Heisenberg uncertainty principle

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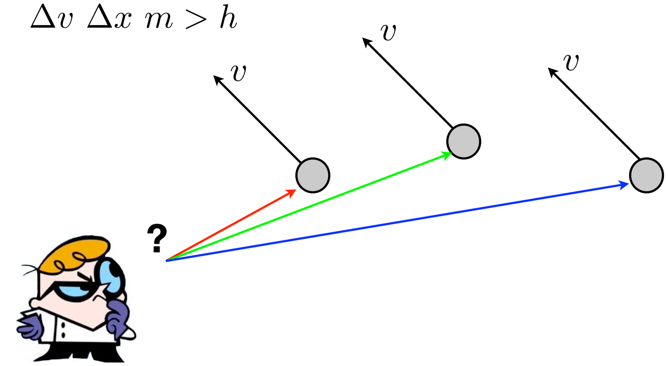
Comments

In the framework of quantum mechanics -one of the most successful theories created- is not possible to know -at the same time- the position and velocity of a given object.

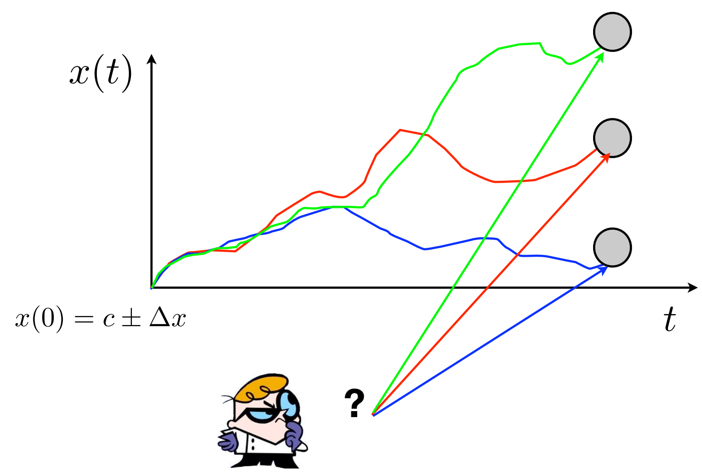
$$\Delta v \; \Delta x \; m > h$$



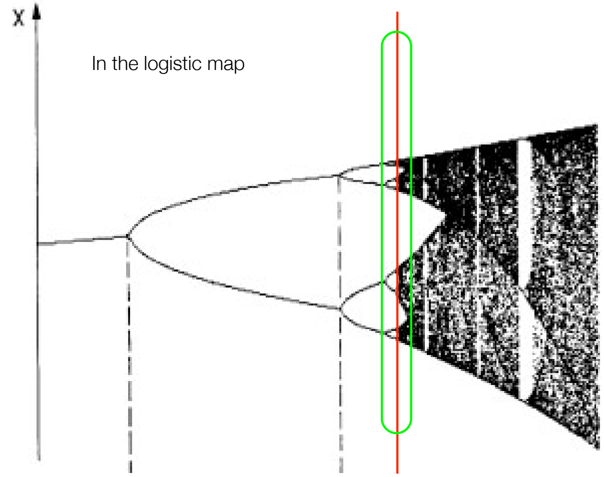
Einstein was very unhappy about this apparent randomness in nature. His views were summed up in his famous phrase, '**God does not play dice.**' (as Wilkins does)

(from a Stephen Hawking lecture)

