Santa Fe

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Evolutionary Game Theory

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Payoff Matrix

$$\begin{array}{cc} & \begin{array}{cc} Split & Steal \end{array} \\ \begin{array}{c} Split \\ Steal \end{array} & \begin{pmatrix} 500 & 0 \\ 1000 & 0 \end{pmatrix} \end{array}$$

Game Theory

Oskar Morgenstern

John von Neumann

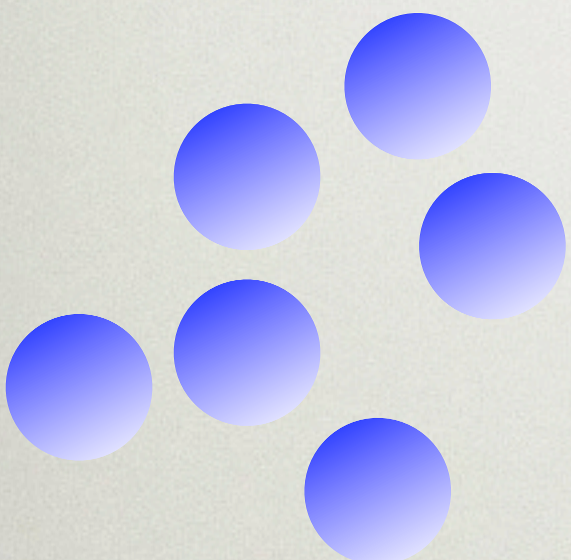
At Spring Lake, ca. 1946.
Courtesy of the Institute for Advanced Study, Princeton

Theory of Games and Economic Behavior, 1944

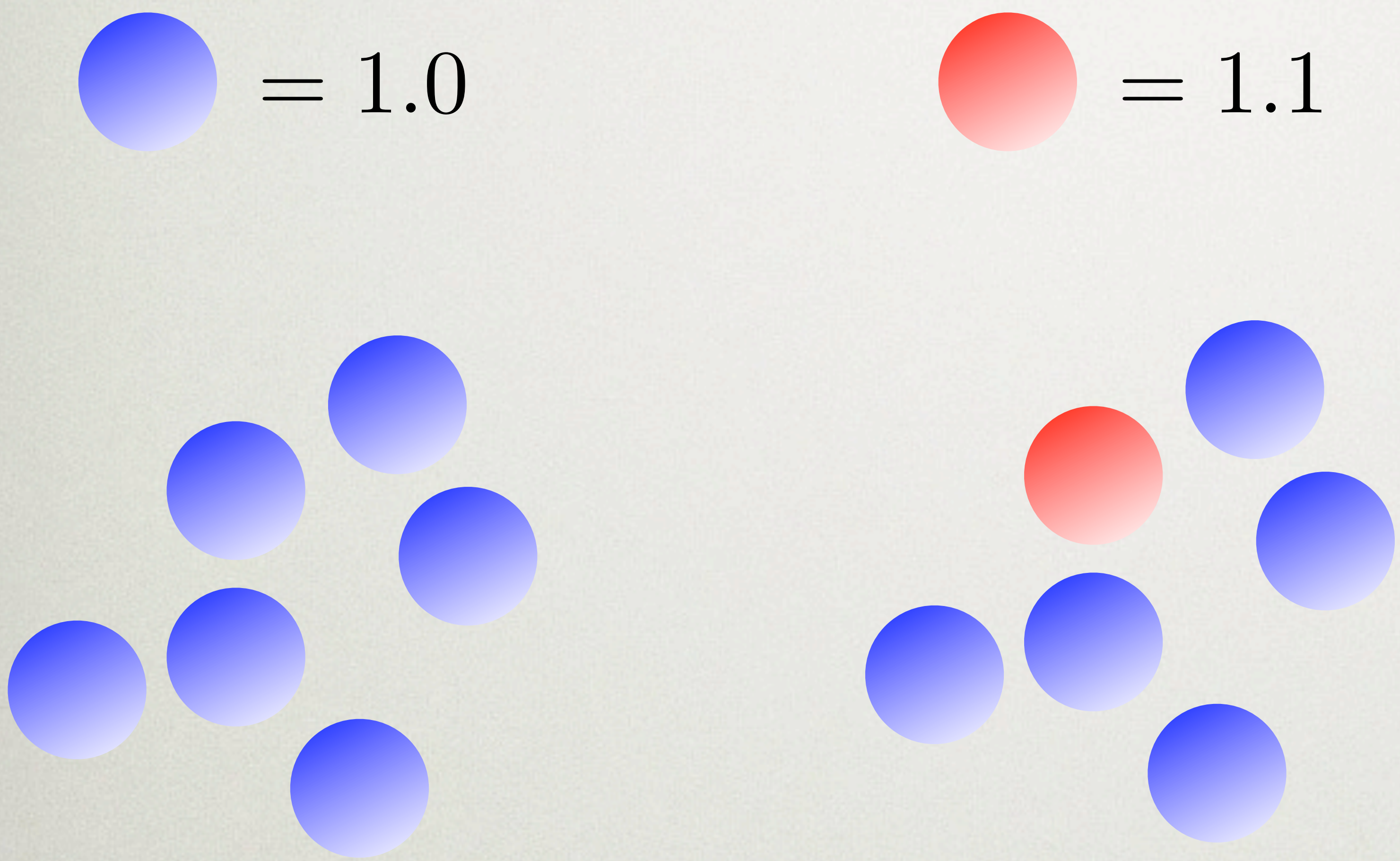
- Deals with human decision-making among interacting individuals.

