

Information Theory: Part II

Applications to Stochastic Processes

- We now consider applying information theory to a long sequence of measurements.

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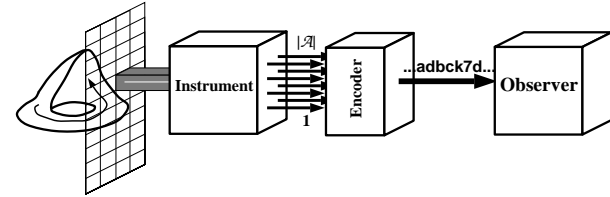
- In so doing, we will be led to two important quantities
 - Entropy Rate:** The irreducible randomness of the system.
 - Excess Entropy:** A measure of the complexity of the sequence.

Context: Consider a long sequence of discrete random variables. These could be:

- A long time series of measurements
- A symbolic dynamical system
- A one-dimensional statistical mechanical system

The Measurement Channel

- Can also picture this long sequence of symbols as resulting from a generalized measurement process:



- On the left is “nature”—some system's state space.
- The act of measurement projects the states down to a lower dimension and discretizes them.
- The measurements may then be encoded (or corrupted by noise).
- They then reach the observer on the right.
- Figure source: Crutchfield, “Knowledge and Meaning ... Chaos and Complexity.” In Modeling Complex Systems. L. Lam and H. C. Morris, eds. Springer-Verlag, 1992: 66-10.

Stochastic Process Notation

- Random variables $S_i, S_i = s \in \mathcal{A}$.
- Infinite sequence of random variables: $\overleftrightarrow{S} = \dots S_{-1} S_0 S_1 S_2 \dots$
- Block of L consecutive variables: $S^L = S_1, \dots, S_L$.
- $\Pr(s_i, s_{i+1}, \dots, s_{i+L-1}) = \Pr(s^L)$
- Assume translation invariance or stationarity:

$$\Pr(s_i, s_{i+1}, \dots, s_{i+L-1}) = \Pr(s_1, s_2, \dots, s_L).$$

- Left half (“past”): $\overleftarrow{S} \equiv \dots S_{-3} S_{-2} S_{-1}$
- Right half (“future”): $\overrightarrow{S} \equiv S_0 S_1 S_2 \dots$

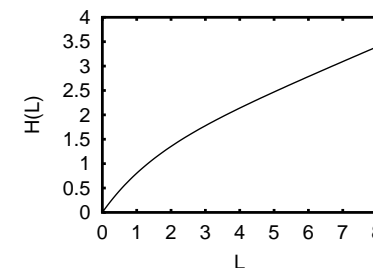
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Entropy Growth

- Entropy of L -block:

$$H(L) \equiv - \sum_{s^L \in \mathcal{A}^L} \Pr(s^L) \log_2 \Pr(s^L).$$

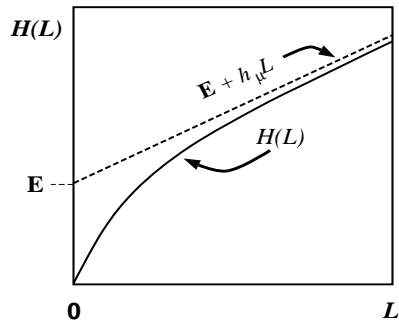
- $H(L)$ = average uncertainty about the outcome of L consecutive variables.



- $H(L)$ increases monotonically and asymptotes to a line
- We can learn a lot from the shape of $H(L)$.

Entropy Rate

- Let's first look at the slope of the line:



- Slope of $H(L)$: $h_\mu(L) \equiv H(L) - H(L-1)$
- Slope of the line to which $H(L)$ asymptotes is known as the *entropy rate*:

$$h_\mu = \lim_{L \rightarrow \infty} h_\mu(L).$$

Entropy Rate, continued

- Slope of the line to which $H(L)$ asymptotes is known as the *entropy rate*:

$$h_\mu = \lim_{L \rightarrow \infty} h_\mu(L).$$

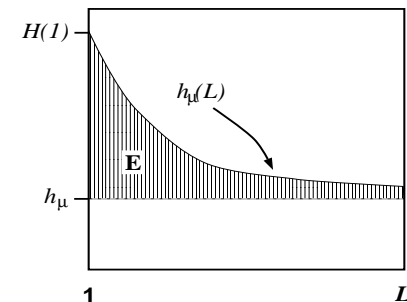
- $h_\mu(L) = H[S_L | S_1 S_1 \dots S_{L-1}]$
- I.e., $h_\mu(L)$ is the average uncertainty of the next symbol, given that the previous L symbols have been observed.