In chaotic (deterministic) systems

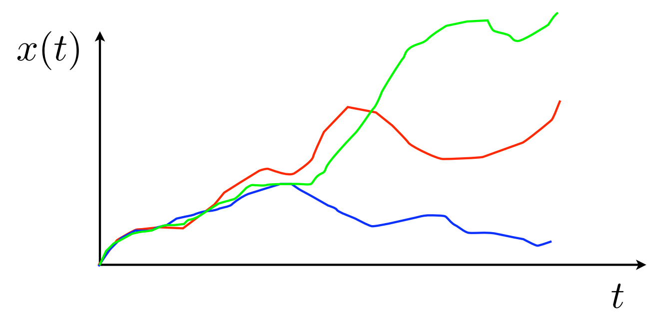


Some process are not chaotic, not regular



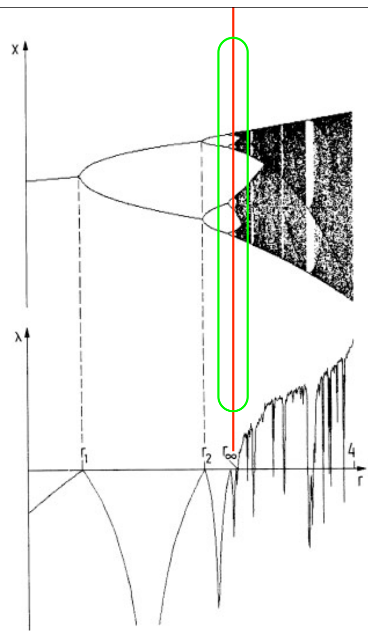
$$|x_{green}(t) - x_{blue}(t)| \sim e^{\lambda t}$$

λ : Lyapunov exponent, positive for chaos

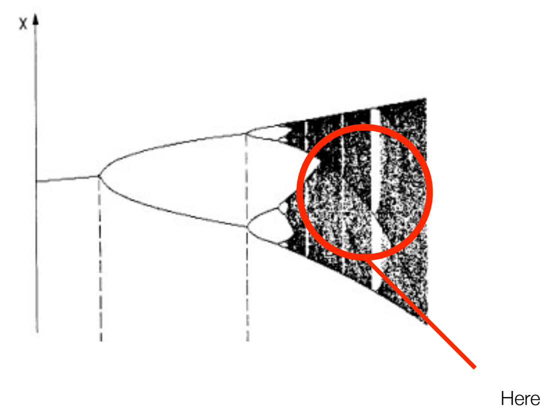


accumulation points

Liapunov exponent



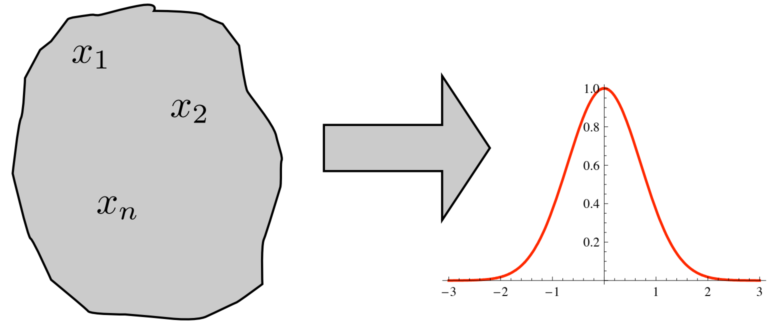
There are important theorems for random variables that are valid for chaotic systems



Abraham de Moivre (1667) a French mathematician famous for de Moivre's formula (friend of Isaac Newton, Edmund Halley, and James Stirling)

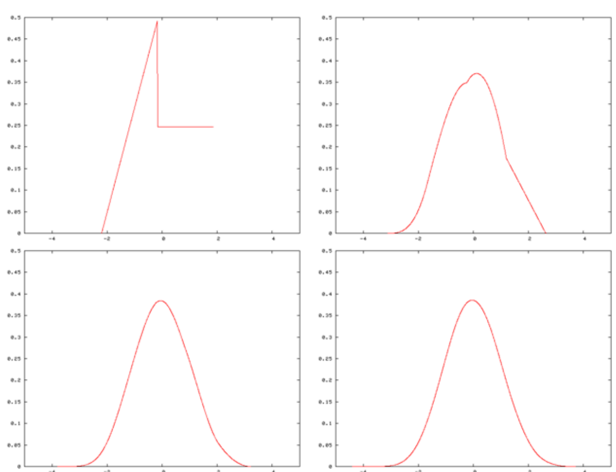
$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

The central limit theorem (CLT) states that the **sum of a large number** of independent and identically-distributed **random variables** will be approximately normally distributed (i.e., **following a Gaussian distribution**, or bell-shaped curve) if the random variables have a finite variance (*1773*).



