Probability Puzzle #3: Dice Exercises

Problem Description

Summary

Can a single fair die – or a combination of fair dice, possibly with different numbers of sides – be used to generate random numbers that are not uniformly distributed, or to generate random numbers whose ranges of possible values do not conform to the natural range of values of the dice?

This puzzle consists of several specific challenges, detailed below.

Definitions

- A *fair die* is one for which all of its sides have equal probabilities of resulting when the die is rolled. For example, with a 6-sided die, where the sides are numbered 1 through 6, we have the following *probability mass function* - i.e. the enumeration of probabilities for all possible outcomes.

  $$p(i) = \begin{cases} \frac{1}{6}, & \text{for } i = 1, 2, \ldots, 6 \\ 0, & \text{otherwise} \end{cases}$$

  Another way to say this is that the results from a fair die are distributed according to a *discrete uniform probability distribution*.

- A discrete triangular distribution is one in which the probability of outcomes over its range increases in a linear fashion to a peak (unless the peak occurs at the start of the range), and then decreases in a linear fashion (unless the peak occurs at the end of the range). For example, in the distribution with the probability mass function

  $$p(i) = \begin{cases} \frac{i}{16}, & \text{for } i = 1, 2, 3, 4 \\ \frac{8-i}{16}, & \text{for } i = 5, 6, 7 \\ 0, & \text{otherwise} \end{cases}$$

  the distribution is symmetric, with the most likely outcome found at $i = 4$, where $p(i) = 1/4$, and the least likely outcomes being $i = 1$ and $i = 7$, both with $p(i) = 1/16$. As hinted at above, we can also have asymmetric discrete triangular distributions, where the most likely value is not at the center of the range of possible values.

- In any discrete probability distribution, the probabilities for all outcomes always add up to 1.

Challenge details

For these problems, you may use a set of 7 dice, of the type commonly used in playing Dungeons & Dragons and many other RPGs. The set includes 1 die of each of the following types (except where specified, a d*N* die is numbered from 1 through *N* on its faces):

- d4
- d6
- d8
- d10 – 10-sided die, numbered 0-9.
- d90 – 10-sided die, numbered 0, 10, 20, …, 90.
- d12
- d20

You have several random number generation/sampling tasks to complete. Some of these specify uniform discrete distributions, and others require non-uniform distributions.

Using any combination of dice (including the possibility of rolling one or more dice multiple times), and applying arithmetic and/or logical operations to the results of the throw(s) as appropriate, complete at least 2 tasks from each of the following 3 groups. (Feel free to complete all of them.)

1. Use the dice to sample from the following, assuming uniform (equal likelihood) probability distributions:
   a) The 4 cardinal compass directions (north, east, south, west), in degrees (0, 90, 180, 270 – or 90, 180, 270, 360).
   b) The 8 cardinal and intercardinal directions (i.e. the 4 cardinal directions, plus the 4 intermediate directions: northeast, southeast, southwest, northwest), in degrees.
   c) The base colors in the Logo color palette, using their numeric values (5, 15, 25, …, 135).
   d) The base colors in the Logo color palette, excluding gray (5), using their numeric values.
   e) The base colors in the Logo color palette, excluding turquiose (75), using their numeric values.
   f) The integers {1, 2, 3}.

2. Use the dice to sample from the following, assuming uniform (equal likelihood) probability distributions:

   a) The integers {1, 2, 3, …, 40}

   b) Hour:minute:second values, in the range 00:00:00 through 23:59:59.

   c) Days of the year, from 1 through 365.

   d) Days of the year, with the added feature that there should be a 0.75 probability that a 365-day year is used, and a 0.25 probability that a 366-day year is used.

3. Use the dice to sample from the following, with probability distributions as specified:

   a) The integers {2, 3, …, 7, …, 11, 12}, with 7 as the most likely, and 2 and 12 the least likely, and the probabilities of intermediate outcomes ascending and descending in linear fashion.

   b) The integers {-5, -4, …, 0, …, 4, 5}, with 0 being the most likely, and -5 and 5 the least likely, and the probabilities of intermediate outcomes ascending and descending in linear fashion.

   c) The integers {1, 2, 3, 4}, where the probability of each outcome is proportional to its value. For example, an outcome of 2 should be twice as likely as an outcome of 1, while 3 should be 3 times as likely as 1, etc.

   d) The integers {1, 2, 3, 4, 5}, where the probability of each is proportional to its value.

   e) The integers {1, 2, …, 8}, where the most likely outcome is 1, and the least likely is 8, with the probability of obtaining a 7 being twice that of obtaining an 8, the probability of a 6 triple the probability of an 8, etc..

Guidelines

Restrictions and assumptions

- The dice in the set are the only sources of randomness you may use.
- Assume that all dice in the set are fair – i.e. all may be used to sample the associated discrete uniform probability distributions.
- You can't employ any special techniques or tricks to bias the outcome of dice rolls.
- If you're tempted to add new markings to the dice, try instead to come up with arithmetic or logical transformations, to map the dice outcomes to the desired set of outcomes. In other words, outcomes from the dice rolls must be taken exactly as they appear – but arithmetic calculations and *if…then…else* rules can be applied to those outcomes.

Hints

- Not all dice in the set need necessarily be used.
- Many of the tasks have multiple acceptable approaches.
- When using arithmetic operations to combine outcomes from multiple dice, it's often helpful to construct a table showing the combinations possible, so that you can be sure that the combined outcomes follow the desired distribution.
- If you find multiple approaches to a task, consider following the one that requires fewer dice rolls and/or the one that has a lower chance of rejecting dice rolls.
- In addition to the usual arithmetic operations (addition, subtraction, multiplication, division), feel free to explore the use of less obvious operations, such as absolute value, modular division, rounding or truncation (including floor and ceiling operations), taking the min or max of multiple numbers, etc.