Foundations and Frontiers of Complex Systems
SFI Complex Systems Summer School, Bariloche 2008

Introduction to Complex Networks

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Introduction to Complex Networks

Introduction to Complex Networks: Overview

- **Concepts**
- **Elements of Graph Theory**
- **Random Networks**
- **Scale Free and Small World Networks**
- **Characterization of Networks**

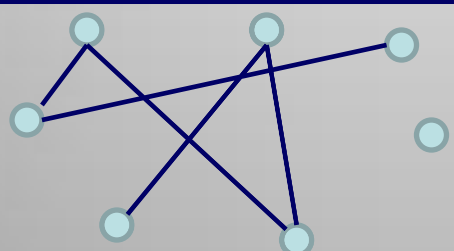
Introduction to Complex Networks: Graphs

Mathematical Concept: Graph

The elements are :

nodes or vertices

links or edges

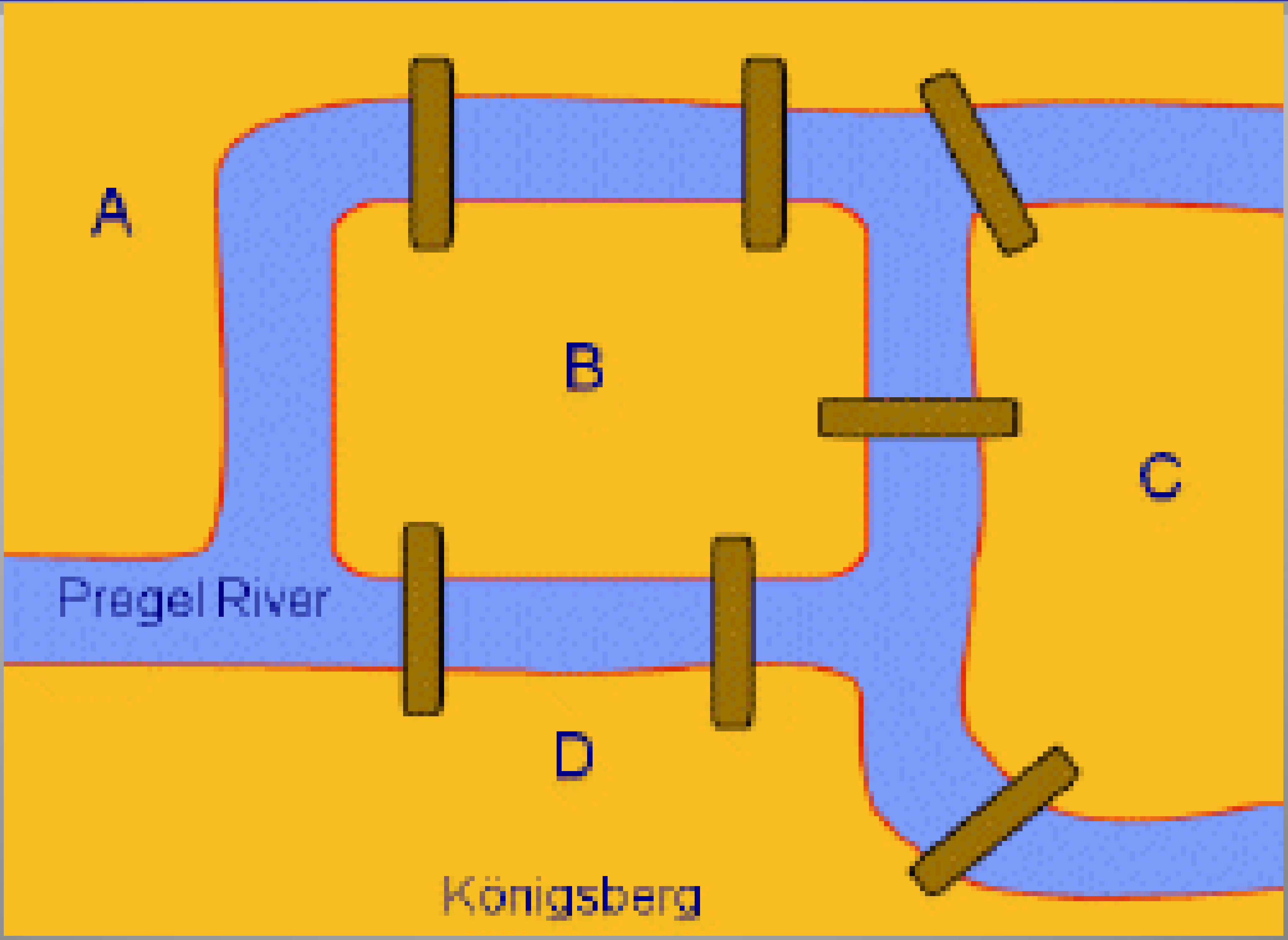


First work on Graph Theory

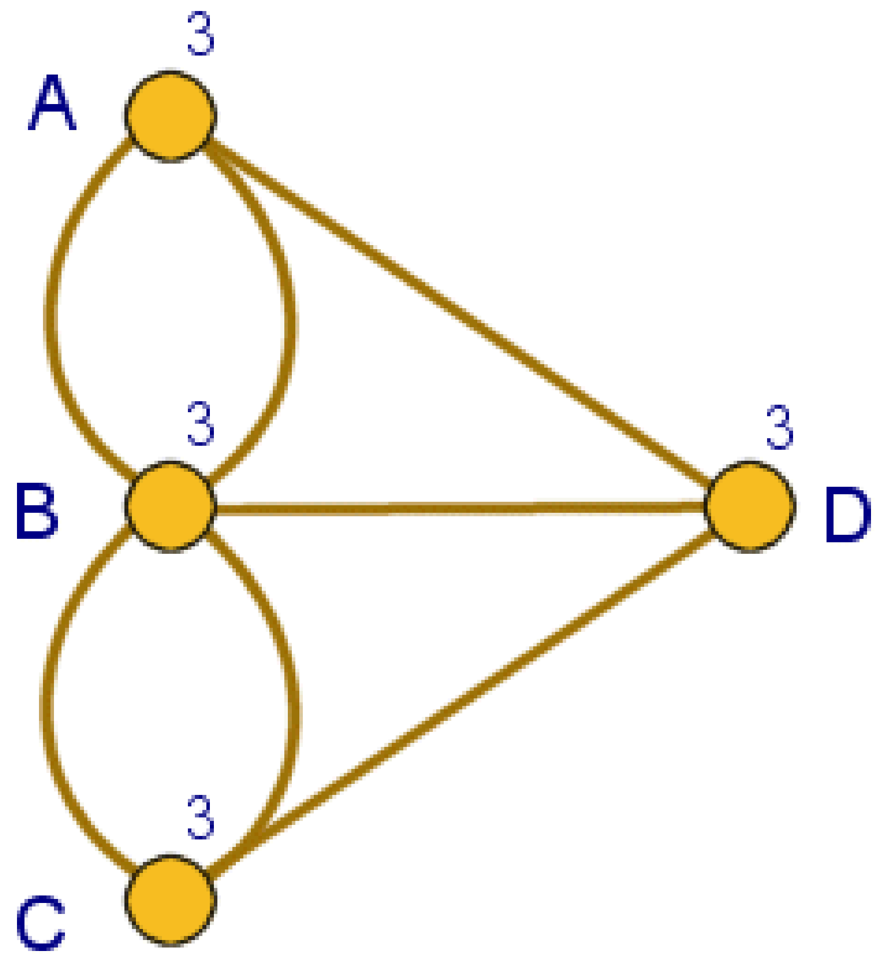
Leonhard Euler, XVIII century

The problem of Königsberg bridges

Introduction to Complex Networks: Graphs



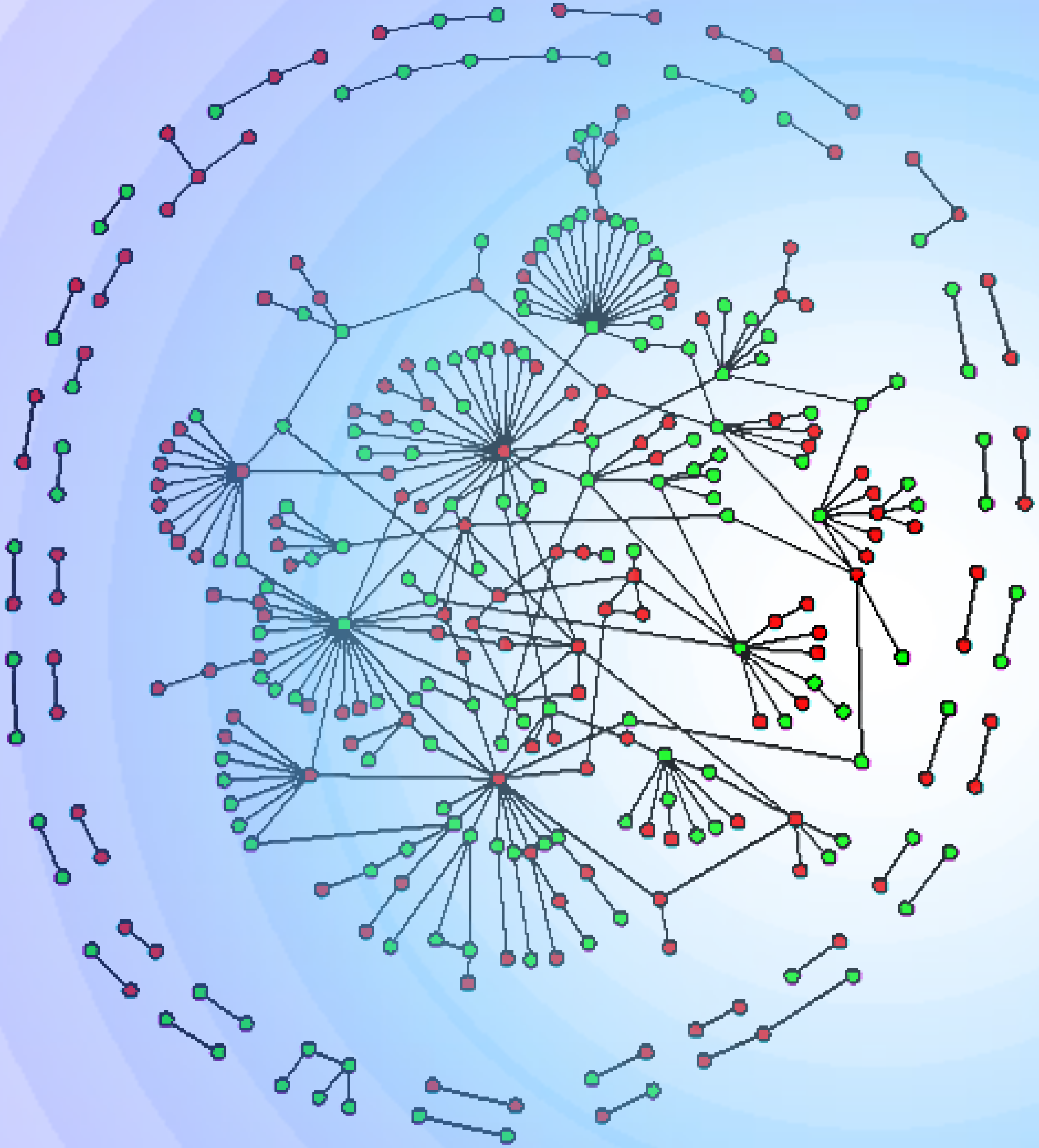
Introduction to Complex Networks: Graphs



Introduction to Complex Networks: Definition

Network:

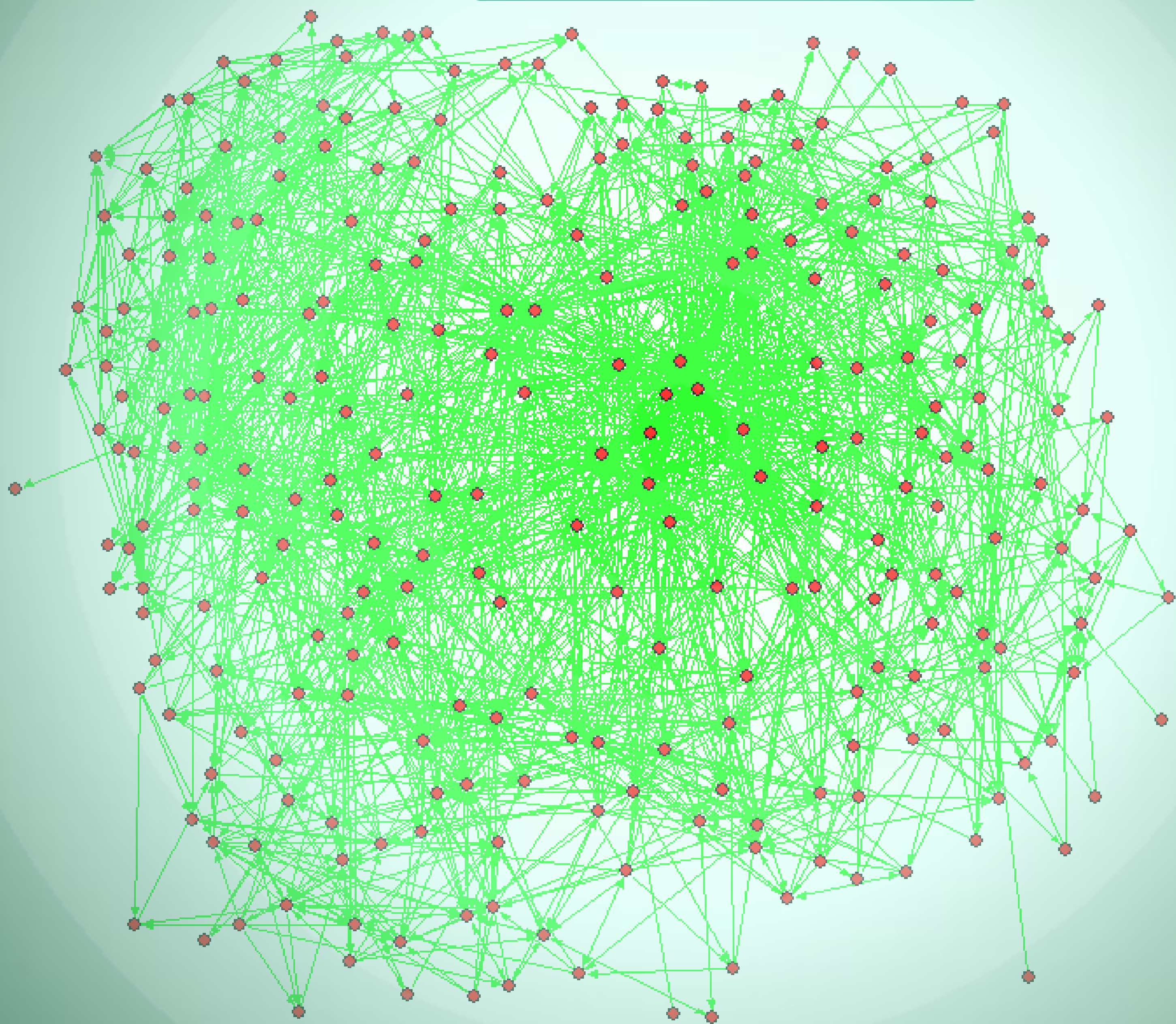
A population of *unities* connected and interacting through links.

