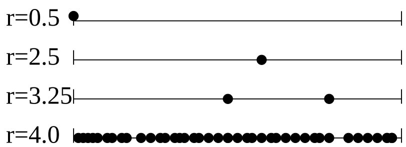
Introduction to Chaos Part II

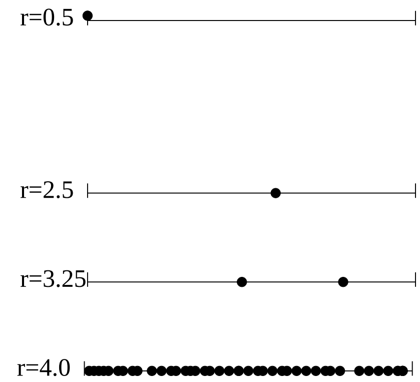
We have seen several possible long-term behaviors for the logistic equation:

1. $r = 0.5$: attracting fixed point at 0.
2. $r = 2.5$: attracting fixed point at 0.6.
3. $r = 3.25$: attracting cycle of period 2.
4. $r = 4.0$: chaos.

Graphically, we can illustrate this as follows:

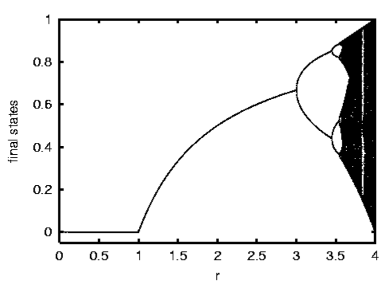


- I.e., for each r, iterate and plot the final x values as dots on the number line.
- What else can the logistic equation do??



- Do this for more and more r values and “glue” the lines together.
- Turn sideways and ...

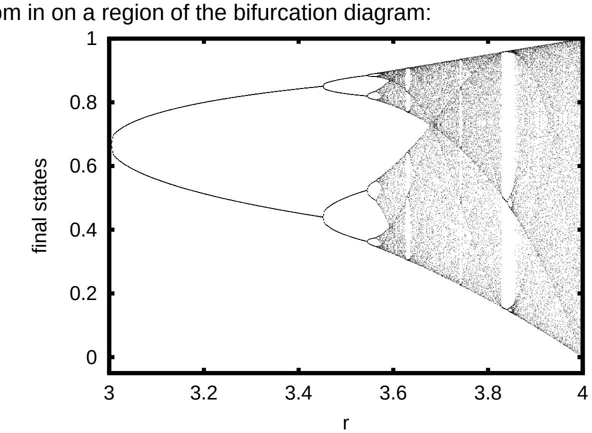
Bifurcation Diagram



- The bifurcation diagram shows the all the the possible long-term behaviors for the logistic map.
- $0 < r < 1$, the orbits are attracted to zero.
- $1 < r < 3$, the orbits are attracted to a non-zero fixed point.
- $3 < r < 3.45$, orbits are attracted to a cycle of period 2.
- Chaotic regions appear as dark vertical lines.

Bifurcation diagram, continued

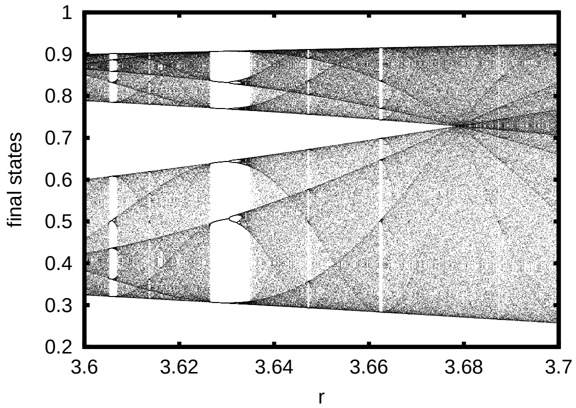
Let’s zoom in on a region of the bifurcation diagram:



- The sudden qualitative changes are known as **bifurcations**.
- There are **period-doubling bifurcations** at $r \approx 3.45$, $r \approx 3.544$, etc.
- Note the window of period 3 near $r = 3.83$.

