From Determinism to Stochasticity

AKA

A Brief Introduction to Symbolic Dynamics

Outline of lectures:

- Probability theory for dynamical systems
- Stochastic processes
- Measurement theory

From Determinism to Stochasticity

History, a sample:

Proofs of chaos:

1. Jacques Hadamard (1898): Geodesics on surfaces of negative curvature
2. Marston Morse (1921): Construction of aperiodic recurrent geodesic
3. Gustav Hedlund (1938): First formal development laid out.

Engineering period:

1. Claude Shannon (1948): Information Theory
2. Many (1970s): Coding theory

Applications and Data analysis (1960s and on to today):

1. Smale (1960s): Differentiable dynamical systems
2. Sinai (1960s): Metric theory of dynamical systems
3. Sarkovski (1964): Ordering of period orbits in 1D maps

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References? Many; for example:

1. M. C. Mackey, **Time's Arrow: Origins of Thermodynamic Behavior**, Springer, New York (1993).

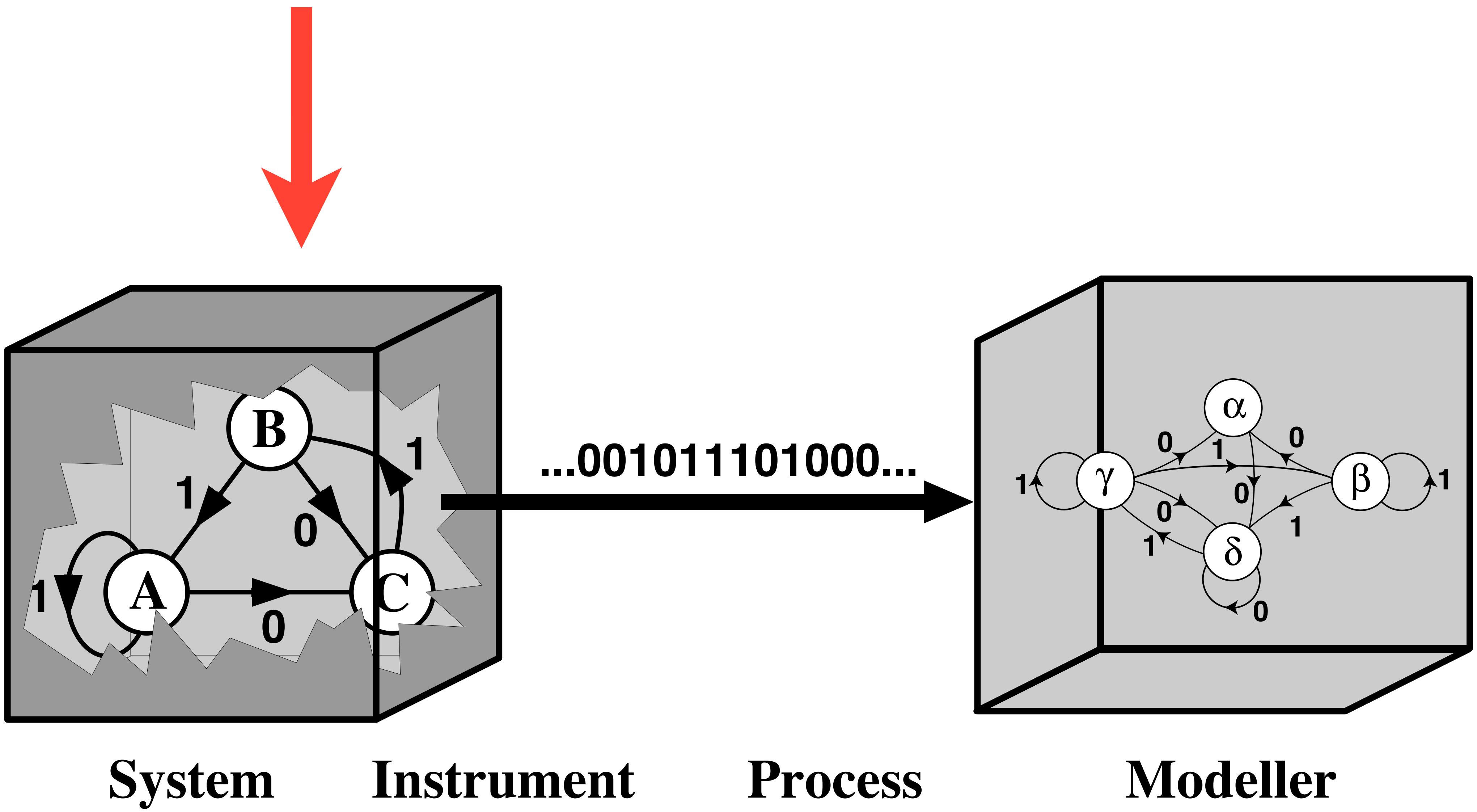
2. D. Lind and B. Marcus, **An Introduction to Symbolic Dynamics and Coding**, Cambridge University Press, New York (1995).

3. Hao Bai-Lin, **Applied Symbolic Dynamics and Chaos**, World Scientific Publishing, Singapore (1998).

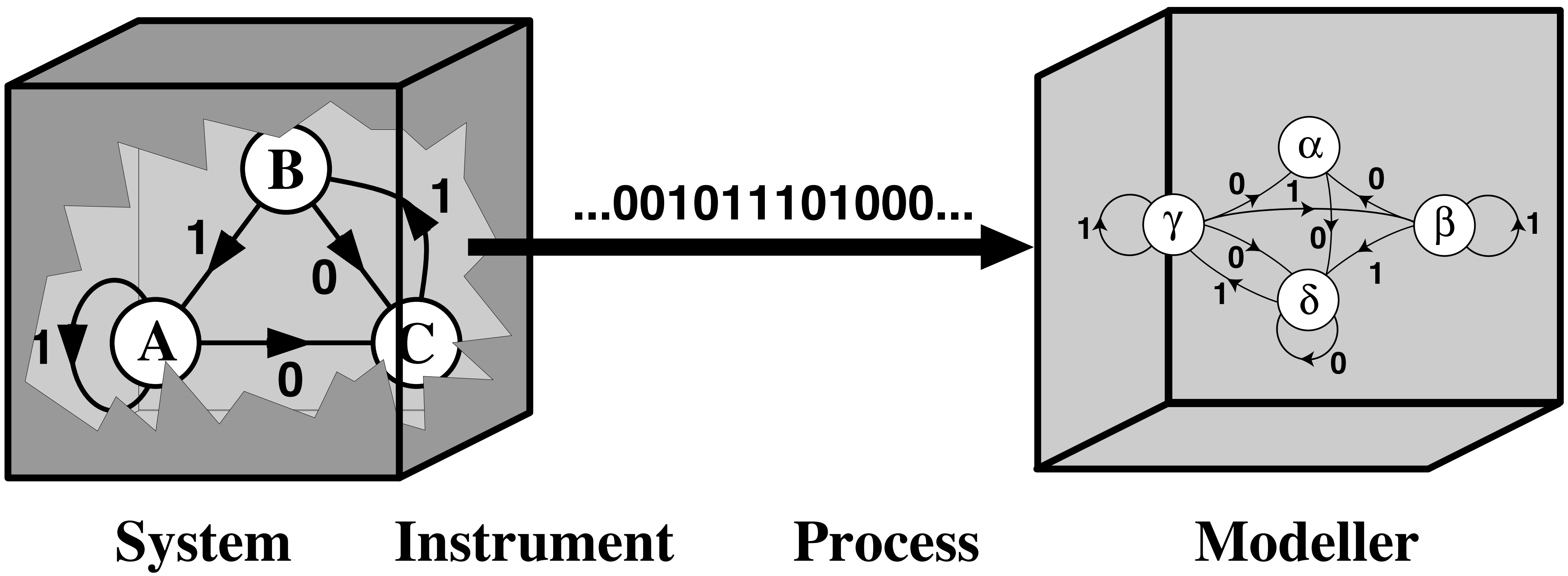
4. J. R. Dorfman, **An Introduction to Chaos in Nonequilibrium Statistical Mechanics**, Cambridge University Press, New York (1999).

