Some Thoughts on Complexity Measures

The term *complexity* has many different meanings. At least one adjective is needed to help distinguish between different uses of the word:

- Kolmogorov-Chaitin Complexity
- Computational Complexity
- Stochastic Complexity
- Statistical Complexity
- Structural Complexity
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Note: Some portions of the first half of this presentation were prepared jointly with Jim Crutchfield.

Deterministic Complexity

The *Kolmogorov-Chaitin* complexity $K(x)$ of an object x is the length, in bits, of the smallest program (in bits) that when run on a *Universal Turing Machine* outputs x and then halts.

References:

- Kolmogorov, *Problems of Information Transmission*, 1:4-7. (1965)
- Kolmogorov, *IEEE Trans. Inform. Theory*, IT-14:662-664. (1968).
- Solomonoff. *Inform. Contr.*, 7:1-22, 224-254. (1964).
- Chaitin, *J. Assoc. Comp. Mach.*, 13:547-569. (1966).
- Martin-Löf, *Inform. Contr.*, 9:602-619. (1966).
- **Books:**
  - Chpt. 7 of: Cover and Thomas, “Elements of Information Theory,” Wiley, 1991.
  - Chaitin, “Information, Randomness and Incompleteness,” World Scientific, 1987.

Kolmogorov Complexity $\approx$ Randomness

- The Kolmogorov complexity $K(x)$ is maximized for random strings, since it requires a deterministic accounting of all symbols in the string.
- The average growth rate of $K(x)$ is equal to the entropy rate h_μ.
- If x = trajectory of a chaotic dynamical system f:

$$K(x(t)) = h_\mu(f) \quad \text{for typical } x(0) \; .$$

(Brudno, *Trans. Moscow Math. Soc.*, 44:127. (1983).)

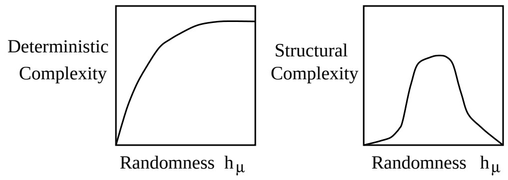
- If a string x is random, then it possesses no regularities. Thus,

$$K(x) = |\mathrm{Print}(x)| \; .$$

- That is, the shortest program to get a UTM to produce x is to just hand the computer a copy of x and say “print this.”

Measures of “Complexity” that Capture a Property Distinct from Randomness

- The entropy rate h_μ and the Kolmogorov Complexity $K(x)$ do not measure pattern or structure or correlation or organization.
- h_μ and $K(x)$ are maximized for random strings.



- Structural complexity or statistical complexity measures are not maximized by random strings.
- The excess entropy and the statistical complexity are examples of structural complexity measures.
- The following slides review a few other statistical complexity measures.

Other Approaches to Structural Complexity

Logical Depth

The **Logical Depth** of x is the **run time** of the shortest program that will cause a UTM to produce x and then halt.

Logical depth is not a measure of randomness; it is small for both trivially ordered and random strings.

References:

- Bennett, *Found. Phys.*, 16:585-592, 1986.
- Bennett, in *Complexity, Entropy and the Physics of Information*, Addison-Wesley, 1990.

Other Approaches to Statistical Complexity, Continued

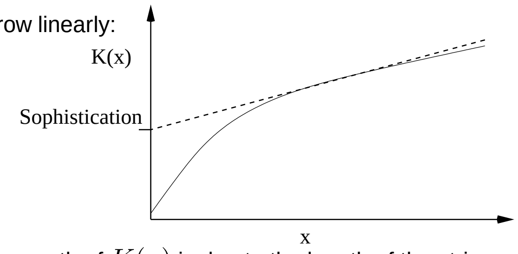
Thermodynamic Depth

Proposed as a measure of structural complexity (Lloyd and Pagels, *Annals of Physics*, 188:186-213). Thermodynamic depth of an object is the total entropy generated in the production of that object. However, thermo. depth depends crucially on the choice of state. Lloyd and Pagels give no general prescription for how states should be chosen. Once states are chosen, thermo. depth is equivalent to the reverse time entropy rate. (Shalizi and Crutchfield, Physical Review E 59 1999.)