Bifurcation diagram, continued

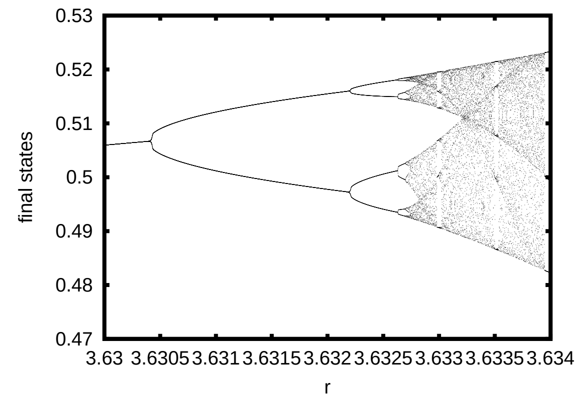
Let's zoom in again:



- Note the sudden changes from chaotic to periodic behavior.

Bifurcation diagram, continued

Let's zoom in once more:



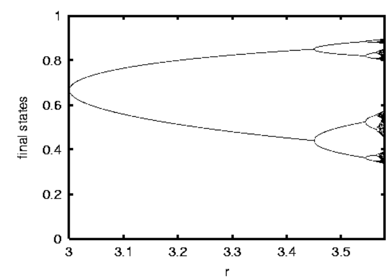
- Note the small scales on the vertical axis, and the tiny scale on the y axis.
- Note the self-similar structure. As we zoom in we keep seeing sideways pitchforks.

Bifurcation Diagram Summary

- As we vary r, the logistic equation shuffles suddenly between chaotic and periodic behaviors, but the bifurcation diagram reveals that these transitions appear in a structured, or regular, way.
- In the next few slides we'll examine one of the regularities in the bifurcation diagram: The **period-doubling route to chaos**.

Period-Doubling Route to Chaos

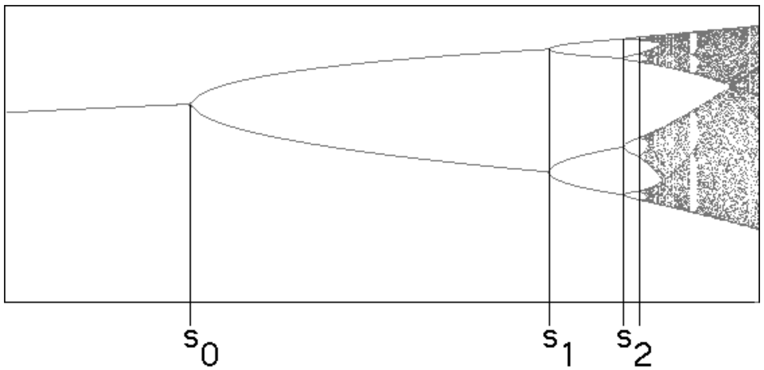
- As r is increased from 3, a sequence of period doubling bifurcations occur.



- At $r = r_\infty \approx 3.569945672$ the periods "accumulate" and the map becomes chaotic.
- For $r > r_\infty$ it has SDIC. For $r < r_\infty$ it does not.
- This is a type of **phase transition**: a sudden qualitative change in a system's behavior as a parameter is varied continuously.

Period-Doubling Route to Chaos: Geometric Scaling

- Let's examine the ratio of the lengths of the pitchfork tines in the bifurcation diagram.

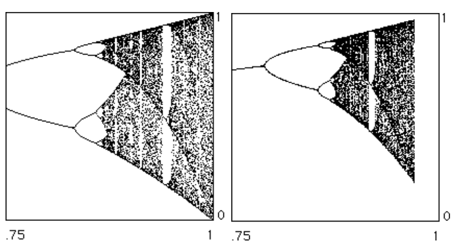


- The first ratio is: $\delta_1 = \frac{s_1 - s_0}{s_2 - s_1}$.
- The n$^{\mathrm{th}}$ ratio is: $\delta_n = \frac{s_n - s_{n-1}}{s_{n+1} - s_n}$.