Interpretations of Entropy Rate

- Uncertainty per symbol.
- Irreducible randomness: the randomness that persists even after accounting for correlations over arbitrarily large blocks of variables.
- The randomness that cannot be "explained away".
- Entropy rate is also known as the Entropy Density or the Metric Entropy.
- h_μ = Lyapunov exponent for many classes of 1D maps.
- The entropy rate may also be written: $h_\mu = \lim_{L \rightarrow \infty} \frac{H(L)}{L}$.
- h_μ is equivalent to thermodynamic entropy.
- These limits exist for all stationary processes.

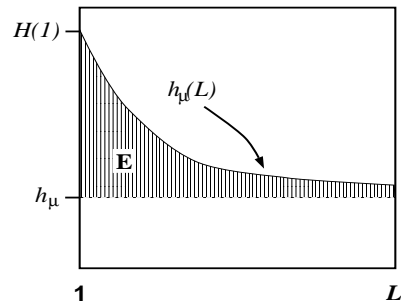
How does $h_\mu(L)$ approach h_μ ?

- For finite L , $h_\mu(L) \geq h_\mu$. Thus, the system appears more random than it is.



- We can learn about the complexity of the system by looking at *how* the entropy density converges to h_μ .

The Excess Entropy



- The **excess entropy** captures the nature of the convergence and is defined as the shaded area above:

$$\mathbf{E} \equiv \sum_{L=1}^{\infty} [h_\mu(L) - h_\mu].$$

- \mathbf{E} is thus the total amount of randomness that is “explained away” by considering larger blocks of variables.

Excess Entropy: Other expressions and interpretations

Mutual information

- One can show that \mathbf{E} is equal to the mutual information between the “past” and the “future”:

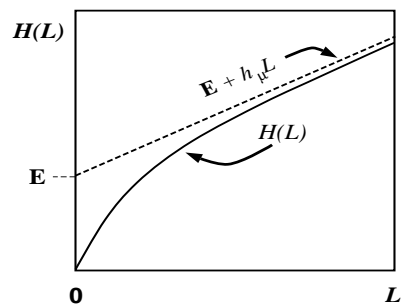
$$\mathbf{E} = I(\vec{S}; \vec{S}) \equiv \sum_{\{\vec{s}\}} \Pr(\vec{s}) \log_2 \left[\frac{\Pr(\vec{s})}{\Pr(\vec{s}) \Pr(\vec{s})} \right].$$

- \mathbf{E} is thus the amount one half “remembers” about the other, the reduction in uncertainty about the future given knowledge of the past.
- Equivalently, \mathbf{E} is the “cost of amnesia:” how much more random the future appears if all historical information is suddenly lost.

Excess Entropy: Other expressions and interpretations

Geometric View

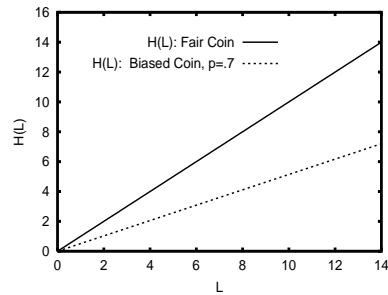
- \mathbf{E} is the y -intercept of the straight line to which $H(L)$ asymptotes.
- $\mathbf{E} = \lim_{L \rightarrow \infty} [H(L) - h_\mu L]$.



