The Market as a Learning Organism

Larry Blume and David Easley

Cornell University
The Market is Smart

Ultimately, the economy always follows the stock market so, somehow, the stock market knows.

– Ted Andros, Wall Street Plus

Amazingly, the stock market knows better than the analysts do.

– Harry Herrmann, CIO, Waddell & Reed
The market is mad.

The professors overlooked the fact that people, traders included, are not always reasonable. This is the true lesson of Long-Term’s demise. No matter what the models say, traders are not machines guided by silicon chips; they are impressionable and imitative; they run in flocks and retreat in hordes.

– Roger Lowenstein

*When Genius Failed: The Rise and Fall of LTCM*

The stock market has predicted nine out of the last five recessions.

– Paul Samuelson

*Newsweek, 19 Sept. 1966*
How Markets Learn

Iowa Presidential Futures Market

Three mechanisms for market learning:

- Markets **exchange** information among traders, and traders come to learn what others know from market prices.
- Markets **select** traders with the most accurate beliefs.
- Markets **balance** the beliefs of differentially informed traders.

- Infinite horizon exchange economy
- Complete markets, Arrow securities
- No learning

We characterize Pareto optimal consumption paths and their supporting prices.
Definitions

Horse Race Economy

The Environment:

\( S \)  States, \( \{1, \ldots, s\} \).

\( \Sigma \)  Paths, \( \sigma = (\sigma_0, \ldots) \). \( \sigma^t = (\sigma_0, \ldots, \sigma_t) \).

\( n_t^s \)  State counts, \( n_t^s = \#\{\tau \leq t : \sigma_\tau = s\} \).

\( \rho \)  True probability distribution on \( S \); State process is iid.
Traders:

- $c$: A consumption plan $\{c_t\}_{t=1}^{\infty}$, $c_t: \sigma^t \mapsto \mathbb{R}^{++}$.
- $e^i$: Trader $i$’s endowment. . .
- $\rho^i$: Trader $i$’s iid beliefs on $S$.
- $\beta^i$: Trader $i$’s discount factor.
- $u^i: \mathbb{R}_+ \rightarrow \mathbb{R}$: Trader $i$’s payoff to consumption at any partial history.

$$U_i(c) = E_{p^i} \left\{ \sum_{t=0}^{\infty} \beta^i_t u_i(c_t(\sigma^t)) \right\}$$
A.1. The payoff functions $u_i$ are $C^1$, strictly concave, strictly monotonic, and satisfy an Inada condition at 0.

A.2. Each trader has a strictly positive at every partial history, and the aggregate endowment is uniformly bounded, below away from zero and from above.

A.3. At every partial history, each trader $i$ believes all truly possible states to be possible.
If $c^*$ is a Pareto optimal allocation of resources, then there is a strictly positive vector of welfare weights $(\lambda^1, \ldots, \lambda^I)$ such that $c^*$ solves

$$\max_{(c^1, \ldots, c^I)} \sum_i \lambda^i U_i(c^i)$$

such that

$$\sum_i c^i - e \leq 0$$

$$\forall t, \sigma \ c^i_t(\sigma^t) \geq 0$$

where $e_t = \sum_i e^i_t$. 

Blume and Easley 

The Market Organism
For all $t$ there is a random variable $\eta_t : \sigma^t \mapsto \mathbb{R}_{++}$ such that

$$\lambda^i \beta_i^t u_i'(c_i^t(\sigma^t)) \prod_s \rho_i(s)^{n_i^s(\sigma)} - \eta_t(\sigma^t) = 0$$

almost surely, and

$$\sum_i c_i^t(\sigma^t) = e_t(\sigma^t)$$
Def. A present value price system is $p = \{p_t\}_{t=0}^{\infty}$, $p_t : \sigma_t \mapsto \mathbb{R}_{++}$ such that, for each trader $i$, $p \cdot e^i < \infty$.

Def. A competitive equilibrium is a present value price system $p^*$ and a consumption plan $c^{i*}$ for each trader such that . . .

Existence is due to Peleg and Yaari (1970), and the first welfare theorem is elementary.