Feigenbaum's Constant

- This ratio approaches a limit: $\lim_{n\to\infty} \delta_n = 4.669201609\ldots$. This is known as **Feigenbaum's constant** δ.
- This means that the bifurcations occur in a regular way.
- Amazingly, the value of δ is **universal**: it is the same for any period-doubling route to chaos!
- Figure Source: `http://classes.yale.edu/fractals/Chaos/Feigenbaum/Feigenbaum.html`

Universality



- The figure on the left is the bifurcation diagram for $f(x) = r\sin(\pi x)$.
- The figure on the right is the bifurcation diagram for $f(x) = \frac{27}{4} r x^2 (1 - x)$.
- The bifurcation diagrams are very similar: **both have** $\delta \approx 4.6692$.
- Mathematically, things are constrained so that there is, in some sense, only one possible way for a system to undergo a period-doubling to chaos.
- Figure Source: `http://classes.yale.edu/fractals/Chaos/LogUniv/LogUniv.html`

Experimental Verification of Universality

- Universality isn't just a mathematical curiosity. Physical systems undergo period-doubling order-chaos transitions. Almost miraculously, these systems also appear to have a universal δ.
- Experiments have been done on fluids, circuits, acoustics:
  - Water: $4.3 \pm .8$
  - Mercury: $4.4 \pm .1$
  - Diode: $4.5 \pm .6$
  - Transistor: $4.5 \pm .3$
  - Helium: $4.8 \pm .6$

  Data from Cvitanović, *Universality in Chaos*, World Scientific, 1989.
- A very simple equation, the logistic equation, has produced a quantitative prediction about complicated systems (e.g., fluid turbulence) that has been verified experimentally.
- Nature is somehow constrained.

Detour: A Little Bit More About Universality

- The order-disorder phase transition in the logistic map is not the only sort of phase transition that is universal.
- Second order (aka continuous) phase transitions are also universal.
- There are several different universality classes, each of which has different values for quantities analogous to δ.
- The symmetry of the order parameter and the dimensionality of the space of the system determine the universality class.
- The order parameter is a quantity which is zero on one side of the transition and non-zero on the other.
- The theory of critical phenomena does not tell one how to find order parameters. Sometimes order parameters are not obvious.

A Little Bit More About Universality, continued

- At the transition point, or **critical point**, some quantities (e.g., specific heat) usually diverge. The divergence is described by a power law. The exponents for these power laws are called **critical exponents**.
- At the critical point, the correlations between components of the system usually decay with a power law as the distance increases. Away from the critical point, the decay is exponential—much faster.

A Little Bit about Power Laws

- At critical points, functions like the specific heat diverge with a power law.
- This divergence arises because the correlations between the system's parts is long range—the corrections decay with a power law, not an exponential.
- Power-law decay of correlations is an indication that the system is organized or complex.
- However, this does not mean that the only way that power law distributions can be formed is via long-range order or correlations or complexity.
- In fact, there are very simple mechanisms for producing power laws.

Simple Ways to Make A Power Law Distribution

Exponentially Observing Exponential Distribution

- Suppose a quantity is growing exponentially: $X(t) = e^{\mu t}$.
- Suppose we measure the quantity at a random time T, obtaining the value $\bar{X} = e^{\mu T}$.
- Let T also be exponentially distributed: $Pr(T > t) = e^{-\nu T}$.
- Then the probability density for $\bar{X}$ is given by $f_{\bar{X}}(x) = kx^{-\mu/\nu - 1}$.
- Like magic, a power law has appeared.
- In general, there are lots of ways to make power laws by combining exponential distributions in different ways.
- See Reed and Hughes, Why power laws are so common in nature. Physical Review E 66:067103. 2002. `http://www.math.uvic.ca/faculty/reed/`.