Evolution

 $= 1.0$

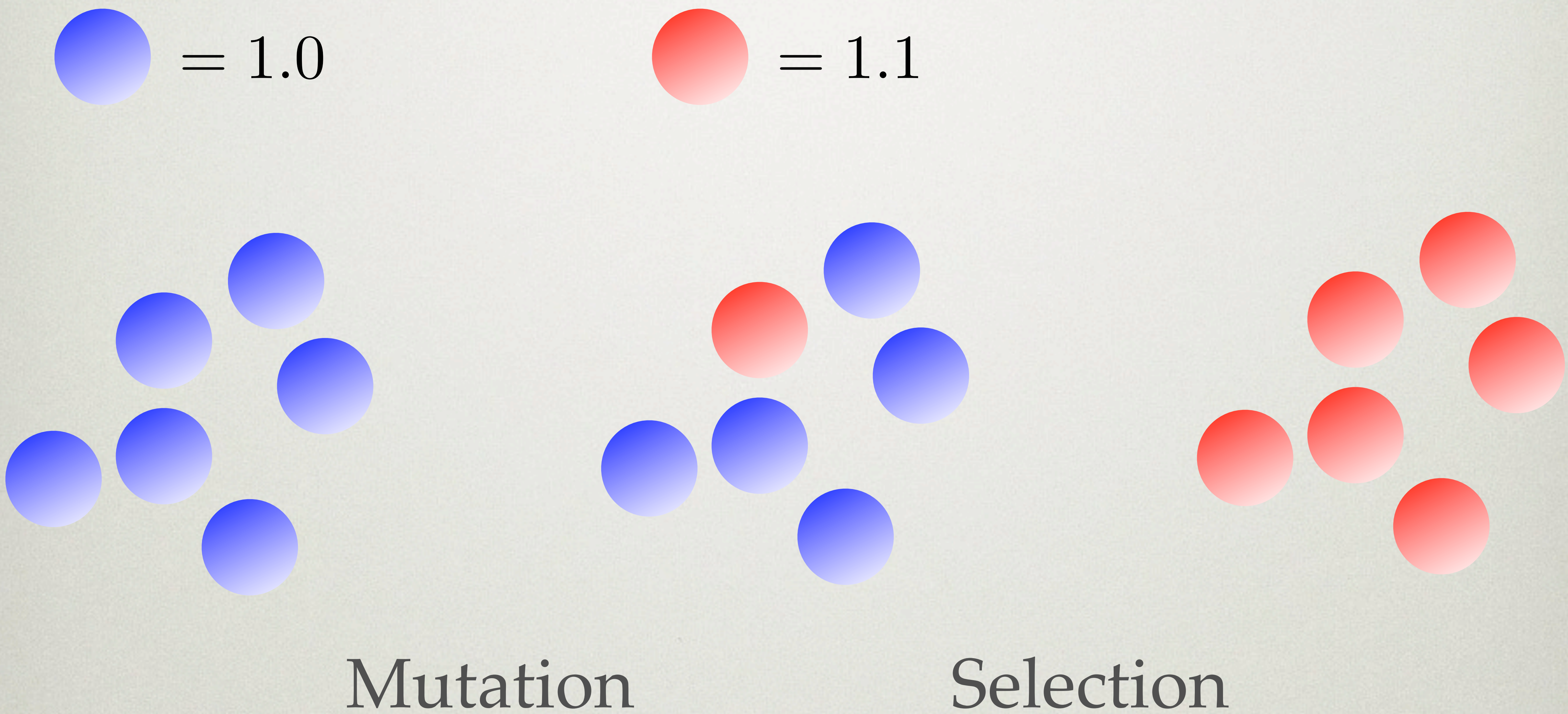


Evolution

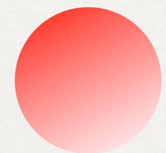


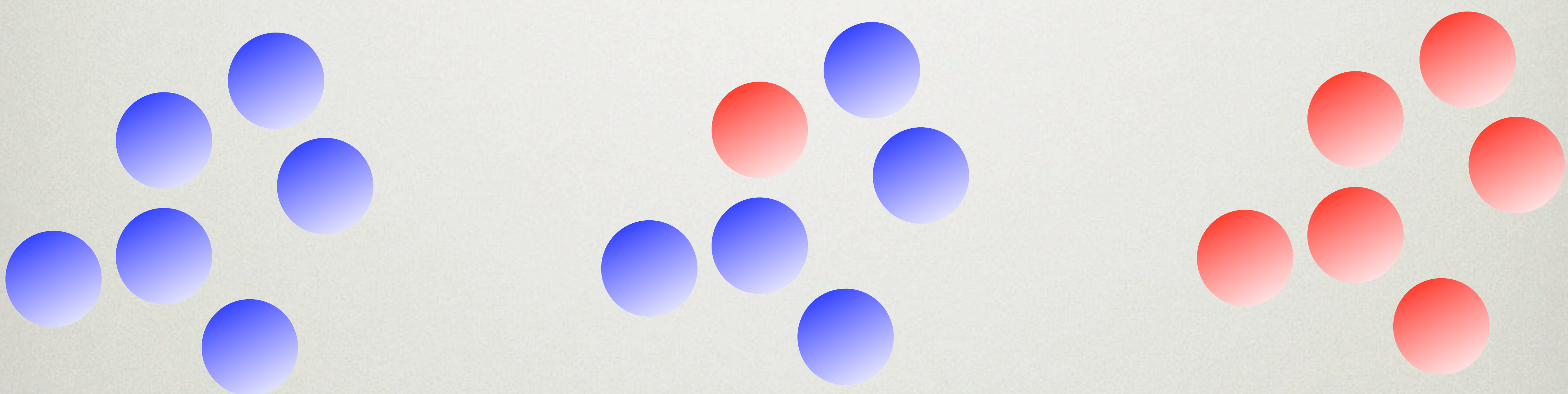
Mutation

Evolution



Evolution

 $= 1.0$  $= 1.1$



Mutation Selection

Constant Selection

Evolution

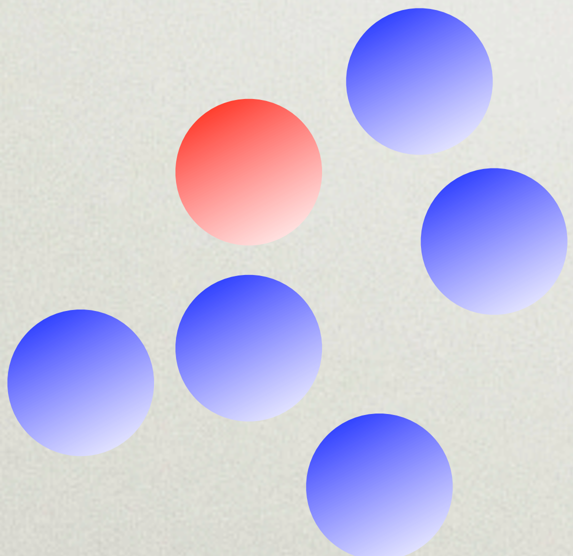
Fitness depends on the relative abundance

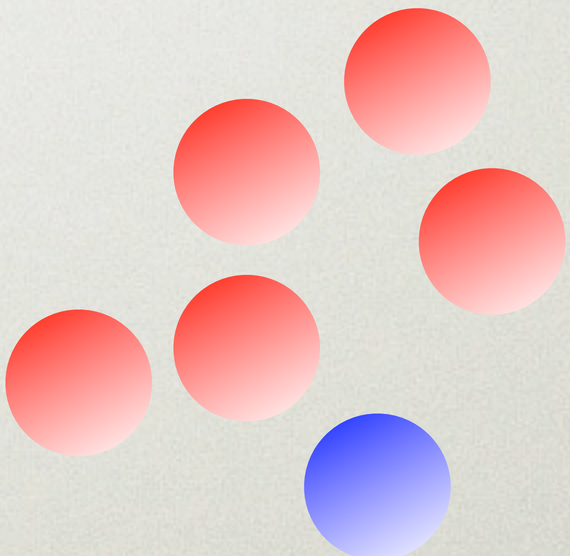
Evolution

Fitness depends on the relative abundance

$= 1.0$ $\quad$ $= 1.0$

$= 1.1$ $\quad$ $= 0.9$





Evolution

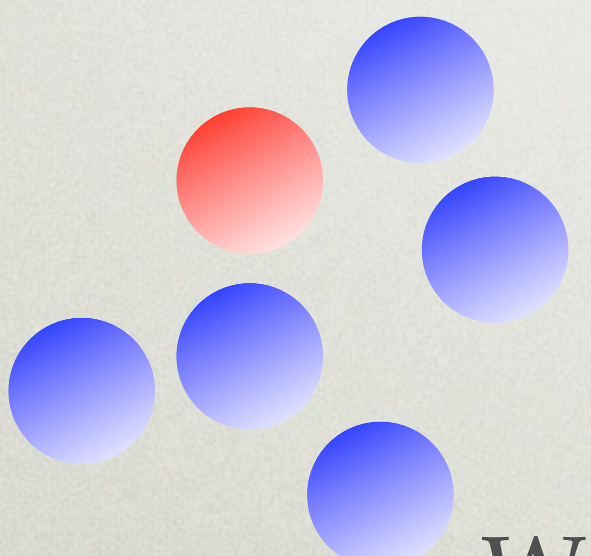
Fitness depends on the relative abundance

