From Determinism to Stochasticity Measurement Theory II

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From Determinism to Stochasticity
Measurement Theory II

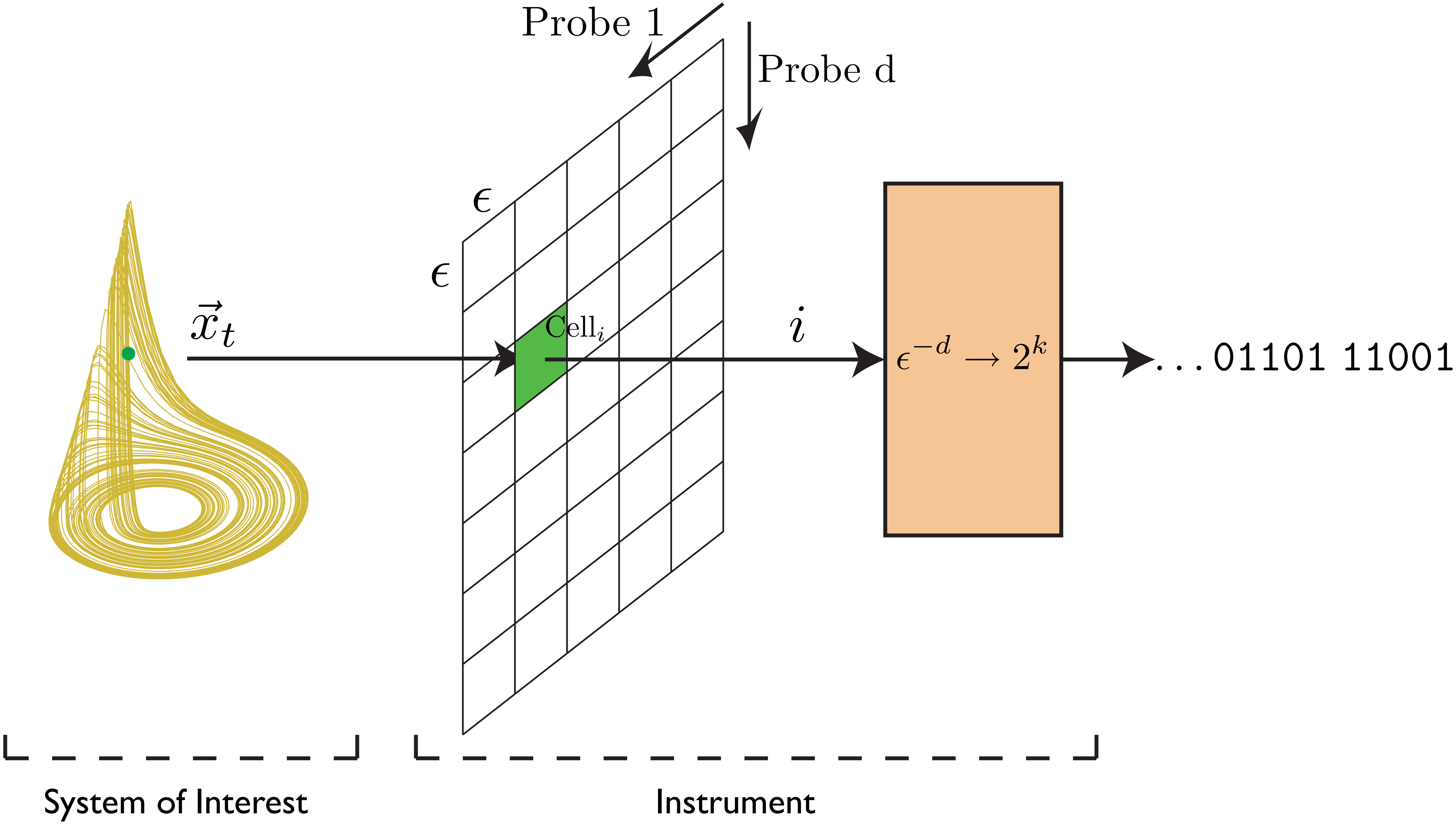
Previous Dynamics Lecture:

When measurements are faithful
study infinite discrete sequences
to learn about continuous-state dynamical system.

Now:

1. Most general faithful instrument
2. When measurements mislead
3. General stochastic processes

From Determinism to Stochasticity ...



Measurement Channel

From Determinism to Stochasticity ...

Measurement Theory ...

Markov partition very stringent.

Generating partitions:

Most general idea of "good" instrument.

Easier to find than Markov partitions, which may not exist.

Only requirement: Sequences track individual orbits

$$||\Delta(s^L)|| \rightarrow 0, \ \forall s^L \in \Sigma_f$$

Requires chaos:

Instability translates into reverse-time shrinking of cells.

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Measurement Theory ...

Some facts about partitions:

Markov partitions are generating.

Markov partition reduces Perron-Frobenius operator to
finite-dimensional stochastic transition matrix.

Resulting process is modeled by finite, probabilistic Markov chain.

Generating partitions may lead to
finite- or infinite-dimensional
hidden Markov models.

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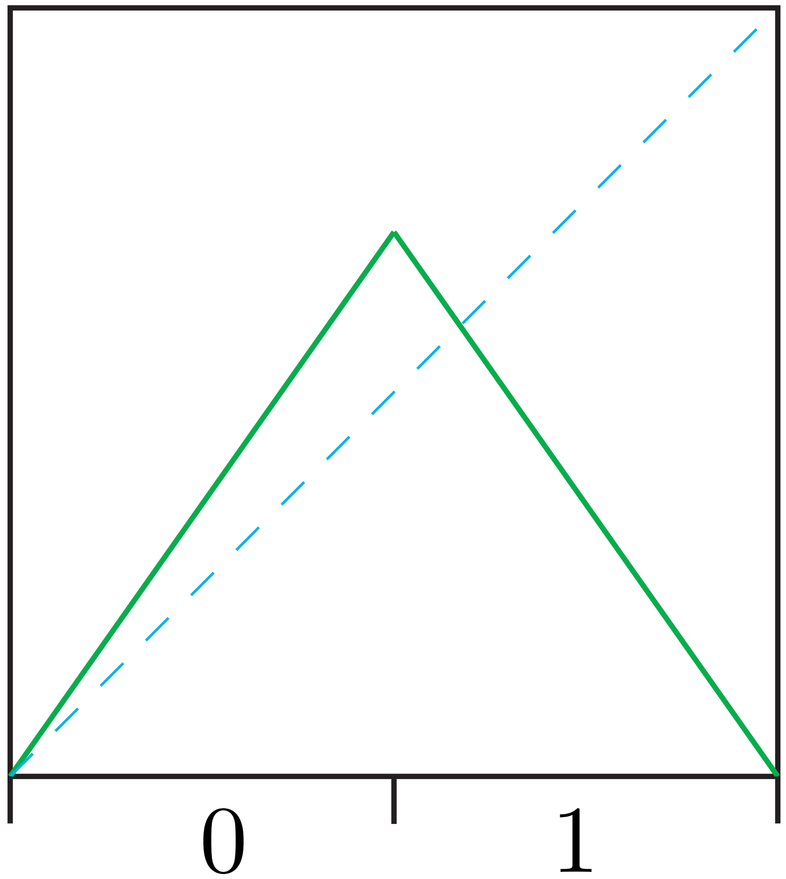
Measurement Theory ...

Generating partition for Tent map:

At any parameter (slope > 1):

$$\mathcal{P} = \{0 \sim [0, \tfrac{1}{2}], 1 \sim [\tfrac{1}{2}, 1]\}$$



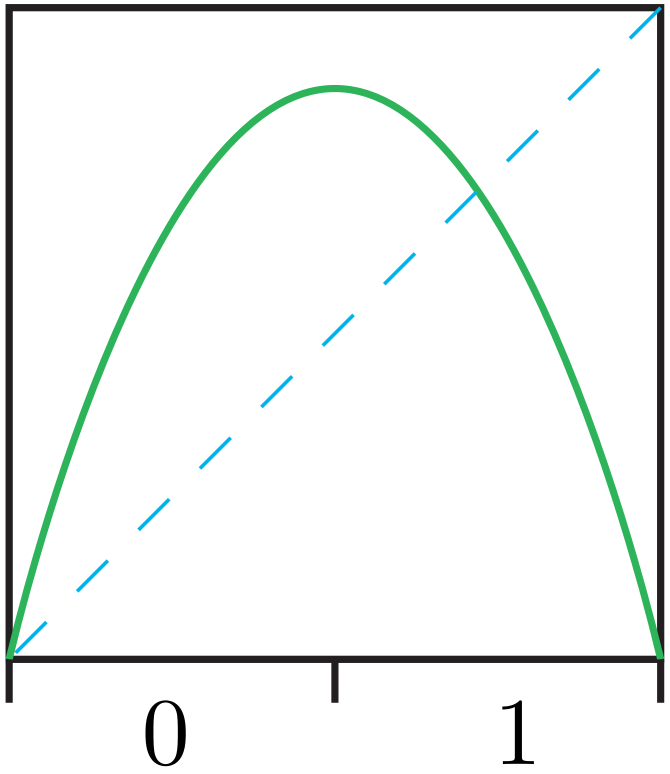
Not Markov!
except at slope = 2.

Measurement Theory ...

Generating partition for Logistic map:

At any parameter:

$$\mathcal{P} = \{0 \sim [0, \tfrac{1}{2}], 1 \sim [\tfrac{1}{2}, 1]\}$$



Not Markov!
except at r = 4.

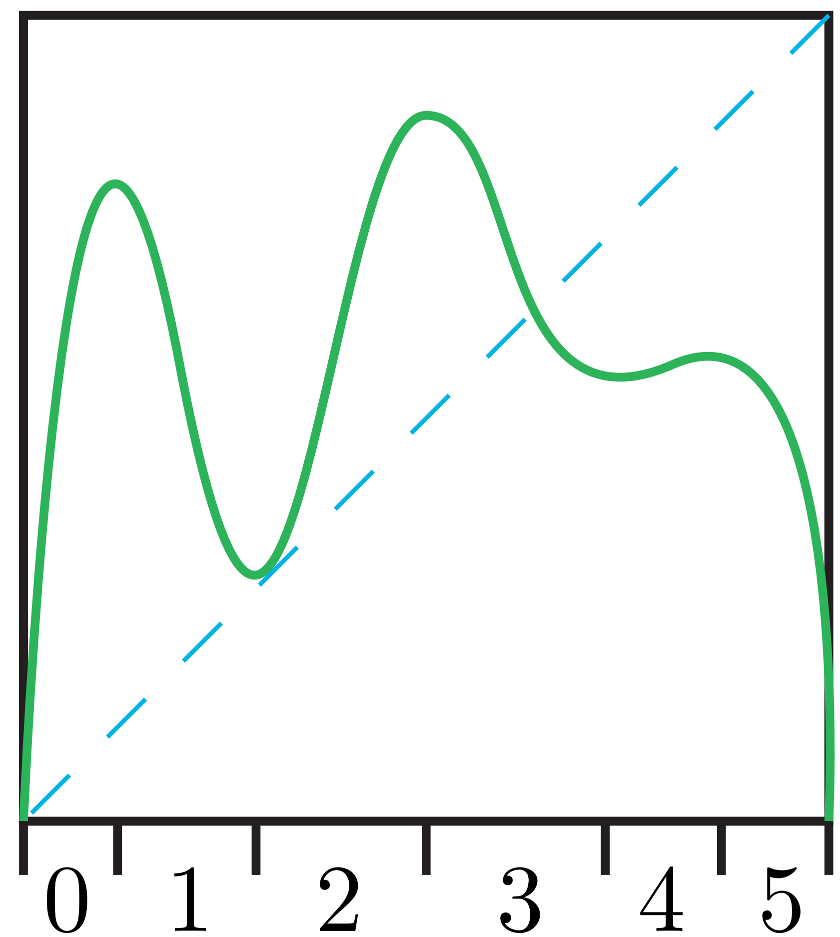
Measurement Theory ...