Excess Entropy Summary

- Is a structural property of the system — measures a feature complementary to entropy.
- Measures memory or spatial structure.
- Lower bound for statistical complexity, minimum amount of information needed for minimal stochastic model of system

Example I: Fair Coin

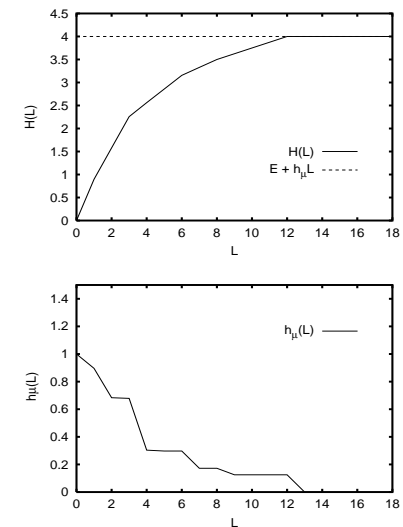


- For fair coin, $h_\mu = 1$.
- For the biased coin, $h_\mu \approx 0.8831$.
- For both coins, $\mathbf{E} = 0$.
- Note that two systems with different entropy rates have the same excess entropy.

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Example II: Periodic Sequence



- Sequence: ...1010111011101110...

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Example II, continued

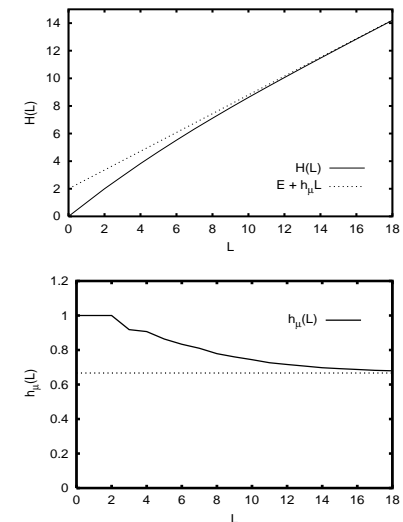
- Sequence: ...1010111011101110...
- $h_\mu \approx 0$; the sequence is perfectly predictable.
- $\mathbf{E} = \log_2 16 = 4$: four bits of phase information
- For any period- p sequence, $h_\mu = 0$ and $\mathbf{E} = \log_2 p$.

For more than you probably ever wanted to know about periodic sequences, see Feldman and Crutchfield, Synchronizing to Periodicity: The Transient Information and Synchronization Time of Periodic Sequences. *Advances in Complex Systems*. 7(3-4): 329-355, 2004.

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Example III: Random, Random, XOR



- Sequence: two random symbols, followed by the XOR of those symbols.

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Example III, continued

- Sequence: two random symbols, followed by the XOR of those symbols.
- $h_\mu = \frac{2}{3}$; two-thirds of the symbols are unpredictable.
- $\mathbf{E} = \log_2 4 = 2$: two bits of phase information.
- For many more examples, see Crutchfield and Feldman, *Chaos*, 15: 25-54, 2003.

Excess Entropy: Notes on Terminology

All of the following terms refer to essentially the same quantity.

- **Excess Entropy**: Crutchfield, Packard, Feldman
- **Stored Information**: Shaw
- **Effective Measure Complexity**: Grassberger, Lindgren, Nordahl
- **Reduced (Rényi) Information**: Szépfalusy, Györgyi, Csordás
- **Complexity**: Li, Arnold
- **Predictive Information**: Nemenman, Bialek, Tishby

Excess Entropy: Selected References and Applications

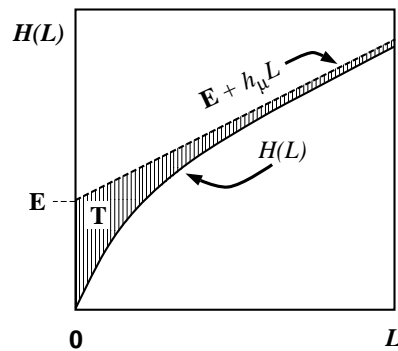
- Crutchfield and Packard, *Intl. J. Theo. Phys.*, 21:433-466. (1982); *Physica D*, 7:201-223, 1983. [Dynamical systems]
- Shaw, "The Dripping Faucet ...," Aerial Press, 1984. [A dripping faucet]
- Grassberger, *Intl. J. Theo. Phys.*, 25:907-938, 1986. [Cellular automata (CAs), dynamical systems]
- Szépfalusy and Györgyi, *Phys. Rev. A*, 33:2852-2855, 1986. [Dynamical systems]
- Lindgren and Nordahl, *Complex Systems*, 2:409-440. (1988). [CAs, dynamical systems]
- Csordás and Szépfalusy, *Phys. Rev. A*, 39:4767-4777. 1989. [Dynamical Systems]
- Li, *Complex Systems*, 5:381-399, 1991.
- Freund, Ebeling, and Rateitschak, *Phys. Rev. E*, 54:5561-5566, 1996.
- Feldman and Crutchfield, SFI:98-04-026, 1998. Crutchfield and Feldman, *Phys. Rev. E* 55:R1239-42. 1997. [One-dimensional Ising models]

