Complexity-Entropy Diagrams:

Exploring the Relationships

between

Complexity and Randomness

David Feldman

College of the Atlantic

`http://hornacek.coa.edu/dave/`

Joint work with: Jim Crutchfield, Carl McTague

Paper: Feldman, McTague, Crutchfield, “Organization in Intrinsic Computation: Complexity-Entropy Diagrams.” 2006.

Outline

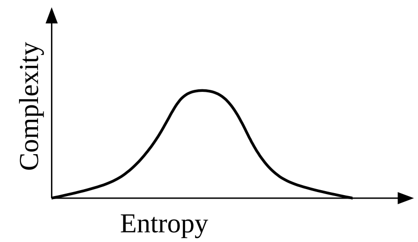
1. Motivation and Background
2. Complexity and Entropy Measures: Entropy Rate, Excess Entropy
3. Complexity-Entropy Diagrams for:
   (a) Dynamical Systems
   (b) Ising Models
   (c) Cellular Automata
   (d) Markov Models
   (e) Topological Processes
4. Conclusions

Why Complexity?

- 1960-1980’s: Study of dynamical systems leads to a number of ways of quantifying randomness or unpredictability: metric entropy, Lyapunov exponents, fractal dimensions, ...
- But, dynamical systems do more than just be unpredictable.
- Dynamical systems produce patterns, organization, structure, complexity...
- These qualities are not captured by a measure of unpredictability.
- This led to a search for measures of complexity that are as general as entropies and dimensions.
- What’s a pattern?

One approach: Prescribing Complexity vs. Entropy Behavior

- Zero Entropy ⟶ Predictable ⟶ simple and not complex.
- Maximum Entropy ⟶ Perfectly Unpredictable ⟶ simple and not complex.
- Complex phenomena combine order and disorder.
- Thus, it must be that complexity is related to entropy as shown:

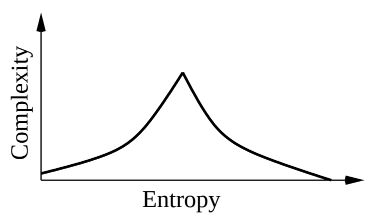


- This plot is often used as the central criteria for defining complexity.

Complexity-Entropy Phase Transition?

Edge of Chaos?

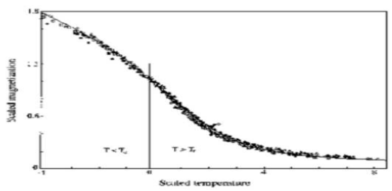
- Additionally, it has been conjectured that there is a sharp transition in complexity as a function of entropy:



- Perhaps this complexity-entropy curve is *universal*—it is the same for a broad class of apparently different systems.
- Part of the motivation for this is the success of *data collapse* in critical phenomena and condensed matter physics.

Data Collapse

- Scaled magnetization vs. scaled temperature for five different magnetic materials: EuO, Ni, YIG, $CrBr_3$, and Pd_3Fe.



- These materials are very different, but clearly possess some deep similarities.
- Figure source: H.E. Stanley, *Rev. Mod. Phys*. **71**:S358. 1999.
- Perhaps there is a similar data collapse for some appropriate definitions of complexity and entropy.
- Note: One could trivially obtain this by simply defining complexity to be a single-valued function of the entropy.

Complexity vs. Entropy: A Different Approach Define Complexity on its own Terms

- Do not prescribe a particular complexity-entropy behavior.
- To be useful, a complexity measure must have a clear interpretation that accounts in a direct way for the correlations and organization in a system.
- Consider a well known complexity measures: excess entropy
- Calculate complexity and entropy for a range of model systems.
- Plot complexity vs. entropy. This will directly reveal how complexity is related to entropy.
- Is there a universal complexity-entropy curve?

Review of Entropy and Complexity Measures

- An infinite sequence of discrete random variables:

$$\cdots S_{-2}\, S_{-1}\, S_0\, S_1\, S_2\, S_3 \cdots$$

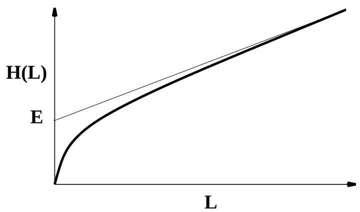
- E.g., Stationary Stochastic Process, A Stationary Time Series, Symbolic Dynamical System, One-Dimensional Equilibrium Spin Chain
- The **Shannon Entropy** H measures the uncertainty associated with a random variable:

$$H[S] \equiv \sum_s -\Pr(s) \log_2 \Pr(s) \; .$$

$\Pr(s) =$ Probability of seeing outcome s.