$$z = \frac{\sum_1^N x_i}{\sqrt{N}}$$

$P_N(z)$

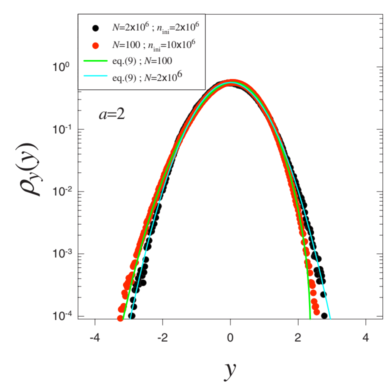


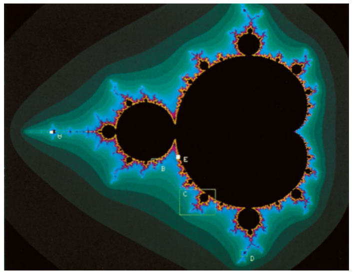
z

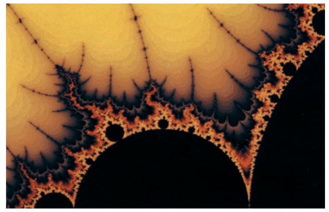
For the logistic map at positive lyapunov exponent (CHAOS)

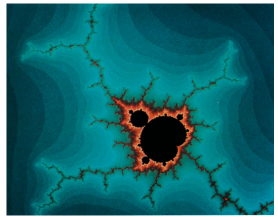
$$x_{i+1} = T(x_i) = 1 - ax_i^2$$

$$y := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(x_i)$$





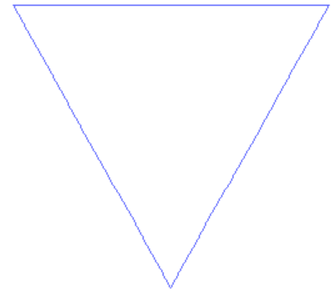




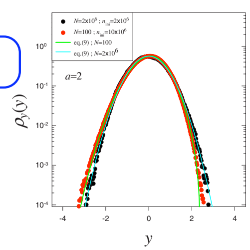
The set M is called "Mandelbrot's set" after B. B. Mandelbrot who first published (1980) a picture of M (see Fig. 94). It shows that M has also a fractal structure (but it is no Julia set). This study was extended by Peitgen and Richter (1984). If c does not belong to M, then $\lim_{n\to\infty} f_c^n(0) \to \infty$. Therefore, they define "level curves" in the following way: color a starting point according to the number of iterations it needs to leave a disk with a given radius R. As shown by Douady and Hubbard (1982), lines of equal color can be interpreted as equipotential lines if the set M is considered to be a charged conductor. Plates VIII–XV show the fascinating results of this procedure which brings us back to Ruelles' remark at the beginning of this section.

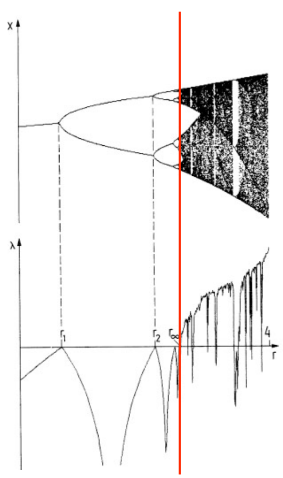
From "Deterministic Chaos", Schuster & Just

Koch curve (after 1904 paper "On a continuous curve without tangents, constructible from elementary geometry" (original French title: "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire") by the Swedish mathematician Helge von Koch.)



