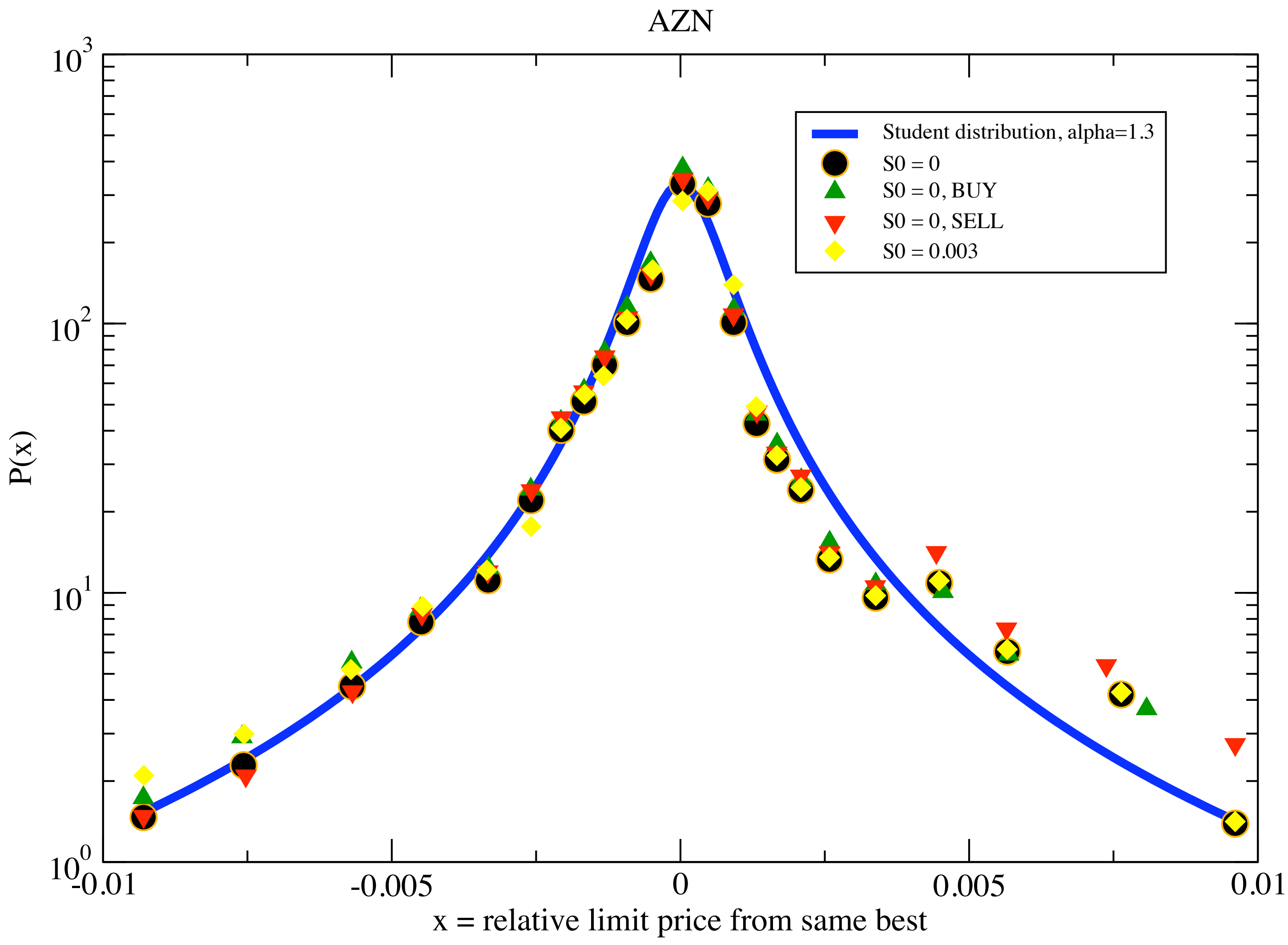


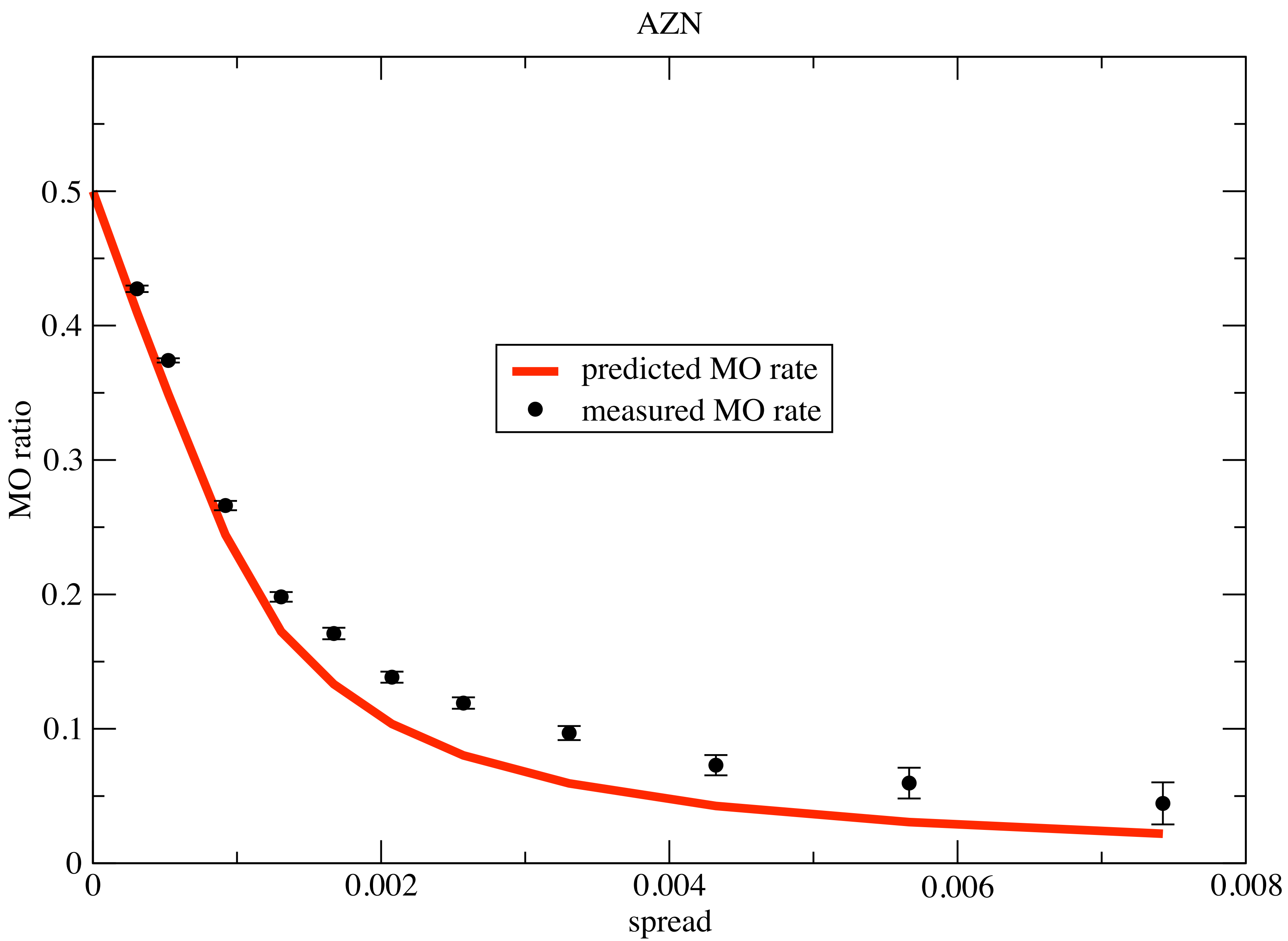
ORA
RTR
CW.
PRU
BARC
LLOY
SB.
VOD
SHEL
AZN
GLXO
-6.0
-6.2
-6.4
-6.6
-6.8
Actual
-6.8
-6.6
-6.4
-6.2
-6.0
Predicted

# Empirical model (with Szabolcs Mike)

- Do same thing but motivate everything based on real data.
- Make phenomenological model of real order flow.
  - Non uniform order placement.
  - Temporally correlated order signs
  - Non-Poisson cancellation process.