5. C. S. Dawa, C. E. A. Finney, and E. R. Tracy, "A review of symbolic analysis of experimental data", Review of Scientific Instruments **74**:2 (2003) 915-930.

You Are Here



The Learning Channel



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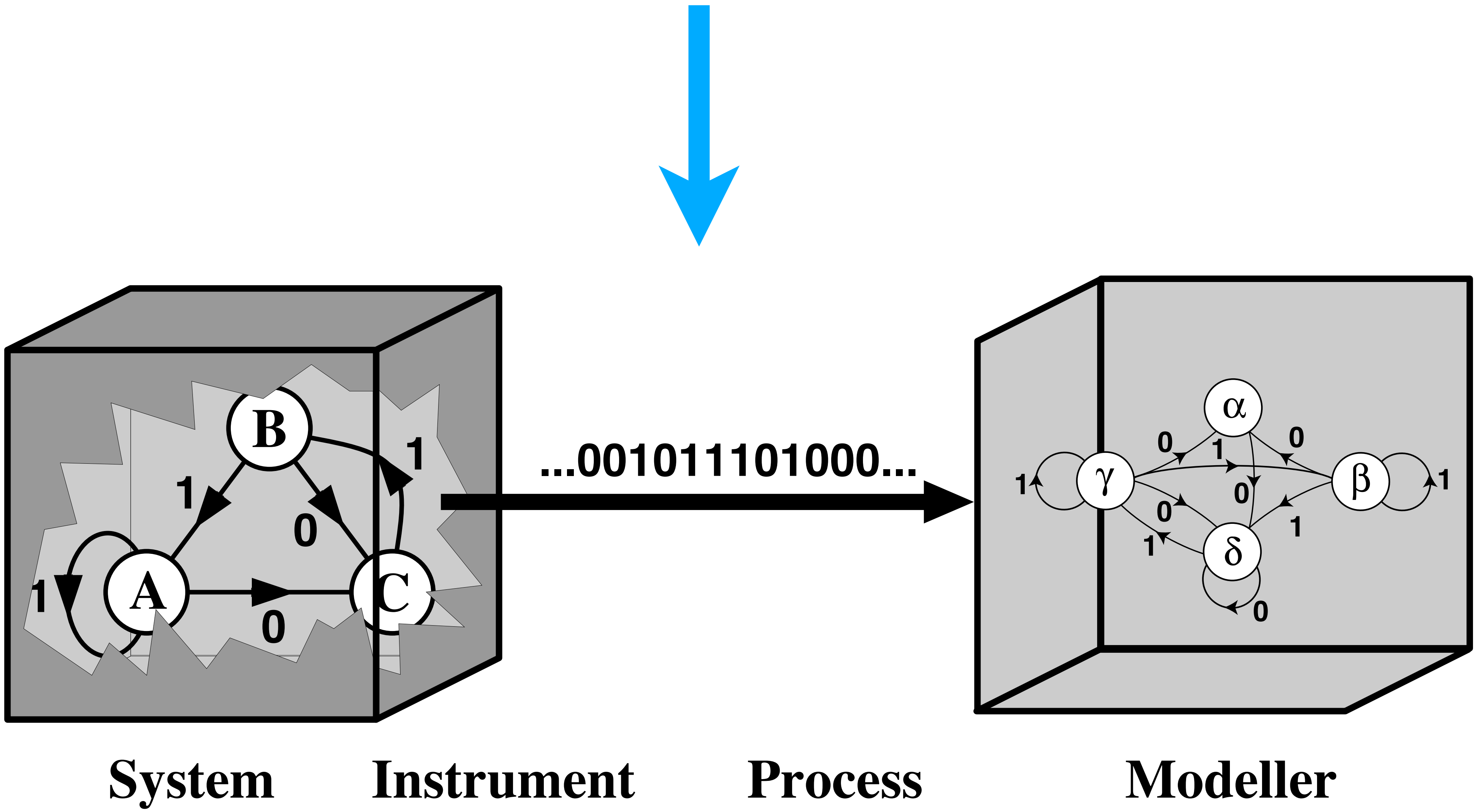
You Will Soon Be Here



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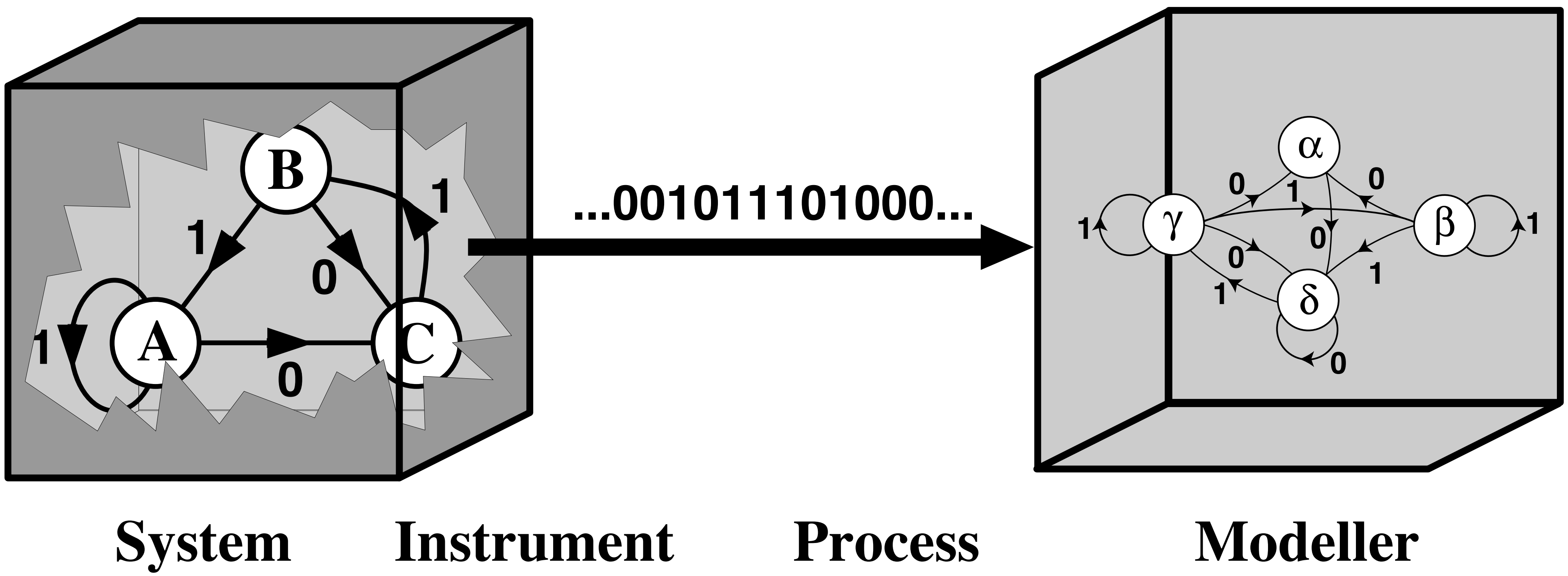