Def. $q^s_t(\sigma^t)$ is the price of the Arrow security that pays off in partial history $(\sigma^t, s)$ in terms of consumption at partial history $\sigma^t$. That is, $q$ is the current value price system.
**A.4.** The aggregate endowment is history-independent. $e_t(\sigma) \equiv e \gg 0$. *For interpretative purposes only.*

**Proposition:** Assume A.1–3. Then for each path $\sigma$ at each date $t$ and for all $\epsilon > 0$ there is a $\delta > 0$ such that if $|c^i_t(\sigma^t) - e| < \delta$, then $||q_t(\sigma^t) - \rho^i|| < \epsilon$. If, in addition, A.4 holds, then if all traders have identical beliefs $\rho^i$, for all dates $t$ and paths $\sigma$, $q_t^s(\sigma^t) = \rho^i$.

If all traders have *correct* beliefs, then the price at $\sigma^t$ of the asset that pays off in state $s$ at date $t + 1$ is $\rho_s$, the probability state $s$ occurs.
Given the uncertainty of the real world, the many actual and virtual traders will have many, perhaps equally many, forecasts. . . If any group of traders was consistently better than average in forecasting stock prices, they would accumulate wealth and give their forecasts greater and greater weight. In this process, they would bring the present price closer to the true value.

– R. Cootner, 1967
Selecting Bettors

\[
\log \frac{u'_i(c^i_t(\sigma^t))}{u'_j(c^j_t(\sigma^t))} = t \log \frac{\beta_j}{\beta_i} + \sum_s n^s_t(\sigma) \log \frac{\rho_j(s)}{\rho_i(s)}
\]

This is something to which the law of large numbers can be applied.
How far are $i$'s beliefs from the truth?

Relative Entropy

$$l_i = \sum_s \rho^s \log \frac{\rho^s}{\rho^s_i}$$

Limit MU ratios:

$$\frac{1}{t} \log \frac{u'_i(c^s_t(\sigma^t))}{u'_j(c^s_t(\sigma^t))} = \log \frac{\beta_j}{\beta_i} - \sum_s \rho_s \left( \log \frac{\rho_s}{\rho^i_s} - \log \frac{\rho_s}{\rho^j_s} \right) -$$

$$\frac{1}{t} \sum_s (n^s_t(\sigma) - t\rho_s) \left( \log \frac{\rho_s}{\rho^i_s} - \log \frac{\rho_s}{\rho^j_s} \right)$$

$$\rightarrow (\log \beta_j - l_j) - (\log \beta_i - l_i)$$
Define the Fitness Index $f_i = \log \beta_i - l_i$.

Assume A.1–3. If $f_i < \max_j f_j$, then trader $i$ vanishes.
Define the **Fitness Index** \( f_i = \log \beta_i - l_i \).

Assume A.1–3. If \( f_i < \max_j f_j \), then trader \( i \) vanishes.

If there is a unique trader \( i \) with maximal survival index \( f_i \) among the trader population, then market prices converge to \( \rho^i \) almost surely.
What is the logical basis for interpreting the price of an all-or-nothing futures contract as a market probability that the event will occur?

Suppose several traders have maximal fitness index.

\[
\log \frac{u'_i(c^i_t(\sigma^t))}{u'_j(c^j_t(\sigma^t))} = - \sum_s (n^s_t(\sigma) - t\rho_s) \left( \log \rho^j_s - \log \rho^i_s \right)
\]

The RHS is a random walk. \( \limsup \text{RHS} = +\infty \), so

\[
\liminf c^i_t = 0.
\]

Are traders with maximal fitness disappearing, or is the market volatile?
For distribution $\theta$ on $S$,

$$\text{lo}(\theta) = \left( \log \frac{\theta(s)}{\theta(1)} \right)_{s=2}^{s}$$

Trader $i$ is interior if $\text{lo}(\rho^i)$ is in the relative interior of $\text{Conv}\{\text{lo}(\rho^j)\}_{j=1}^{l}$. She is extremal if $\text{lo}(\rho^i)$ is an extreme point, that is, not a non-negative linear combination of the other $\text{lo}(\rho^j)$, and boundary otherwise.
If $s \leq 3$,

- All maximally fit traders survive.
- For extremal traders, $\limsup_t c^i_t/e = 1$.

If $s > 3$,

- Interior traders vanish.
- For extremal traders, $\limsup_t c^i_t/e = 1$. 
If $s \leq 3$, 
- All maximally fit traders survive.
- For extremal traders, $\limsup_t c_t^i/e = 1$.

If $s > 3$, 
- Interior traders vanish.
- For extremal traders, $\limsup_t c_t^i/e = 1$.