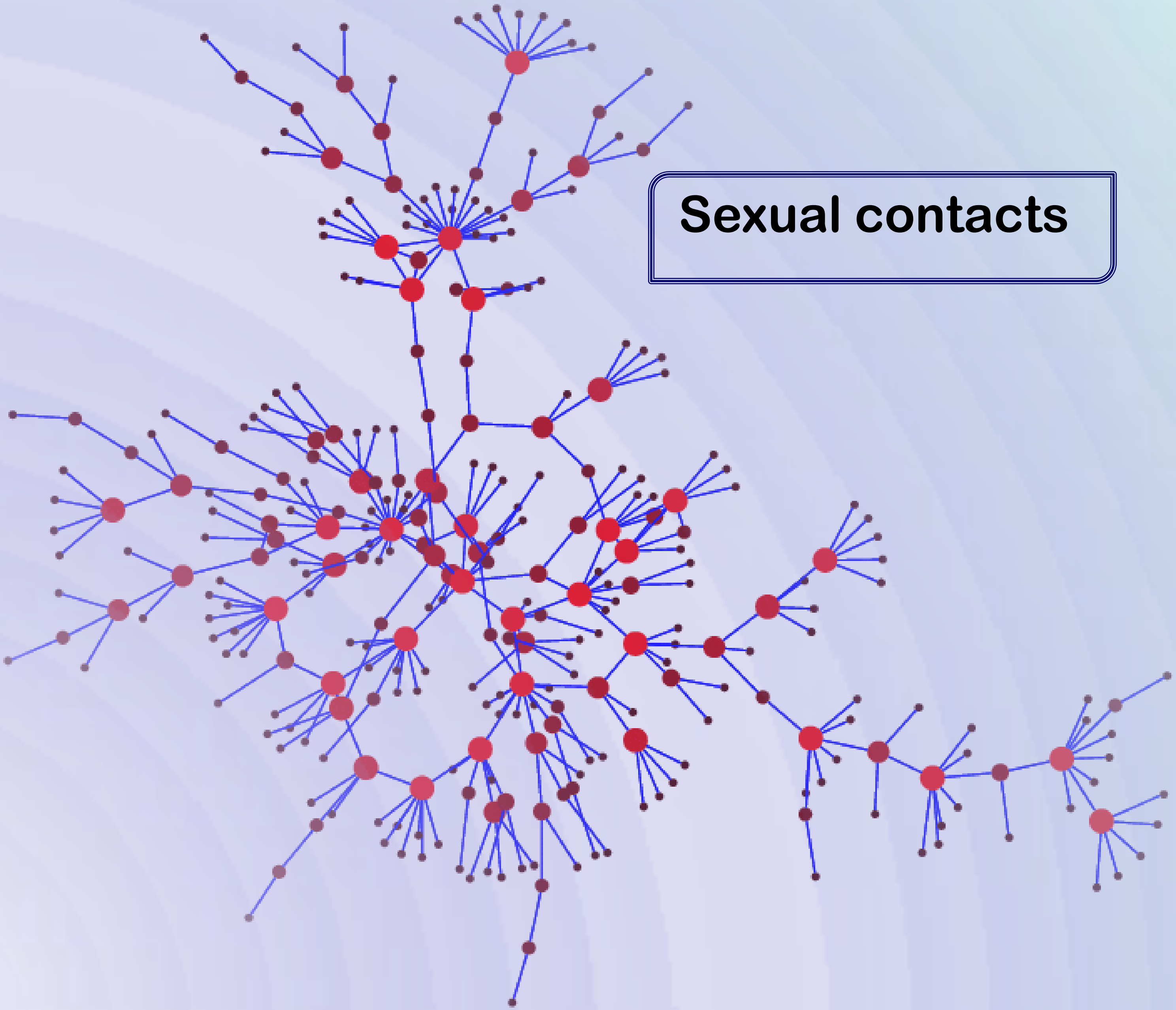


Protein binding networks

Baker's yeast
S. cerevisiae

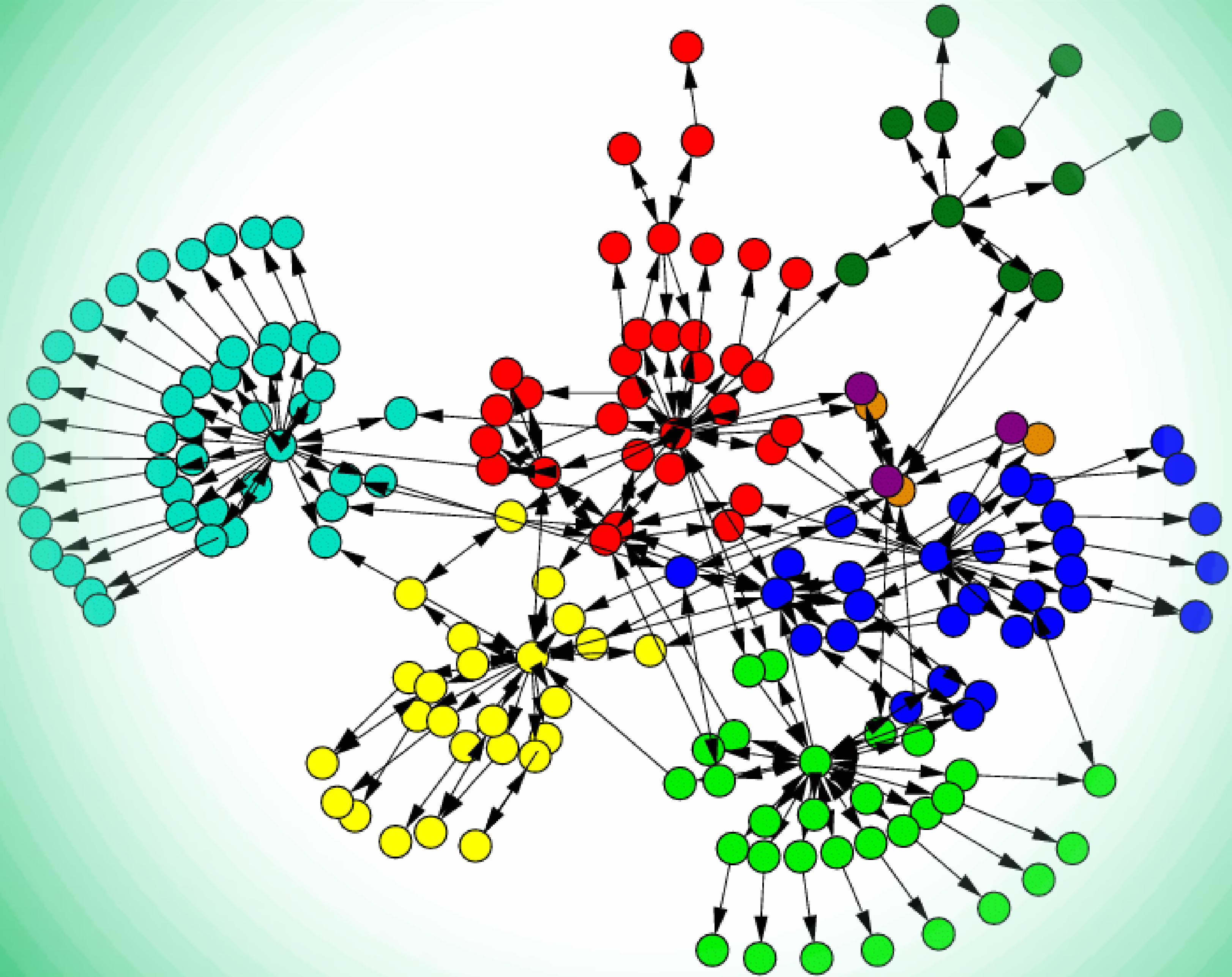
C. elegans neurons

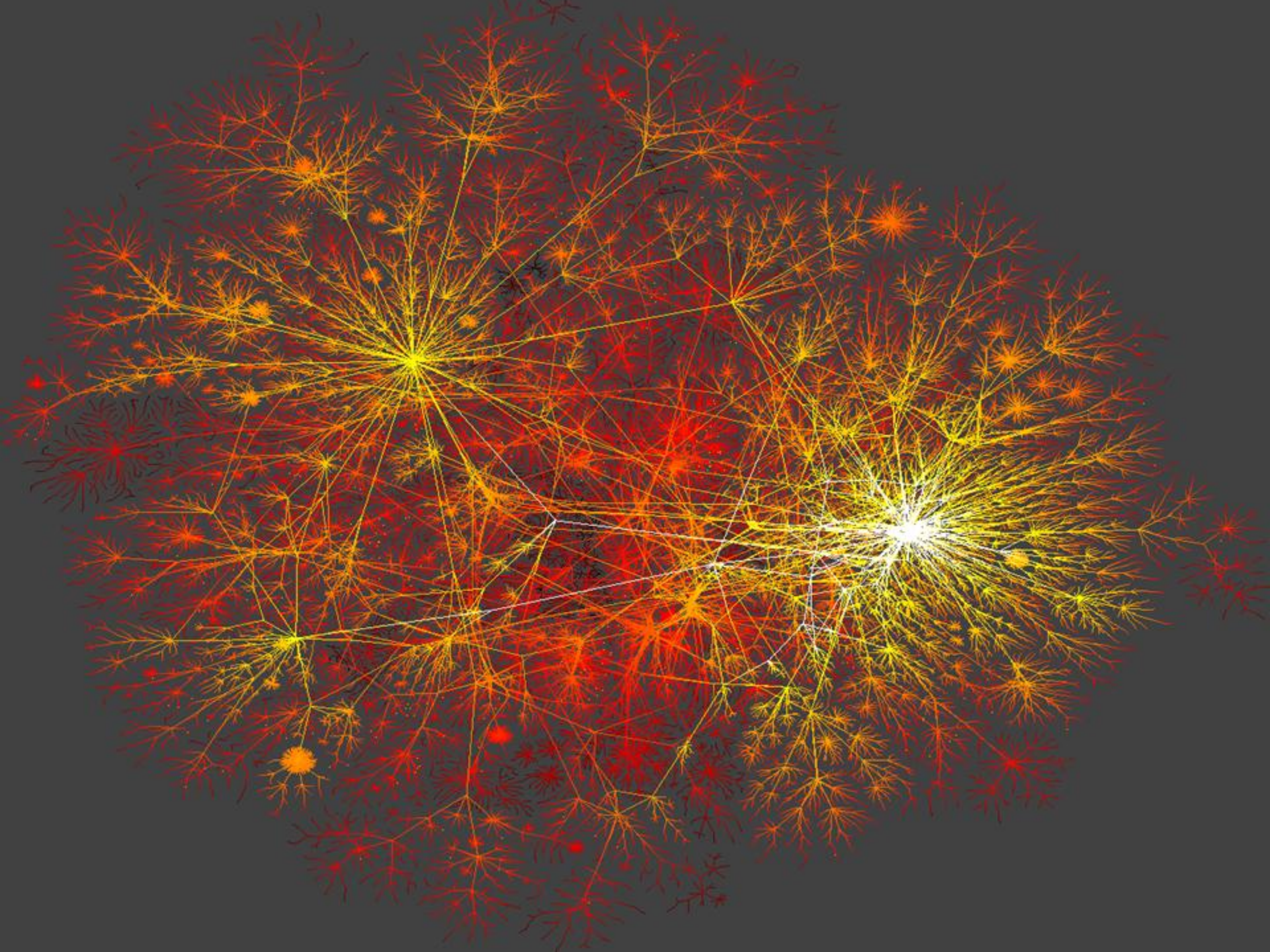




Sexual contacts

Webpages connected by hyperlinks





KINGDOM
Glasgow
Edinburgh
Newcastle
Teesside
Leeds/Bradford
Humberside
Manchester
Birmingham
Norwich
Cardiff
Bristol
Southampton
REPUBLIC OF IRELAND
Dublin
Cork
DENMARK
Copenhagen
Billund
LATVIA
Riga
LITHUANIA
Vilnius
BELARUS
Hamburg
Bremen
Amsterdam
Hannover
Berlin
POLAND
Warsaw
Rotterdam
Eindhoven
Dortmund
Düsseldorf
Antwerp
Cologne
Brussels
Maastricht
Prague
Krakow
Ostrava
Kiev
Le Havre
Rouen
Caen
Luxembourg
Frankfurt
Nuremberg
Brno
Kosice
UKRAINE
Lannion
Brest
Rennes
Paris
Paris (Orly)
Stuttgart
Strasbourg
Munich
Vienna
Bratislava
Budapest
Nantes
Basel/Mulhouse
Zurich
Salzburg
Bern
Geneva
Annecy
Milan
Ljubljana
Zagreb
Odesa
ROMANIA
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Bordeaux
Turin
Belgrade
Bucharest
Genoa
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Ancona
Split
Sarajevo
Biarritz
Toulouse
Montpellier
Avignon
Nice
Santiago de Compostela
Asturias
Bilbao
Pau
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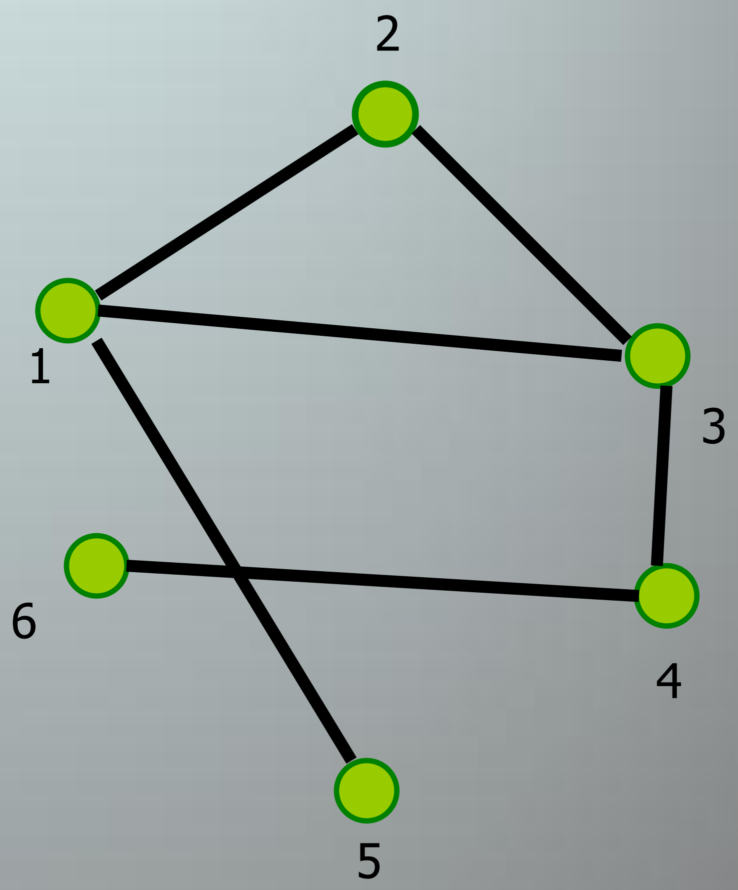
Skopje
Porto
SPAIN
Figari
Tirana
Bari
Brindisi
Naples
Thessaloniki
Istanbul
Madrid
Menorca (Mahon)
PORTUGAL
Valencia
Palma de Mallorca
Ibiza
Cagliari
Crotone
Lamezia Terme
Lisbon
Alicante
GREECE
TURKEY
Palermo
Reggio Calabria
Catania
Athens
Seville
Granada
Malaga
Algiers
Tunis
Pantelleria
Malta
Lampedusa
TUNISIA
Kriti
Paphos
Larnaca
CYPRUS
Rabat
Casablanca
MOROCCO
Beirut
LEBANON
Damascus
Tripoli
Marrakech
Tel Aviv
ISRAEL
Amman
Cairo
Tenerife
Lanzarote
Fuerteventura
Las Palmas
ALGERIA
WESTERN

- Graph theory concepts
- Matrices
- Probability theory

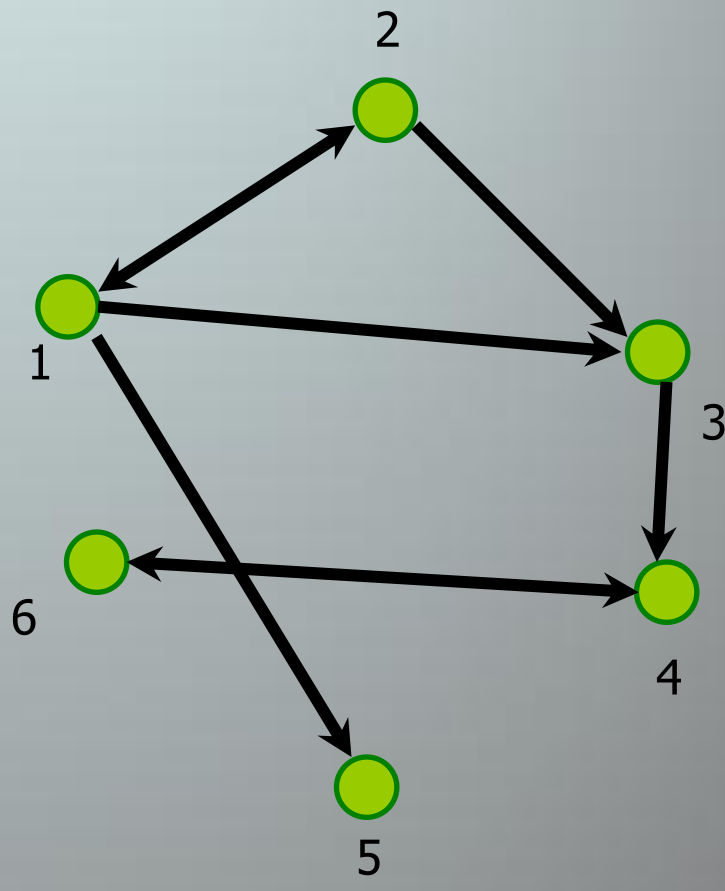
- **Graph G=(V,E)**
  - **V = set of vertices**
  - **E = set of edges**

Undirected graph
V={1,2,3,4,5,6}
E={(1,2),(1,3),(1,5),(2,3),(3,4),(4,6)}



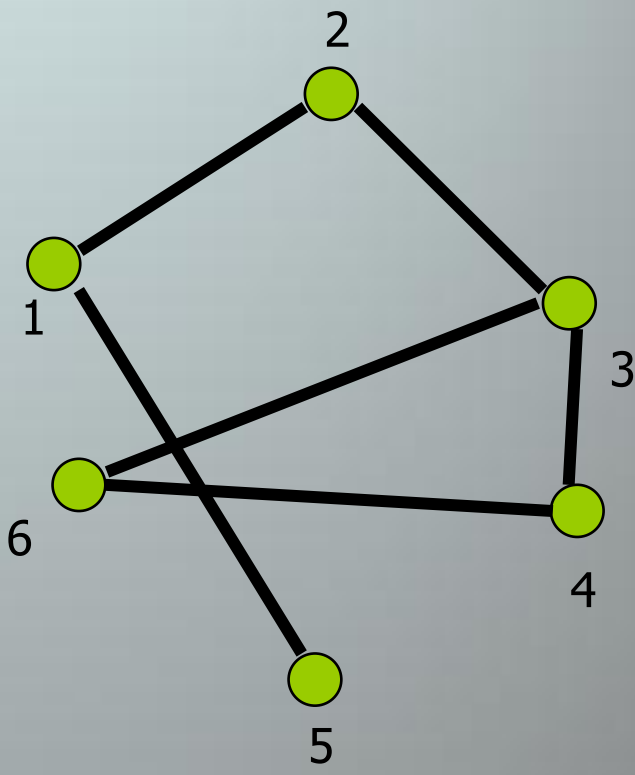
- Graph G=(V,E)
  - V = set of vertices
  - E = set of edges

Directed graph
V={1,2,3,4,5,6}
**E={1,2›, ‹2,1› ,‹1,3›, ‹1,5›,
‹2,3›, ‹3,4›, ‹4,6›, ‹6,4›}**



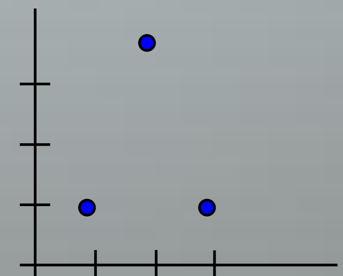
Degree d(i) of node I

Number of edges incident on node i



Degree distribution

f_k = fraction of nodes with degree k



In-degree $d_{in}(i)$ of node

Number of edges pointing to node i

In-degree sequence

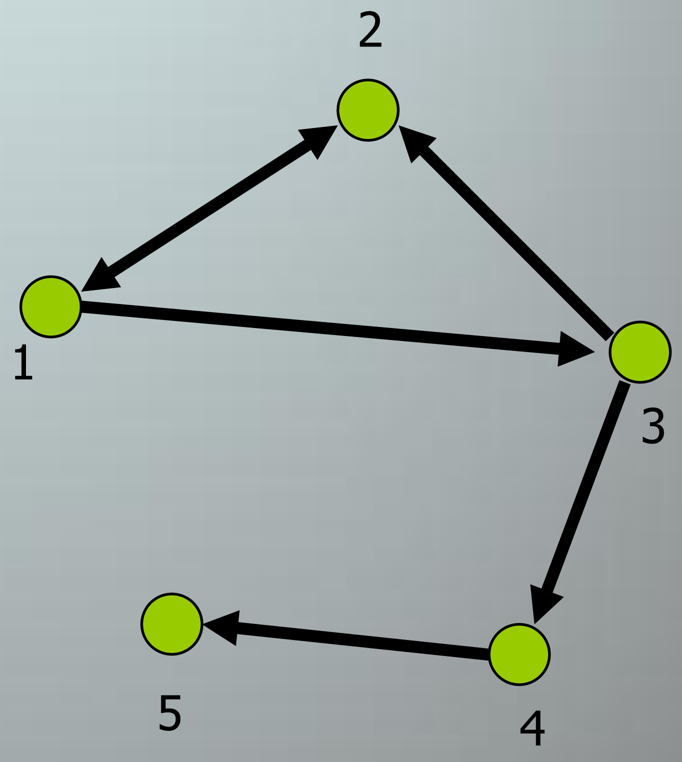
[1,2,1,1,1]

Out-degree $d_{out}(i)$ of node i

Number of edges leaving node i

Out-degree sequence

[2,1,2,1,0]