Generating partition for general 1D maps:

Lap: Monotone piece of $f(x)$

Partition:

$$\mathcal{P} = \{\text{domain of lap}(f)\}$$



Theorem: If map is chaotic, P is generating.

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Measurement Theory ...

What happens when there is

no Markov partition and

no generating partition?

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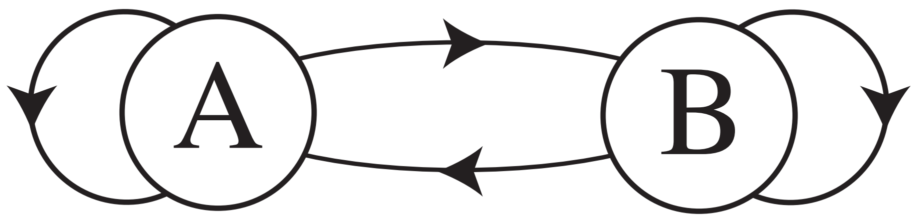
Example of a nongenerating partition:

Logistic map at 2 onto 1

Internal Markov Model:



Measurement alphabet:

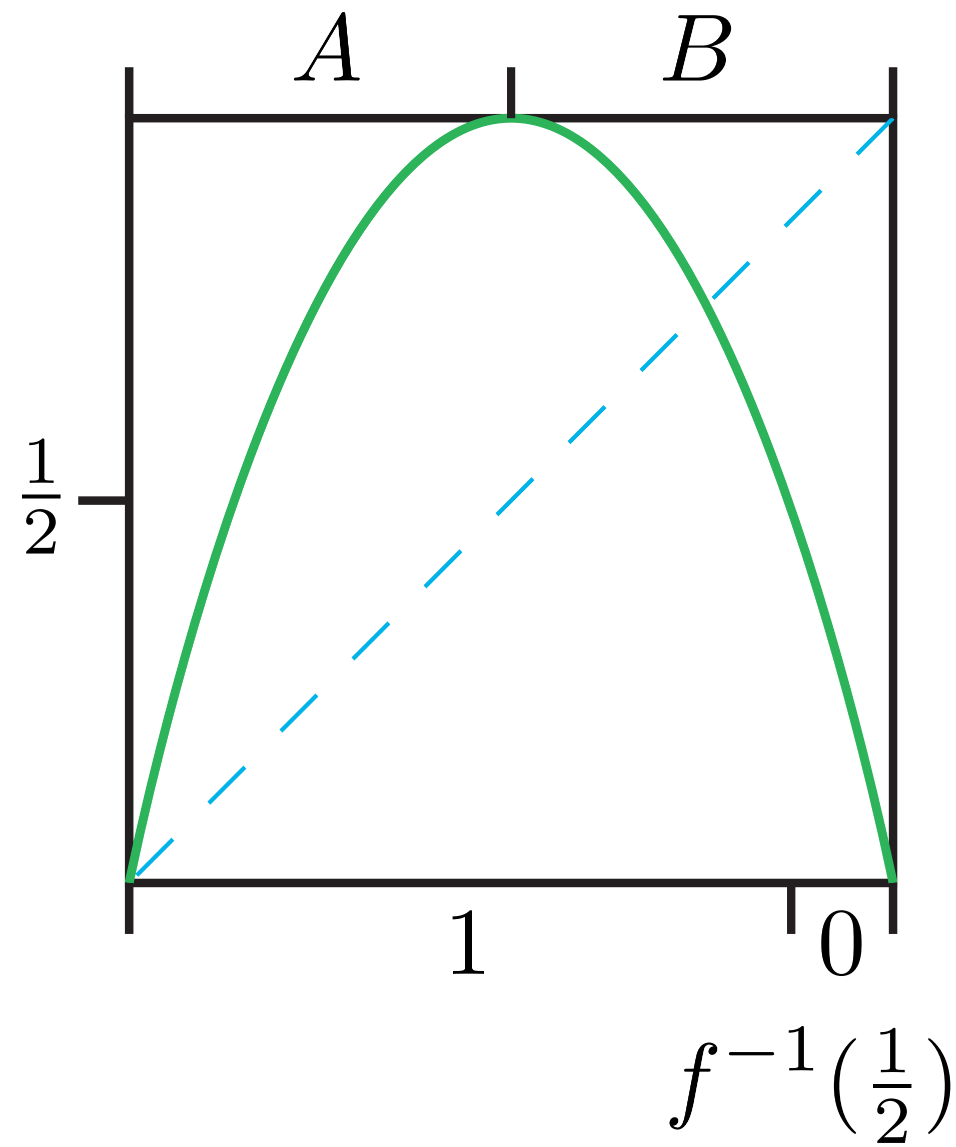
$\{0, 1\}$

Measurement partition:

$$\mathcal{P} = \{1 \sim [0, d], 0 \sim (d, 1]\}$$

Decision point:

$$d = \max \{f^{-1}(\tfrac{1}{2})\}$$



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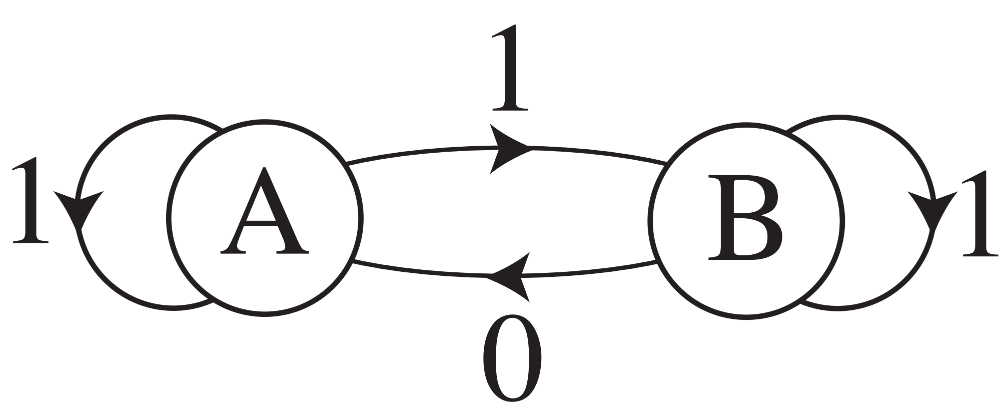
Measurement Theory ...

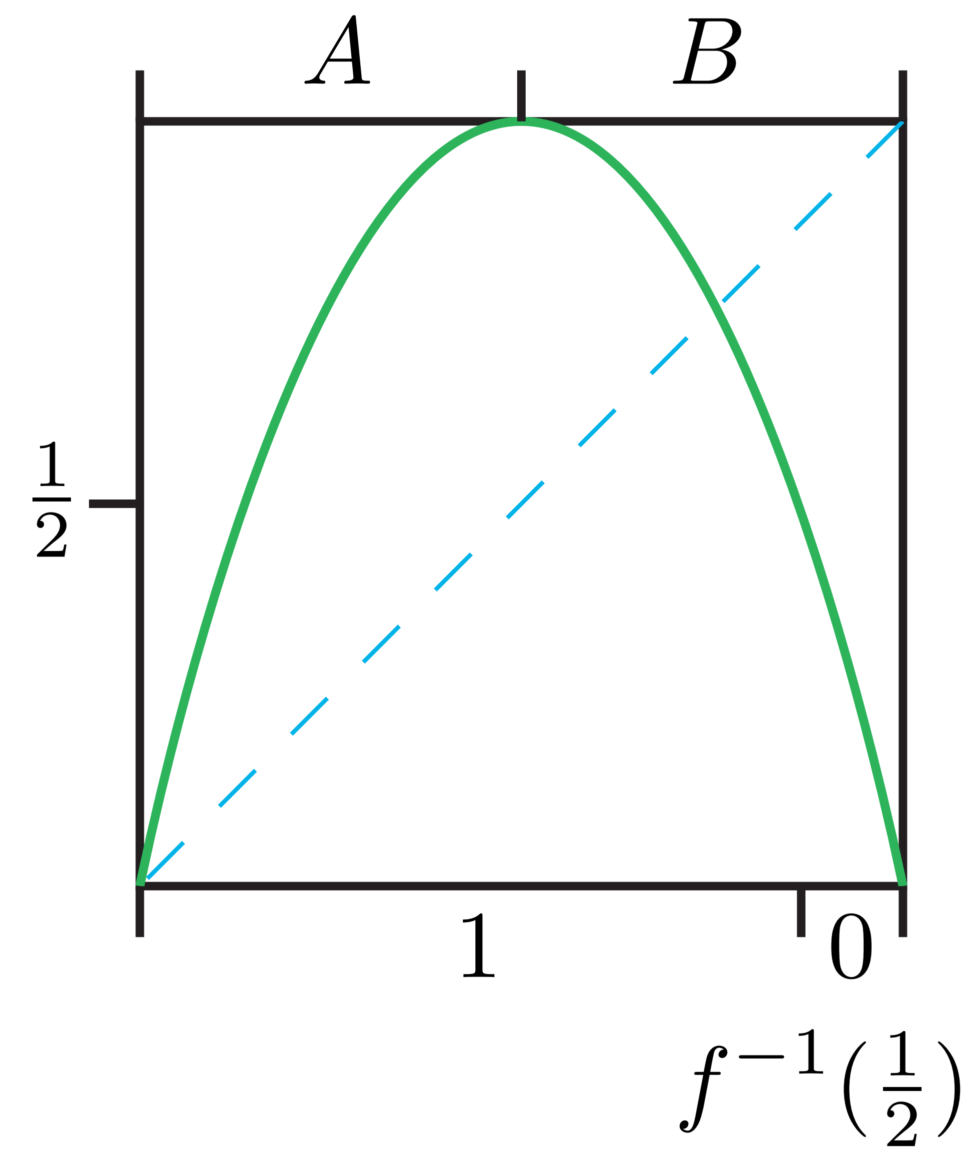
Example of a nongenerating partition ...

Decision point:

$$d = \max \{f^{-1}(\tfrac{1}{2})\}$$

Hidden Markov Model:





The Simple Nonunifilar Source.

Nonunifilar: Map from A-B sequences to observed 0-1 sequences throws away information!

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Measurement Theory ...

Synopsis:

How to model measurement process.

How chaos interacts with measurement partition.

End up with discrete-valued sequences.

What kind of stochastic process is the result?

Markov process? Sometimes.
Hidden Markov process? Sometimes.
Sometimes finite, sometimes infinite order.

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Measurement Theory ...

Main questions now:

How do we characterize the resulting process?

Measure degrees of unpredictability & randomness.

What correlational structure is there?

How do we build a model from the process itself?

How much can we reconstruct about the hidden internal dynamics?

Stochastic Processes:

Chain of random variables:

$$\overleftrightarrow{S} \equiv \ldots S_{-2} S_{-1} S_0 S_1 S_2 \ldots$$

Random variable: S_t

Alphabet: $\mathcal{A}$

Realization:

$$\cdots s_{-2} s_{-1} s_0 s_1 s_2 \cdots \; ; \; s_t \in \mathcal{A}$$

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Stochastic Processes:

Chain of random variables: $\overleftrightarrow{S} = \overleftarrow{S}_t \overrightarrow{S}_t$

Past: $\overleftarrow{S}_t = \ldots S_{t-3} S_{t-2} S_{t-1}$

Future: $\overrightarrow{S}_t = S_t S_{t+1} S_{t+2} \ldots$

L-Block: $S_t^L \equiv S_t S_{t+1} \ldots S_{t+L-1}$

Word: $s_t^L \equiv s_t s_{t+1} \ldots s_{t+L-1} \in \mathcal{A}^L$

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Stochastic Processes ...