- Let $H(L)$ be the Shannon entropy of L consecutive random variables.
- How does $H(L)$ grow with L?

Entropy rate h_μ



- The asymptotic slope is denoted h_μ:

$$h_\mu \equiv \lim_{L\to\infty} \Delta H(L) \; .$$

- h_μ is known as: **entropy rate**, **metric entropy**, and **entropy density**.
- h_μ is the irreducible randomness: the randomness that persists even after statistics over arbitrarily long sequences are taken into account
- The entropy rate may also be written: $h_\mu = \lim_{L\to\infty} \frac{H(L)}{L}$.

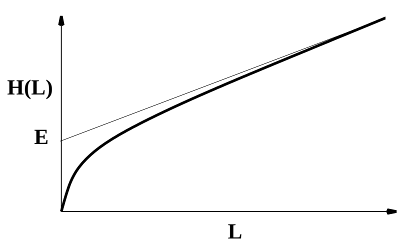
Excess Entropy

- $\mathbf{E}$ is thus the total amount of randomness that is "explained away" by considering larger blocks of variables.
- One can also show that $\mathbf{E}$ is equal to the mutual information between the "past" and the "future":

$$\mathbf{E} = I(\overleftarrow{S};\overrightarrow{S}) \equiv H[\overleftarrow{S}] - H[\overleftarrow{S} \mid \overrightarrow{S}] \; .$$

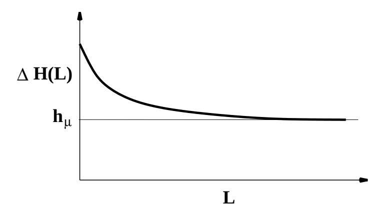
- $\mathbf{E}$ is thus the amount one half "remembers" about the other, the reduction in uncertainty about the future given knowledge of the past.
- Equivalently, $\mathbf{E}$ is the "cost of amnesia:" how much more random the future appears if all historical information is suddenly lost.

Entropy growth



- Slope $= \Delta H(L) = H(L) - H(L-1) = H[S_L|S_{L-1}S_{L-2}\ldots S_1]$
- $H[X|Y] =$ entropy of X given that Y is known.
- The slope of $H(L)$ tells you how uncertain you are about the next measurement, given that the previous L symbols have been seen.
- Eventually, $H(L)$ is a straight line — keeping track of more measurements doesn't reduce uncertainty at all.

How does $\Delta H(L)$ approach h_μ?



- For finite L , $\Delta H(L) \geq h_\mu$. Thus, the system appears more random than it is.
- We can learn about the complexity of the system by looking at *how* the entropy density converges to h_μ.
- The **excess entropy** captures the nature of the convergence and is defined as the area between the two curves above:

$$\mathbf{E} \equiv \sum_{L=1}^{\infty} [\Delta H(L) - h_\mu] \; .$$

Iterated Map: Logistic Equation

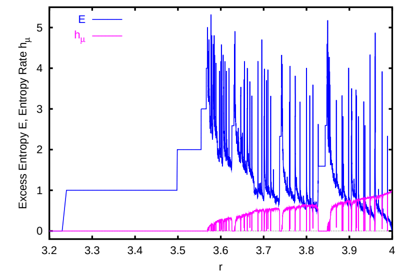
- Iterate the logistic equation: $x_{n+1} = f(x_n)$, where $f(x) = rx(1-x)$.
- Generate symbol sequence via:

$$s_i = \begin{cases} 0 & x \le \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} .$$

- As the parameter r is varied, the system exhibits a wide range of behavior: periodic and chaotic.

Complexity and Entropy: Logistic Equation

Plot of the excess entropy $\mathbf{E}$ and the entropy rate h_μ for the logistic equation as a function of the parameter r:



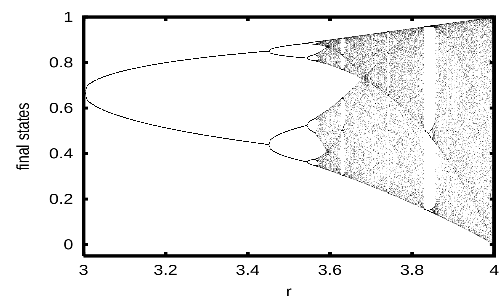
- Note that $\mathbf{E}$ and h_μ depend in a complicated way on r.
- Hard to see how complexity and entropy are related.