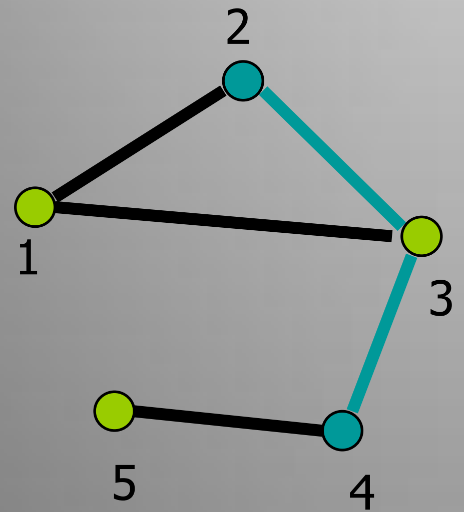


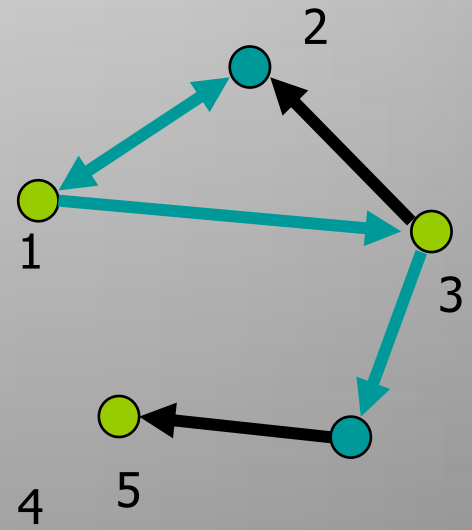
I. Graph Theory – Paths length

Path (from node i to node j): any sequence of links from node i to node j

Path length: number of links on the path

Connected nodes: If there is a path nodes i and j are connected



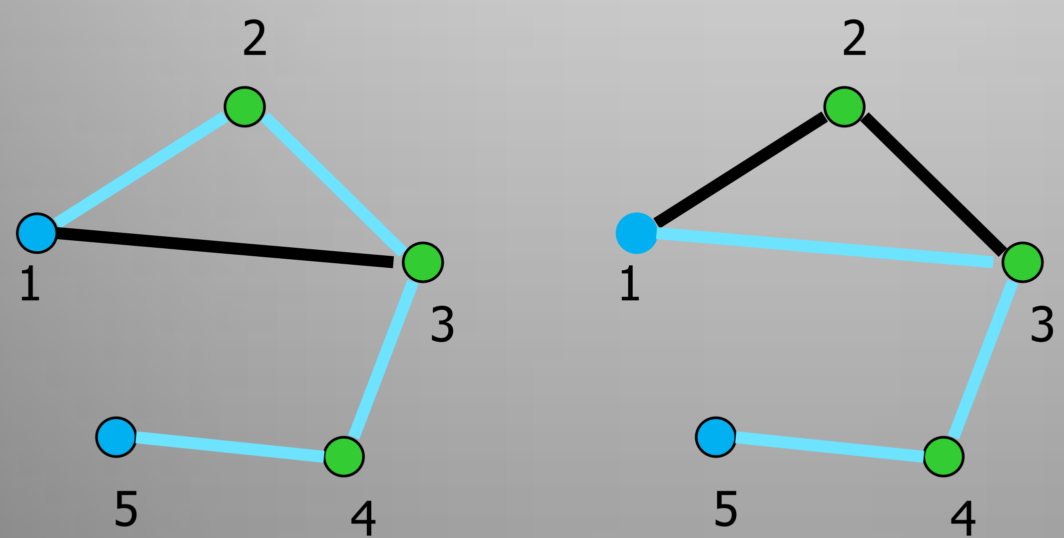


I. Graph Theory – Paths length

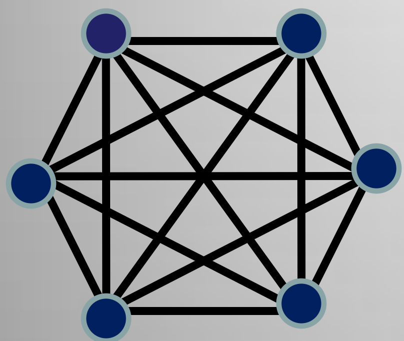
Shortest Path (from node i to node j):

the path with the lowest path length

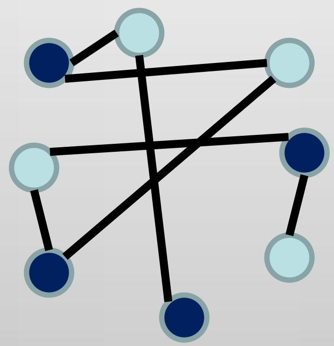
Diameter: The longest among the shortest path in the graph



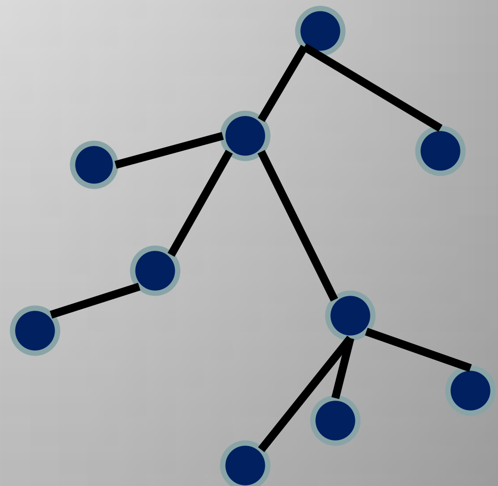
Introduction to Complex Networks: Mathematical Tools
I. Graph Theory – Particular cases



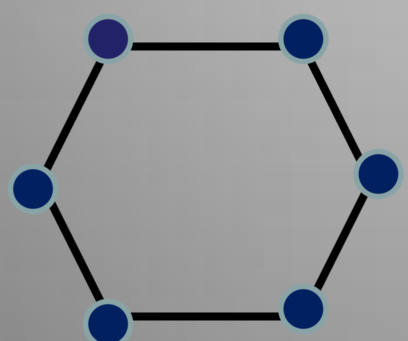
Fully Connected Clique



Bipartite



Tree



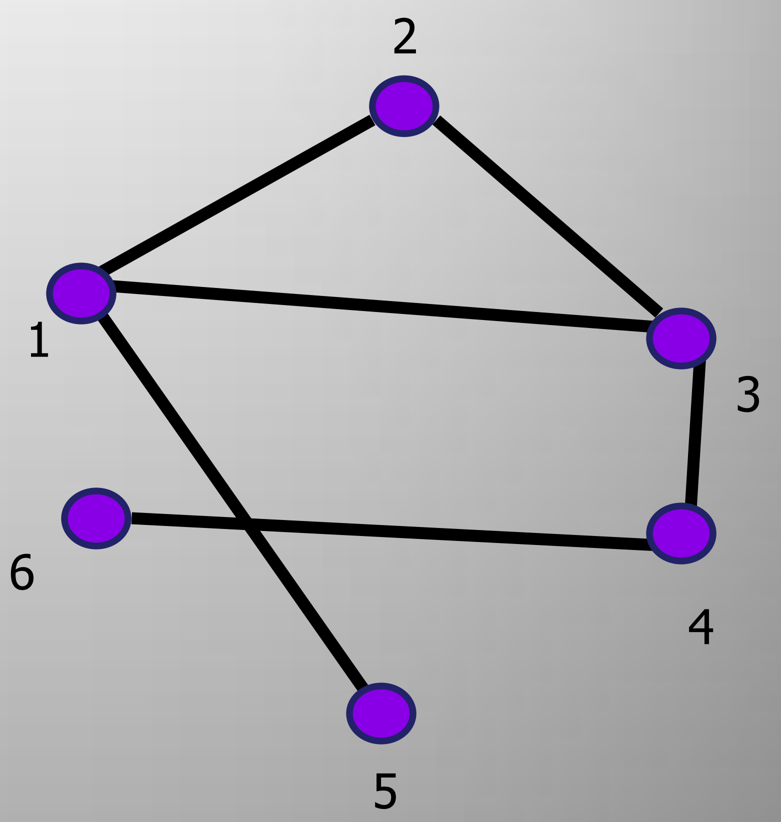
Cycle

Adjacency Matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Laplacian Matrix

$$L = \begin{pmatrix} -3 & 1 & 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$



Introduction to Complex Networks: Random networks

Also called Poisson networks - P. Erdös and A. Rényi in 60s

Take N vertices

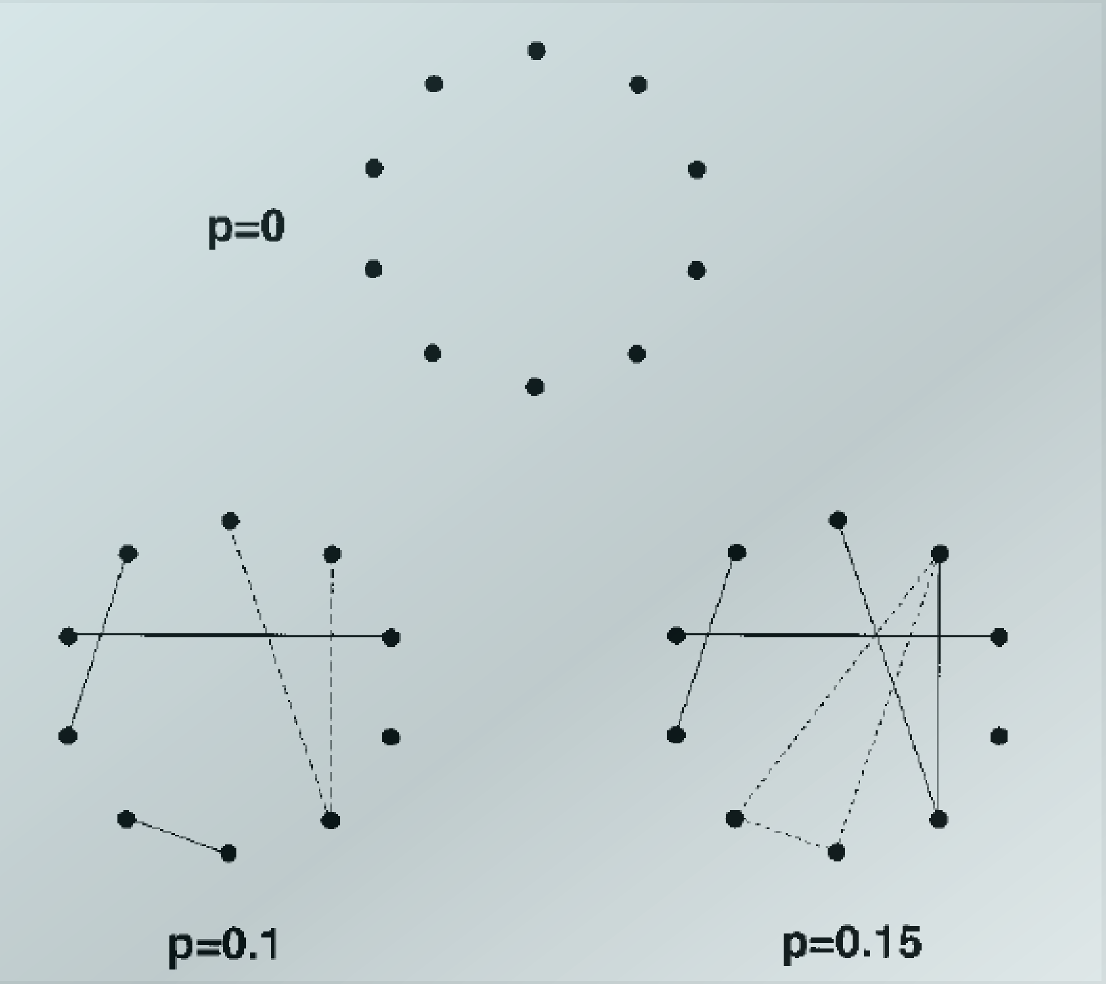
Connect each pair of vertices with an edge with some probability p

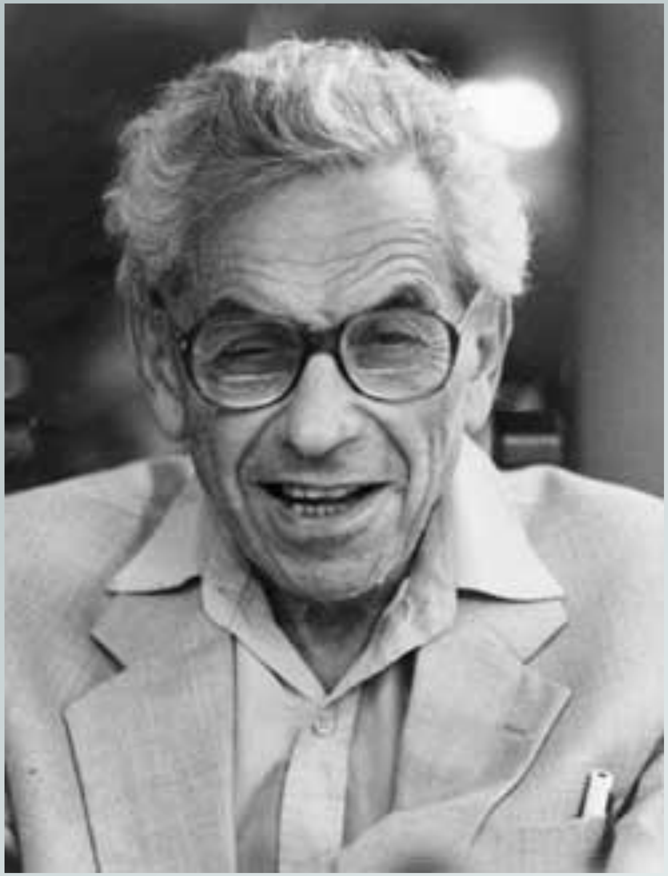
There are $N(N-1)/2$ possible edges

The obtained graph has $p\,N(N-1)/2$ edges

The probability that a vertex has degree k follows *a binomial (Poisson)* distribution