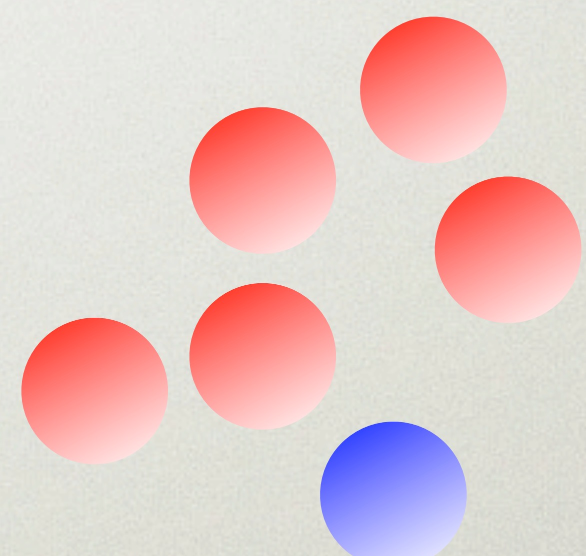
 $= 1.0$

 $= 1.1$

 $= 1.0$

 $= 0.9$





Wolves and hares

Evolution

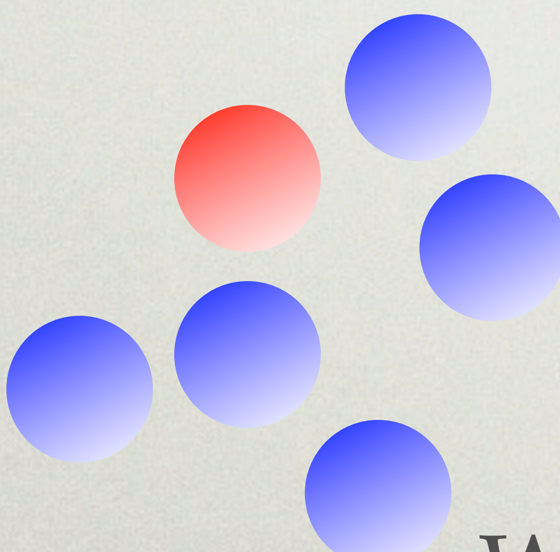
Fitness depends on the relative abundance

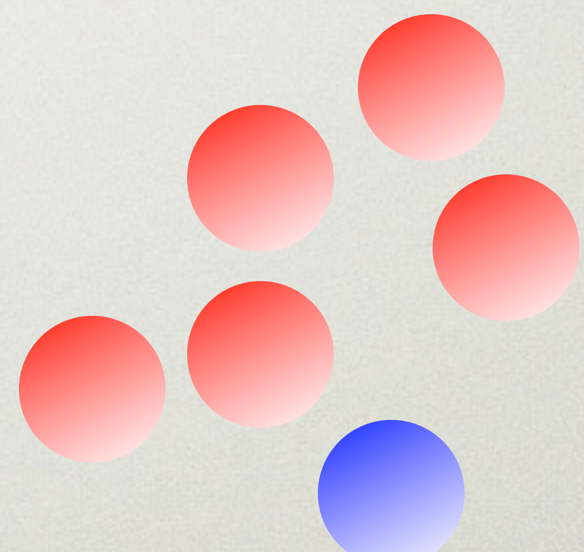
 $= 1.0$

 $= 1.1$

 $= 1.0$

 $= 0.9$





Wolves and hares

Frequency dependent Selection

Evolution

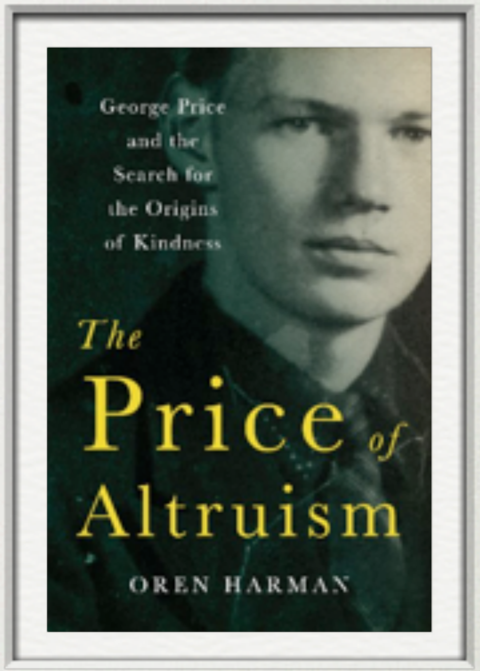
Fitness depends on the relative abundance

$$\begin{array}{c c} & \begin{array}{cc} Mutant & Wildtype \end{array} \\ \begin{array}{c} Mutant \\ Wildtype \end{array} & \left(\begin{array}{cc} 0.9 & 1.1 \\ 1.0 & 1.0 \end{array} \right) \end{array}$$

Frequency dependent Selection

Evolutionary Game Theory

John Maynard Smith

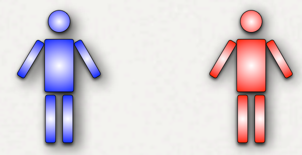


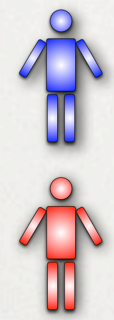
George Price

- Frequency dependent fitness introduces a strategic aspect to evolution.