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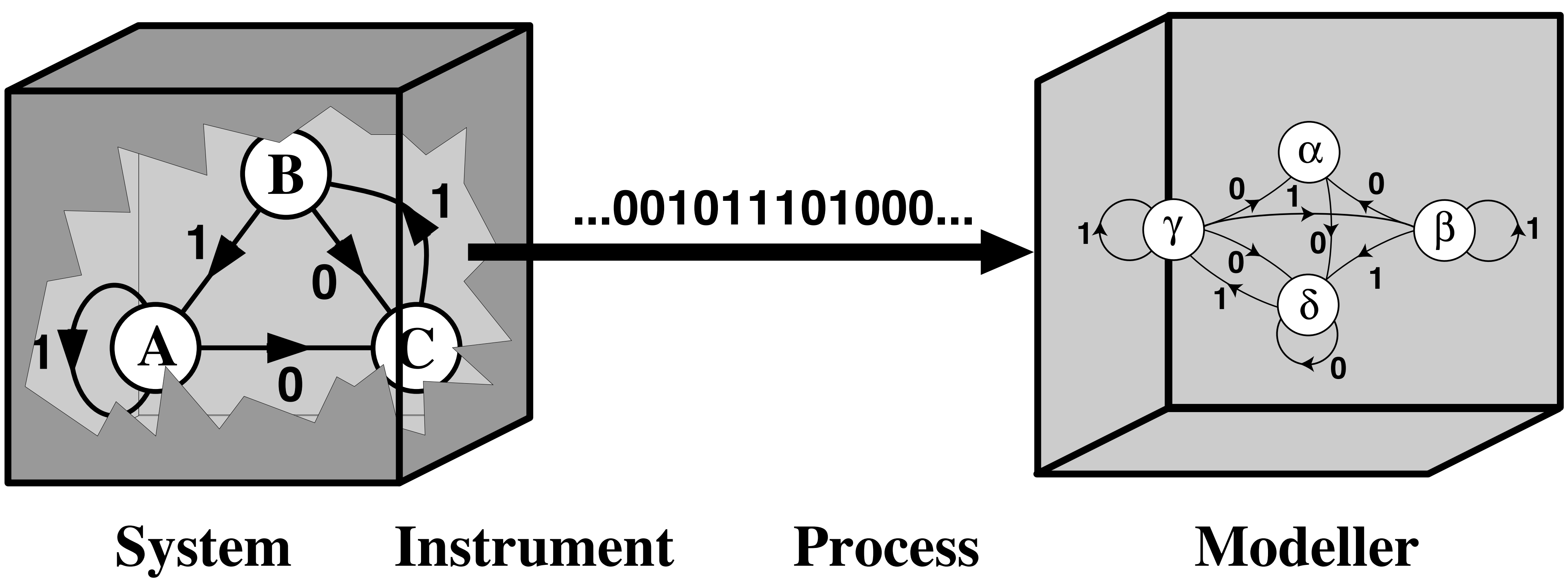
In Two Weeks



The Learning Channel

From Determinism to Stochasticity ...

Probability Theory of Dynamical Systems ...

Dynamical Evolution of Distributions:

Dynamical system: $\{\mathcal{X}, \mathcal{T}\}$

State density: $p(x) \quad x \in \mathcal{X}$

Can evolve individual states and sets: $\mathcal{T} : x_0 \rightarrow x_1$

Initial density: $p_0(x)$ Model of measuring a system

Evolve a density? $p_0(x) \rightarrow_{\mathcal{T}} p_1(x)$

What happens with localized initial density?

$$p_0(x) = \begin{cases} 20, & |x - 1/3| \leq 0.025 \\ 0, & \text{otherwise} \end{cases}$$

Consider a set of increasingly more complicated systems and how they evolve distributions ...

From Determinism to Stochasticity ...

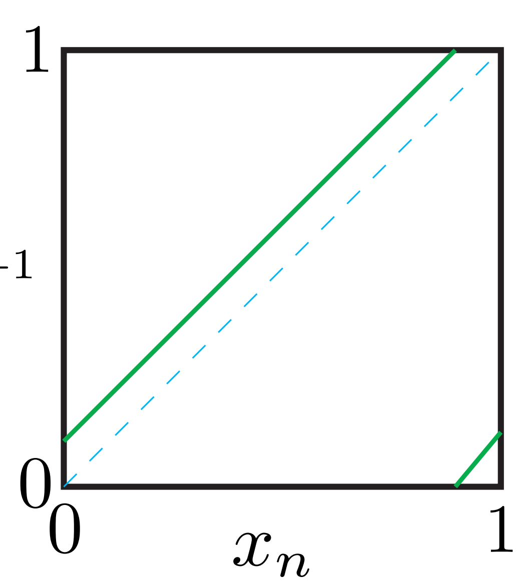
Probability Theory of Dynamical Systems ...

Dynamical Evolution of Distributions ...

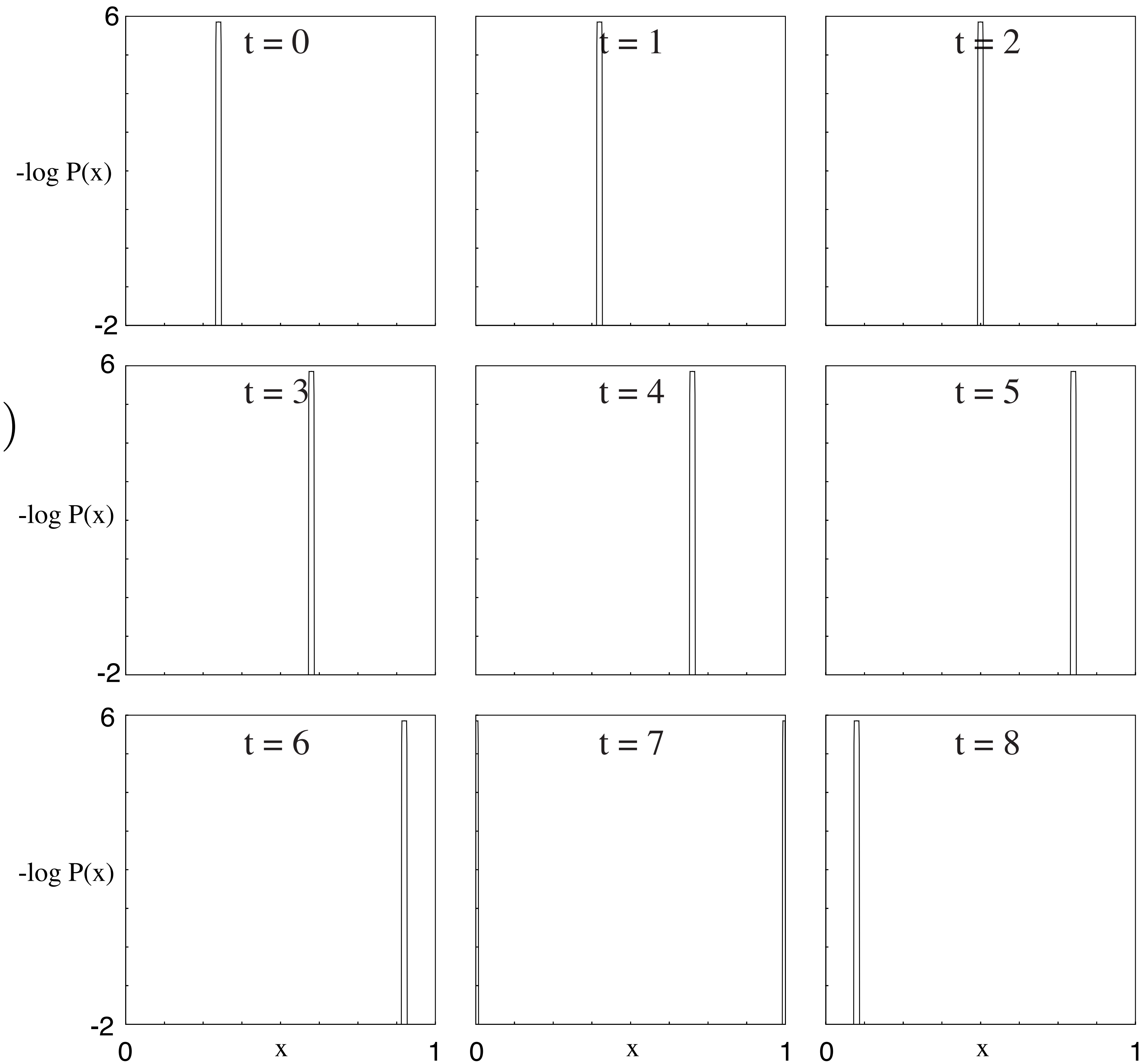
Example:

Linear circle map

$$x_{n+1} = 0.1 + x_n \pmod 1$$



$$f'(x) = 1$$



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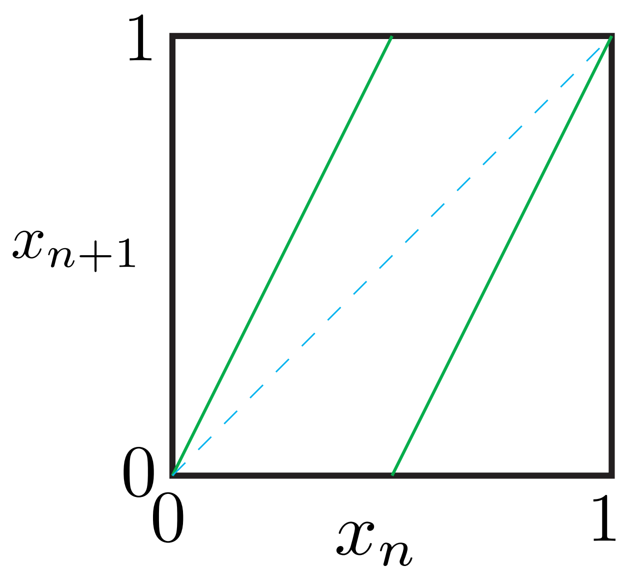
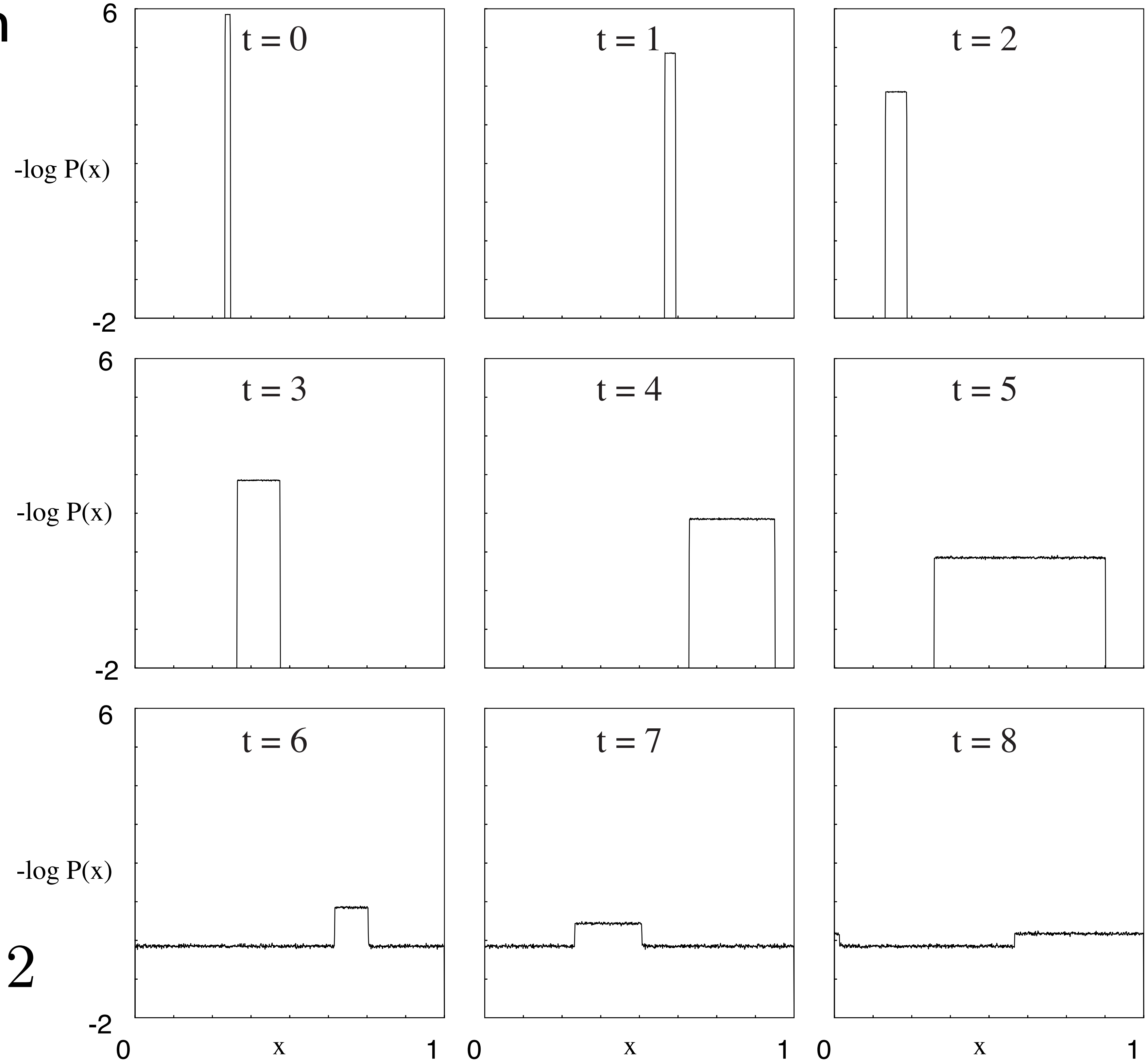
Probability Theory of Dynamical Systems ...

Dynamical Evolution of Distributions ...