Introduction to Complex Networks: Random networks
Erdös - Rényi Model

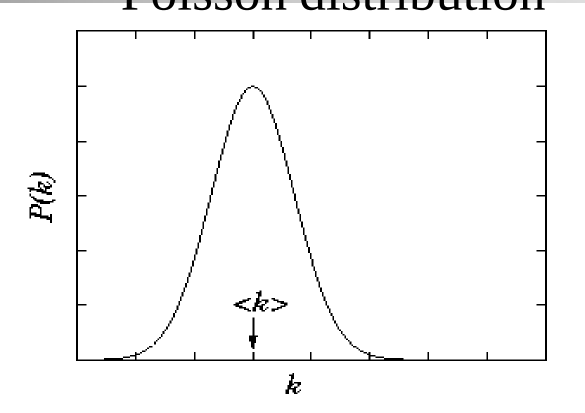




N nodes, every pair of nodes being connected with probability *p*

Introduction to Complex Networks: Random networks Degree distribution

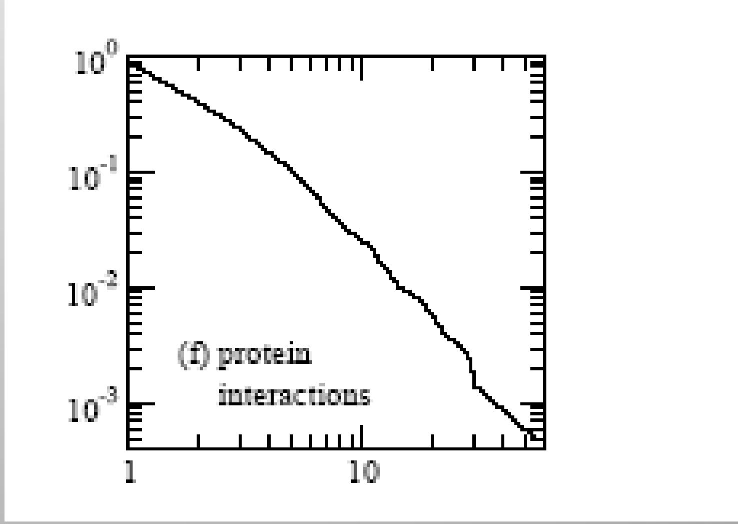
Poisson distribution



$$N(K) = N \frac{\lambda^K}{K!} \exp(-\lambda)$$

$$\lambda = \langle K \rangle = 2E / N$$

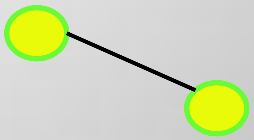
BUT Most real networks do NOT follow Poisson law



Real life networks are not "strictly random"

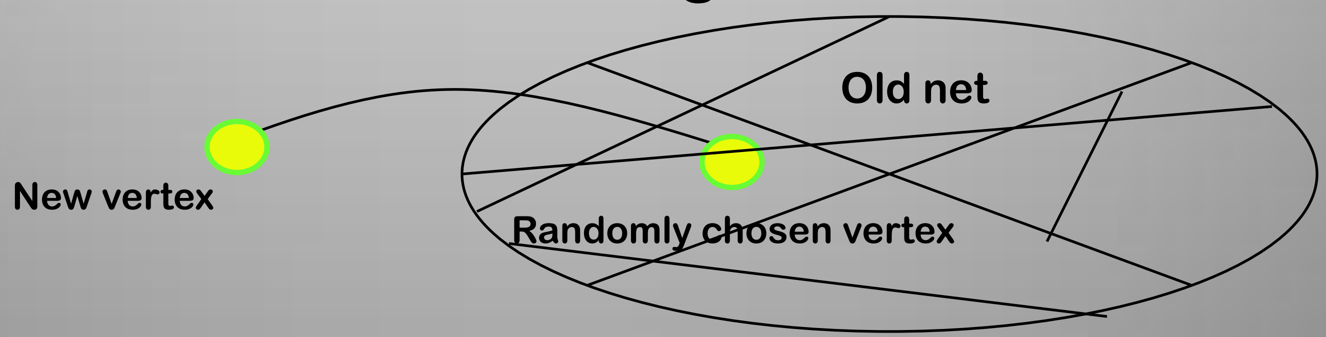
Is it possible to define a model that generates graphs with statistical properties similar to those in real life?

Initial network



At each time step t

- A new vertex is attached to a randomly chosen vertex
- There are t vertices and t edges



Introduction to Complex Networks: Exponential networks

P(k,t): probability that a randomly chosen vertex has degree k at time t

$$P(k,t) = \frac{1}{t}\sum_{s=1}^{t} p(k,s,t)$$

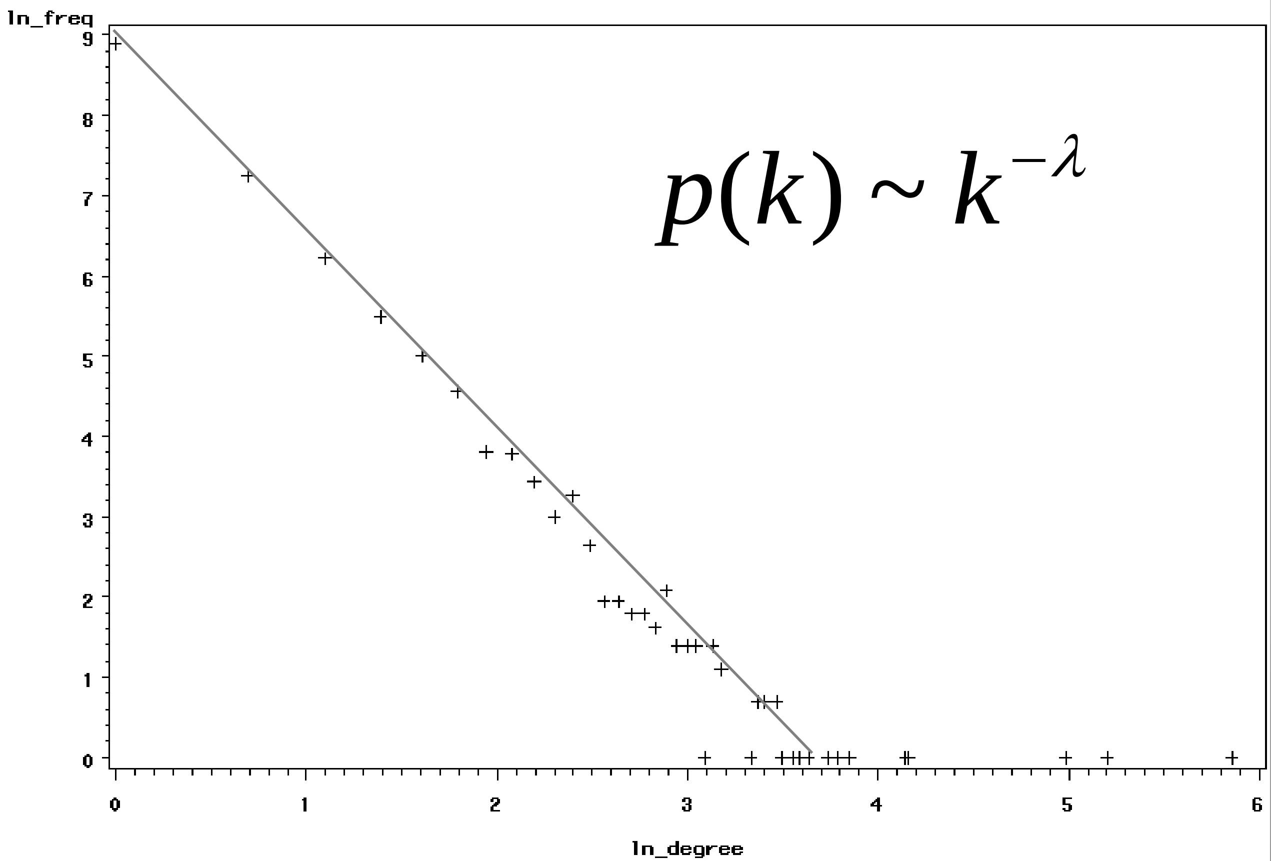
Initial condition: $P(k,t=2) = \delta_{k,2}$

- $P(k) = P(k,t \to \infty)$**: stationary degree distribution**
- **Stationary equation:**

$$2P(k) - P(k-1) = \delta_{k,1} \longrightarrow P(k) = 2^{-k}$$

Introduction to Complex Networks: BA-model

Many large networks are characterized by a highly skewed distribution of the number of neighbors (degree)

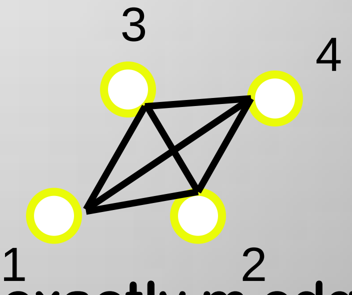


Introduction to Complex Networks: BA-model

NETWORK	exponent γ
film actors	2.3
telephone call graph	2.1
email networks	1.5/2.0
sexual contacts	3.2
WWW	2.3/2.7
internet	2.5
peer-to-peer	2.1
metabolic network	2.2
protein interactions	2.4

Introduction to Complex Networks: BA-model

1. Start with an initial set of m_0 fully connected nodes: $m_0 = 4$



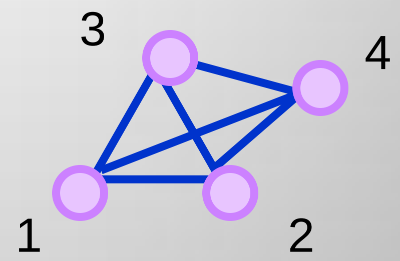
2. Next, add new vertices, contributing with exactly m edges, one by one.

3. Each new edge connects to an existing vertex with a probability proportional to the number of edges that vertex already has

Preferential attachment: The rich gets richer

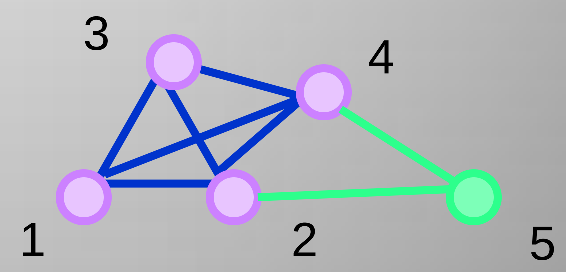
Introduction to Complex Networks: BA-model

1. At the beginning each vertex has an equal number of edges (links). The probability of choosing any vertex (node) is 1/4



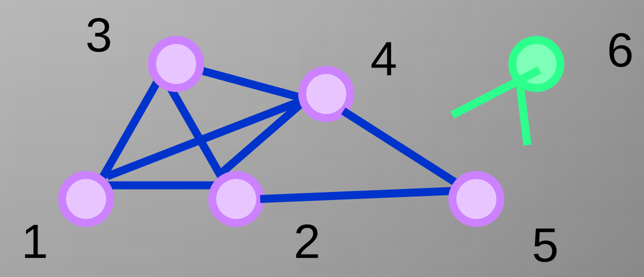
2. We add a new vertex with m=2

Choose 2 elements at random, e.g. 2 and 4



3. Now the probabilities of selecting nodes 1,2,3,4 or 5 are 3/16, 1/4, 3/16, 1/4, 1/8 respectively

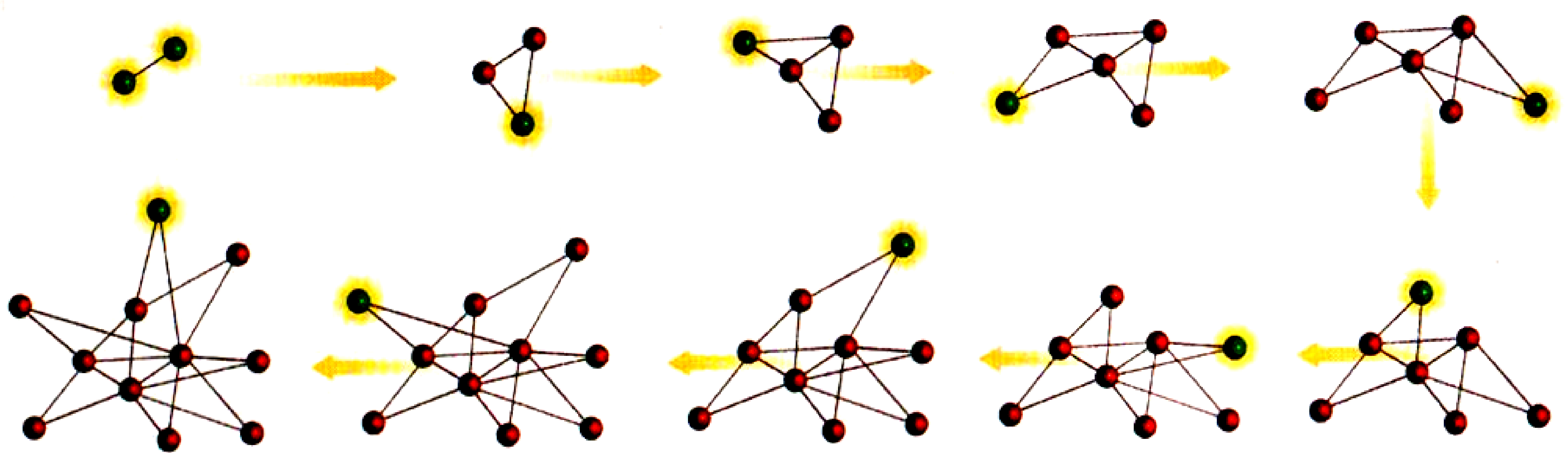
4. Add a new vertex with m=2, and connect each edge to a different node chosen according to its probability of being selected.