Survival requires enormous patience and extreme beliefs.
If multiple traders are maximally fit, then for all extremal traders $i$ and all $\epsilon > 0$, $|q_t - \rho^i| < \epsilon$ infinitely often. If $s > 3$ it is possible that for $\epsilon > 0$ sufficiently small, the event $|q_t - \rho| < \epsilon$ is transient, even if some survivor has rational expectations.
Even with corners, $f_i < f_j$ implies $c_t^i \to 0$ a.s.

CARA utility: $u_i(c) = -\frac{1}{\gamma_i} \exp -c/\gamma_i, l_i = l_j$. 
Even with corners, $f_i < f_j$ implies $c_t^i \to 0$ a.s.

CARA utility: $u_i(c) = -\frac{1}{\gamma_i} \exp(-c/\gamma_i), \ l_i = l_j.$

$$\log \frac{u_i'(c_t^i(\sigma^t))}{u_j'(c_t^j(\sigma^t))} = -\sum_s (n_t^s(\sigma) - t\rho_s) \left( \log \rho_s^j - \log \rho_s^i \right)$$
Even with corners, \( f_i < f_j \) implies \( c_t^i \to 0 \) a.s.

CARA utility: \( u_i(c) = -\frac{1}{\gamma_i} \exp -c/\gamma_i, i = i. \)

\[
\frac{c_j}{\gamma_j} - \frac{c_i}{\gamma_i} = -\sum_s \left(n_s^x(\sigma) - t\rho_s\right) \left(\log \rho_s^j - \log \rho_s^i\right)
\]
True probability versus beliefs such that
\[ l_i = 0.2. \]

Long run average price is correct only when \( \rho = 1/2. \)
Concluding Comments

- Noise Traders
Concluding Comments

- Noise Traders
- Selection in Other Environments
Concluding Comments

- Noise Traders
- Selection in Other Environments
- Other Pricing Implications
Concluding Comments

- Noise Traders
- Selection in Other Environments
- Other Pricing Implications
- Other Selection Results
Random Walks

D \rightarrow \text{Minus} \rightarrow A

D \rightarrow \text{Minus} \rightarrow B

D \rightarrow \text{Minus} \rightarrow C

D \rightarrow \text{Minus} \rightarrow E

E \rightarrow \text{Minus} \rightarrow A

E \rightarrow \text{Minus} \rightarrow B

E \rightarrow \text{Minus} \rightarrow C

E \rightarrow \text{Minus} \rightarrow D
$u^i(c) = \log c$, identical discount factors.

$$\log \frac{c^i_t(\sigma^t)}{c^j_t(\sigma^t)} = \log \frac{\lambda_i}{\lambda_j} + \sum_s n_s^t(\sigma^t) \log \frac{\rho_i(s)}{\rho_j(s)}$$

Consumption shares evolve as do \textit{Bayesian posteriors}.
Limit log MU ratios:

\[
\log \frac{u_i'(c_t^i(\sigma^t))}{u_j'(c_t^j(\sigma^t))} = \log \frac{\beta_j}{\beta_i} - \sum_s \rho_s \left( \log \frac{\rho_s}{\rho_s^i} - \log \frac{\rho_s}{\rho_s^j} \right)
- \sum_s \left( n_t^s(\sigma) - t\rho_s \right) \left( \log \frac{\rho_s}{\rho_s^i} - \log \frac{\rho_s}{\rho_s^j} \right)
\]
Limit log MU ratios:

\[
\log \frac{u'_i(c^i_t(\sigma^t))}{u'_j(c^j_t(\sigma^t))} = \log \frac{\beta_j}{\beta_i} - \sum_s \rho_s \left( \log \frac{\rho_s}{\rho^i_s} - \log \frac{\rho_s}{\rho^j_s} \right) \\
- \sum_s (n^s_t(\sigma) - t\rho_s) \left( \log \frac{\rho_s}{\rho^i_s} - \log \frac{\rho_s}{\rho^j_s} \right)
\]
Multiple Survivors

Limit log MU ratios:

$$\log \frac{u'_i(c^i_t(\sigma^t))}{u'_j(c^j_t(\sigma^t))} = 0$$

$$- \sum_s (n^s_t(\sigma) - t\rho_s) \left( \log \frac{\rho^i_s}{\rho^j_s} - \log \frac{\rho^i_s}{\rho^j_s} \right)$$