Process:

$$\Pr(\overleftrightarrow{S}) = \Pr(\ldots S_{-2} S_{-1} S_0 S_1 S_2 \ldots)$$

Sequence (or word) distributions:

$$\{\Pr(S_t^L) = \Pr(S_t S_{t+1} \ldots S_{t+L-1}) : S_t \in \mathcal{A}\}$$

Process:

$$\{\Pr(S_t^L) : \forall t, L\}$$

Consistency conditions:

$$\Pr(S_t^{L-1}) = \sum_{S_{t+L-1}} \Pr(S_t^L) \qquad \Pr(S_{t+1}^{L-1}) = \sum_{S_t} \Pr(S_t^L)$$

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Types of Stochastic Process:

Stationary process:

$$\Pr(S_t S_{t+1} \ldots S_{t+L-1}) = \Pr(S_0 S_1 \ldots S_{L-1})$$

Assume stationarity, unless otherwise noted.

Notation: Drop time indices.

Types of Stochastic Process ...

Uniform Process:

Equal-length sequences occur with same probability

$$U^L : \ \Pr(s^L) \ = \ 1/|\mathcal{A}|^L$$

Example: Fair coin

$$\mathcal{A} = \{H, T\}$$

$$\Pr(H) = \Pr(T) = 1/2$$

$$\Pr(s^L) = 2^{-L}$$

Types of Stochastic Process ...

Independent, Identically Distributed (IID) Process:

$$\Pr(\overleftrightarrow{S}) = \ldots \Pr(S_t) \Pr(S_{t+1}) \Pr(S_{t+2}) \ldots$$

$$\Pr(S_t) = \Pr(S_\tau), \ \forall \ t, \tau$$

Example: Biased coin

$$\Pr(H) = p$$

$$\Pr(T) = 1 - p = q$$

$$\Pr(s^L) = p^n q^{L-n}$$

Number of heads in sequence: n

Types of Stochastic Process ...

Markov Process:

$$\Pr(\overleftrightarrow{S}) = \ldots \Pr(S_{t+1}|S_t)\Pr(S_{t+2}|S_{t+1})\Pr(S_{t+3}|S_{t+2})\ldots$$

Example: No Consecutive 0s (Golden Mean Process)

$$\mathcal{A} = \{0, 1\}$$

$$\Pr(0|0) = 0$$

$$\Pr(1|0) = 1$$

$$\Pr(0|1) = 1/2$$

$$\Pr(1|1) = 1/2$$

Not Noisy Period-2 Process: GMP @ L = 4 has 0110.

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Types of Stochastic Process ...

Hidden Markov Process:

Internal Order-R Markov Process: $\Pr(\overleftrightarrow{S})$

$$\Pr(S_t|\ldots S_{t-2}S_{t-1}) = \Pr(S_t|S_{t-R}\ldots S_{t-1})$$

$$s_t \in \mathcal{A}$$

Observed via a function of the internal sequences

$$\overleftrightarrow{Y} = f(\overleftrightarrow{S})$$

Measurement alphabet: $y_t \in \mathcal{B}$

Measurement random variables: $\overleftrightarrow{Y} = \ldots Y_{-2}Y_{-1}Y_0Y_1\ldots$

Observation process: $\Pr(\overleftrightarrow{Y} \mid \overleftrightarrow{S})$

Observed process: $\Pr(\overleftrightarrow{Y})$

Block Distribution: $\Pr(Y^L)$

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Types of Stochastic Process ...

Hidden Markov Process ...

Example: The Even Process

Internal Process: Golden Mean

$$s_t \in \{0, 1\}$$

Observation Process: $y_t \in \{a, b\}$

$$Y_t = f(S_{t-1} S_t)$$

$$y_t = \begin{cases} a, & s_{t-1} s_t = 11 \\ b, & s_{t-1} s_t = 01 \text{ or } 10 \end{cases}$$

$$\overleftrightarrow{s} = \texttt{1101110111101011111011} \ldots$$

$$\overleftrightarrow{y} = .\ \texttt{abbaabbaaabbbbaaaabba} \ldots$$

Models of Stochastic Processes:

Markov chain model of a Markov process:

States: $v \in \mathcal{A} = \{1, \ldots, k\}$

$$\overleftrightarrow{V} = \ldots V_{-2} V_{-1} V_0 V_1 \ldots$$

Transition matrix: $T_{ij} = \Pr(v_{t+1} | v_t) \equiv p_{vv'}$

$$T = \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix}$$

Stochastic matrix: $\sum_{i=1}^{k} T_{ij} = 1$

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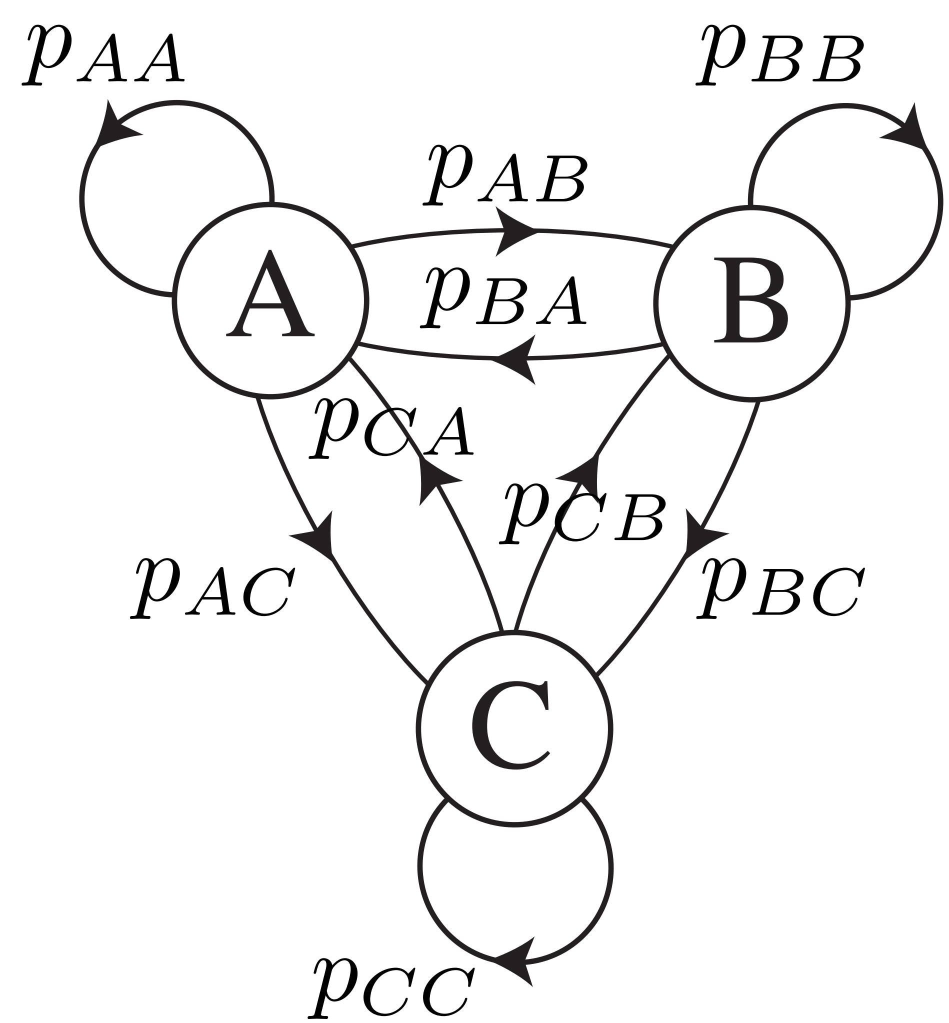
Models of Stochastic Processes ...

Markov chain ...

Example: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}$$

$$p_{AA} + p_{AB} + p_{AC} = 1$$
$$p_{BA} + p_{BB} + p_{BC} = 1$$
$$p_{CA} + p_{CB} + p_{CC} = 1$$



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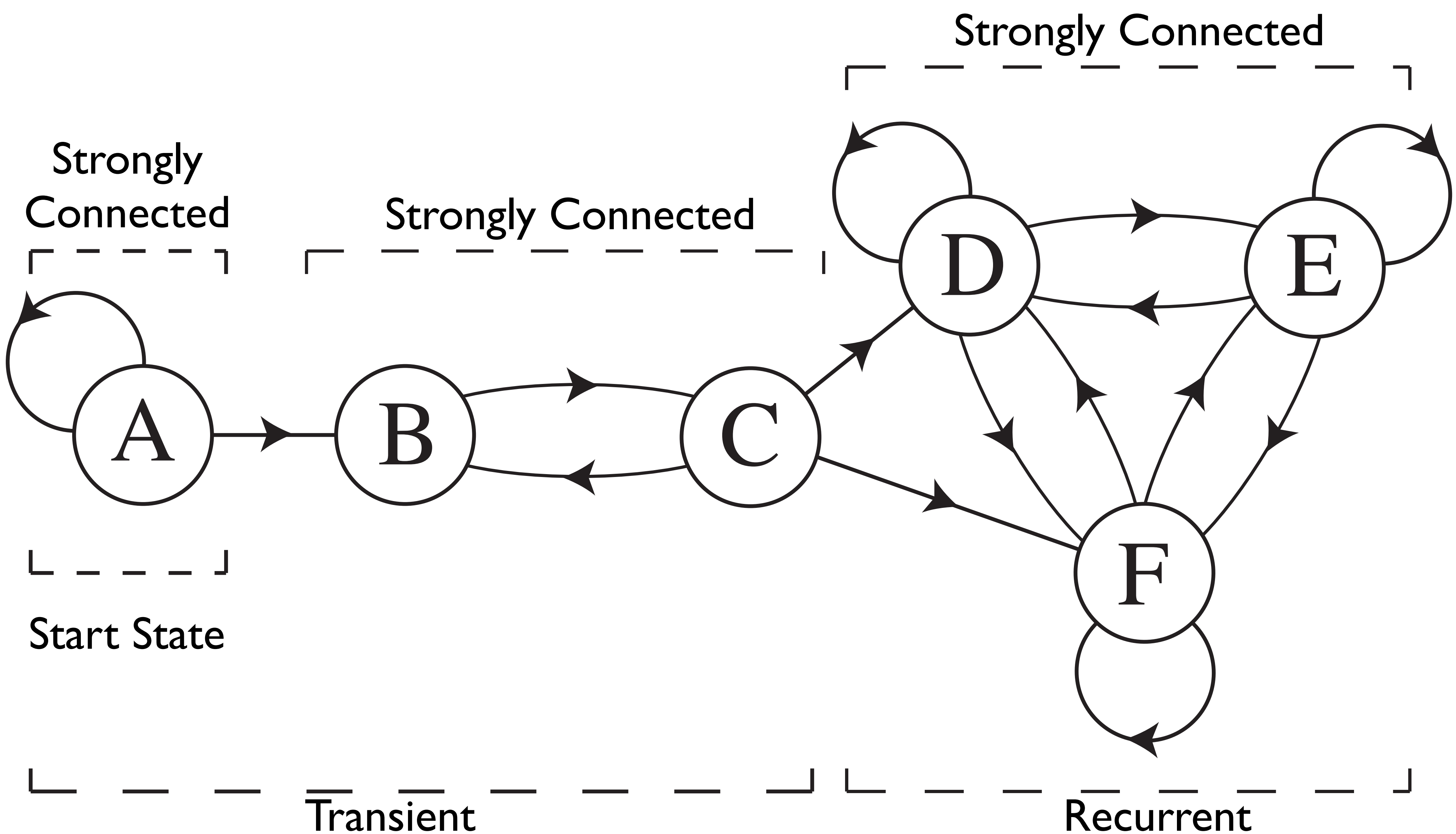
Models of Stochastic Processes ...