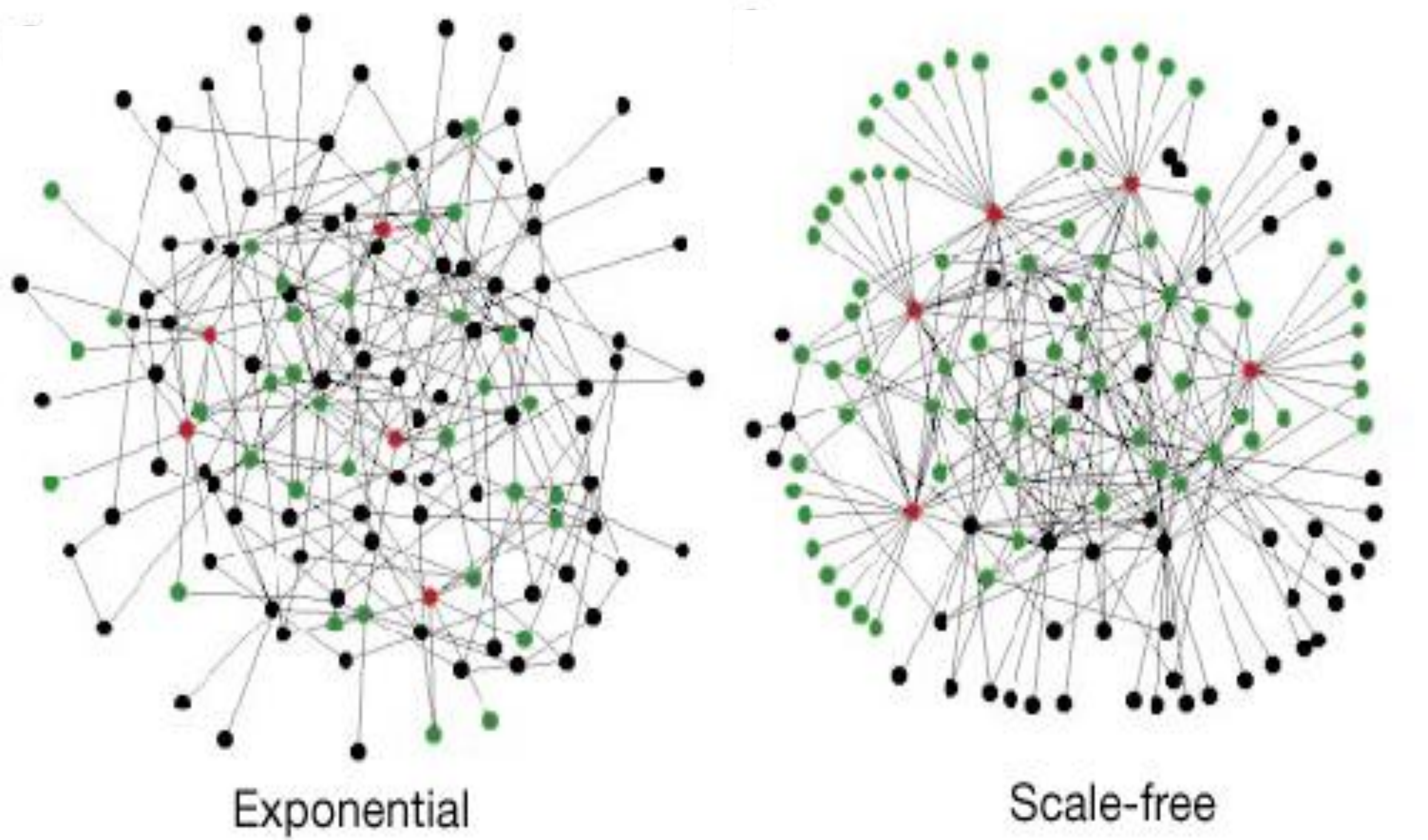


Introduction to Complex Networks: BA-model

Two main features

1. Growth : networks expand continuously by the addition of new vertices
2. Preferential-attachment (rich gets richer) : new vertices attach preferentially to sites that are already well connected.





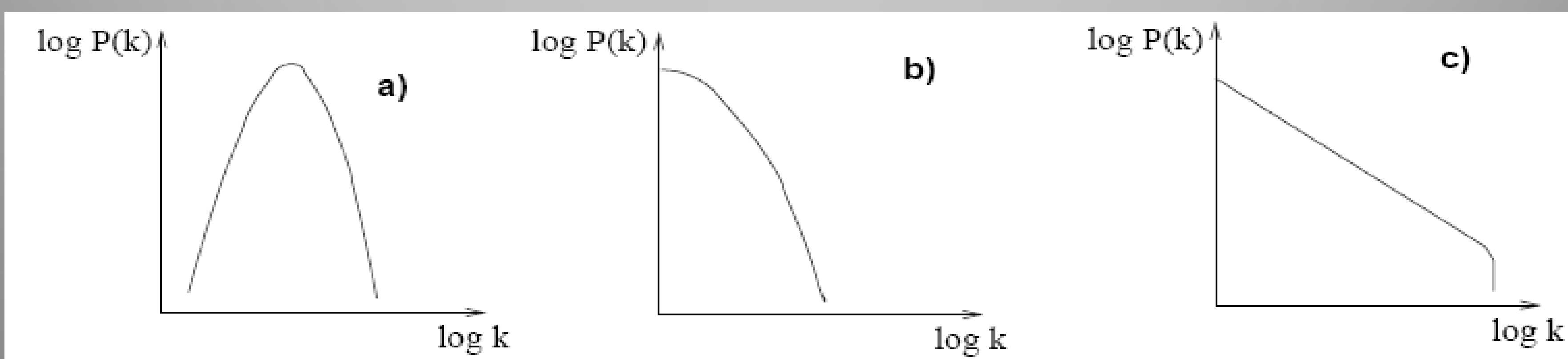
130 nodes and 215 links

Introduction to Complex Networks: Degree distribution comparison

a) Poisson: $P(k) = e^{-\bar{k}} \bar{k}^{k} / k!$, e.g. a classical random equilibrium graph of Erdos and Renyi when the total number of vertices is infinite

b) Exponential: $P(k) \approx e^{-k}$, e.g. a citation graph with attachment of new vertices to randomly chosen old ones

c) Power-law: $P(k) \approx k^{-\gamma}$, e.g. a citation graph with attachment of new vertices to preferentially chosen old ones



But…

the power-law degree distribution is not the only interesting property

Short paths: real-life networks are “Small Worlds”

Clustering coefficient: real-life networks tend to have high clustering coefficient

Are my friends also friends among them?

Average length

Distance between two vertices = length of the shortest path between them

Distances l are distributed with some distribution function P(l)

P(l): the probability that the length of the shortest path between two randomly chosen vertices is l

Average length of the shortest path

$$\bar{l} = \sum_{l} l P(l)$$

Introduction to Complex Networks: Small World Networks - Clustering

The clustering coefficient characterizes the "connectedness" of the environment close to a vertex.

$$C_i = \frac{n_i}{\frac{k_i(k_i - 1)}{2}}$$

Where

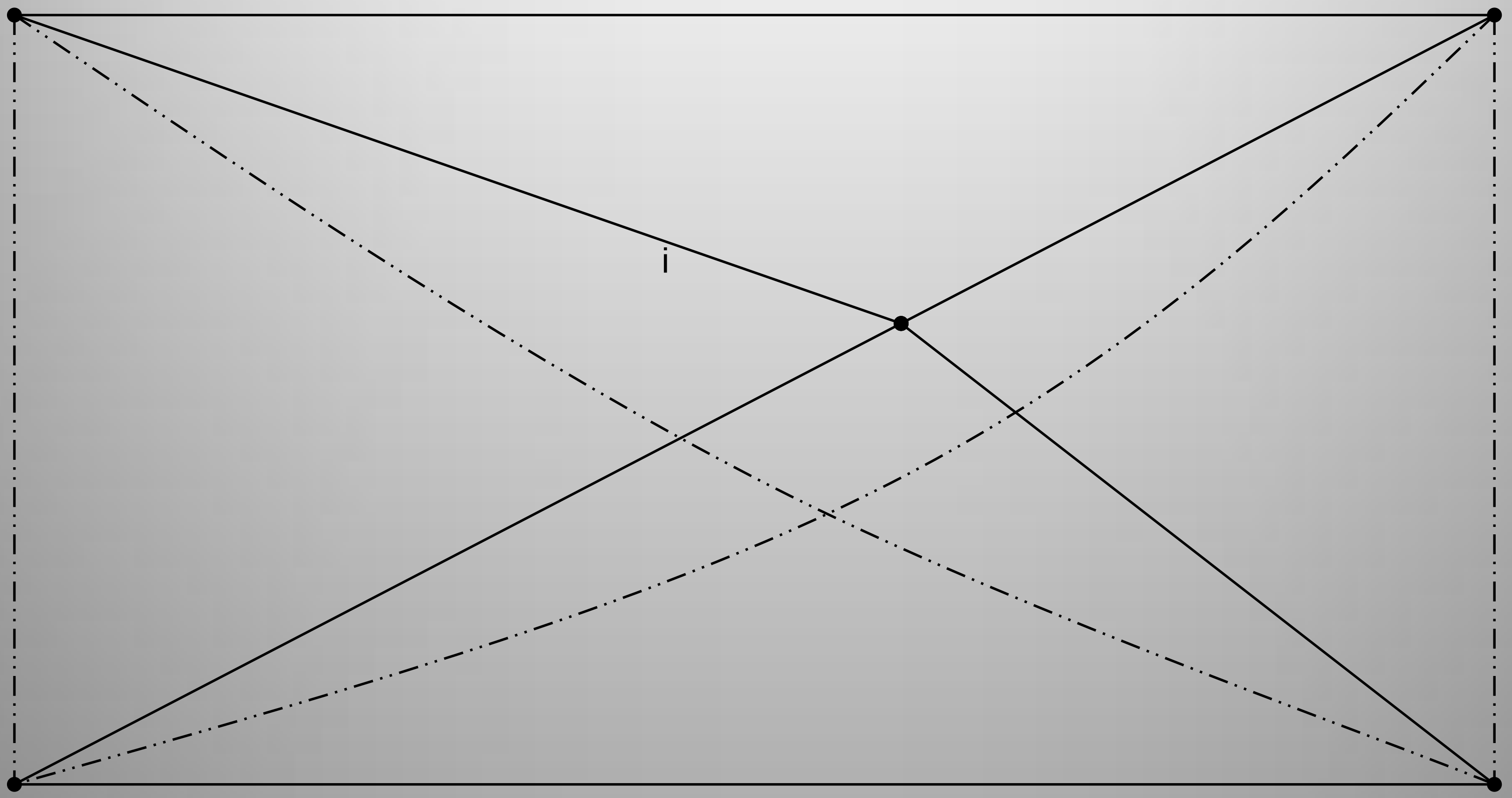
n_i: number of connections among the neighbors

$k_i(k_i$-1)/2: number of possible connections among the neighbors.

Or..Measures the density of triangles (local clusters) in the graph

$$C = \frac{1}{n}\sum_i \frac{\text{present triangles centered at node i}}{\text{possible triangles centered at node i}}$$

Introduction to Complex Networks: Small World Networks - Clustering



Present triangles =2

Possible triangles $\binom{4}{2}$**=6**

Clustering=1/3

Millgram's small world experiment

Letters were handed out to people in Nebraska to be sent to a target in Boston

People were instructed to pass on the letters to someone they knew on first-name basis

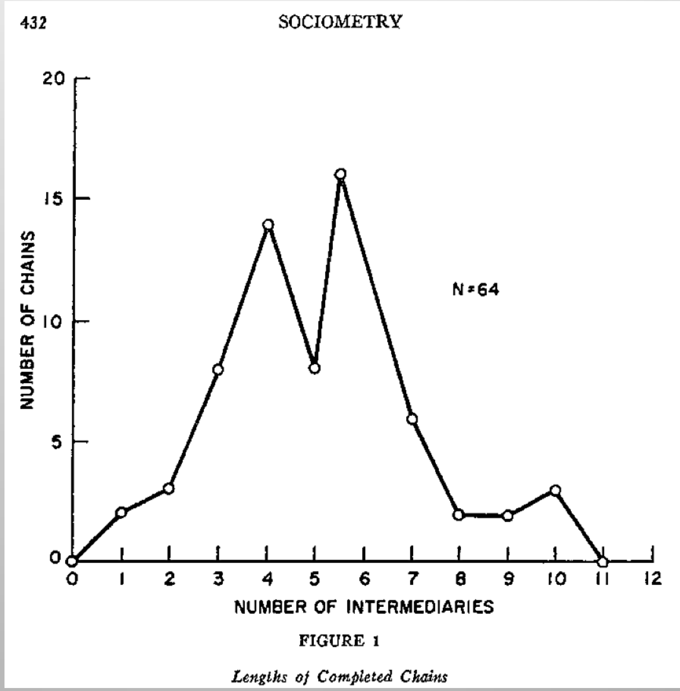
The letters that reached the destination followed paths of mean length 6