Simple Ways to Make A Power Law Distribution, Continued

Multiplicative Noise:

- Define a random variable X as the product of a number of other random variables. In many cases X will be distributed with a power law.
- See, e.g., Sornette, Multiplicative processes and power laws. Physical Review E 57, 4811. 1998. `arXiv.org/cond-mat/9708213`.

Warning about Empirical Power Laws!

- Concluding that a distribution is actually a power is a potentially subtle matter.
- It is **not** a good idea to do a linear fit on a log-log plot!!
- For much more, see A. Clauset, C.R. Shalizi, and M.E.J. Newman, ”Power-law distributions in empirical data” E-print (2007). `arXiv.org/physics/0706.1062`. This is an extraordinary paper.
- Source code in matlab and R for Clauset, et al, is available at `http://www.santafe.edu/~aaronc/powerlaws/`.
- For additional commentary, see Shalizi, “So you think you have a power law—Well isn’t that special?” `http://cscs.umich.edu/~crshalizi/weblog/491.html`.
- See also, M.E.J. Newman, Power laws, Pareto distributions and Zipf’s law, *Contemporary Physics* **46**, 323-351 (2005). `arXiv.org/cond-mat/0412004`.

Power Law Conclusions

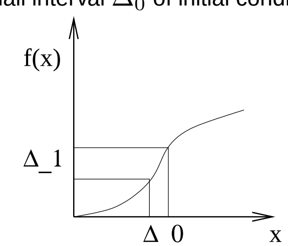
- There are many simple, non-complex ways to make power laws.
- They are not necessarily an indicator of complexity or correlation or organization.
- They are not necessarily an indicator of criticality—of a system on the edge of a phase transition.
- Many of the claims in the literature for the existence of power laws may be based on faulty data analysis.

This ends the interlude on power laws. We now return to chaotic dynamics...

Measuring Sensitive Dependence: Lyapunov Exponent

SDIC arises because the function pushes nearby points apart. The Lyapunov exponent measures this pushing.

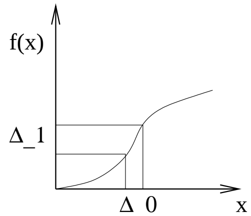
- Consider an initial small interval Δ_0 of initial conditions centered at x_0.



- After one iteration, this interval becomes $\Delta_1 \approx |f'(x_0)|\Delta_0$.
- $|f'(x_0)|$ is the local stretch (or shrink) factor.
- After n iterations, $\Delta_n = \prod_n |f'(x_n)|\Delta_0$.
- The idea is that for the n$^{\rm th}$ iterate interval is getting stretched (shrunk) by the stretch factor $f'(x)$ evaluated at the x_n, the location of the n$^{\rm th}$ iterate of x_0.

Lyapunov Exponent, continued

- We expect the growth of the interval Δ_0 to be exponential, since we're multiplying the interval at each time step.



- That is, we expect that $\frac{\Delta_n}{\Delta_0} = e^{\lambda n}$, where λ is the exponential growth rate.
- The exponential growth is just the the product of all the local stretch factors along an itinerary:

$$e^{\lambda n} = \prod_n |f'(x_n)| \ .$$

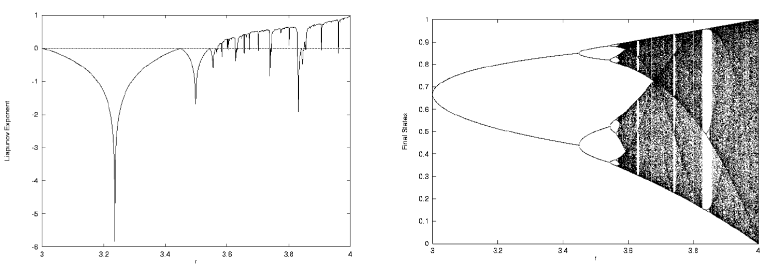
Lyapunov Exponent, continued

- Solving for λ:

$$\lambda \equiv \lim_{N\to\infty} \left[\frac{1}{N} \sum_n \ln |f'(x_n)| \right] \ . \quad (1)$$

- λ is the **Lyapunov exponent**. It measures the degree of SDIC.
- If $\lambda > 0$, the function has SDIC.