# Distribution of order placement relative to best price

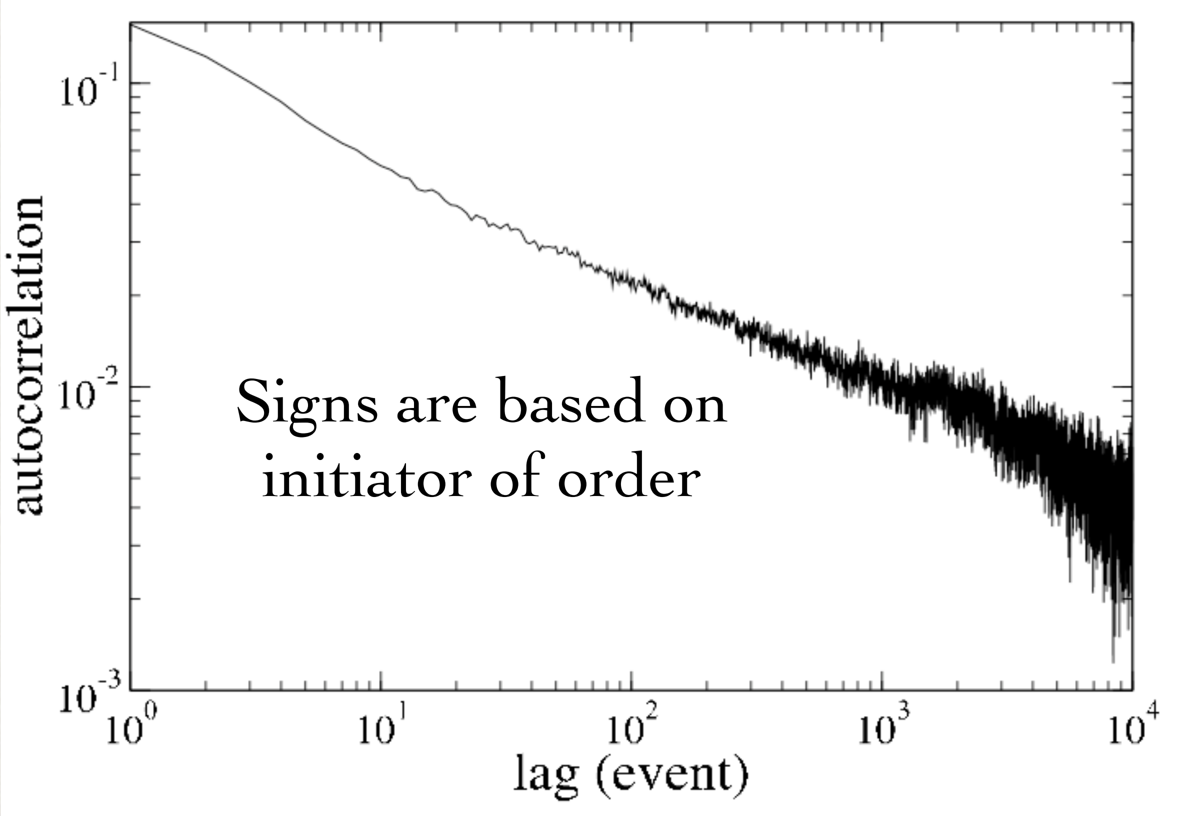




AZN
0.5
0.4
0.3
0.2
0.1
0
MO ratio
0
0.002
0.004
0.006
0.008
spread
predicted MO rate
measured MO rate