Excess Entropy and Entropy Rate Summary

- Excess entropy $\mathbf{E}$ is a measure of complexity (order, pattern, regularity, correlation ...)
- Entropy rate h_μ is a measure of unpredictability.
- Both $\mathbf{E}$ and h_μ are well understood and have clear interpretations.
- Both $\mathbf{E}$ and h_μ are functions of the distribution over sequences.
- For a periodic sequence, $\mathbf{E} = \log_2(\text{Period})$, and $h_\mu = 0$.
- For more, see, e.g., Crutchfield and Feldman, *Chaos*. **15**:23. 2003.

Let's calculate h_μ and $\mathbf{E}$ for some systems and see what the complexity-entropy diagram looks like...

Logistic Equation: Bifurcation Diagram



- For a given r (horizontal axis), the "final states" are shown.
- Chaotic behavior appears as a solid vertical line.
- Examples:
  - $r = 3.2$: Period 2.
  - $r = 3.5$: Period 5.
  - $r = 3.7$: Chaotic.

Ising Models

Consider a one- or two-dimensional Ising system with nearest and next nearest neighbor interactions:

- This system is a one- or two-dimensional lattice of variables $s_i \in \{\pm 1\}$.
- The energy of a configuration is given by:

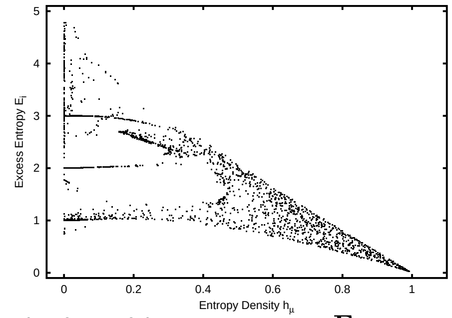
$$\mathcal{H} \equiv -J_1 \sum_i s_i s_{i+1} - J_2 \sum_i s_i s_{i+2} - B \sum s_i \ .$$

- The probability of observing a configuration $\mathcal{C}$ is given by the Boltzmann distribution:

$$\Pr(\mathcal{C}) \propto e^{-\frac{1}{T}\mathcal{H}(\mathcal{C})} \ .$$

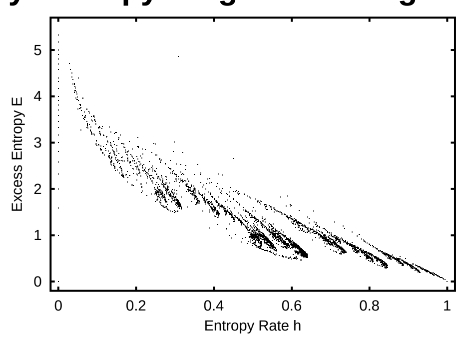
- Ising models are very generic models of spatially extended, discrete degrees of freedom that have some interaction that makes them want to either do the same or the opposite thing.

Complexity-Entropy Diagram for 2D Ising Models



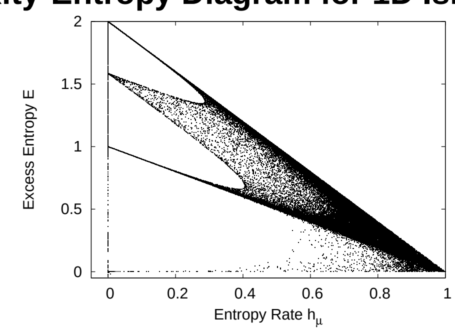
- Mutual information form of the excess entropy $\mathbf{E}_i$ vs. entropy density h_μ for the two-dimensional Ising model with AFM couplings
- Model parameters are chosen uniformly from the following ranges: $J_1 \in [-3, 0]$, $J_2 \in [-3, 0]$, $T \in [0.05, 4.05]$, and $B = 0$.
- Surprisingly similar to the one-dimensional Ising model.
- Results via Monte Carlo simulation of 100x100 lattices.