Kinds of state:



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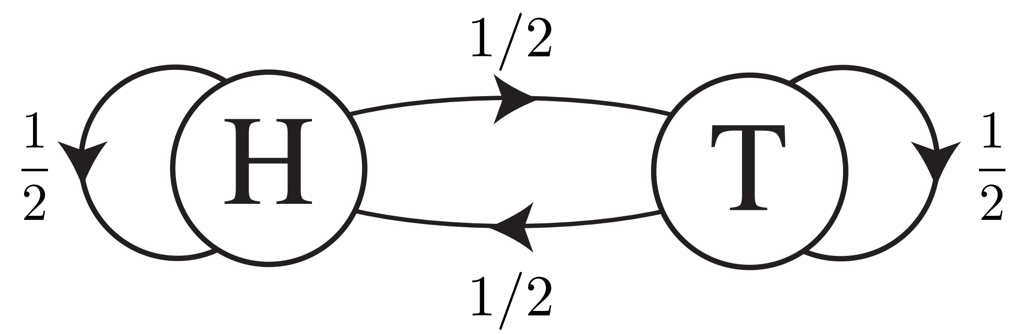
Models of Stochastic Processes ...

Example:

Fair Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



$$\pi = (1/2, 1/2)$$

$$\Pr(H) = \Pr(T) = 1/2$$

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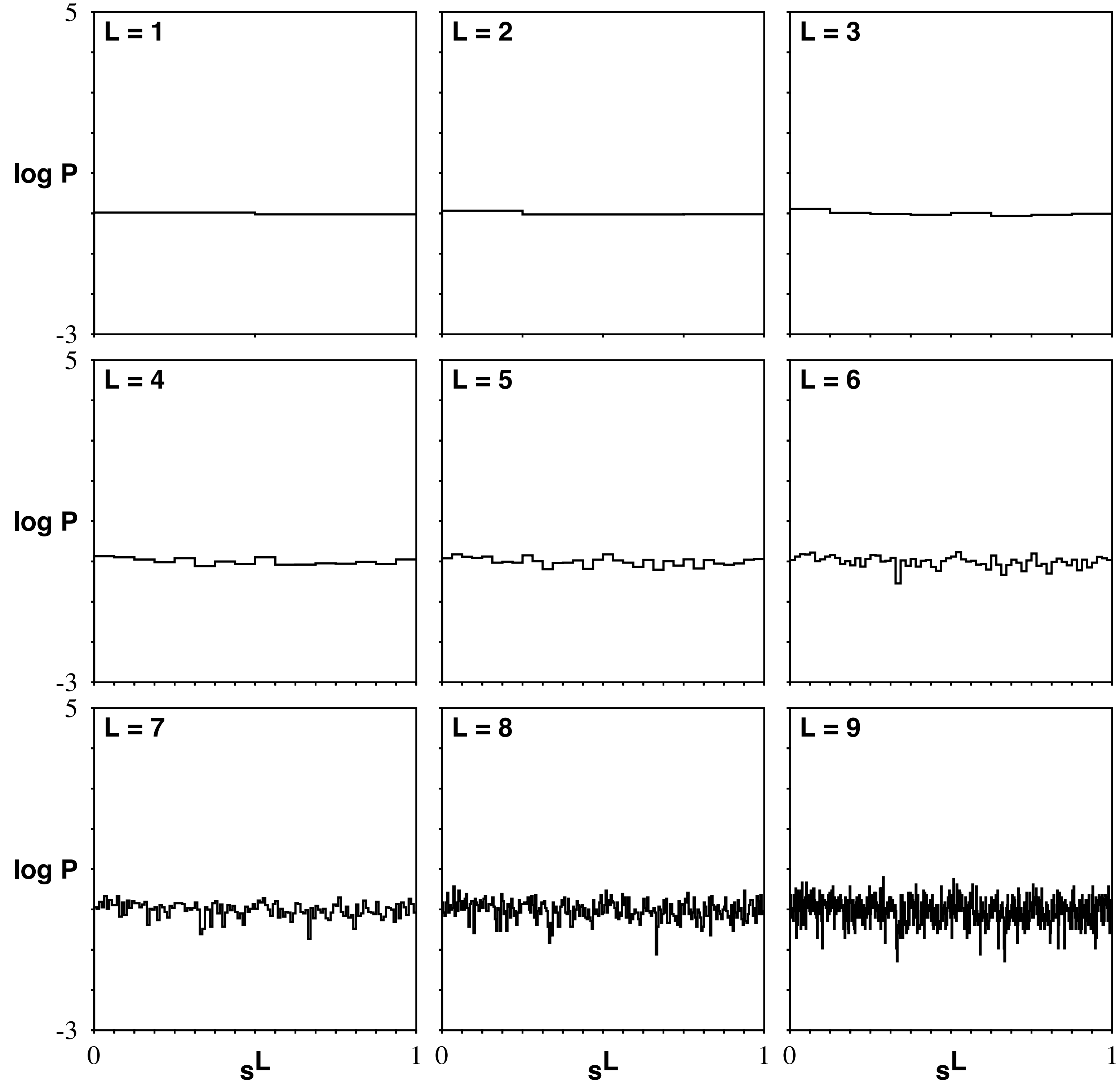
Example:

Fair Coin ...

Sequence Distribution: $\Pr(v^L) = 2^{-L}$



Word as binary fraction:

$$s^L = s_1 s_2 \ldots s_L$$

$$\text{“}s^L\text{”} = \sum_{i=1}^{L} \frac{s_i}{2^i}$$

$$s^L \in [0, 1]$$

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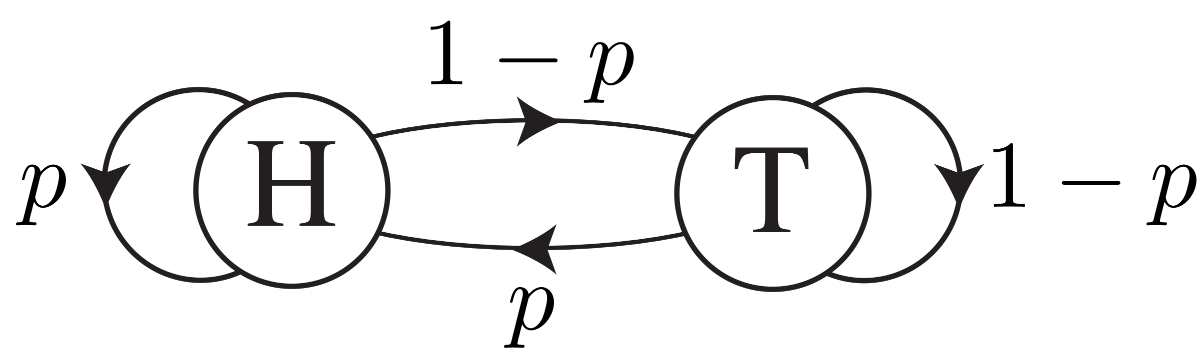
Models of Stochastic Processes ...

Example:

Biased Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$



$$\pi = (p, 1-p)$$

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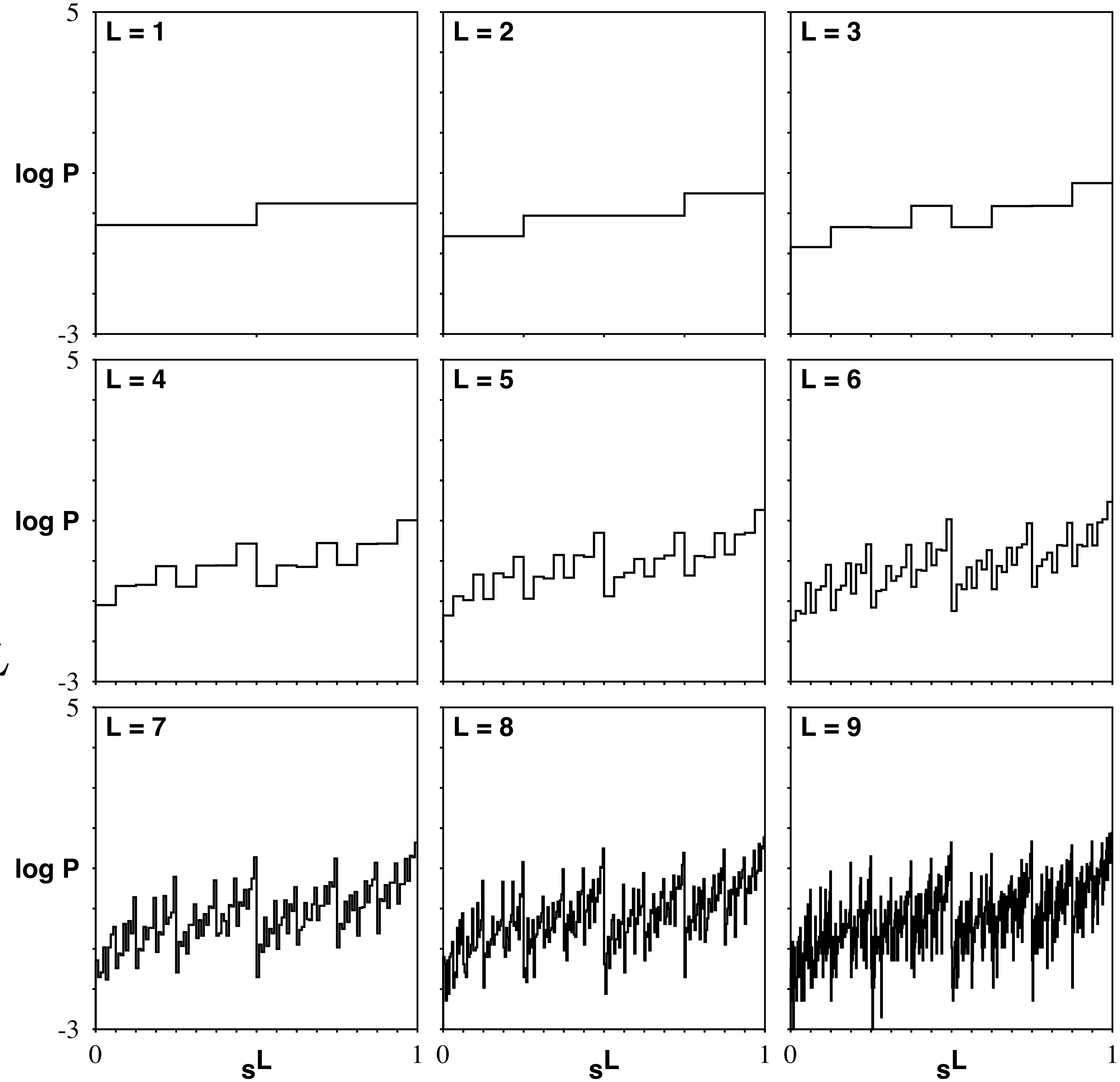
Example:

Biased Coin ...



Sequence Distribution:

$$\Pr(s^L) = p^n (1-p)^{L-n},$$

$$n = \text{Number } H\text{s in } s^L$$

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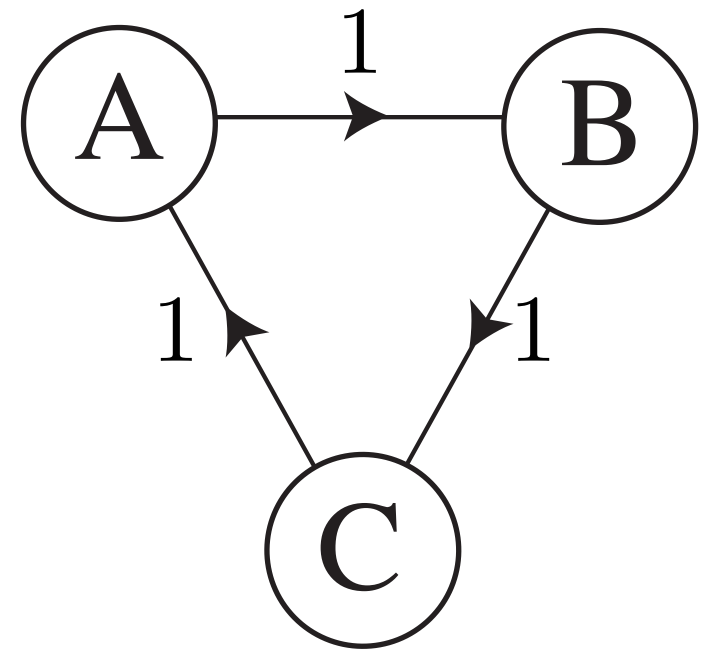
Example:

Periodic: $\mathcal{A} = \{A, B, C\}$



$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ Careful!