Maynard Smith, J. and Price, G.R. The logic of animal conflict, Nature (1973)

Evolutionary Game Theory





$$\begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}$$

Frequency of A players: x

Payoff of A players π_A,

$$a_1 x + a_0(1-x)$$

Similarly for B players,

$$\pi_B = b_1 x + b_0(1-x)$$

Fitness = Payoff

Replicator Dynamics

Change in number of A players, $\dot{x}$

Average payoff of the population, $\bar{\pi} = x\pi_A + (1-x)\pi_B$

$$\dot{x} = x(\pi_A - \bar{\pi})$$

Replicator Dynamics

Change in number of A players, $\dot{x}$

Average payoff of the population, $\bar{\pi} = x\pi_A + (1-x)\pi_B$

$$\dot{x} = x(\pi_A - \bar{\pi})$$

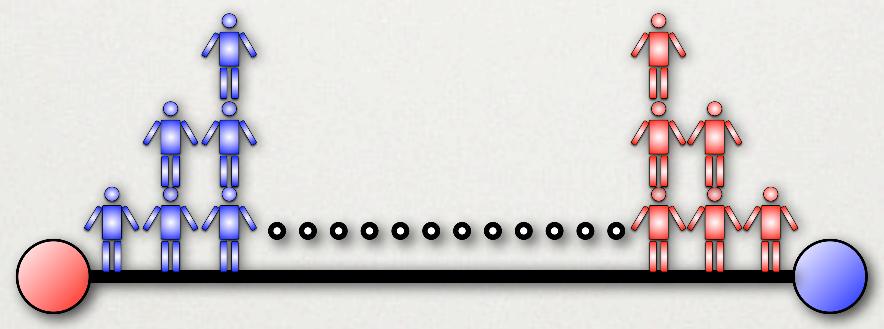
$$\boxed{\dot{x} = x(1-x)(\pi_A - \pi_B)}$$

Replicator Equation

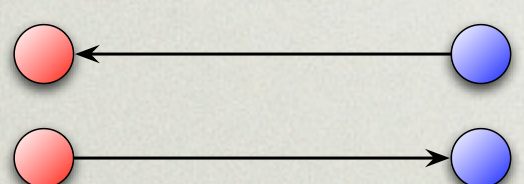
Replicator Dynamics

$$\begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}$$

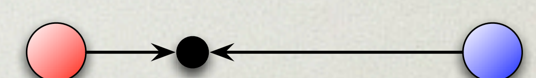
$$\dot{x} = x(1-x)(\pi_A - \pi_B)$$



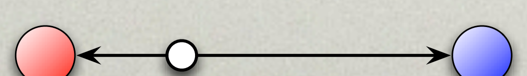
Dominance



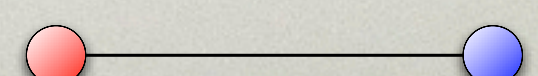
Co-existence



Bi-stability



Neutrality



Replicator Dynamics

General form

$$\dot{x}_i = x_i(\pi_i - \bar{\pi}) \qquad \bar{\pi} = \sum_{i=1}^{n} x_i \pi_i$$

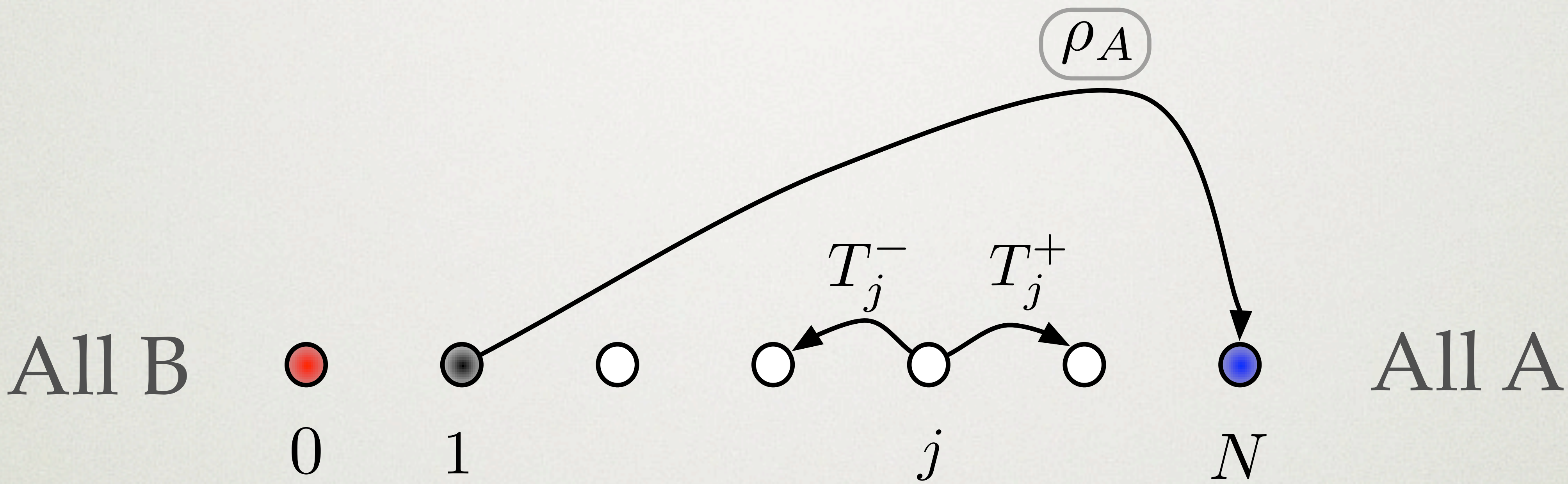
Where x_i gives the frequency of players playing strategy i where, $\sum_i x_i = 1$.

Equivalent to Lotka-Volterra in $n-1$ dimensions

The discrete version, $x_i^{'} = x_i \frac{\pi_i}{\bar{\pi}}$

Relation to population genetics $\longrightarrow$ R.Cressman, Evolutionary Dynamics and Extensive Form Games, MIT Press (2003)

Fixation Probability



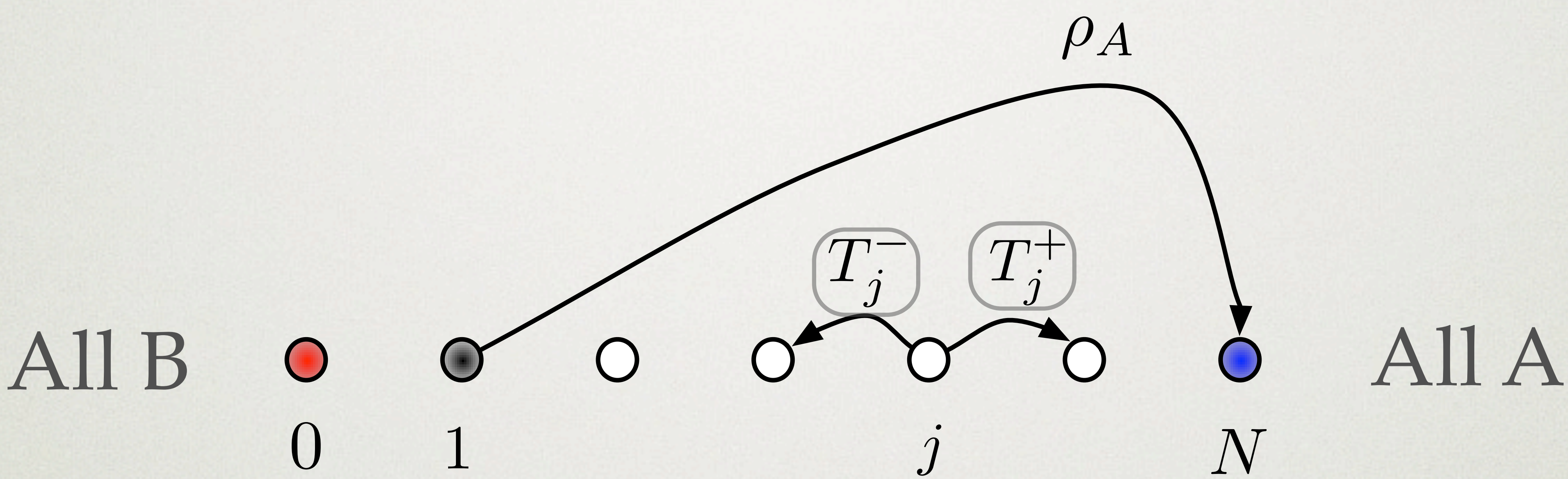
$$\rho_A = \frac{1}{1 + \sum_{m=1}^{N-1} \prod_{j=1}^{m} \frac{T_j^-}{T_j^+}}$$