Example:
Shift map





Spreading: $f'(x) = 2$

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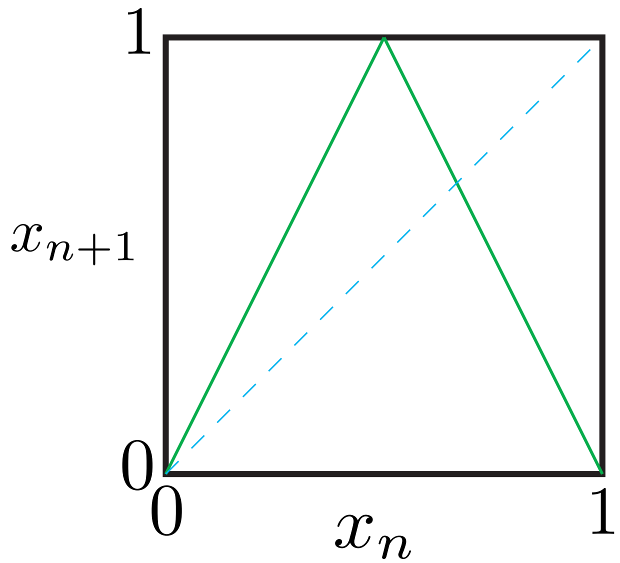
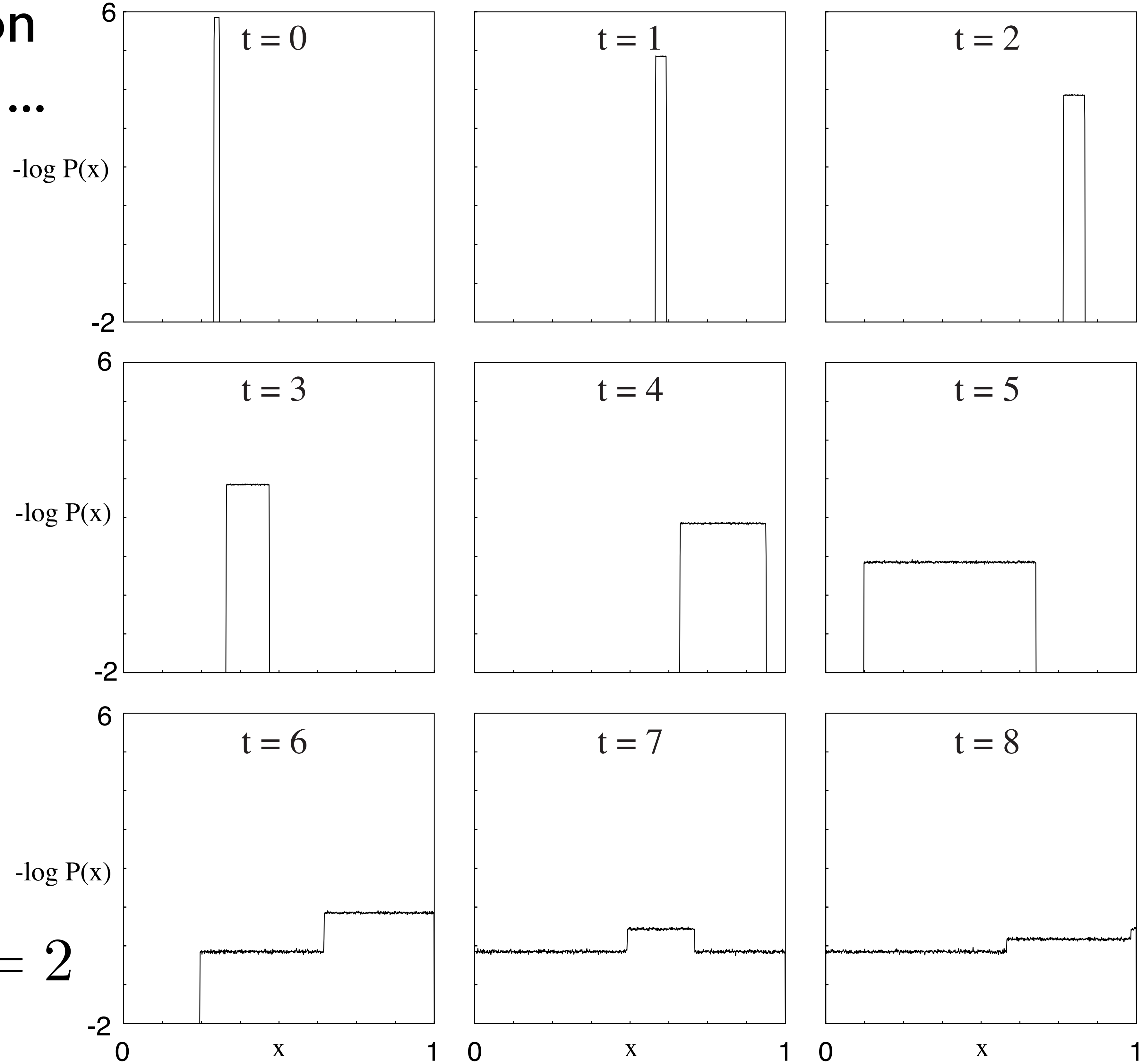
Probability Theory of Dynamical Systems ...

Dynamical Evolution of Distributions ...

Example:

Tent map a = 2.0





Spreading: $|f'(x)| = 2$

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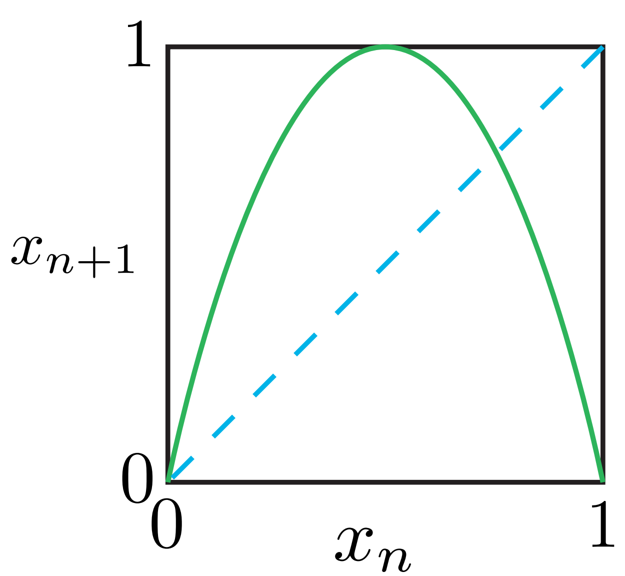
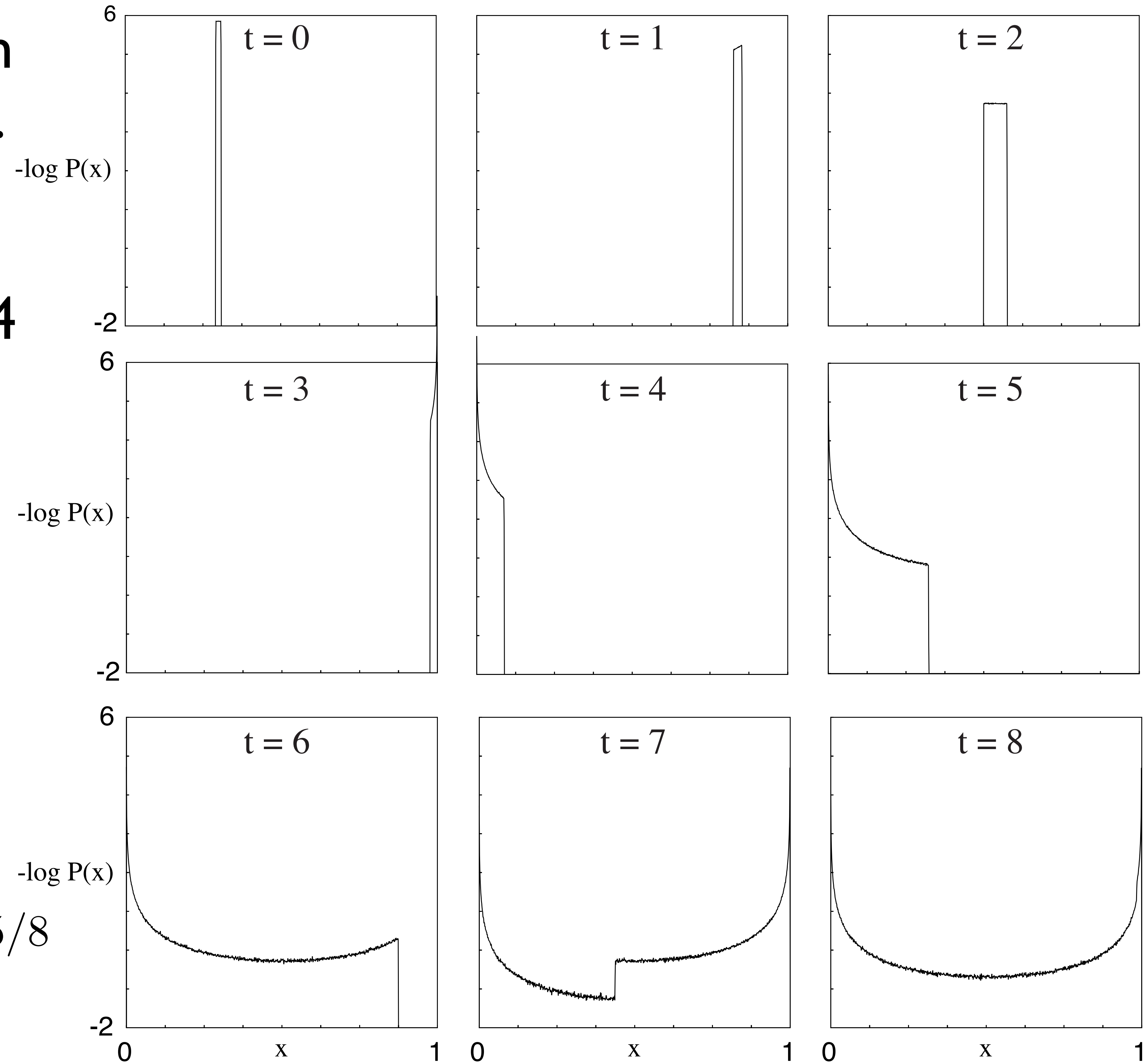
Probability Theory of Dynamical Systems ...