Six degrees of separation

See
The Kevin Bacon game
The Erdös number
Babel - Crash

Introduction to Complex Networks: Small World Networks

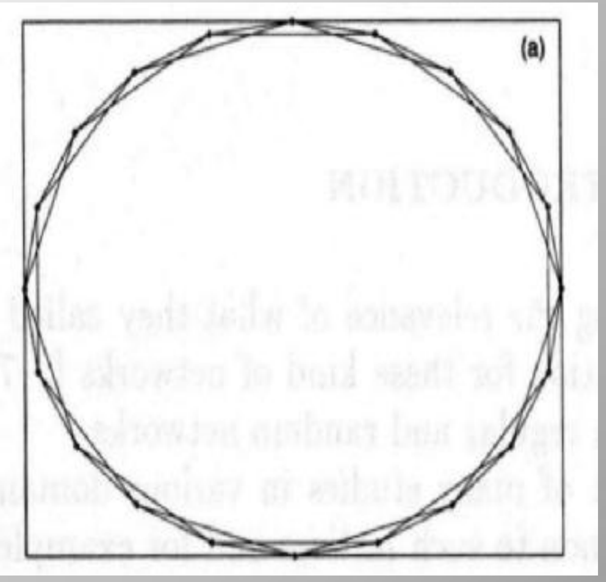
Stanley Milgram



Introduction to Complex Networks: Small World Networks

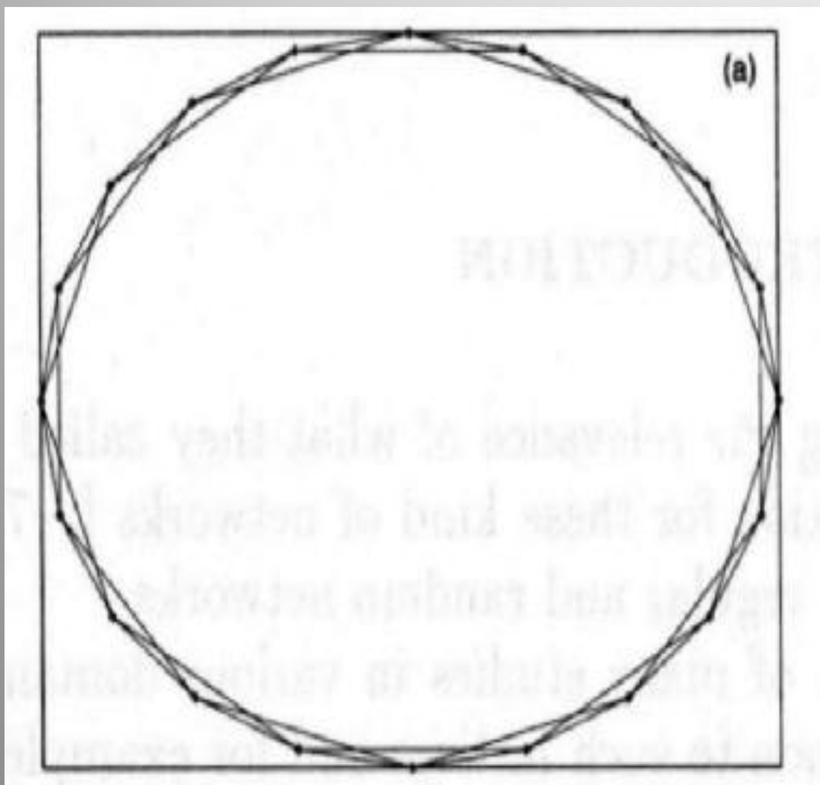
Features

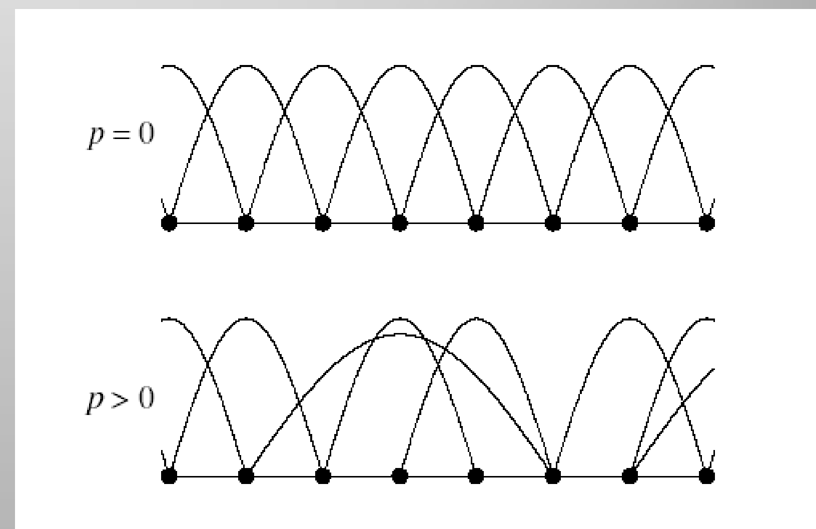
High clustering of regular lattices



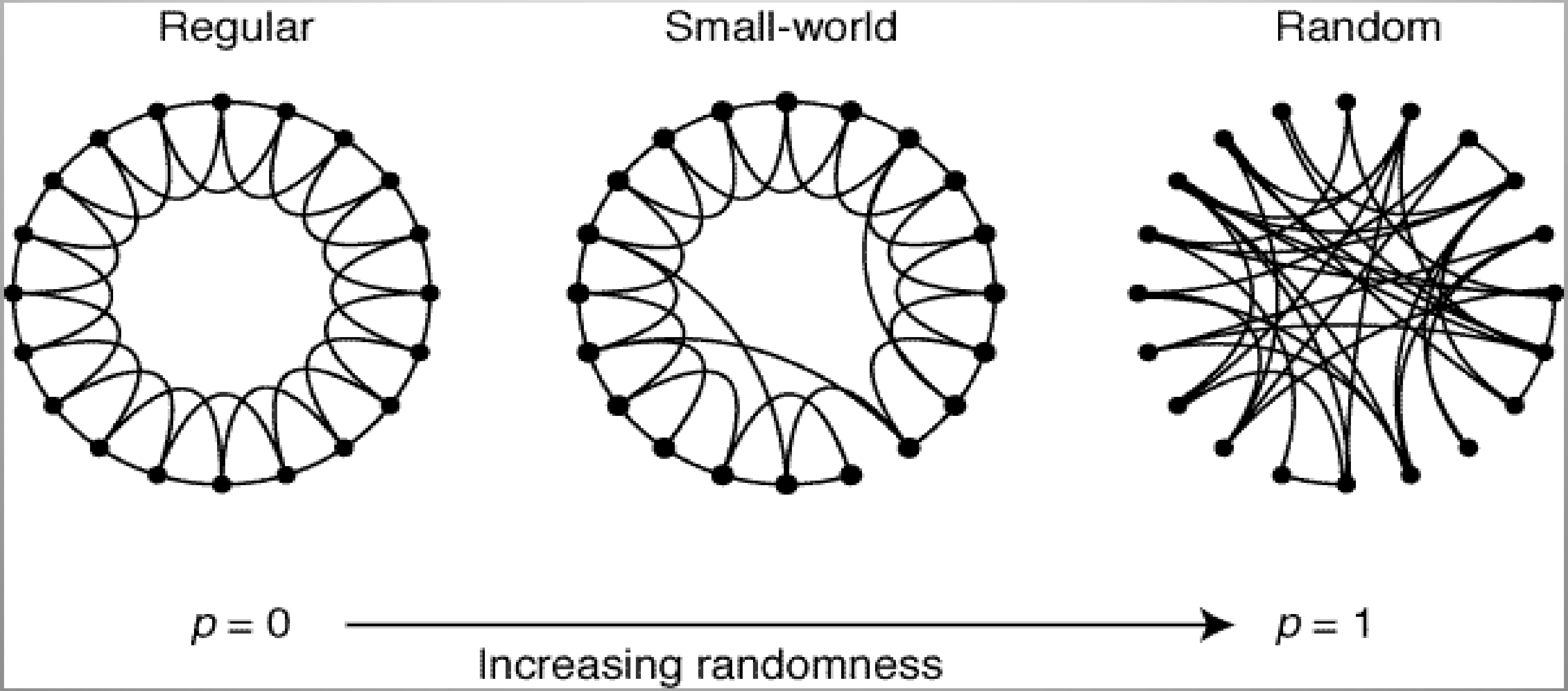
Small-world effect" (small average shortest-path length) of random networks

Construction





Interpolating between regular and random networks



Introduction to Complex Networks: Small World Networks

Large networks ($n >> 1$)

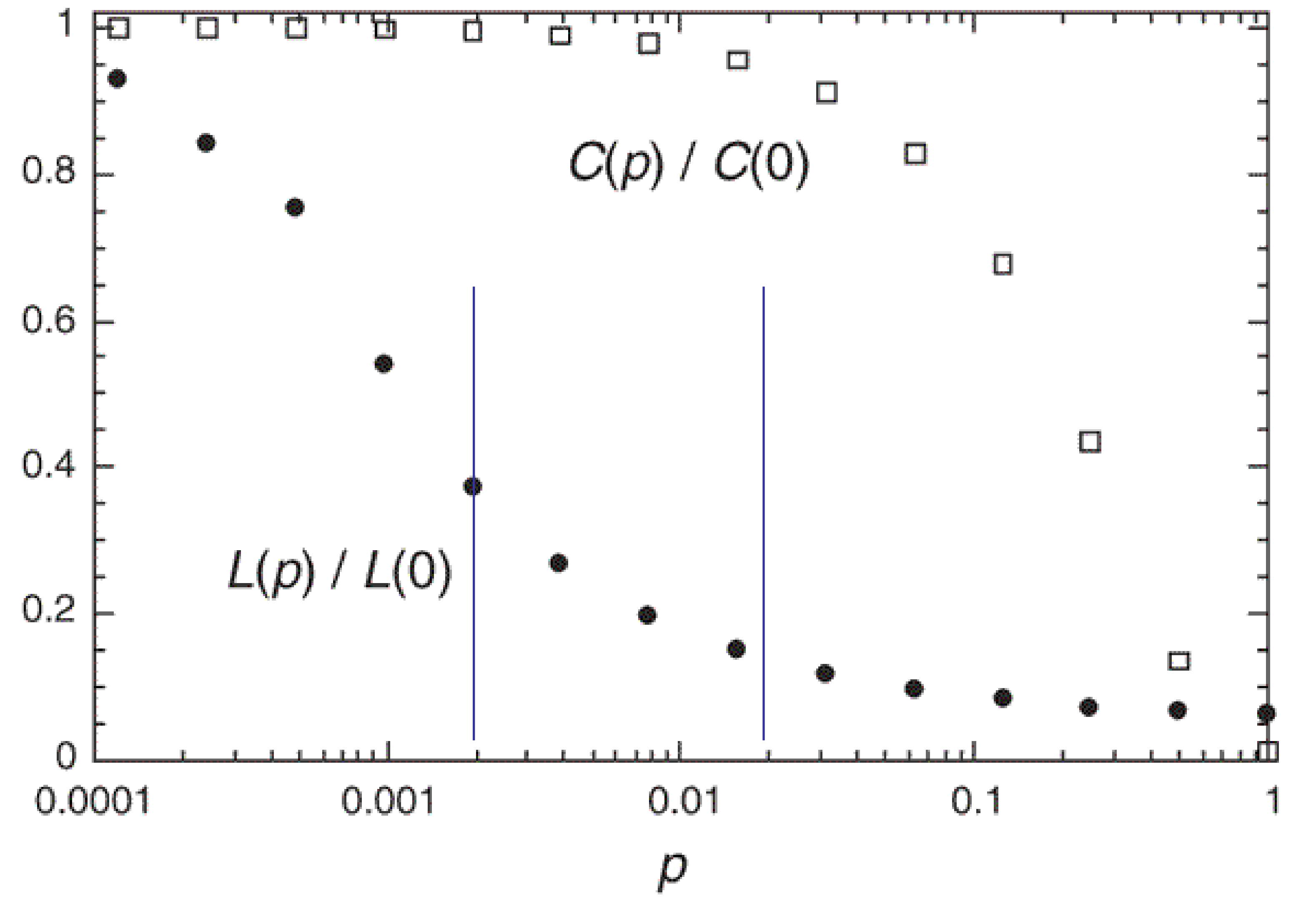
Sparse connectivity (avg degree $z << n$)

No central node ($k_{max} << n$)

Large clustering coefficient (larger than in random graphs of same size)

Short average paths (~$\log n$, close to those of random graphs of the same size)