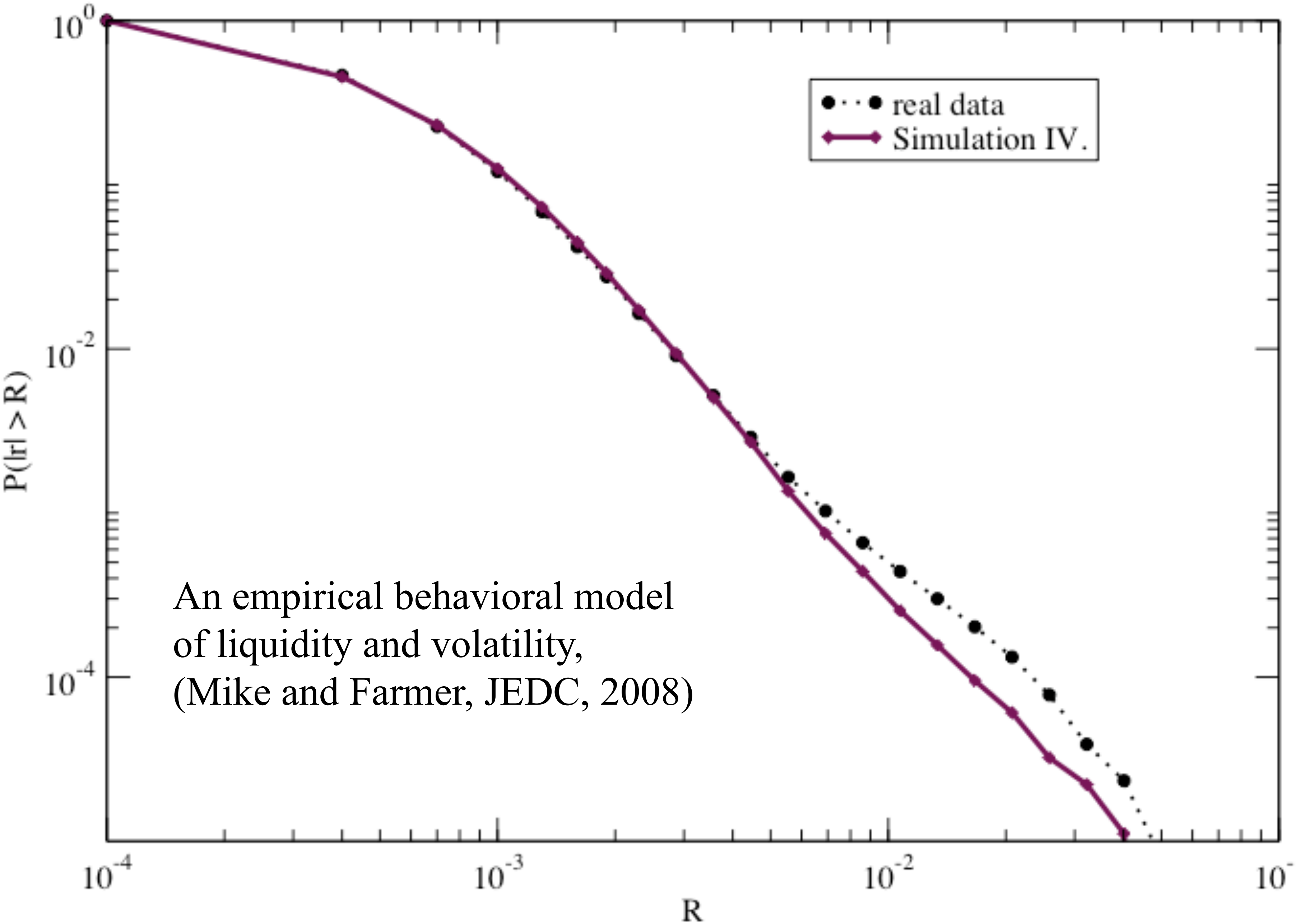
# Autocorrelation of order flow (long memory of supply and demand)