For neutral evolution,

$$T_j^+ = T_j^- \Rightarrow \rho_A = \frac{1}{N}$$

Moran, Patrick Alfred Pierce, The Statistical Processes of Evolutionary Theory. Oxford, Clarendon Press (1962)

Fixation Probability



$$T_j^+ = \frac{j f_A}{j f_A + (N-j) f_B} \frac{N-j}{N}$$

$$T_j^- = \frac{(N-j) f_B}{j f_A + (N-j) f_B} \frac{j}{N}$$

$$f_A = \exp(+w\pi_A)$$

$$f_B = \exp(+w\pi_B)$$

$$\pi_A = \frac{j}{N} a_1 + \frac{N-j}{N} a_0$$

$$\pi_B = \frac{j}{N} b_1 + \frac{N-j}{N} b_0$$

$$\begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}$$

Fixation Probability

- For weak selection,

$$w \ll 1 \quad \Rightarrow \quad \rho_A \approx \frac{1}{N} + \frac{w}{N} \underbrace{\sum_{m=1}^{N-1} \sum_{j=1}^{m} (\pi_A - \pi_B)}_{\Gamma}$$

- Strategy A is favored by selection, i.e.

$$\rho_A > \frac{1}{N} \quad \Leftrightarrow \quad \Gamma > 0$$

Fixation Probability

$$2a_0 + a_1 > 2b_0 + b_1$$

$$\frac{b_0 - a_0}{a_1 - a_0 - b_1 + b_0} < \frac{1}{3} \qquad \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}$$

One-third law

Nowak, Sasaki, Taylor, Fudenberg Nature. 2004

Who replaces whom?

- A replaces B if, $\frac{\rho_B}{\rho_A} < 1$

$$\frac{\rho_B}{\rho_A} = \prod_{j=1}^{N-1} \frac{T_j^-}{T_j^+} = \exp\left[-w \underbrace{\sum_{j=1}^{N-1} (\pi_A - \pi_B)}_{\Phi} \right]$$

$$\frac{\rho_B}{\rho_A} < 1 \quad \Leftrightarrow \quad \Phi > 0$$

- In particular, for weak selection,

$$w \ll 1 \quad \Rightarrow \quad \frac{\rho_B}{\rho_A} \approx 1 - w\Phi$$

Who replaces whom?

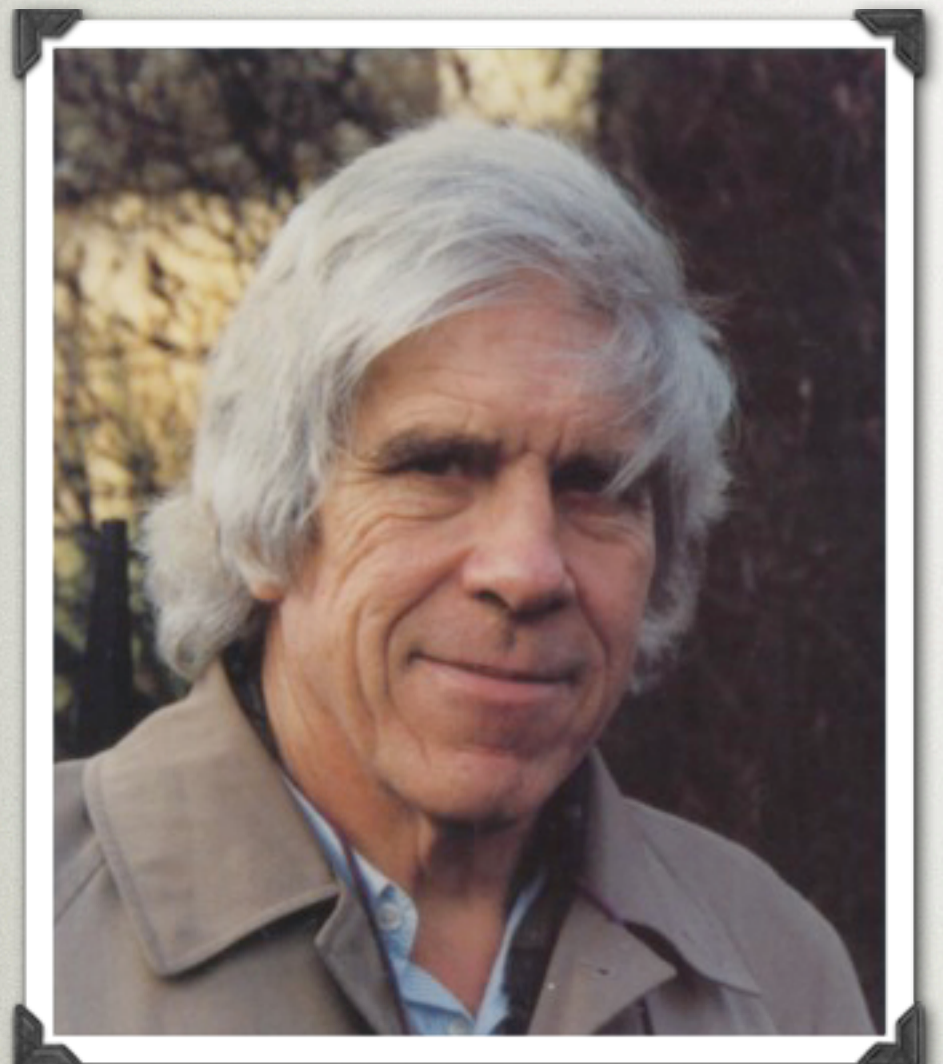
$$a_1 + a_0 > b_1 + b_0$$

$$\begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}$$

Risk Dominance

Harsanyi, Selten. A general theory of equilibrium selection in games, MIT Press, Cambridge, 1988