In chaos the central limit distribution holds

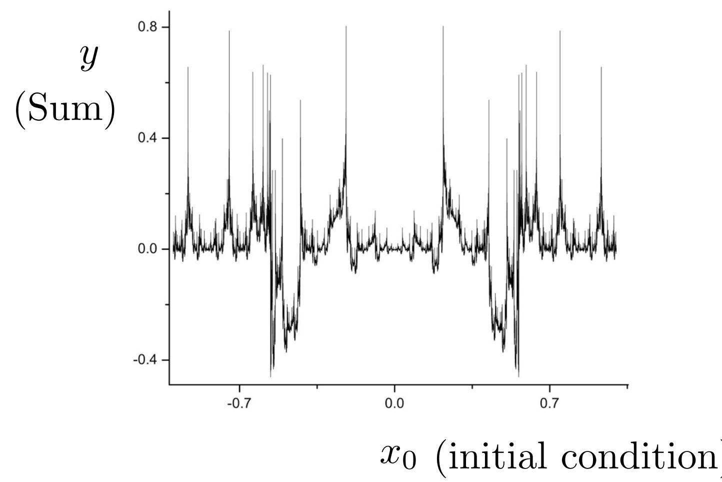




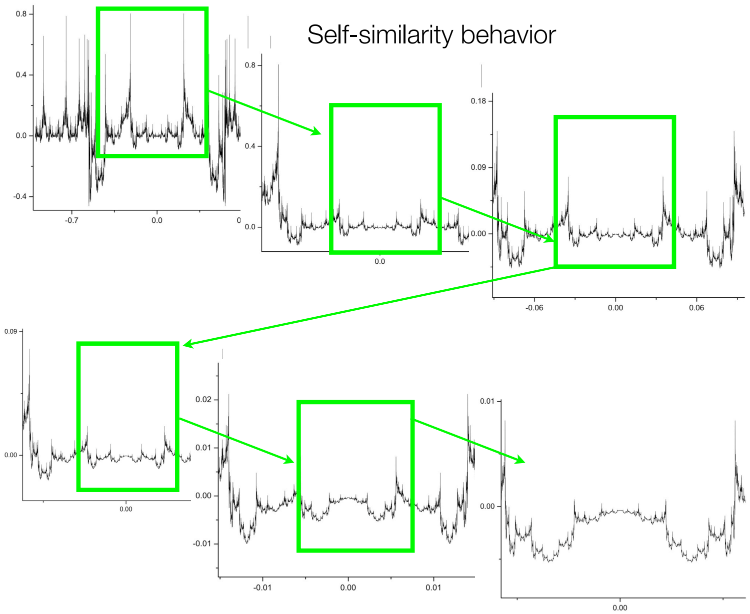
What happens at the edge of chaos?

At zero Lyapunov exponent ?

$$y := \frac{1}{\sqrt{N}}\sum_{i=1}^{N} f(x_i)$$



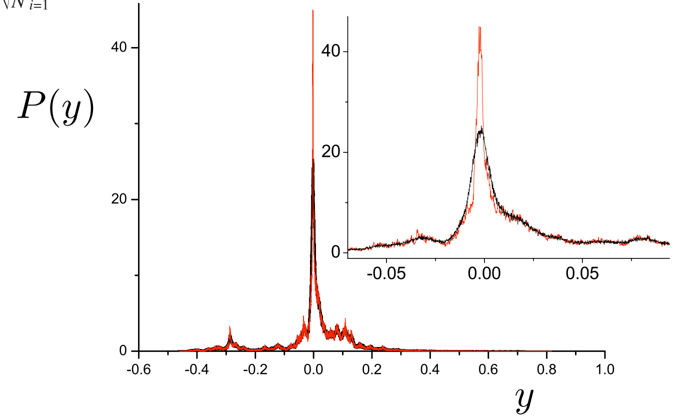
Self-similarity behavior



Limit distribution **at zero Lyapunov exponent**

(and near zero Lyap. exponent)

$$y := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(x_i)$$



Patterns

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Themes

- Patterns
- Tools for pattern descriptions
- Reaction diffusion equation
- Waves and excitable media
- Non local mechanism
- Beyond standard analysis
- D’Arcy 1917

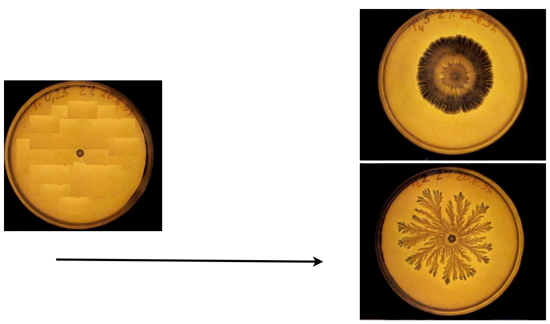
PDE, Reaction Diffusion Models:

"Under certain conditions spatially inhomogeneous patterns can evolve by diffusion driven instability" A. M. Turing, 1952

The chemical basis of morphogenesis, A. M. Turing, Phil. Trans. Roy. Soc. London, B237, 37-72, 1952.