Introduction to Complex Networks: Small World Networks



Introduction to Complex Networks: Other Networks

Cooper Frieze model

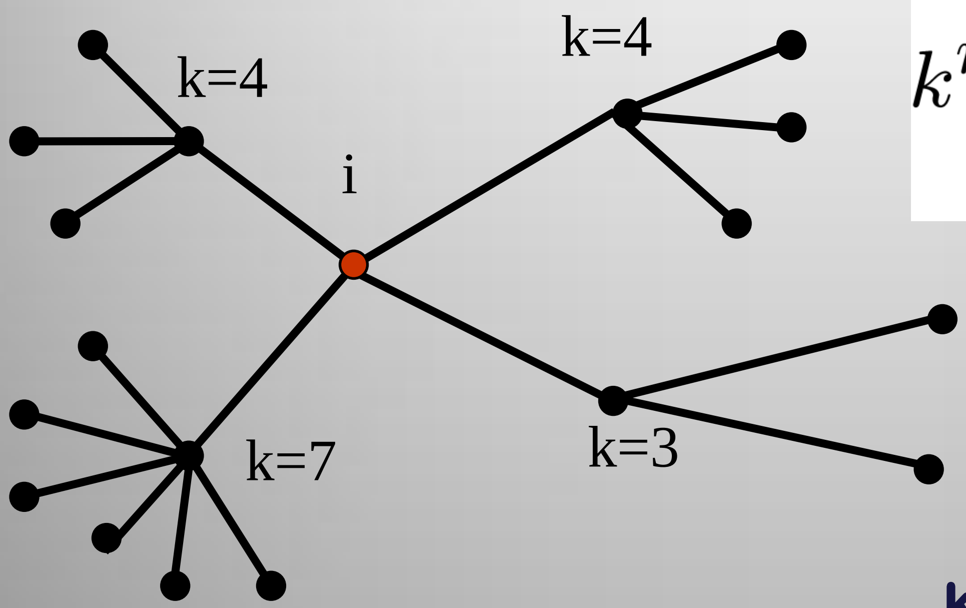
Multiple parameters that allow for adding vertices, edges, preferential attachment, uniform linking

Directed graphs [Bollobas et al]

Allow for preferential selection of both the source and the destination

Allow for edges from both new and old vertices

Introduction to Complex Networks: Topological characterization - Assortativity



$$k^{nn}(i) = \frac{1}{k_i} \sum_{j \in \mathcal{V}(i)} k_j$$

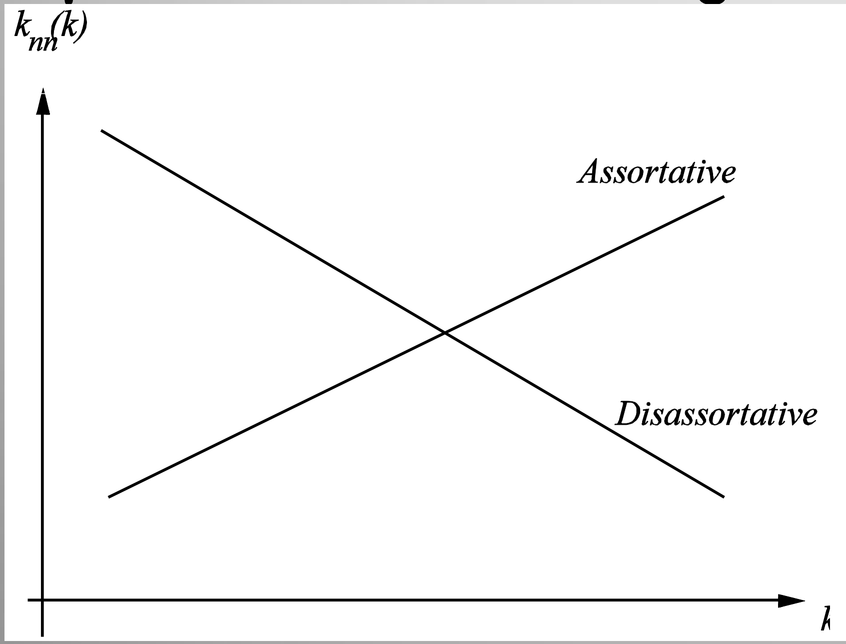
$k^{nn}(i)=(3+4+4+7)/4=4.5$

Assortativity: how similar I am with my friends

Introduction to Complex Networks: Topological characterization - Assortativity

Assortative behaviour: growing $k_{nn}(k)$

Example: social networks :Large sites are connected with large sites



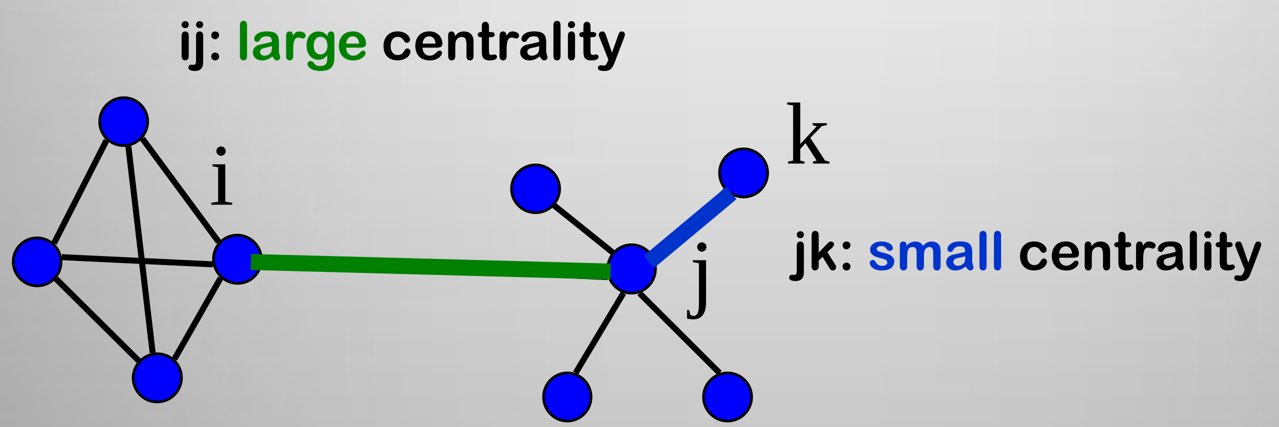
$$k^{nn}(i) = \frac{1}{k_i} \sum_{j \in \mathcal{V}(i)} k_j$$

Disassortative behaviour: decreasing $k_{nn}(k)$

Example: internet

Large sites connected with small sites, hierarchical structure

Introduction to Complex Networks: Topological characterization – Betweeness centrality



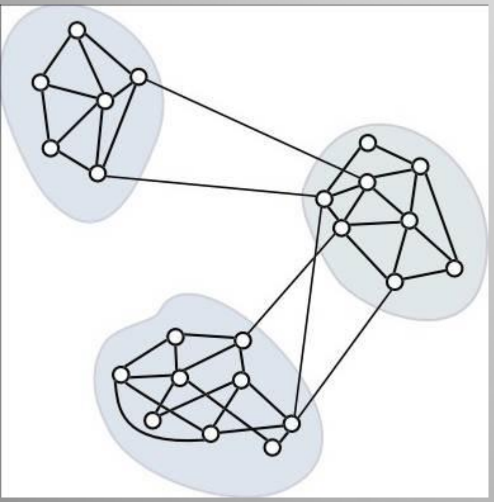
$$g(ij) = \sum_{s,t} \frac{\sigma_{st}(ij)}{\sigma_{st}}$$

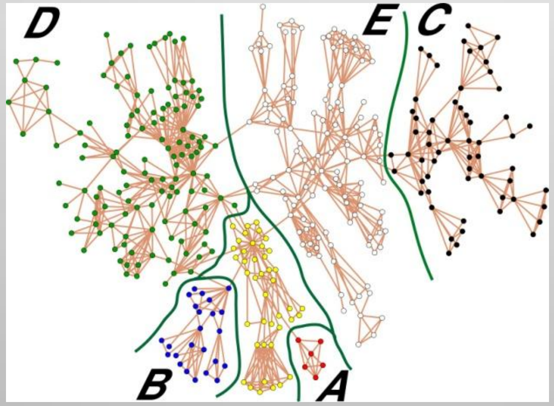
σ_{st} = # of shortest paths from s to t

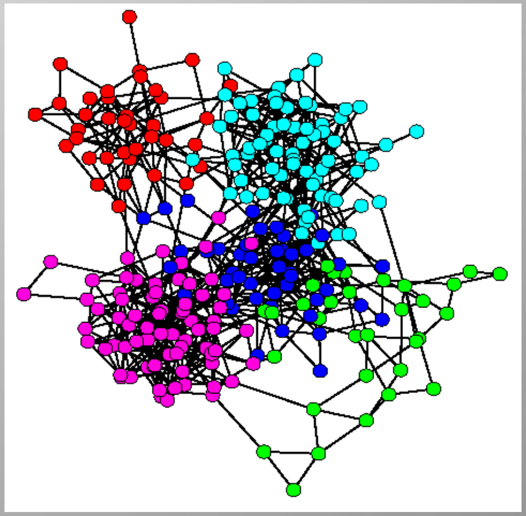
$\sigma_{st}(ij)$= # of shortest paths from s to t via (ij)

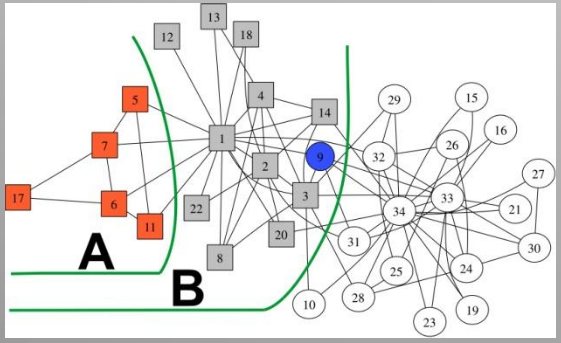
Introduction to Complex Networks: Topological characterization – Modularity

Real networks are fragmented into communities or modules









Coming soon…

Foundations and Frontiers of Complex Systems, SFI Complex Systems Summer School, Bariloche 2008

Collective behavior in complex networks

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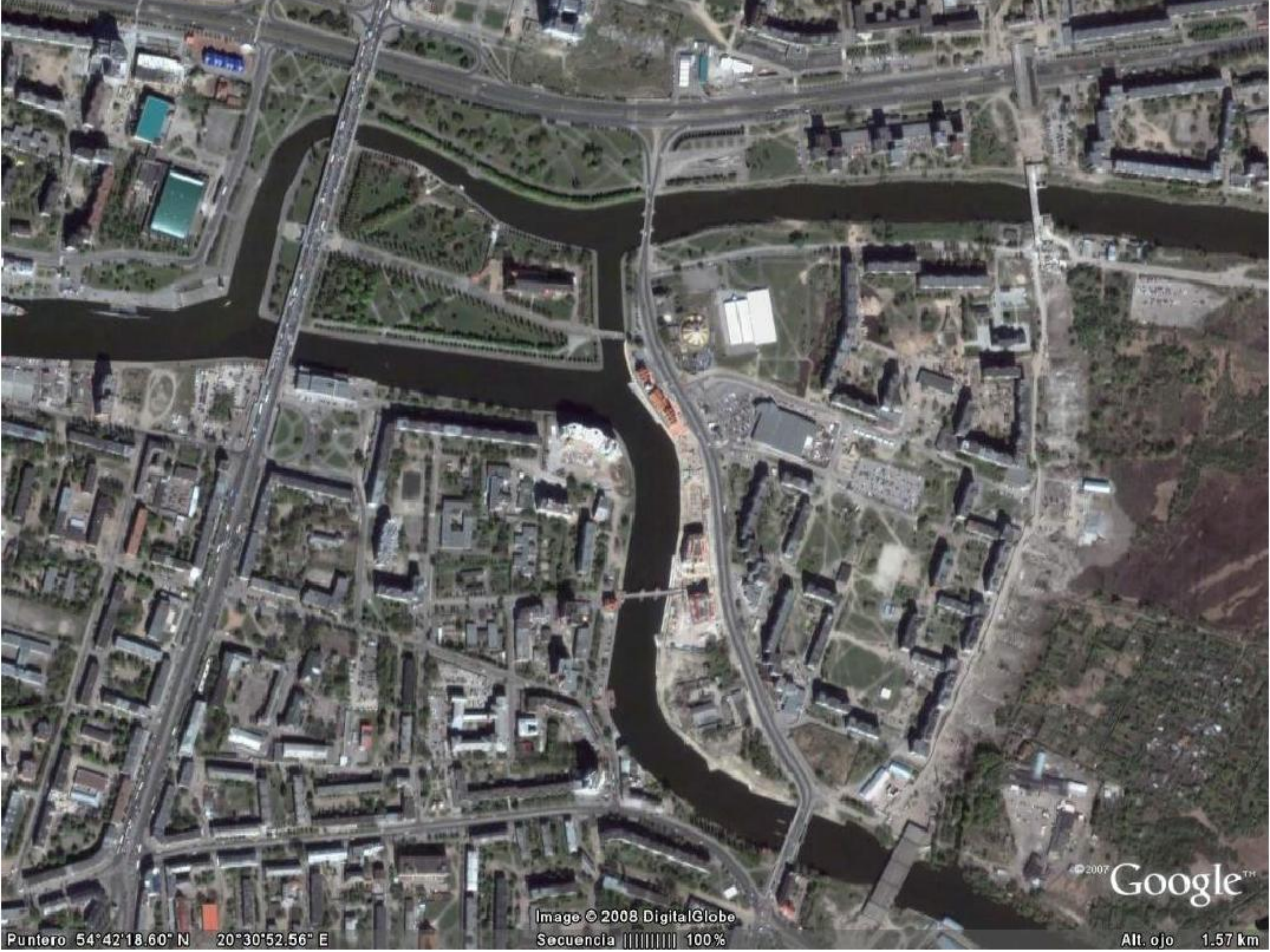
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