Other Approaches to Structural Complexity, continued

Sophistication:

- Koppel, *Complex Systems*, 1:1087-91, 1987.
- The Kolmogorov complexity $K(x)$ of an object will grow linearly in x. The sophistication is essentially the size of the model—the part of $K(x)$ that doesn't grow linearly:



- The linear growth of $K(x)$ is due to the length of the string.
- The constant part $K(x)$ describes the regularities of the string.
- Roughly speaking, the sophistication can be thought of as the Kolmogorov version of the excess entropy.

Other Approaches to Structural Complexity, continued

Non-Linear Modeling

- Wallace and Boulton, 1968.
- Crutchfield and McNamara, *Complex Systems* 1: 417-452, 1987.
- Rissanen, *Stochastic Complexity in Statistical Inquiry*, World Scientific, 1989.
- Crutchfield and Young, in *Complexity, Entropy and the Physics of Information*, Addison-Wesley, 1990.

Model Convergence and Hierarchical Grammatical Complexities

- Badii and Politi, *Complexity: Hierarchical Structures and Scaling in Physics*, Cambridge, 1997.
- Badii and Politi, *Phys. Rev. Lett.*, 78:444-447, 1997.

Note: Badii and Politi's book contains a solid discussion of many different structural complexity measures.

Early Uses of Mutual Information

Probably only of historical interest. (?)

- Rothstein, in *The Maximum Entropy Formalism*, MIT Press, 1979.
- Chaitin, in *Information, Randomness, and Incompleteness*, World Scientific, 1987.
- Gatlin, *Information Theory and the Living System*, Columbia University Press. 1972.
- Watanabe, *Knowing and Guessing: A Quantitative Study of Inference and Information*, Wiley, 1969.

Other Approaches to Structural Complexity, continued

Miscellaneous References:

- Kolmogorov, *Russ. Math. Surveys*, 38:29, 1983.
- Wolfram, *Comm. Math. Phys.*, 96:15-57, 1984.
- Wolfram, *Physica D*, 10:1-35, 1984.
- Hubermann and Hogg, *Physica D*, 22:376-384, 1986.
- Bachas and Hubermann, *Phys. Rev. Lett.*, 57:1965, 1986
- Peliti and Vulpiani, eds., *Measures of Complexity*, Springer-Verlag, 1988.
- Wackerbauer, et. al., *Chaos, Solitons & Fractals*, 4:133-173, 1994.
- Bar-Yam, *Dynamics of Complex Systems*, Addison-Wesley, 1997.

Non-constructive Complexity: The Road Untakable

All Universal Turing Based complexity measures suffer from several drawbacks:

1. They are uncomputable.
2. By adopting a UTM, the most powerful discrete computation model, one loses the ability to distinguish between systems that can be described by computational models less powerful than a UTM.

UTM-based "complexity" measures include:

- **Logical Depth:** Bennett, *Found. Phys.*, 16:585-592, 1986.
- **Sophistication:** Koppel, *Complex Systems*, 1:1087-91, 1987.
- **Effective Complexity:** Gell-Mann and Lloyd, *Complexity*, 2:44-52, 1996.

On the other hand UTM-based arguments are useful for providing a clear framework for expressing notions of complexity.

Complexity = Order $\times$ Disorder?

- There are a number of complexity measures of the form:

$$\text{Complexity} = \text{Order} \times \text{Disorder}$$

- Disorder is usually some form of entropy.
- Sometimes "order" is simply $(1 - h_\mu)$.
- Often, "order" is taken to be some measure of "distance from equilibrium," where equilibrium and equiprobability are sometimes considered to be synonymous.

In my view these sorts of complexity measures have some serious shortcomings:

- Lack a clear interpretation and direct accounting of structure.
- Unclear that distance from equilibrium is equivalent to order.
- Assign a value of zero complexity to all systems with vanishing entropy.