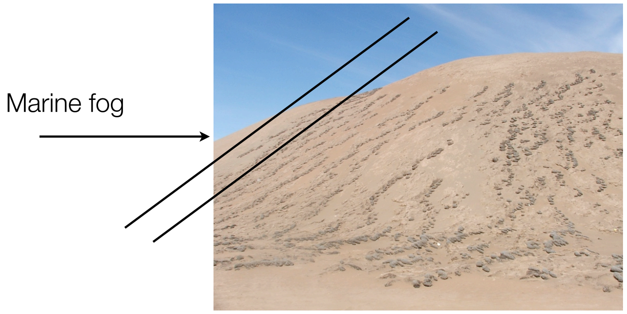
Minimal model

$$\begin{aligned}\partial_t u &= f(u,v) + d_u \nabla^2 u \\ \partial_t v &= g(u,v) + d_v \nabla^2 v\end{aligned}$$



from Ben-Jacob et al.

Atacama desert (one of the driest on earth)



Linear stability analysis

(u_0, v_0): homogenous steady state, i. e.:

$$f(u_0, v_0) = g(u_0, v_0) = 0$$

$$\mathbf{w} = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix}$$

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} \qquad \mathbf{A} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

$$\mathbf{w} \sim e^{\lambda t} \quad \begin{cases} \lambda > 0 & \mathbf{w} \to \infty \\ \lambda < 0 & \mathbf{w} \to 0 \end{cases}$$

Conditions for stability of the homogeneous state

The eigenvalues are given by the solution of (exercise 1)

$$\begin{vmatrix} f_u - \lambda & f_v \\ g_u & g_v - \lambda \end{vmatrix} = 0$$

Then there are two solution: λ_1, λ_2 from the equation:

$$\lambda^2 - (f_u + g_v)\lambda + (f_u g_v - f_v g_u) = 0$$

Linear stability, $\Re[\lambda] < 0$,is guaranteed if (exercise 2)

$$tr\mathbf{A} = f_u + g_v < 0, \quad |\mathbf{A}| = f_u g_v - f_v g_u > 0 \qquad \mathbf{A} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

Conditions for instability of the homogeneous state

$$\begin{aligned} \partial_t u &= f(u,v) + d_u \nabla^2 u \\ \partial_t v &= g(u,v) + d_v \nabla^2 v \end{aligned}$$

Linear version

$$\partial_t \mathbf{w} = \mathbf{A}\mathbf{w} + \mathbf{D}\nabla^2 \mathbf{w} \qquad \mathbf{w} = \sum_k c_k e^{\lambda t} \mathbf{W}_k(\mathbf{r})$$

Using cos functions

$$\begin{aligned} \lambda \mathbf{W}_k &= \mathbf{A}\mathbf{W}_k + \mathbf{D}\nabla^2 \mathbf{W}_k \\ &= \mathbf{A}\mathbf{W}_k - \mathbf{D}k^2 \mathbf{W}_k \end{aligned}$$

Eigenvalues equation

$$|\lambda \mathbf{I} - \mathbf{A} + \mathbf{D}k^2| = 0 \qquad \mathbf{w} = \sum_k c_k e^{\lambda t} \mathbf{W}_k(\mathbf{r})$$

$$\lambda^2 + \lambda[k^2(1+d) - (f_u + g_v)] + h(k^2) = 0$$

$$h(k^2) = dk^4 (df_u + g_v) k^2 + |\mathbf{A}|$$

Linear stability, $\Re[\lambda] > 0$,is guaranteed if (exercise 3)

$$df_u + g_v > 0, \qquad (df_u + g_v)^2 - 4d|\mathbf{A}| > 0$$

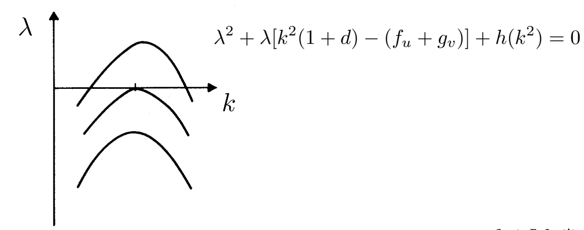
Conditions

Stability of the homogeneous state

$$tr\mathbf{A} = f_u + g_v < 0, \quad |\mathbf{A}| = f_u g_v - f_v g_u > 0$$

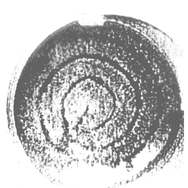
Instability of some modes

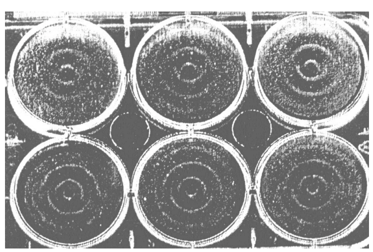
$$df_u + g_v > 0, \qquad (df_u + g_v)^2 - 4d|\mathbf{A}| > 0$$