Complexity-Entropy Diagram for Logistic Equation



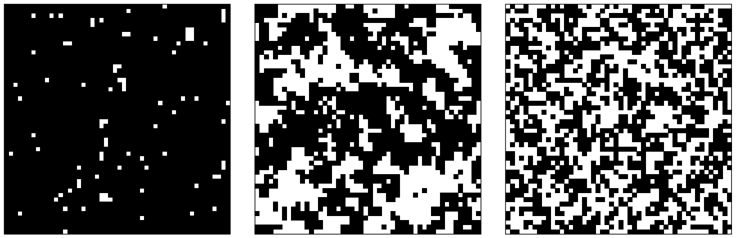
- Structure is apparent in this plot that isn't visible in the previous one.
- Not all complexity-entropy values can occur.
- Maximum complexity occurs at zero entropy.
- Note self-similar structure. Not surprising, since the bifurcation diagram is self-similar.

Complexity-Entropy Diagram for 1D Ising Models



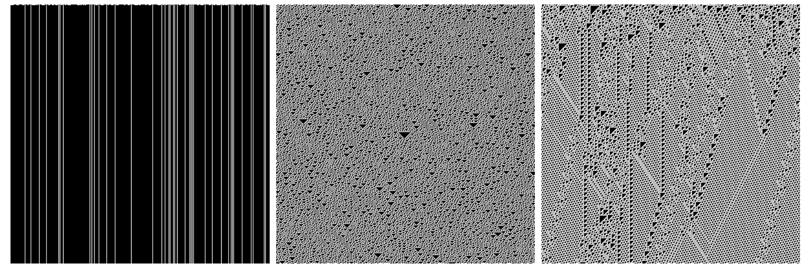
- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ for the one-dimensional Ising model with anti-ferromagnetic couplings.
- Model parameters are chosen uniformly from the following ranges: $J_1 \in [-8, 0]$, $J_2 \in [-8, 0]$, $T \in [0.05, 6.05]$, and $B \in [0, 3]$.
- Note how different this is from the logistic equation.
- These are exact transfer-matrix results.

Ising Model Configurations



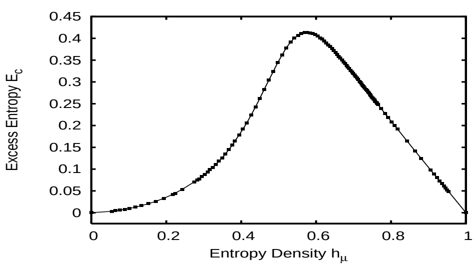
- Typical configurations for the 2D Ising model below, at, and above the critical temperature.

Different Rules Yield Different Patterns



- Each pattern is for a different rule.

Complexity-Entropy Diagram for 2D Ising Model Phase Transition

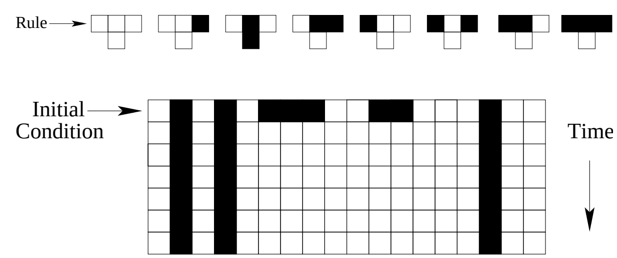


- Convergence form of the excess entropy $\mathbf{E}_c$ vs. entropy density h_μ for the two-dimensional Ising model with NN couplings and no external field.
- Model undergoes phase transition as T is varied at $T \approx 2.269$.
- There is a peak in the excess entropy, but it is somewhat broad.
- Results via Monte Carlo simulation of 100x100 lattice.

Cellular Automata

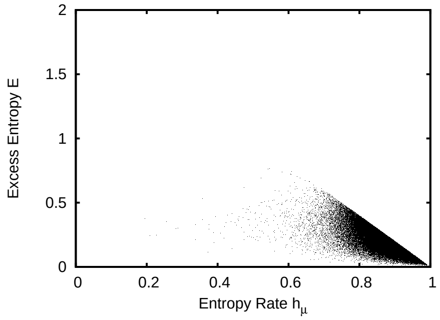
- The next row in the grid is determined by the row directly above it according to a given rule
- Start with a random initial condition

Example:



- The number of cells away from the center cell that the rule considers is known as the radius of the CA.