Excess Entropy: Selected References and Applications, continued

- Feldman and Crutchfield. *Physical Review E*, 67:051104. 2003. [Two-dimensional Ising models]
- Feixas, et al, *Eurographics*, Computer Graphics Forum, 18(3):95-106, 1999. [Image processing]
- Ebeling. *Physica D*, 1090:42-52. 1997. [Dynamical systems, written texts, music]
- Bialek, et al, *Neur. Comp.*, 13:2409-2463. 2001. [Long-range 1D Ising models, machine learning]

Transient Information T

- $T \equiv \sum_{L=1}^{\infty} [E + h_{\mu}L - H(L)]$.
- T is related to the total uncertainty experienced while synchronizing to a process.



- The shaded area is the transient information T .
- T measures how difficult it is to synchronize to a sequence.

Some Applications in Agent-Based Modeling Settings

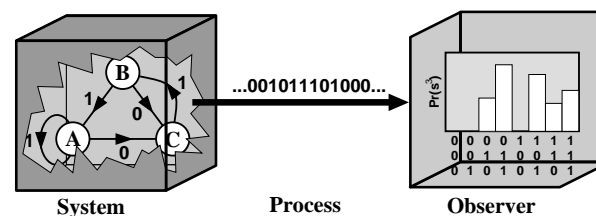
1. If an agent doesn't have sufficient memory, its environment will appear more random. In a quantitative sense, regularities that are missed (as measured by the excess entropy) are converted into randomness (as measured by the entropy rate).
 - Crutchfield and Feldman, Synchronizing to the Environment: Information Theoretic Constraints on Agent Learning. *Advances in Complex Systems*. 4. 251–264. 2001.
2. The average-case difficulty for an agent to synchronize to a periodic environment is measured by the transient information.
 - Feldman and Crutchfield. Synchronizing to a Periodic Signal: The Transient Information and Synchronization Time of Periodic Sequences. *Advances in Complex Systems*. 7. 329–355. 2004.

Some Applications in Agent-Based Modeling Settings, continued

3. More generally it seems likely that the entropy and mutual information are useful tools for quantifying
 - (a) properties of agents: e.g., how much memory they have
 - (b) the behavior of agents: e.g., how unpredictably they act
 - (c) properties of the environment: e.g., how structured it is

Estimating Probabilities

- E and h_{μ} can be estimated empirically by observing a process.



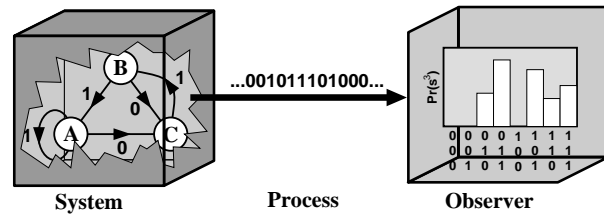
- One simply forms histograms of occurrences of particular sequences and uses these to estimate $\Pr(s^L)$, from which E and h_{μ} may be readily calculated.

For more sophisticated and accurate ways of inferring h_{μ} , see, e.g.,

- Schürmann and Grassberger. *Chaos* 6:414-427. 1996.
- Nemenman. <http://arXiv.org/physics/0207009>. 2002.

A look ahead

- Note that the observer sees measurement symbols: 0's and 1's.



- It doesn't see inside the "black box" of the system.
- In particular, it doesn't see the internal, hidden states of the system, A , B , and C .
- Is there a way an observer can infer these hidden states?
- What is the meaning of *state*?