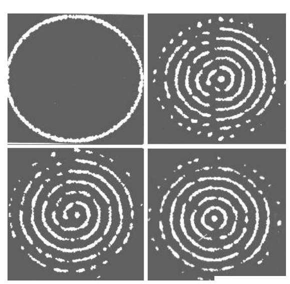
Examples of patterns in cultured brain capillary endothelial cells.

experiments





numerical simulations

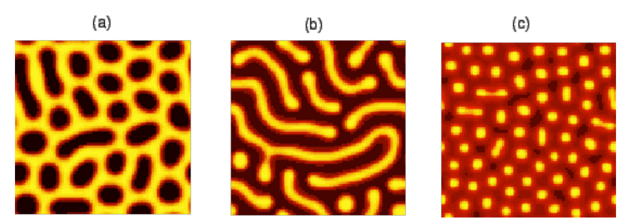


Discussion

Easy work frame

Two diffusion coefficient

Extensive literature



Waves

Minimal non linear model with diffusion

$$\partial_t u = ku(1-u) + D\partial_{xx}u$$

Fisher, 1937

$$\partial_t u = u(1-u) + \partial_{xx}u$$

Static frame

$$u(x,t) = U(z), \qquad z = x - ct$$

$$U'' + cU' + U(1-U) = 0$$

(Exercise 4)

$$\lim_{z\to\infty} U(z) = 0, \qquad \lim_{z\to-\infty} = 1$$

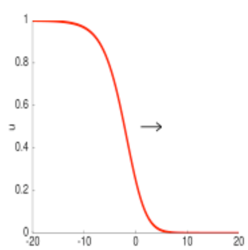
In the plane (U,V)

$$U' = V, \qquad V' = -cV - U(1-U)$$

Linear stability analysis

$$(0,0): \qquad \lambda_\pm = [-c \pm (c^2-4)^{1/2}]/2 \quad \begin{cases} \text{stable node if} & c^2 \geq 4 \\ \text{stable spiral if} & c^2 < 4 \end{cases}$$

$$(1,0): \qquad \lambda_\pm = [-c \pm (c^2+4)^{1/2}]/2$$



For the dimensional model

$$c \geq c_{min} = 2\sqrt{kD}$$

(Exercise 5, why the symbol $\geq$?)