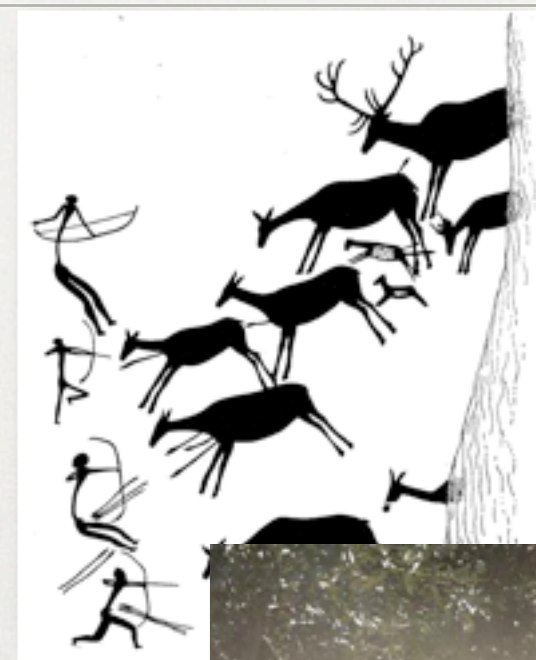
Harsanyi, Selten. Nobel Prize in Economics, 1994



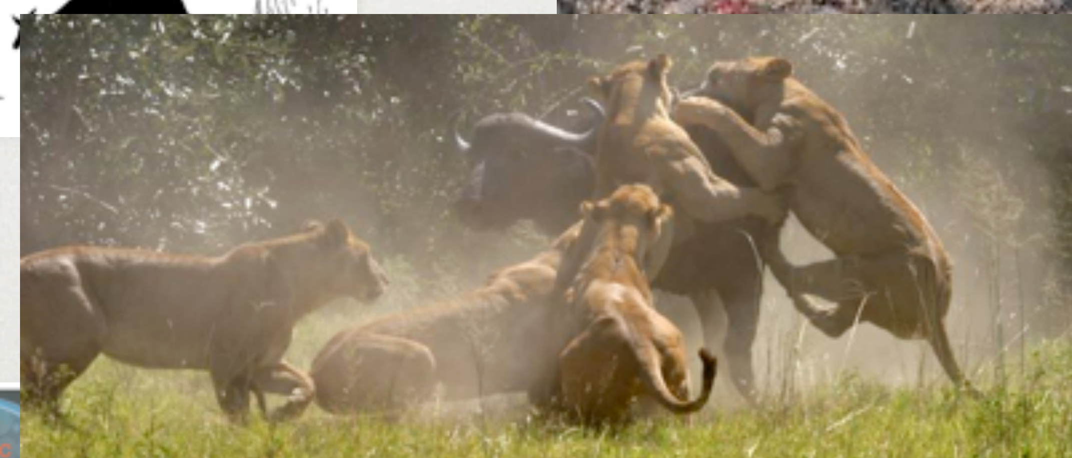
“....human life is a
‘many-person game’
and not just a disjoined
collection of ‘two-person
games’.....”

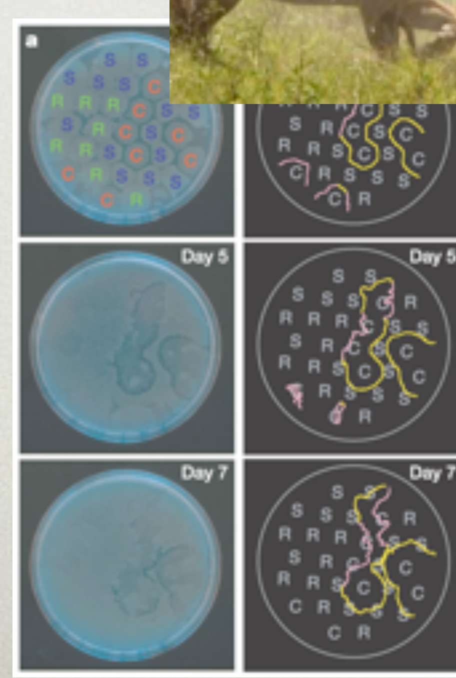
- W. D. Hamilton

Why d player games



- Group hunting
- Territorial defence
- Quorum sensing
- Climate preservation





B. Kerr et al., Nature 418,

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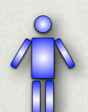
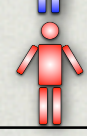
Payoff Matrix ?

For d player games,

Opposing  players	$d-1$	$d-2$	$\ldots$	k	$\ldots$	0
	a_{d-1}	a_{d-2}	$\ldots$	a_k	$\ldots$	a_0
	b_{d-1}	b_{d-2}	$\ldots$	b_k	$\ldots$	b_0

Fixation Probability

- For weak selection, $w \ll 1$ strategy A is favored by selection, if

$$\Gamma > 0$$

$$\sum_{m=1}^{N-1} \sum_{j=1}^{m} (\pi_A - \pi_B) > 0$$

Fixation Probability

- For weak selection, $w \ll 1$ strategy A is favored by selection, if

$$\Gamma > 0$$

$$\sum_{m=1}^{N-1} \sum_{j=1}^{m} (\pi_A - \pi_B) > 0$$

$$\sum_{m=1}^{N-1} \sum_{j=1}^{m} \left(\sum_{k=0}^{d-1} \frac{\binom{j-1}{k}\binom{N-j}{d-1-k}}{\binom{N-1}{d-1}} a_k - \sum_{k=0}^{d-1} \frac{\binom{j}{k}\binom{N-j-1}{d-1-k}}{\binom{N-1}{d-1}} b_k \right) > 0$$

Fixation Probability

- Vandermonde's convolution and other binomial properties.

$$\boxed{\sum_{k=0}^{d-1}(d-k)a_k > \sum_{k=0}^{d-1}(d-k)b_k}$$

$$d = 2 \Rightarrow \frac{b_0 - a_0}{a_1 - a_0 - b_1 + b_0} < \frac{1}{3}$$ One-third law

Kurokawa & Ihara, Proc B (2009); Gokhale & Traulsen, PNAS (2010)

Who replaces whom?

- For weak selection, A replaces B, $\frac{\rho_B}{\rho_A} < 1$ if,

$$\Phi > 0$$

$$\sum_{j=1}^{N-1} (\pi_A - \pi_B) > 0$$

Who replaces whom?

- For weak selection, A replaces B, $\frac{\rho_B}{\rho_A} < 1$ if,

$$\Phi > 0$$

$$\sum_{j=1}^{N-1} (\pi_A - \pi_B) > 0$$

$$\sum_{j=1}^{N-1} \left(\sum_{k=0}^{d-1} \frac{\binom{j-1}{k}\binom{N-j}{d-1-k}}{\binom{N-1}{d-1}} a_k - \sum_{k=0}^{d-1} \frac{\binom{j}{k}\binom{N-j-1}{d-1-k}}{\binom{N-1}{d-1}} b_k \right) > 0$$

Who replaces whom?