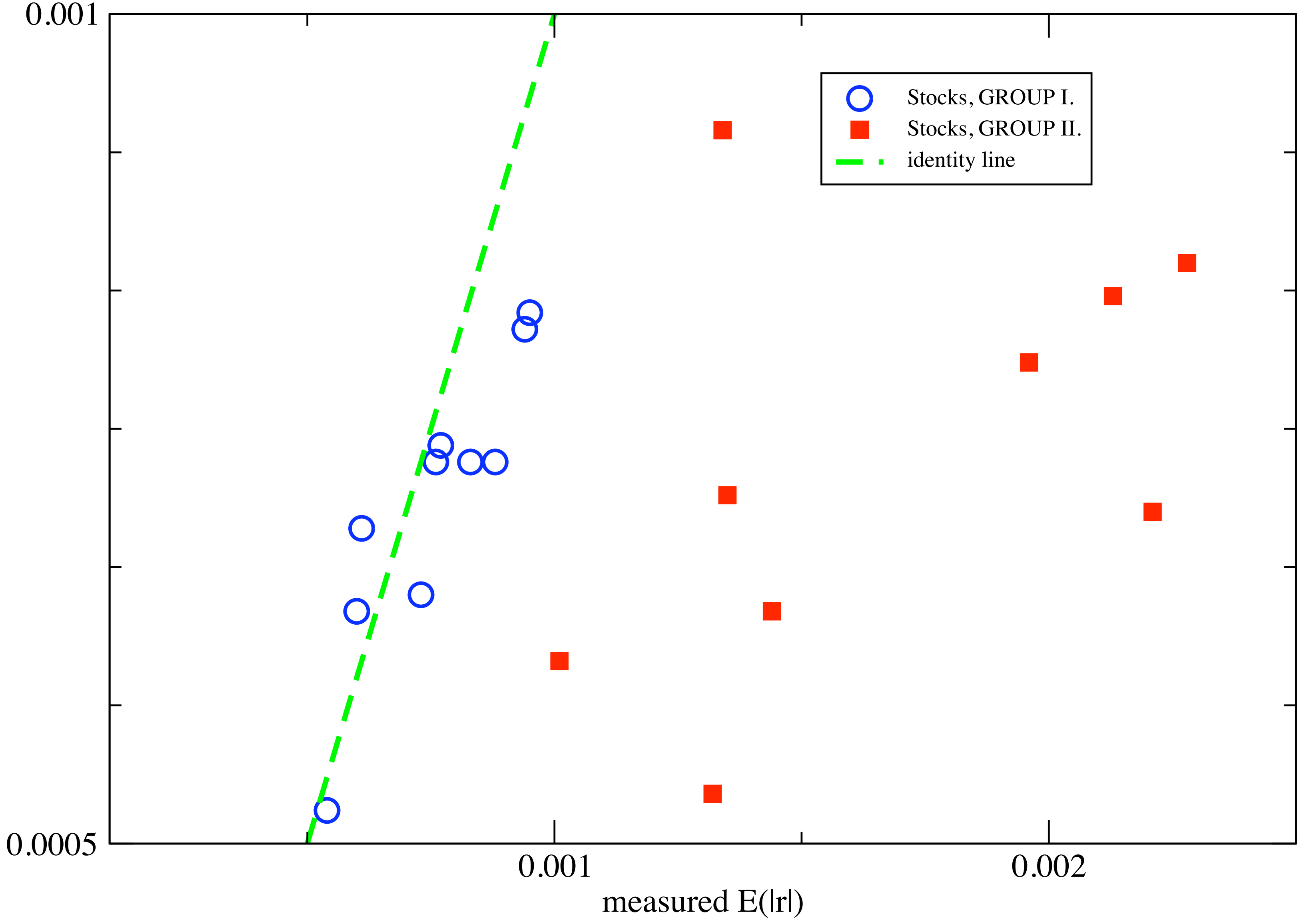
Autocorrelation of trade signs




Bouchaud, Gefen, Potters, and Wyart (2004)
Lillo and Farmer (2004)

# RETURN


real data
Simulation IV.
An empirical behavioral model
of liquidity and volatility,
(Mike and Farmer, JEDC, 2008)
P(|r| > R)
10^0
10^-2
10^-4
10^-4
10^-3
10^-2
10^-1
R



0.001
0.0005
0.001
0.002
measured E(|r|)
predicted E(|r|)
Stocks, GROUP I.
Stocks, GROUP II.
identity line

# Conclusions

- Can treat many financial economic problems with statistical mechanical - like methods.
- Unlike physical problems, particles think - have to put in human behavior. Principles used here:

  ~ market efficiency

  ~ zero intelligence

  ~ behavioral regularities