Sequence distribution:

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{3}$$

$\Pr(AB) = \Pr(BC) = \Pr(CA) = \frac{1}{3}$ $\Pr(s^2) = 0$ otherwise

$\Pr(ABC) = \Pr(BCA) = \Pr(CAB) = \frac{1}{3}$ $\Pr(s^3) = 0$ otherwise

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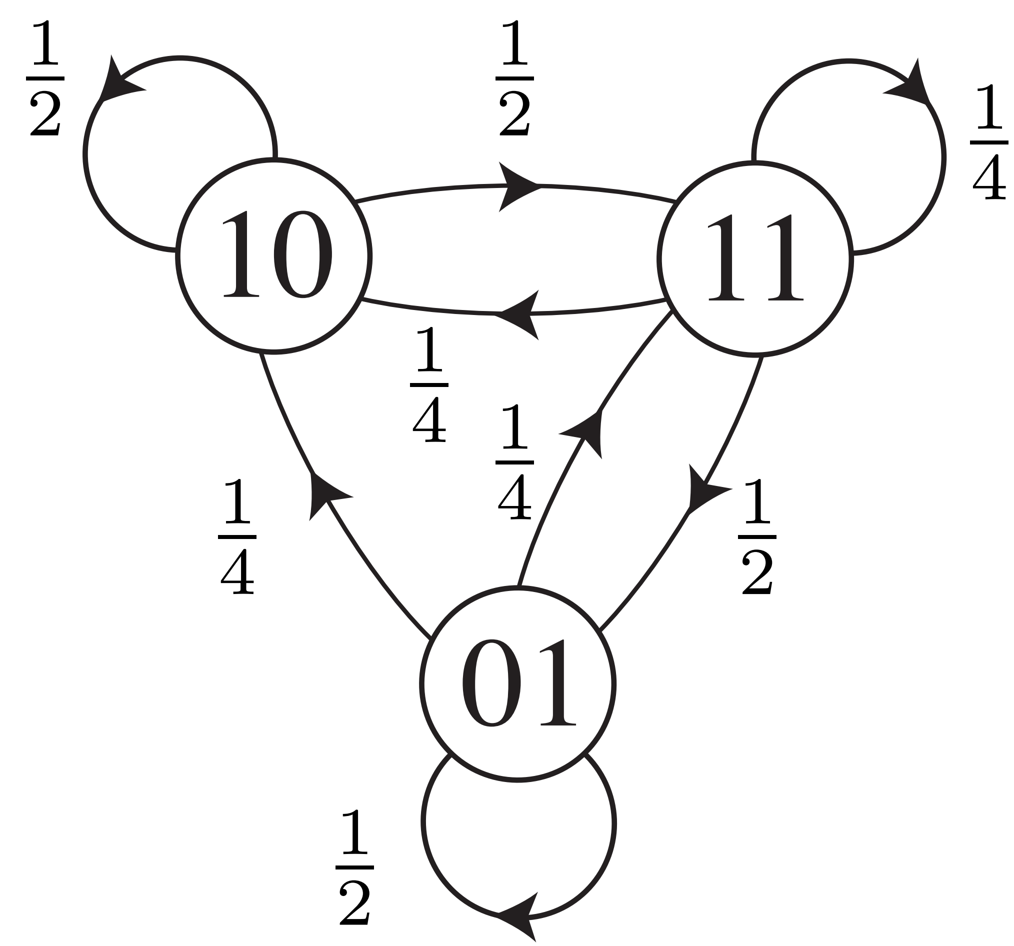
Example:

Golden Mean over 2-Blocks: $\mathcal{A} = \{10, 01, 11\}$

$$T = \begin{array}{c} \begin{matrix} \quad 10 & \quad 01 & \quad 11 \end{matrix} \\ \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \end{array}$$

$$\pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$



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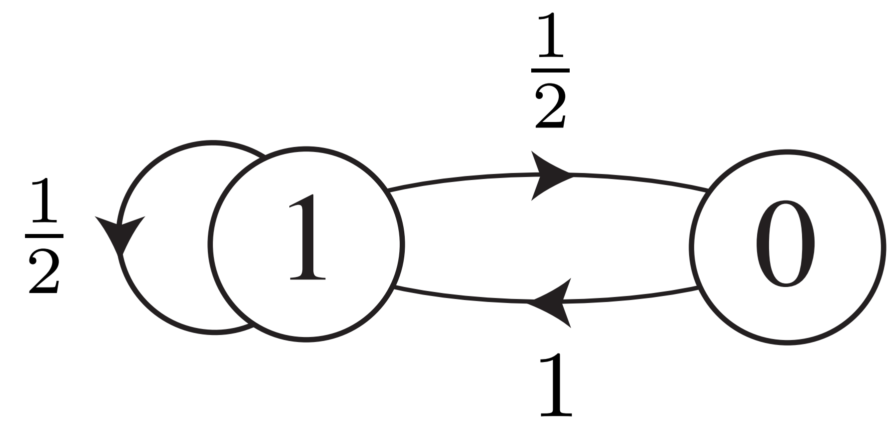
Example ...

Golden Mean over 1-Blocks: $\mathcal{A} = \{0, 1\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\pi = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$



Also an order-1 Markov chain. Minimal order.

Previous model and this:

Different presentations of the Golden Mean Process

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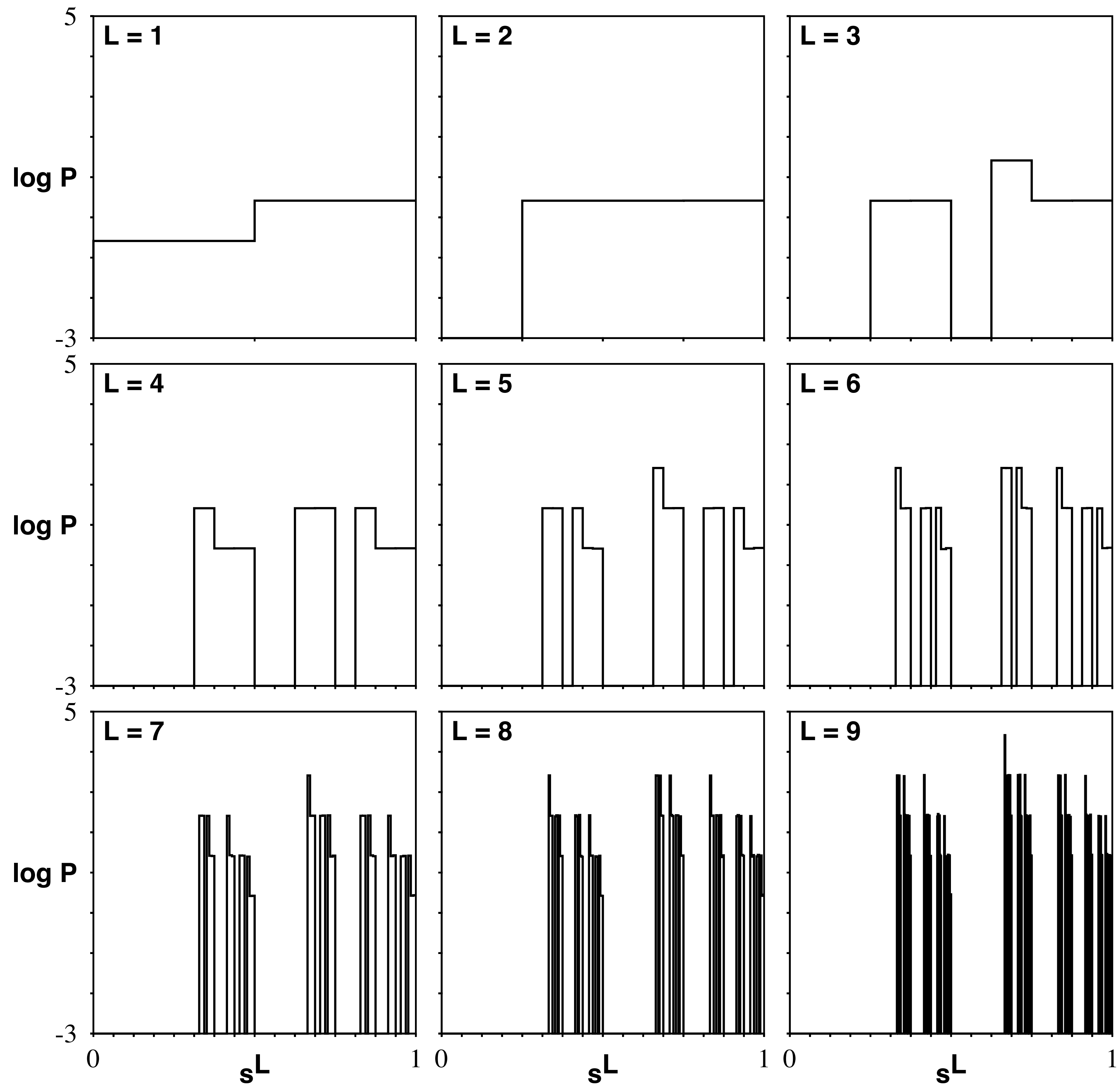
Example:

Golden mean:



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Models of Stochastic Processes ...

Two Lessons:

Structure in the behavior: $\mathrm{supp}\ \Pr(s^L)$

Structure in the distribution of behaviors: $\Pr(s^L)$

Models of Stochastic Processes ...

Hidden Markov Models of Processes:

Internal states: $v \in \mathcal{A}$

Transition matrix: $T = \Pr(v'|v), \; v, v' \in \mathcal{A}$

Observation: Symbol-labeled transition matrices

$$T^{(s)} = \Pr(v', s|v), \; s \in \mathcal{B}$$

$$T = \sum_{s \in \mathcal{B}} T^{(s)}$$

Stochastic matrices:

$$\sum_j T_{ij} = \sum_j \sum_s T_{ij}^{(s)} = 1$$

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Hidden Markov Models ...

Internal state distribution: $\vec{p}_V = (p_1, p_2, \ldots, p_k)$

Evolve internal distribution: $\vec{p}_n = \vec{p}_0 T^n$

State sequence distribution: $v^L = v_0 v_1 v_2 \ldots v_{L-1}$

$$\Pr(v^L) = \pi(v_0) p(v_1|v_0) p(v_2|v_1) \cdots p(v_{L-1}|v_{L-2})$$

Observed sequence distribution: $s^L = s_0 s_1 s_2 \ldots s_{L-1}$

$$\Pr(s^L) = \sum_{v^L \in \mathcal{A}^L} \pi(v_0) p(v_1, s_1|v_0) p(v_2, s_2|v_1) \cdots p(v_{L-1}, s_{L-1}|v_{L-2})$$

No longer 1-1 map between internal & observed sequences:

Multiple state sequences can produce *same* observed sequence.

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Models of Stochastic Processes ...

Hidden Markov Models ...