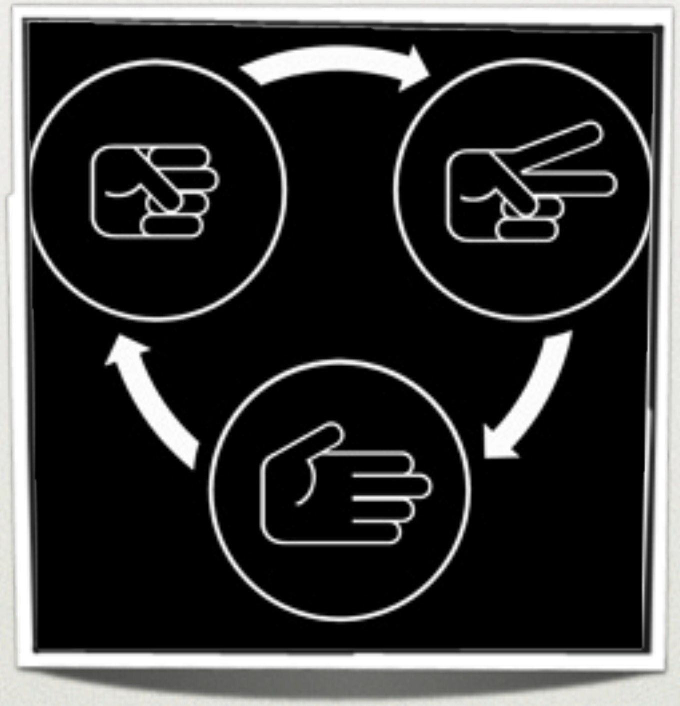
- Using Vandermonde's convolution and other binomial properties

$$\boxed{\sum_{k=0}^{d-1} a_k > \sum_{k=0}^{d-1} b_k}$$

$$d = 2 \Rightarrow \quad a_1 + a_0 > b_1 + b_0$$ Risk dominance

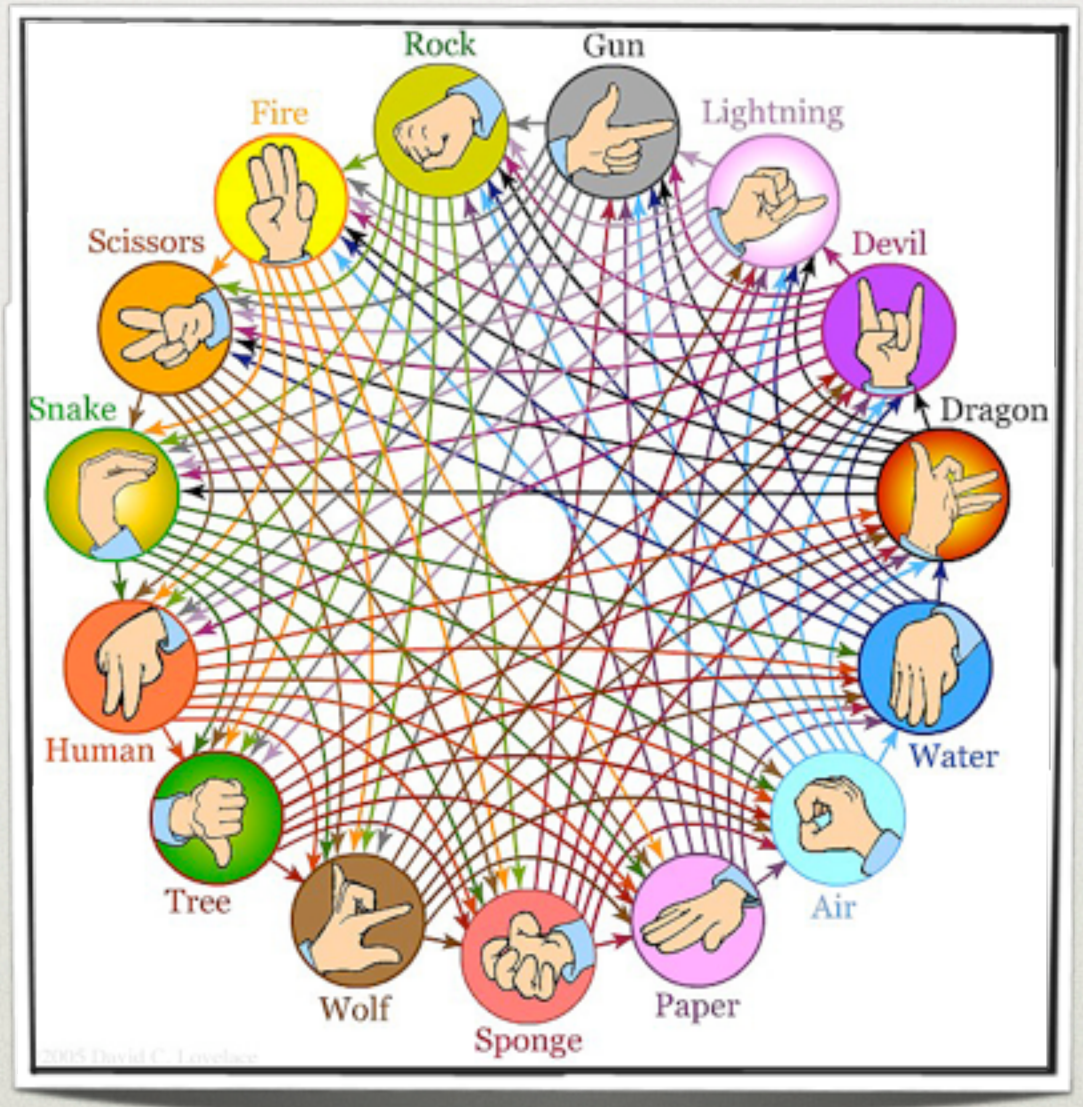
Kurokawa & Ihara, Proc B (2009); Gokhale & Traulsen, PNAS (2010)

What about multiple strategies?