Discussion

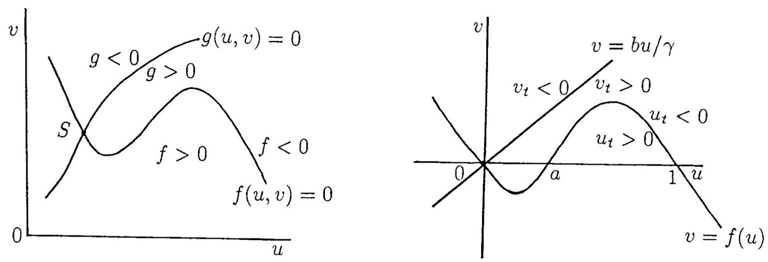
Finite domain: localized structure

Critical Size

Excitable media

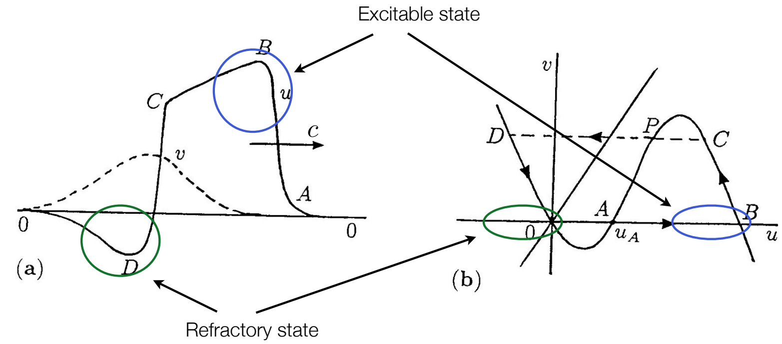
$$u_t = f(u,v), \qquad v_t = g(u,v)$$

Nullclines

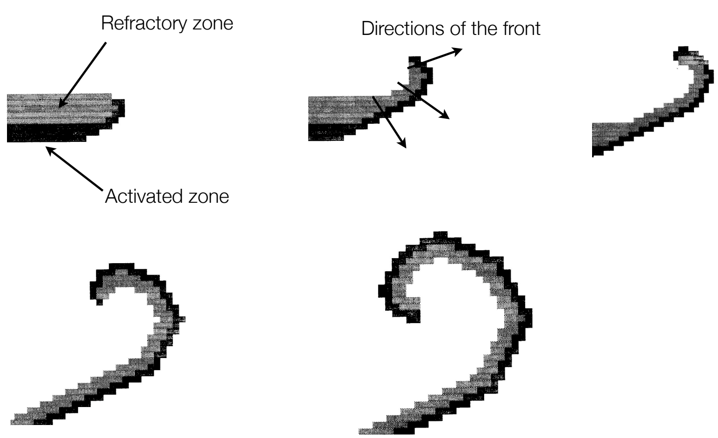


Excitable media with space

$$\partial_t u = f(u,v) + \partial_{xx} u \qquad \partial_t v = g(u,v)$$



Spiral waves:
self sustained spatiotemporal structures



A. S. Mikailov, Foundation of Synergetics

Maree and Panfilov, Physical Review Letter 78, 1997

More dimensions

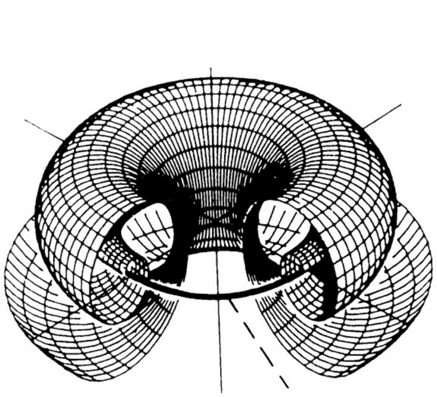


Fig. 3.33. Scroll ring

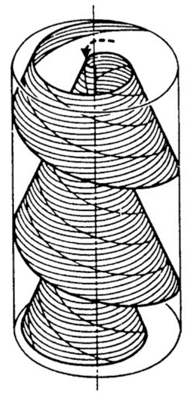
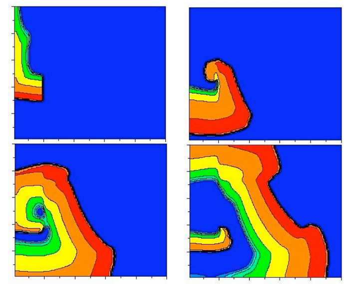


Fig. 3.34. Straight twisted scroll

A. S. Mikailov, Foundation of Synergetics

Example

In 1991 Luo and Rudy (1991) presented an ionic model (LR-I) for the cardiac potential in ventricular cells.



N. Sarvazyan, M. Fuentes, V. Krinsky

http://www.lab.sarvazyan.com/pacing.shtml

Discussion

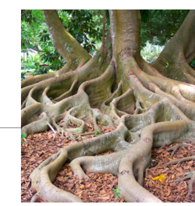
Analytical method for velocity, geometrical properties of the spiral, etc.