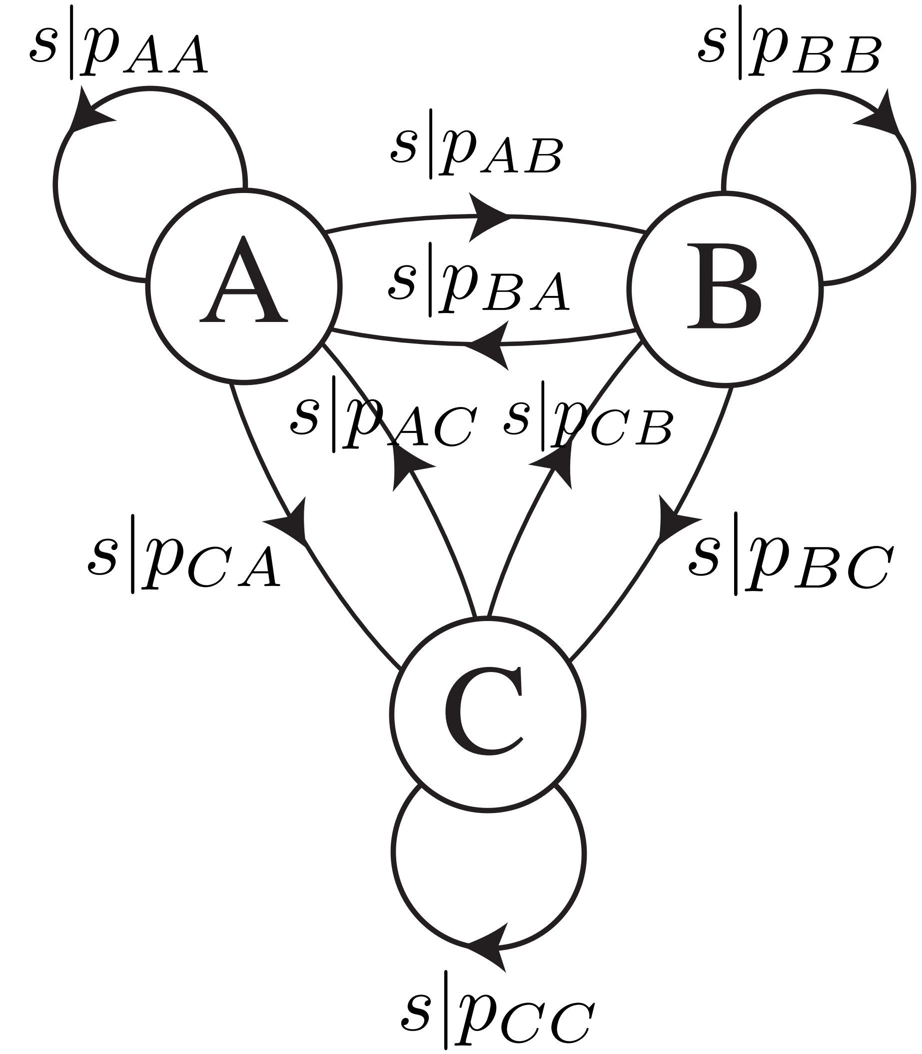
Internal: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(s)} = \begin{pmatrix} p_{AA;s} & p_{AB;s} & p_{AC;s} \\ p_{BA;s} & p_{BB;s} & p_{BC;s} \\ p_{CA;s} & p_{CB;s} & p_{CC;s} \end{pmatrix}$$

$$p_{AA} = \sum_{s \in \mathcal{B}} p_{AA;s}$$



symbol | transition probability

Models of Stochastic Processes ...

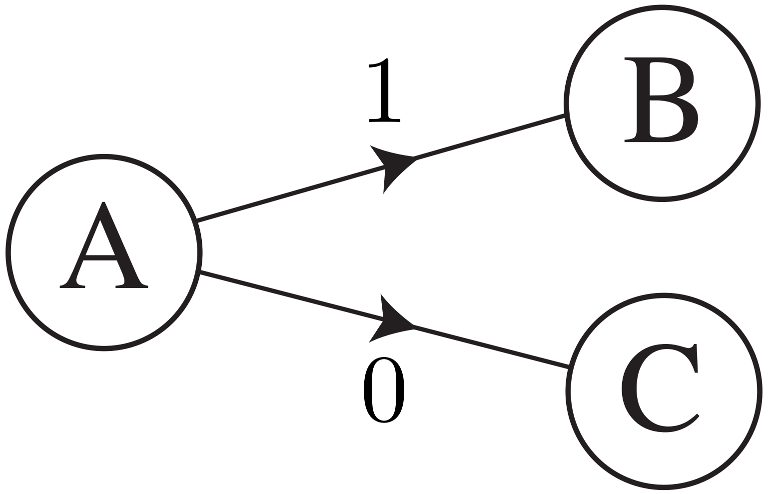
Types of Hidden Markov Model:

"Unifilar": current state + symbol "determine" next state

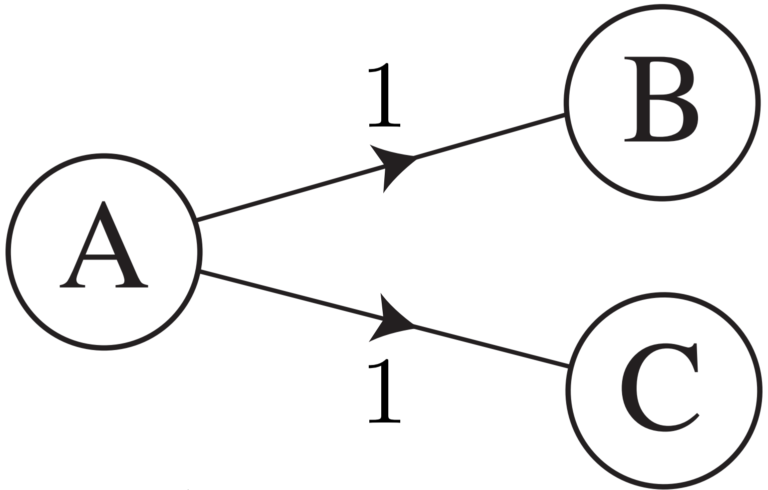
$$\Pr(v'|v,s) = \begin{cases} 1 \\ 0 \end{cases}$$

$$\Pr(v',s|v) = p(s|v)$$

$$\Pr(v'|v) = \sum_{s \in \mathcal{A}} p(s|v)$$



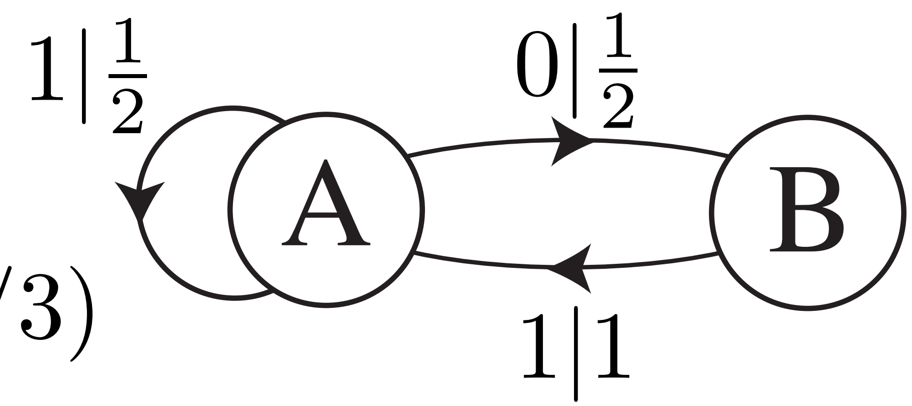
"Nonunifilar": no restriction



Multiple internal edge paths can generate same observed sequence.

Example:

Golden Mean Process as a unifilar HMM:



Internal: $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

Initial ambiguity only: At most 2-to-1 mapping

$$BA^n = 1^n$$

$$AA^n = 1^n$$

Sync'd: $s = 0 \Rightarrow v = B$
$s = 1 \Rightarrow v = A$

Irreducible forbidden words: $\mathcal{F} = \{00\}$

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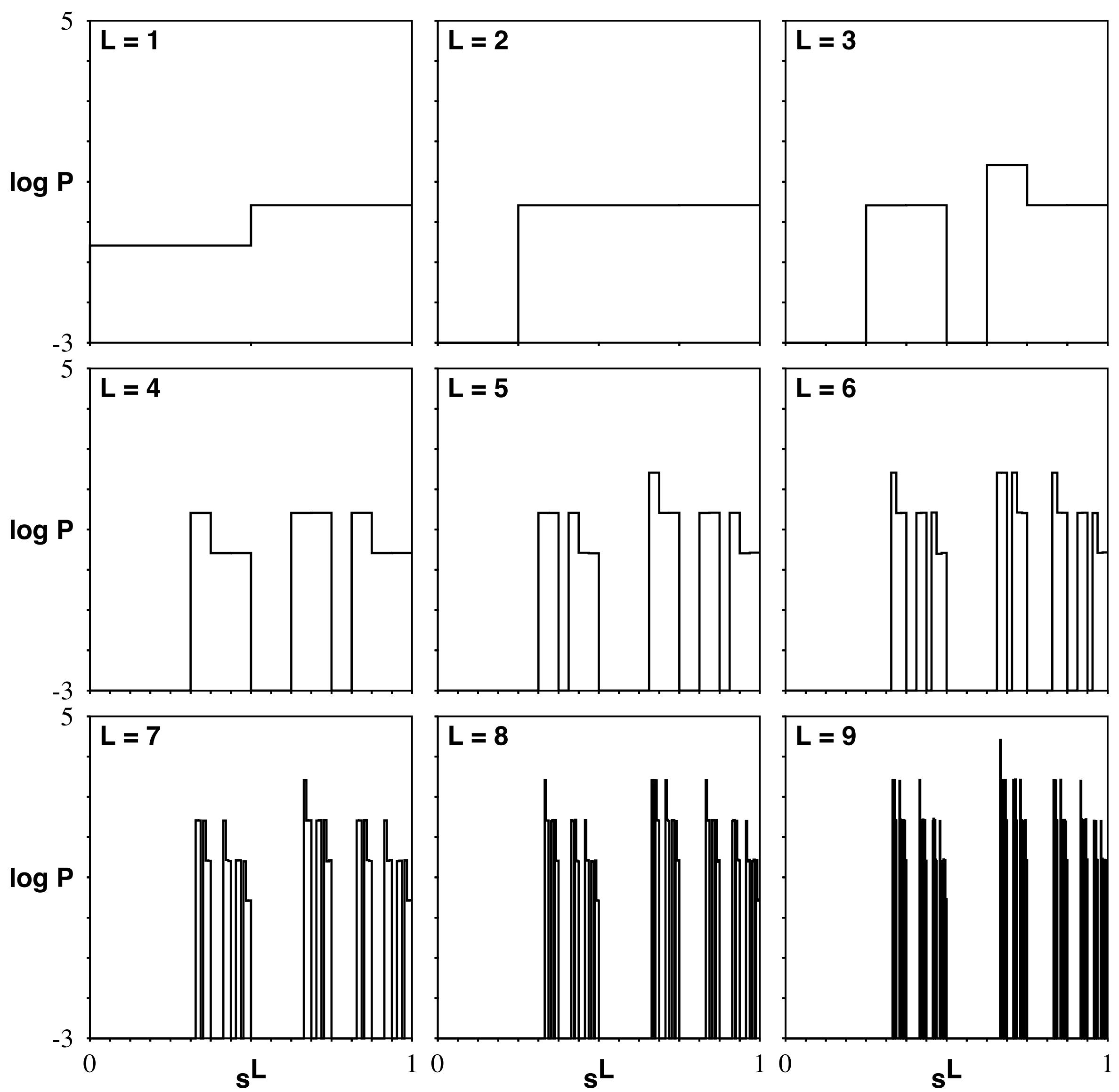
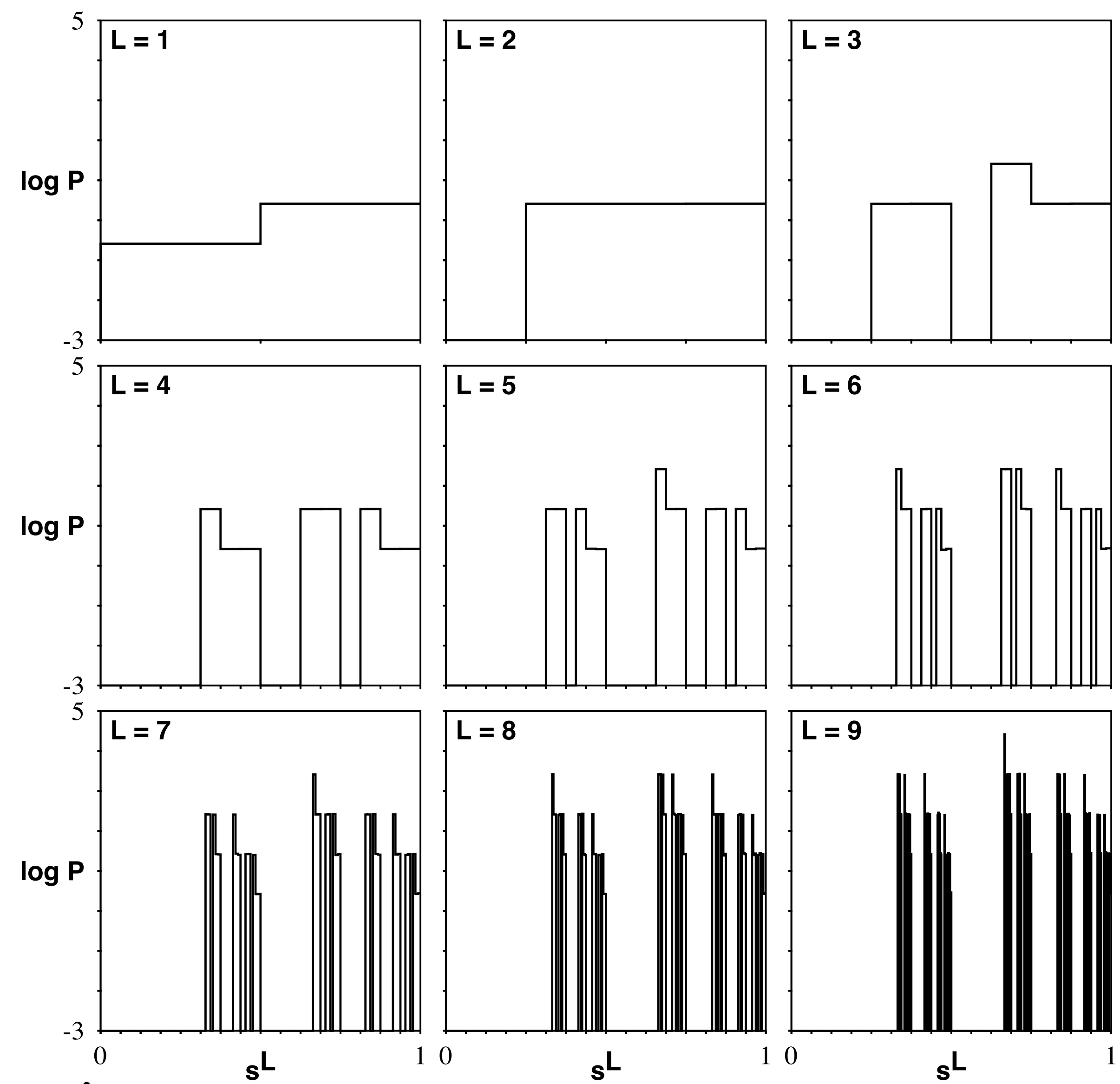
Example:

Golden Mean Process ... Sequence distributions:

Internal state sequences
(A = 1; B = 0)



Observed sequences



Same!

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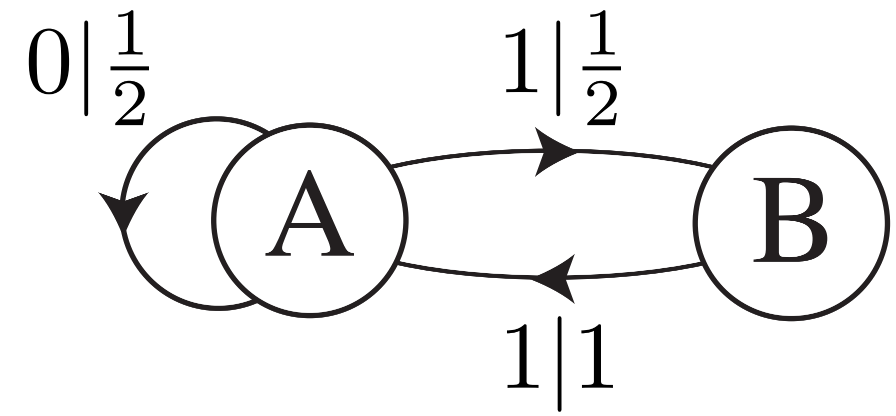
Models of Stochastic Processes ...

Example:

Even Process as a unifilar HMM:



Internal (= GMP): $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$v^L = \ldots AABAABABAA \ldots$$

$$s^L = \ldots .\ \ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0 \ldots \qquad s^L = \{\ldots 01^{2n}0 \ldots\}$$

Irreducible forbidden words: $\mathcal{F} = \{010, 01110, 0111110, \ldots\}$

No finite-order Markov process can model the Even process!

Lesson: Finite Markov Chains are a subset of HMMs.

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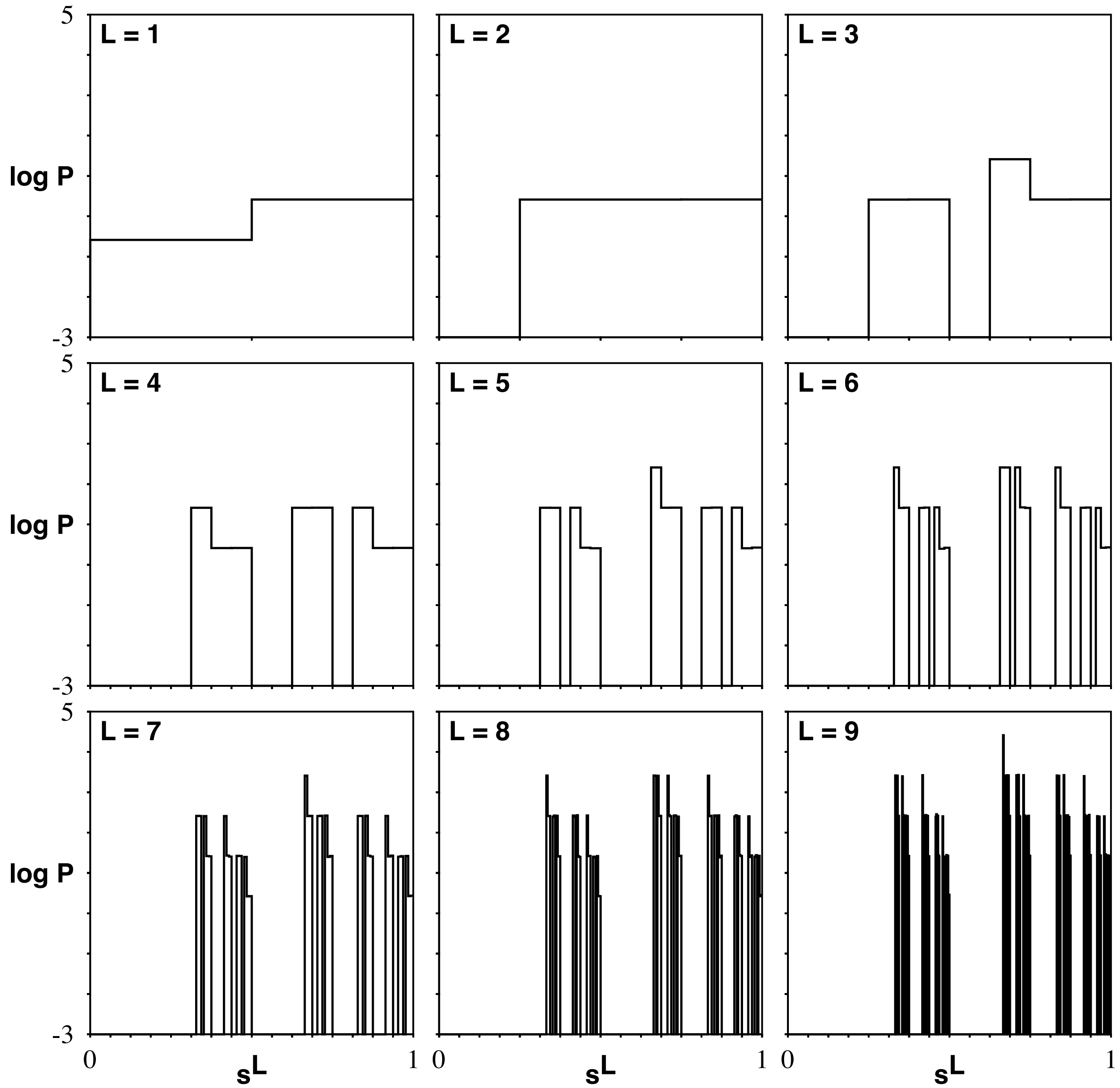
Models of Stochastic Processes ...

Example:

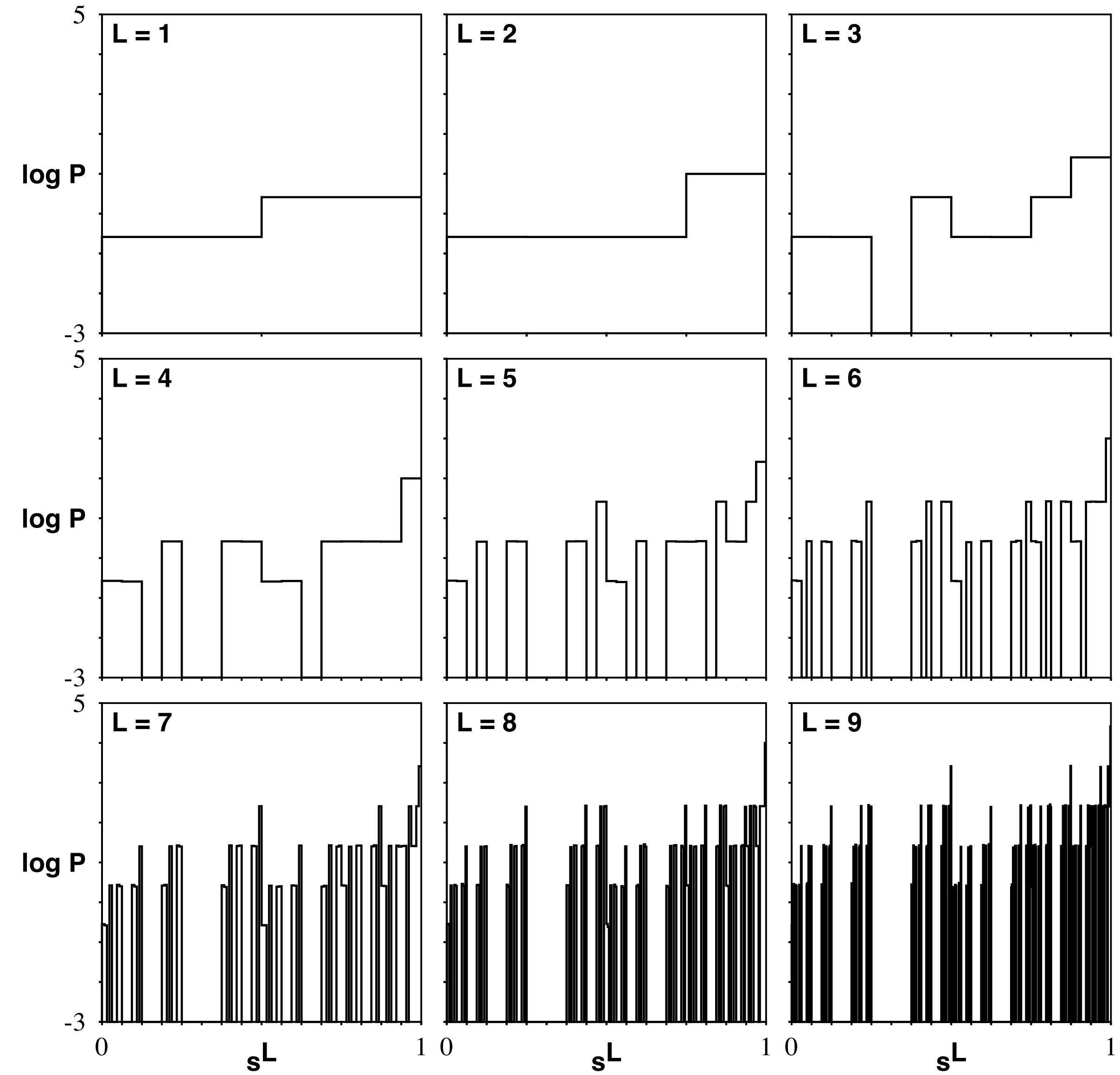
Even Process ... Sequence distributions:

Internal states (= GMP)

(A = 1; B = 0)



Observed sequences



Rather different!