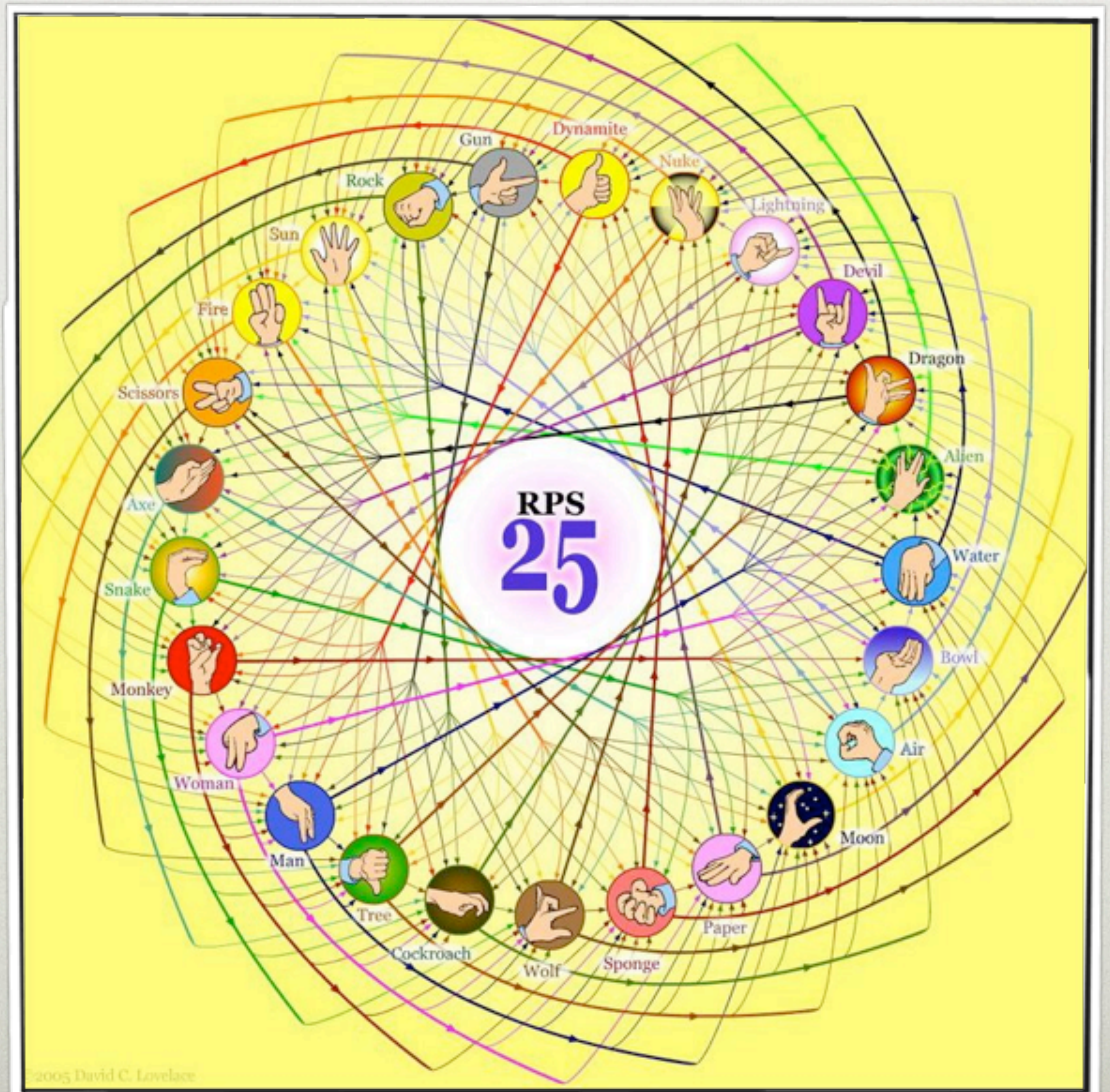
RPS

What about multiple strategies?



RPS

What about multiple strategies?

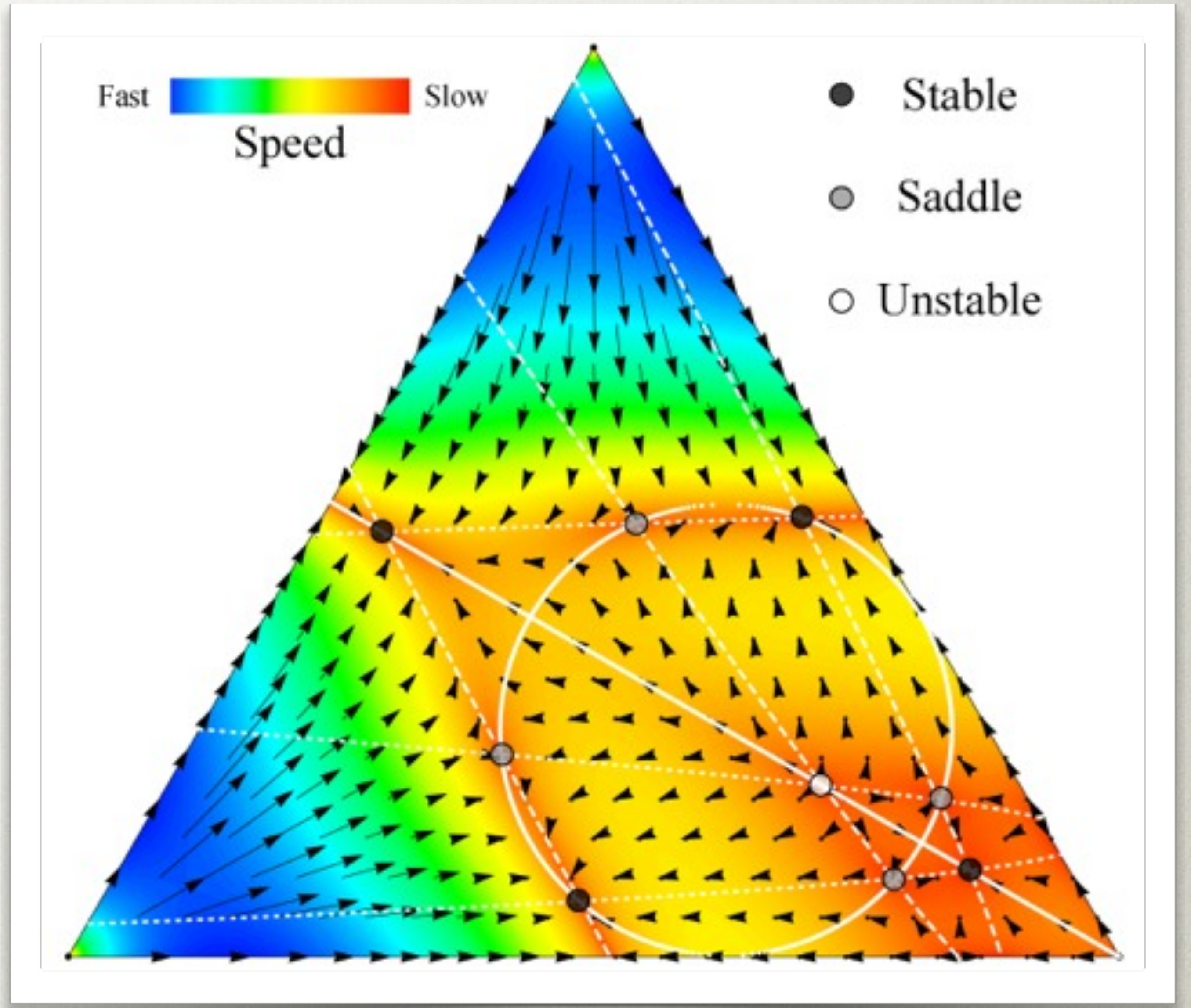


RPS

Maximum number of isolated internal equilibria

For a d player game with n strategies the maximum number of internal equilibria is give by,

$$\boxed{(d-1)^{(n-1)}}$$



e.g. for d=4;n=3

Gokhale & Traulsen, PNAS (2010)

Summary

	Comparison to neutrality	Comparison of strategies to each other	Maximum number of internal equilibria
2×2	☑ Finite ☑ Infinite	☑ Finite ☑ Infinite	1
$\underbrace{2 \times 2 \ldots \times 2}_{d}$	☑ Finite ☑ Infinite	☑ Finite ☑ Infinite	$d - 1$
$\underbrace{n \times n \ldots \times n}_{d}$			$(d-1)^{(n-1)}$

Other applications

- Evolution of language
- Evolutionary psychology
- Economics
- Sociology
- Anthropology
- Political Science
- Population genetics
- ....

- http://www.univie.ac.at/virtuallabs/