Multi-arms spirals

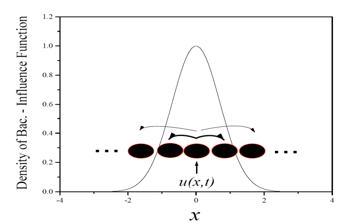
Interaction of spirals

Cellular automata methods

Non local mechanisms



$$\partial_t u = f(u) + g(u) \int_\Omega G(x, x')\ u(x')\ dx', \qquad \int_\Omega G = 1$$



$$\partial_t u = f(u) + g(u) \int_\Omega G(x,x')\ u(x')\ dx', \qquad \int_\Omega G = 1$$

Near the stable point

$$\partial_t u = f'(u_0)u + g'(u_0)u_0 u + g(u_0) \int G\ u\ dx', \qquad u_0 \text{ is a stable point}$$

(Exercise 6)

(Exercise 7, condition for stability for local interaction see the limit: $G \longrightarrow \delta$)

Doing the Fourier transform, i. e. $u \sim e^{ikx+\lambda t}$

$$\lambda = f'(u_0) + g'(u_0)u_0 + g(u_0)F(k), \qquad \text{with} \ \ F(k) = \int G e^{iky} dy$$

Equation with space variables

$$\partial_t u = f(u) + g(u) \int_\Omega G(x,x')\ u(x')\ dx' + d\nabla^2 u$$

Condition for unstable modes

$$F(k) > \frac{Dk^2 - f'(u_0) - g'(u_0)u_0}{g(u_0)}$$

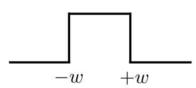
(Exercise 8)

Example

$$\partial_t u = f(u) + g(u) \int_\Omega G(x,x')\ u(x')\ dx' + d\nabla^2 u$$

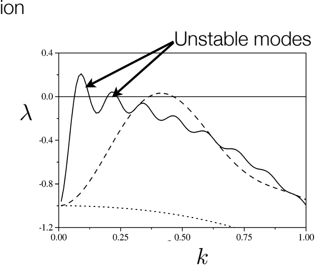
$$f(u) = au, \qquad g(u) = -bu$$

$$G(x,x') = \frac{\theta[w-(x-x')]\theta[w+(x-x')]}{2w}$$



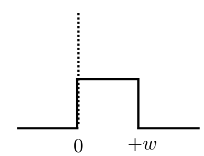
Note: $G \longrightarrow \delta$, is the Fisher equation

$$\lambda = -a\frac{\sin(kw)}{kw} - dk^2$$



Traveling waves - Non local Interactions

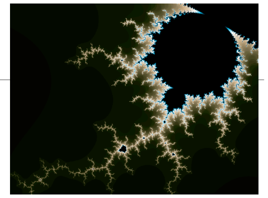
$$G(x,x') = \frac{\theta[x]\theta[w+(x-x')]}{w}$$



$$\Re[F(k)] \ = \ \frac{sin(kw)}{kw}$$ ⟵ Patters

$$\Im[F(k)] \ = \ \frac{1}{kw} - \frac{sin(kw)}{kw} \neq 0$$ ⟵ Oscillations

Frontiers



1- Lacking of first principles descriptions

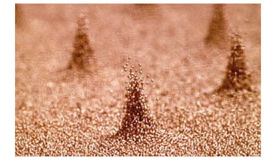
2- A different geometry

Lacking of first principles descriptions

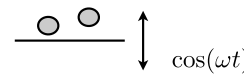
Example

Non continuos Systems

Granular system



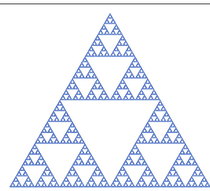
(Exercise 9, discuss what happens in Turing models)

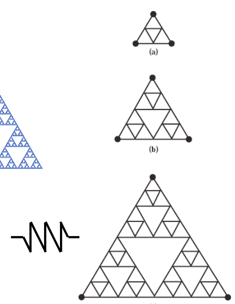


Roughly speaking

- Markovian
- Non Markovian, memory

A different geometry





Example

The Sierpinski gasket (and its wave equation)

Non continuos operator

$$\begin{cases} \partial_t^2 u &= \nabla^2 u \\ \partial_t^2 u &= \lim_{m\to\infty} \frac{3}{2} 5^m H_m u, \quad H_m u = \sum \Delta u \end{cases}$$

Yamaguti et al., Translations of mathematical monographs, American Mathematical Society, 1997

Different solutions (eigenfunctions)

Not only different patterns, but physics etc

Conclusions

- Linear stability analysis is a fundamental tool
- Turing patterns has many application, we must have a critic view
- In the literature: studies for ‘amplitude equations’
- Non continuos systems need special attention
- Dynamical systems knowledge is also necessary
- Geometry
- Calculus