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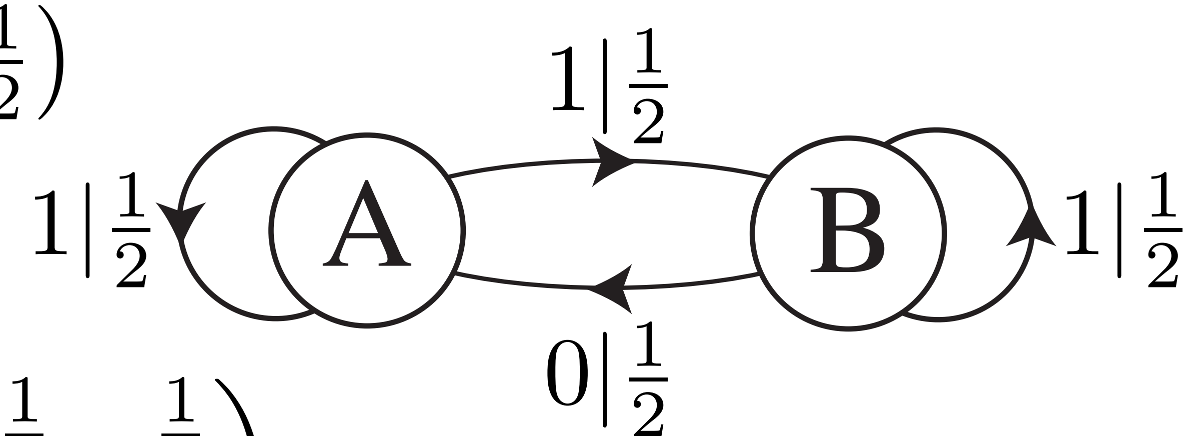
Models of Stochastic Processes ...

Example:

Simple Nonunifilar Source:

Internal (= Fair Coin): $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \pi_V = \left(\frac{1}{2}, \frac{1}{2}\right)$$



Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Many to one: $1111111 \Leftarrow \begin{cases} AAAAAAAA\ldots \\ ABBBBBBB\ldots \\ AABBBBBB\ldots \\ AAABBBBB\ldots \\ \ldots \\ BBBBBBBB\ldots \end{cases}$

Is there a unifilar HMM presentation of the observed process?

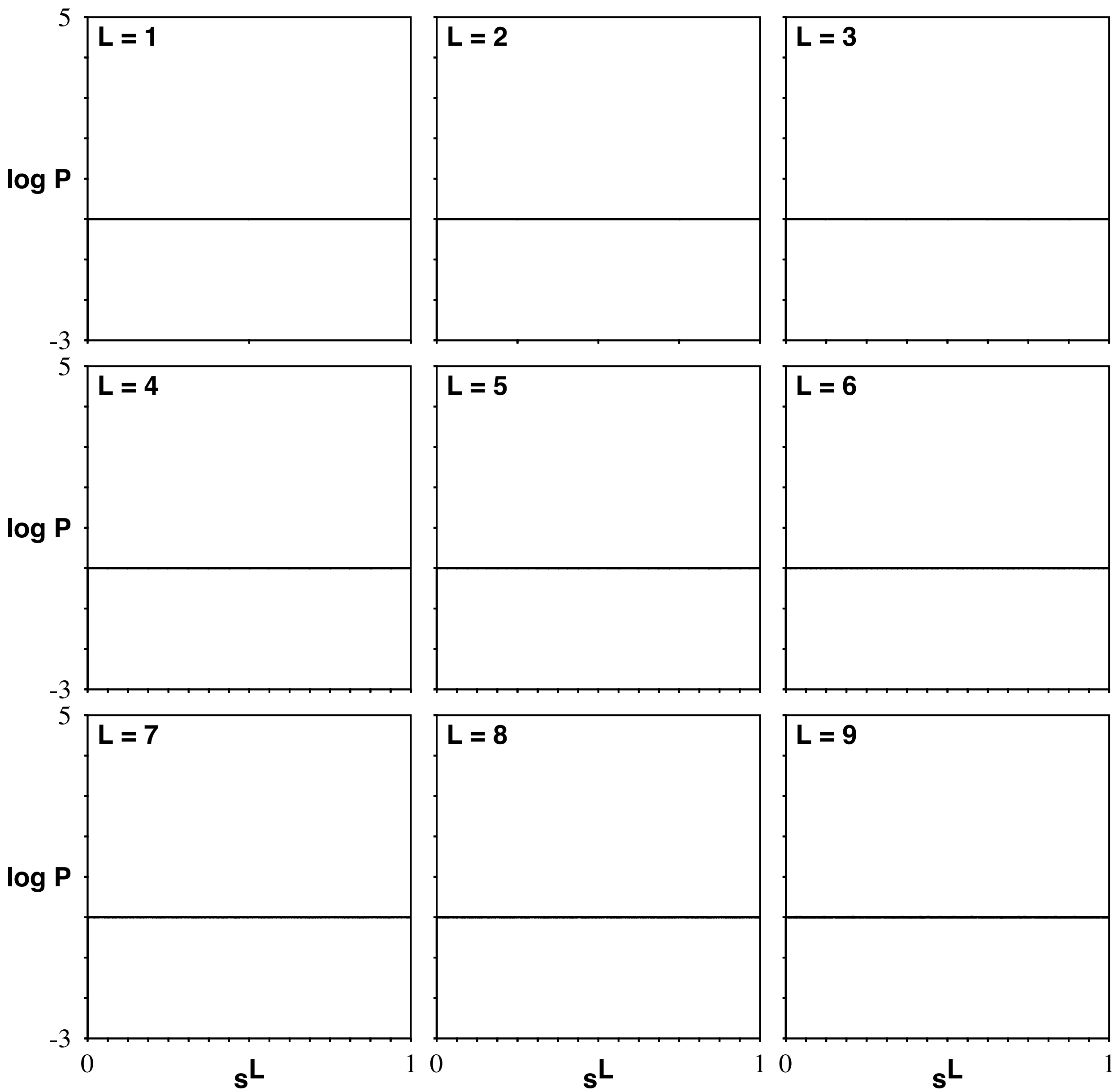
From Determinism to Stochasticity ...

Models of Stochastic Processes ...

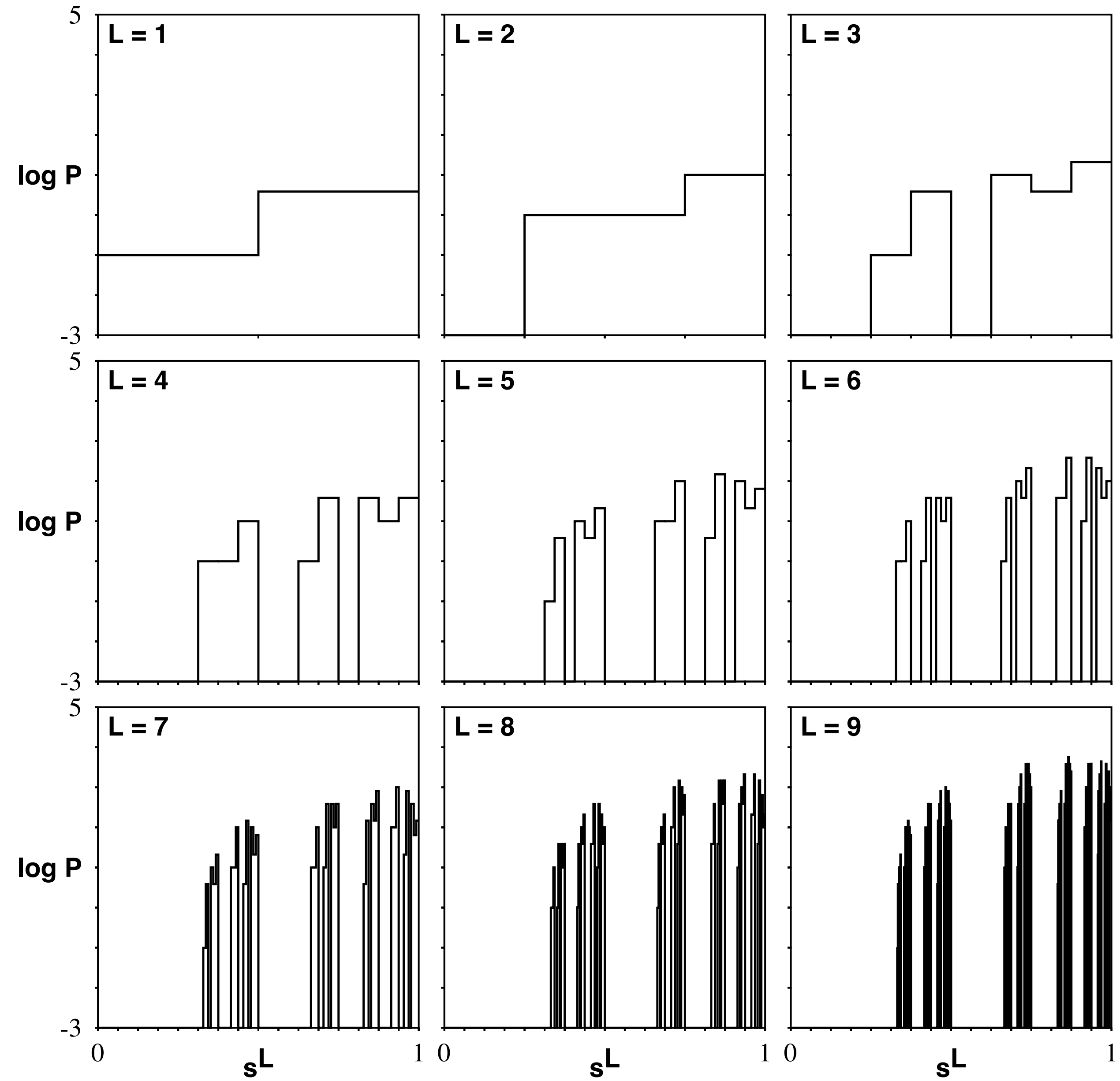
Example:

Simple Nonunifilar Process ...

Internal states (= Fair coin)
(A = 1; B = 0)



Observed sequences



From Determinism to Stochasticity ...

Next lectures: “Complexity Module”

1. Stochastic processes and models of them

2. Information theory for general stochastic processes

3. Measures of complexity

4. Optimal model and how to build them