Complexity = Order × Disorder?, continued

But, you can read the papers and decide for yourself. See, e.g.,

- Shiner, et al. *Phys. Rev. E*, 59:1459. 1999.
- Lopez-Ruiz, et al., *Phys. Lett. A*, 209:321. 1995.
- Piasecki, et al., *Physica A*, 307:157. 2002.

For some critiques, see:

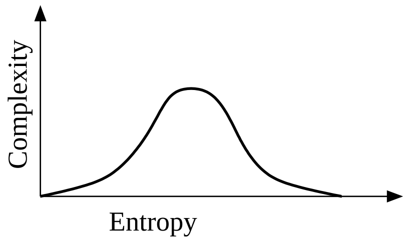
- Feldman and Crutchfield, *Phys. Lett. A*, 238:244. 1998.
- Crutchfield, et al., *Phys. Rev. E*, 62:2996. 2000.
- Binder. *Phys. Rev. E*, 62:2998. 2000.

Complexity vs. Entropy

- What is the relationship between complexity and entropy?
- The rest of these slides are a very condensed version of a talk I gave February 2006. The slides for this talk are posted on the 2006 Beijing CSSS wiki.

One approach: Prescribing Complexity vs. Entropy Behavior

- Zero Entropy ⟶ Predictable ⟶ simple and not complex.
- Maximum Entropy ⟶ Perfectly Unpredictable ⟶ simple and not complex.
- Complex phenomena combine order and disorder.
- Thus, it must be that complexity is related to entropy as shown:

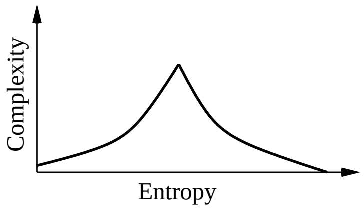


- This plot is often used as the central criteria for defining complexity.

Complexity-Entropy Phase Transition?

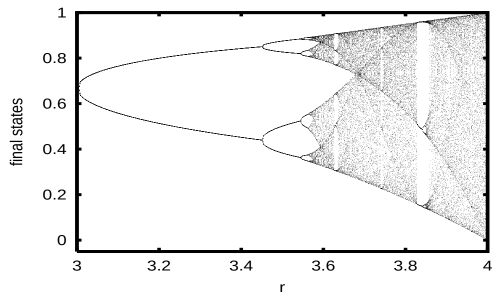
Edge of Chaos?

- Additionally, it has been conjectured that there is a sharp transition in complexity as a function of entropy:



- Perhaps this complexity-entropy curve is *universal*—it is the same for a broad class of apparently different systems.
- Part of the motivation for this is the remarkable success of universality in critical phenomena and condensed matter physics.

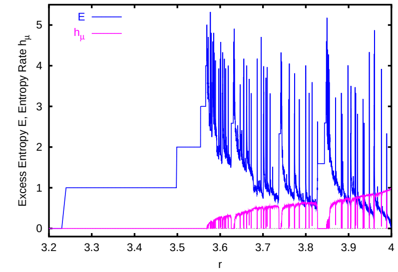
Logistic Equation: Bifurcation Diagram



- For a given r (horizontal axis), the "final states" are shown.
- Chaotic behavior appears as a solid vertical line.
- Examples:
  - $r = 3.2$: Period 2.
  - $r = 3.5$: Period 5.
  - $r = 3.7$: Chaotic.

Complexity vs. Entropy: Logistic Equation

Plot the excess entropy $\mathbf{E}$ and the entropy rate h_μ for the logistic equation as a function of the parameter r.



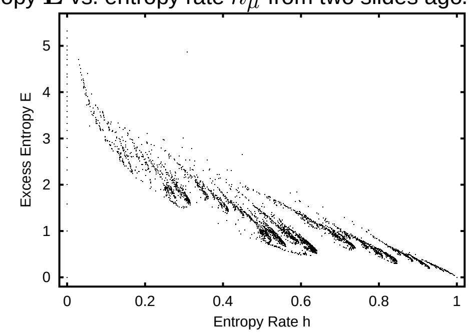
- Note that $\mathbf{E}$ and h_μ depend on a complicated way on r.
- Hard to see how complexity and entropy are related.