Dynamical Evolution of Distributions ...

Example:
Logistic map r = 4





$$f'(x) = 4(1 - 2x)$$

Spreading: $x < 3/8$ or $x > 5/8$

Contraction: $3/8 < x < 5/8$

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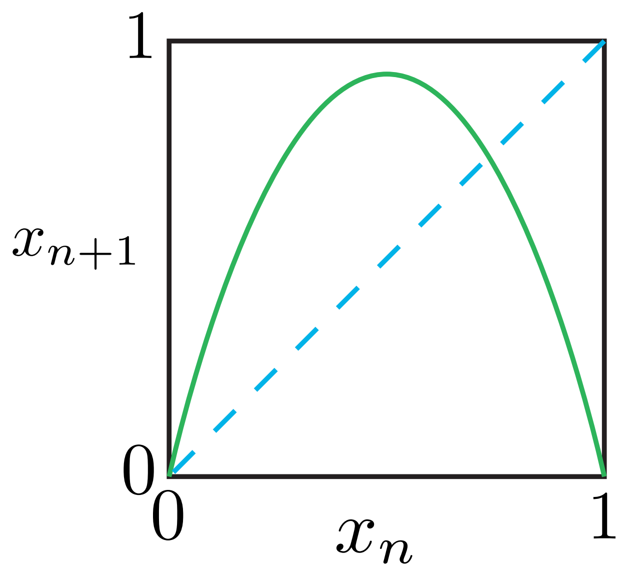
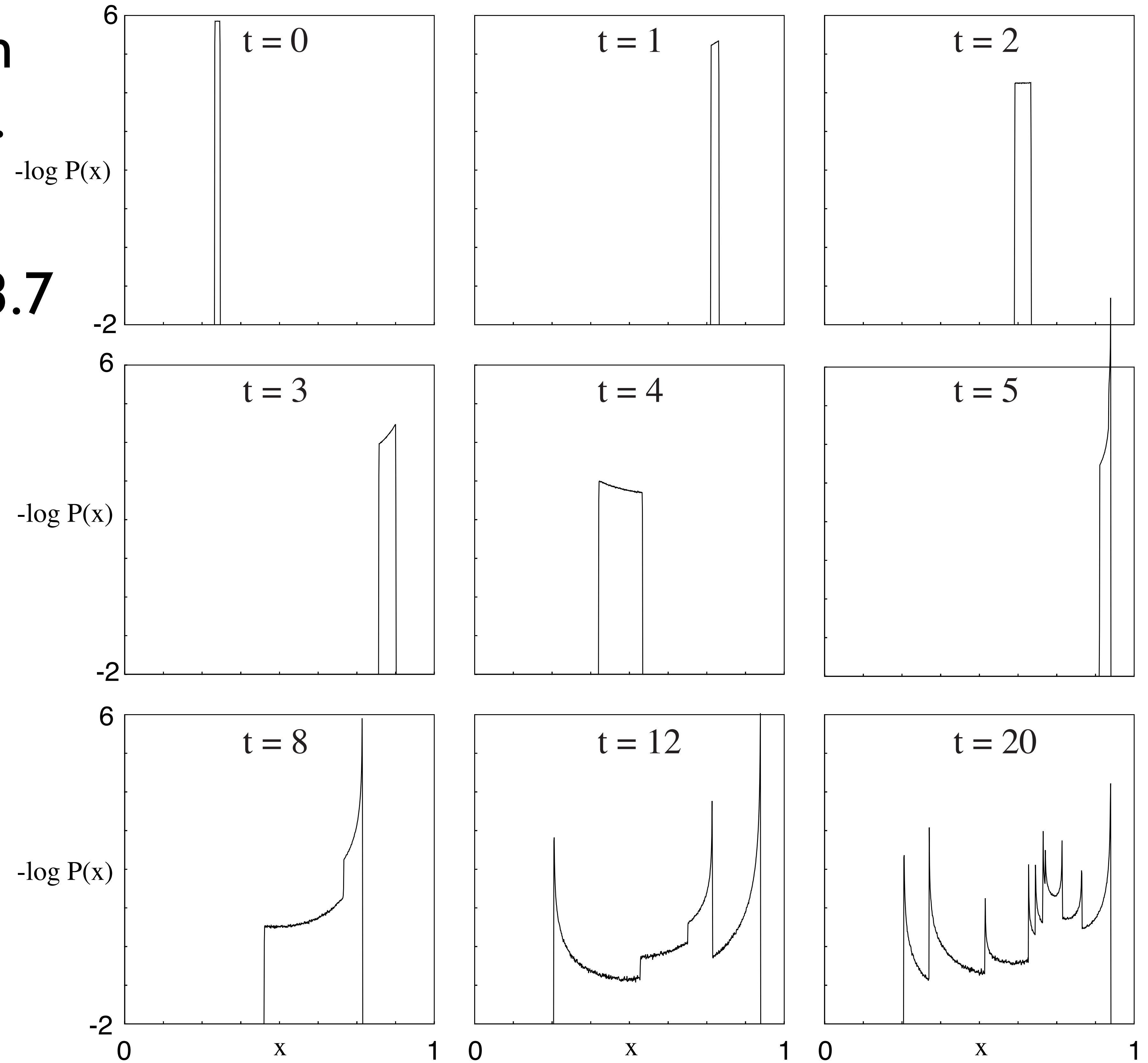
Probability Theory of Dynamical Systems ...

Dynamical Evolution of Distributions ...

Example: Logistic map r = 3.7





Peaks in distribution are images of maximum

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Probability Theory of Dynamical Systems ...

Time-asymptotic distribution: What we observe

How to characterize?

Invariant measure:

A distribution that maps "onto" itself

Analog of invariant sets

Stable invariant measures:

Stable in what sense?

Robust to noise or parameters or ???