Complexity-Entropy Diagram for Markov Models

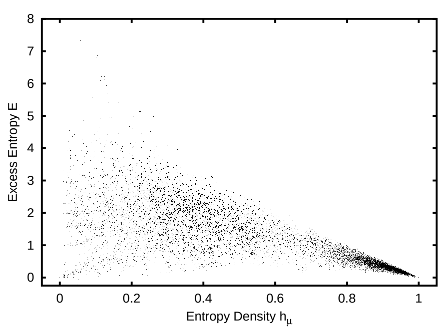


- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ for $100,000$ random Markov models.
- The Markov models here have four states, corresponding to dependence on the previous two symbols, as in the 1D NNN Ising model.
- Transition probabilities chosen uniformly on $[0, 1]$ and then normalized.
- Note that these systems have no forbidden sequences.

Topological Processes and Statistical Complexity

- These topological processes can be exhaustively enumerated for any finite number of states.
- We now use a different measure of complexity: the *statistical complexity* C_μ
- C_μ is the Shannon entropy of the asymptotic distribution over states.
- We consider only minimal machines.
- $C_\mu \geq \mathbf{E}$.

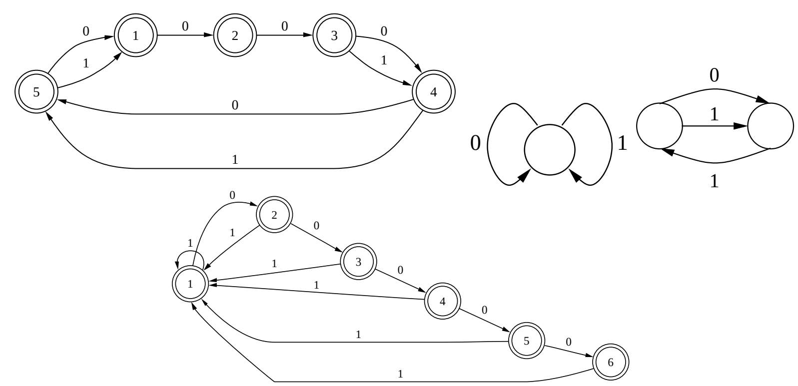
Complexity-Entropy Diagram for Radius-2, 1D CAs



- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ for $10,000$ radius-2, binary CAs.
- $\mathbf{E}$ and h_μ from the spatial strings produced by the CAs.
- The CAs were chosen uniformly from the space of all such CAs.
- There are around $10^{30,000}$ such CAs, so it is impossible to sample the entire space.

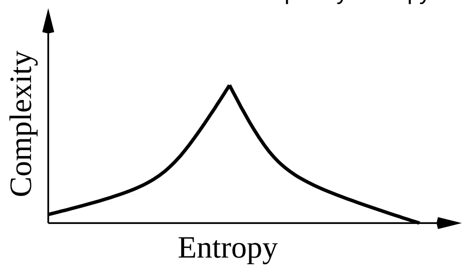
Topological Markov Chain Processes

- Consider finite-state machines that produce 0's and 1's.
- Assume all branching transitions are equally probable
- Examples:



Complexity-Entropy Diagrams: Summary

- Is it the case that there is a universal complexity-entropy diagram?

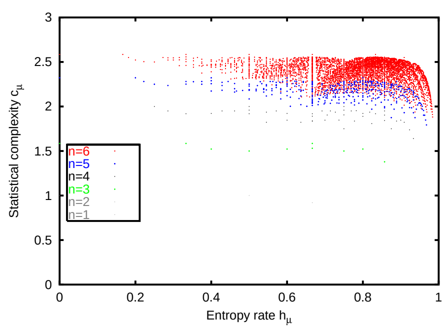


- No.
- However, because of this non-universality, complexity-entropy diagrams provide a useful way to compare the information processing abilities of different systems.
- Complexity-entropy plots allow comparisons across a broad class of systems.

Complexity-Entropy Diagrams: Conclusions

- There is not a universal complexity-entropy curve.
- Complexity is not necessarily maximized at intermediate entropy values.
- It is not always the case that there is a sharp complexity-entropy transition.
- Complexity-entropy diagrams provide a way of comparing the information processing abilities of different systems in a parameter-free way.
- Complexity-entropy diagrams allow one to compare the information processing abilities of very different model classes on similar terms.
- There is a considerable diversity of complexity-entropy behaviors.

Complexity-Entropy Diagram for Topological Processes



- h_μ, C_μ pairs for all $14,694$ distinct topological processes of $n = 1$ to $n = 6$ states.
- Note the prevalence of high-entropy, high-complexity processes.

A Mosaic of Complexity-Entropy Diagrams