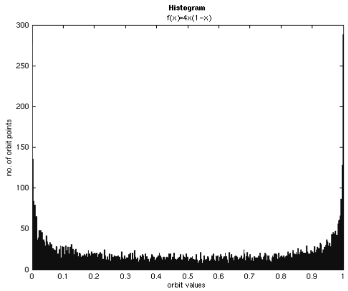
Lyapunov Exponent for the Logistic Equation



- The top graph shows the Lyapunov exponent as a function of r.
- Note that $\lambda > 0$ in the chaotic regions of the bifurcation diagram.

Initial Conditions?

- It seems like the definition of λ depends on the initial condition. If so, λ is a property of x_o, and not a global property of f.
- It turns out that for many dynamical systems you will get the same λ for almost all x_0. Why is this?
- Imagine building a histogram for orbits. For $r = 4$, this will look like:



- http://www-m8.mathematik.tu-muenchen.de/personen/hayes/chaos/Hist.html

Natural Invariant Densities and Ergodicity

- The distribution in this histogram $\rho(x)$ will be obtained by iterating almost any initial condition x_0.
- This distribution is known as the **Natural Invariant Density**.
- If we can figure it out, we can determine the Lyapunov exponent by integrating:

$$\lambda = \int \ln(|f'(x)|)\rho(x)\, dx \ .$$

- In general, if a dynamical property like the lyapunov exponent can be determined by integrating over x instead of performing a dynamical average, the system is **ergodic**.
- Proving that a system is ergodic is usually very hard.
- Trivia: for $r = 4$, $\rho(x) = \frac{\pi}{\sqrt{x(1-x)}}$.
- For other r values, an expression for $\rho(x)$ is not known. Generally, $\rho(x)$ is non-smooth.

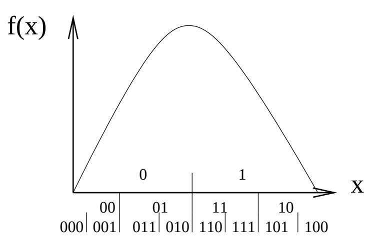
Symbolic Dynamics

- It is often easier to study dynamical systems via symbolic dynamics.
- The idea is to encode the continuous variable x with a discrete variable in some clever way that doesn't entail a loss of information.
- For the logistic equation:

$$s_i = \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} .$$

- Why is this ok? It seems that we're throwing out a lot of information!
  - The the function is deterministic, the initial condition contains all information about the itinerary.
  - For the coding above, longer and longer sequences of 1's and 0's code for smaller and smaller regions of initial conditions.
  - Codings that have this property are known as **generating partitions**.

Symbolic Dynamics, continued



- If we find a generating partition, we can use the symbols to explore the function's properties.
- The symbol sequences are "the same" as the orbits of x: they have the same periodic points, the same stability, etc.
- For $r = 4$, the symbolic dynamics of the logistic equation produce a sequence of 0's and 1's that is indistinguishable from a fair coin toss.
- For more, see the talks by Hao Bai-lin and Emily Burkhead.

Chaos Conclusions

- Deterministic systems can produce random, unpredictable behavior. E.g., logistic equation with $r = 4$.
- Simple systems can produce complicated behavior. E.g., long periodic behavior in logistic equation.
- Some features of dynamical systems are universal—the same for many different systems.
- More generally, dynamics are important. Considering only static averages can be misleading.

Chaos ⇒ Complex Systems

Some of the roots of complex systems are in chaos:

- Universality gives us some reason to believe that we can understand complicated systems with simple models.
- Appreciation that complex behavior can have simple origins.
- Awareness that there's more to dynamical systems than randomness. These systems also make patterns, organize, do cool stuff.
- Is there a way we can describe or quantify these patterns?
- Is there a quantity like the Lyapunov exponent that measures complexity or pattern or structure?
- What is a pattern?