Complexity-Entropy Diagrams

- Plot complexity vs. entropy. This will directly reveal how complexity is related to entropy.
- This is similar to the idea behind phase portraits in differential equations: plot two variables against each other instead of as a function of time. This shows how the two variables are related.
- It provides a parameter-free way to look at the intrinsic information processing of a system.
- Complexity-entropy plots allow comparisons across a broad class of systems.

Complexity-Entropy Diagram for Logistic Equation

- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ from two slides ago.



- Structure is apparent in this plot that isn't visible in the previous one.
- Not all complexity-entropy values can occur; there is a forbidden region.
- Maximum complexity occurs at zero entropy.
- Note the self-similar structure. This isn't surprising, since the bifurcation diagram is self-similar.

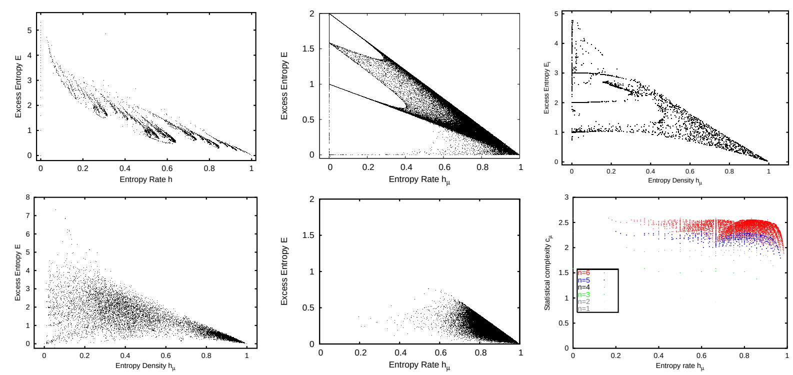
A Gallery of Complexity-Entropy Diagrams

The next slide shows, left to right, top to bottom, complexity-entropy diagrams for:

1. Logistic Equation
2. One-Dimensional Ising model with nearest- and next-nearest-neighbor interactions
3. Two-Dimensional Ising model with nearest- and next-nearest-neighbor interactions
4. One-Dimensional radius-2 cellular automata
5. Random Markov chains
6. All 6-state topological processes

In all cases parameters are sampled uniformly over a fairly wide range. See Feb. 2006 slides for details.

A Mosaic of Complexity-Entropy Diagrams



Complexity-Entropy Diagrams: Conclusions

- There is not a universal complexity-entropy curve.
- Complexity is not necessarily maximized at intermediate entropy values.
- It is not always the case that there is a sharp complexity-entropy transition.
- Complexity-entropy diagrams provide a way of comparing the information processing abilities of different systems in a parameter-free way.
- Complexity-entropy diagrams allow one to compare the information processing abilities of very different model classes on similar terms.
- There is a considerable diversity of complexity-entropy behaviors.

Edge of Chaos?

Is there an edge of chaos to which systems naturally evolve? My very strong hunch is no, not in general. See the following pair of papers.

- Packard, “Adaptation to the Edge of Chaos” in *Dynamic Patterns in Complex Systems*, Kelso et.al, eds., World Scientific, 1988
- Mitchell, Hraber, and Crutchfield “Revisiting the ‘Edge of Chaos’ ” *Complex Systems*, 7:89-130, 1993. (Response to Packard, 1988).

Transitions in CA Rule Space?

- Is there a sharp complexity transition in CA rule space? No, unless you parametrize the space of CAs in a very particular way. The “transition,” then, is a result of the parametrization and not the space itself.

Transitions in CA Rule Space References

- Langton. “Computation at the Edge of Chaos,” *Physica* D (1990).
- Li, Packard and Langton, “Transition Phenomena in Cellular Automata Rule Space” *Physica D* 45 (1990) 77.
- Wooters and Langton, “Is there a Sharp Phase Transition for Deterministic Cellular Automata?”, *Physica D* 45 (1990) 95.
- Crutchfield, “Unreconstructible at Any Radius”, *Phys. Lett.* A 171: 52-60, 1992.
- Feldman, et al, “Organization of Intrinsic Computation.” In preparation.