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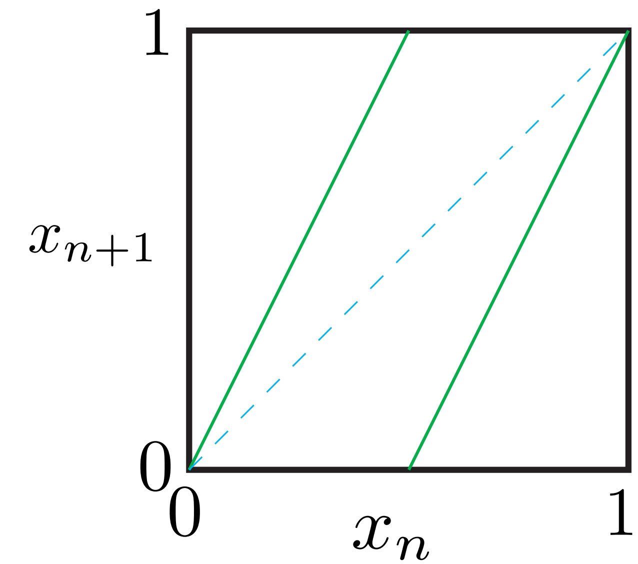
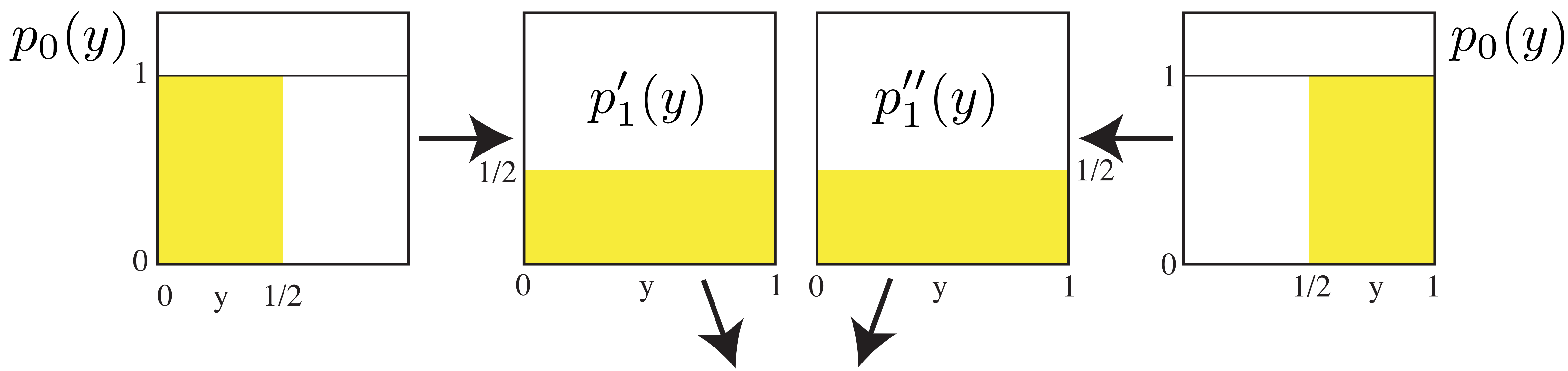
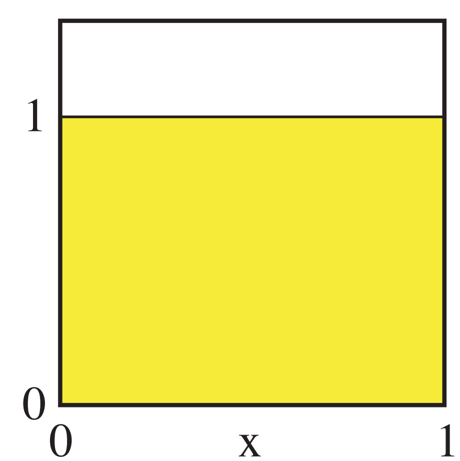


Probability Theory of Dynamical Systems ...

Example: Shift map invariant distribution

Uniform distribution: $p(x) = 1, \; x \in [0, 1]$





$$p_1(y) = p_1'(y) + p_1''(y)$$

$$p_1(x) = p_0(x) \qquad (y \Rightarrow x)$$

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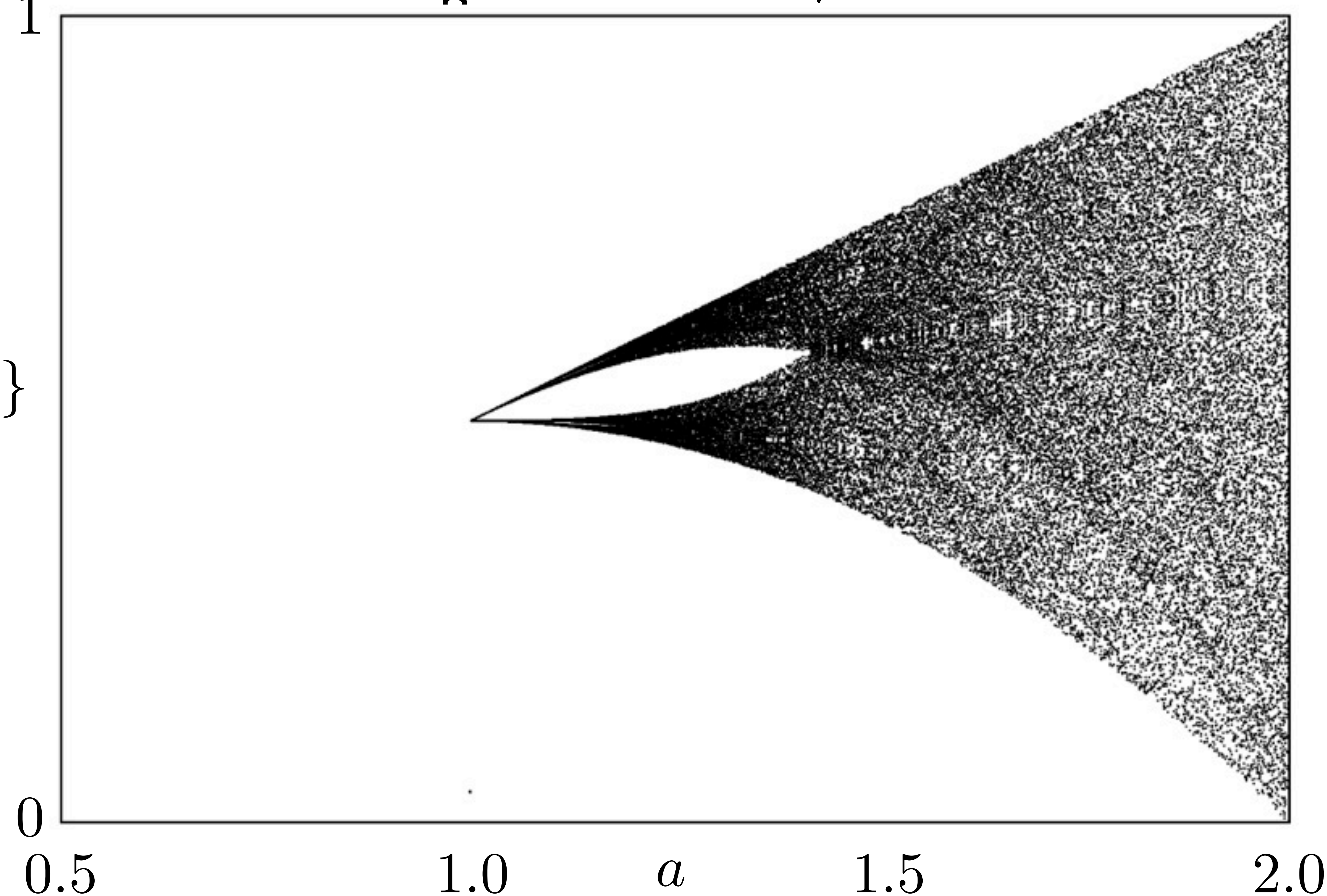
Probability Theory of Dynamical Systems ...
Numerical Example: Tent map
2 bands merge to 1 at $a = \sqrt{2}$.



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Probability Theory of Dynamical Systems ...

Numerical Example: Tent map

Typical chaotic parameter:

$$a = 1.75$$

Two bands merge to one:

$$a = \sqrt{2}$$

From Determinism to Stochasticity ...

Probability Theory of Dynamical Systems ...

Numerical Example: Tent map

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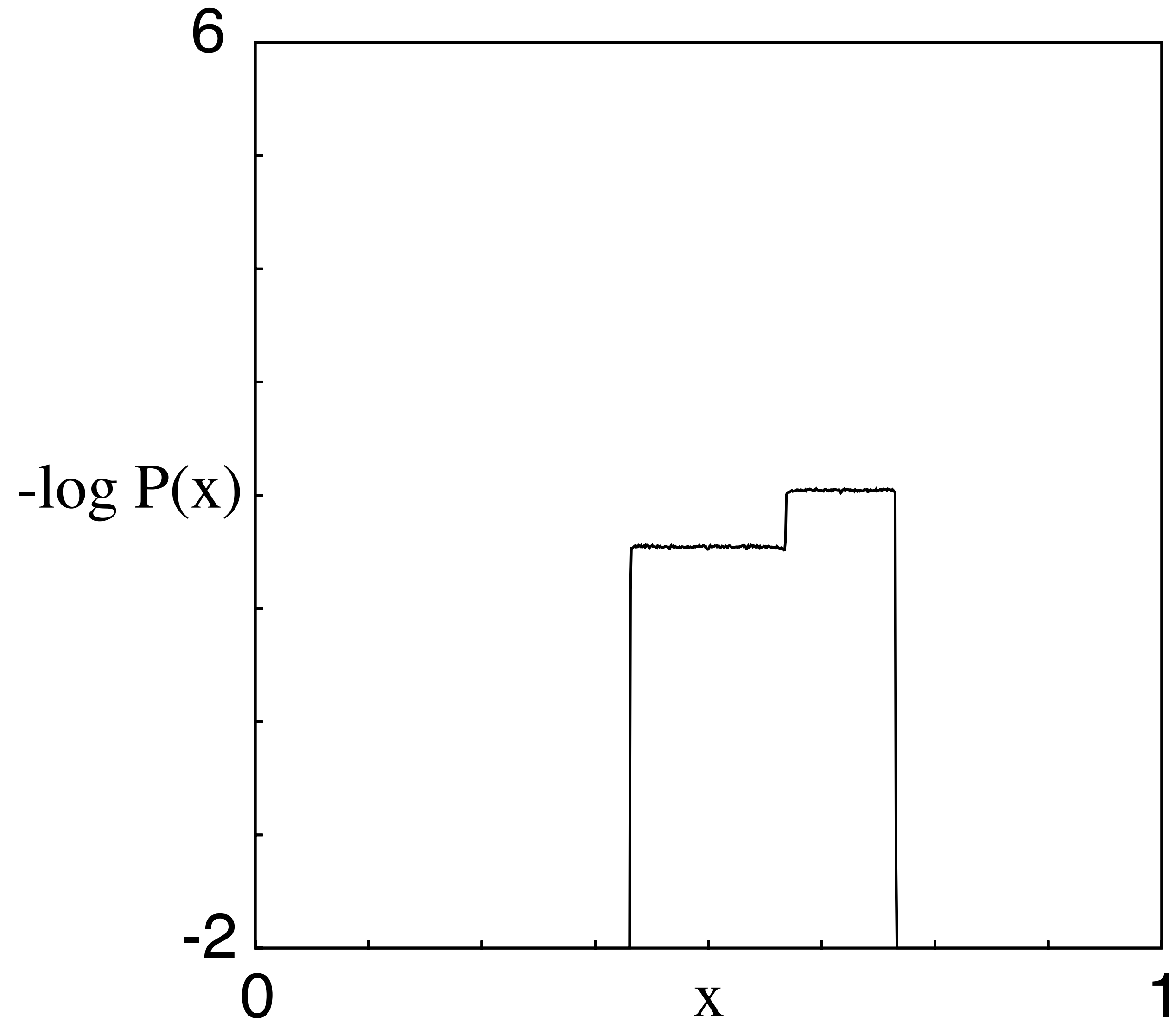

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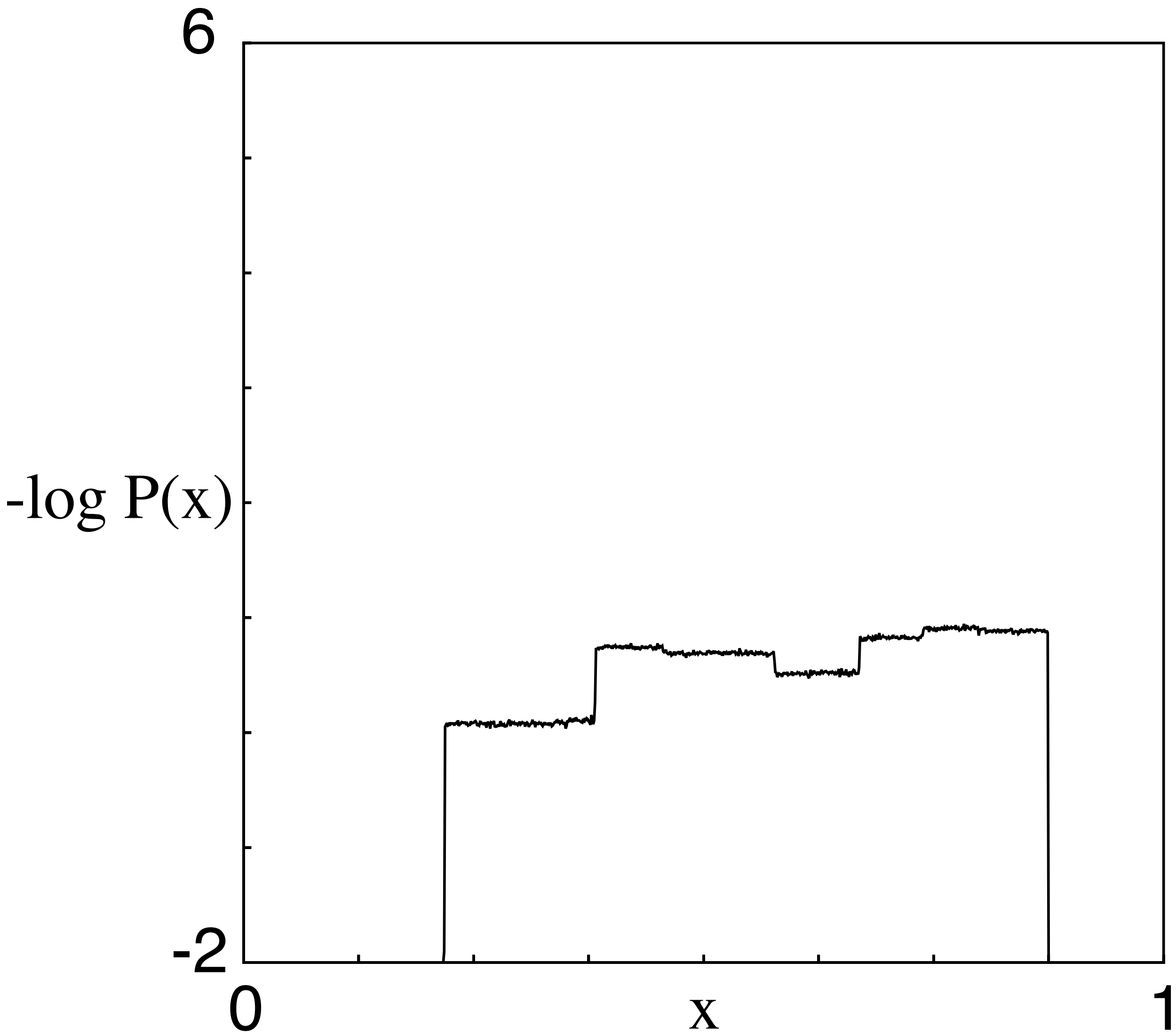
Probability Theory of Dynamical Systems ...

Numerical Example: Tent map

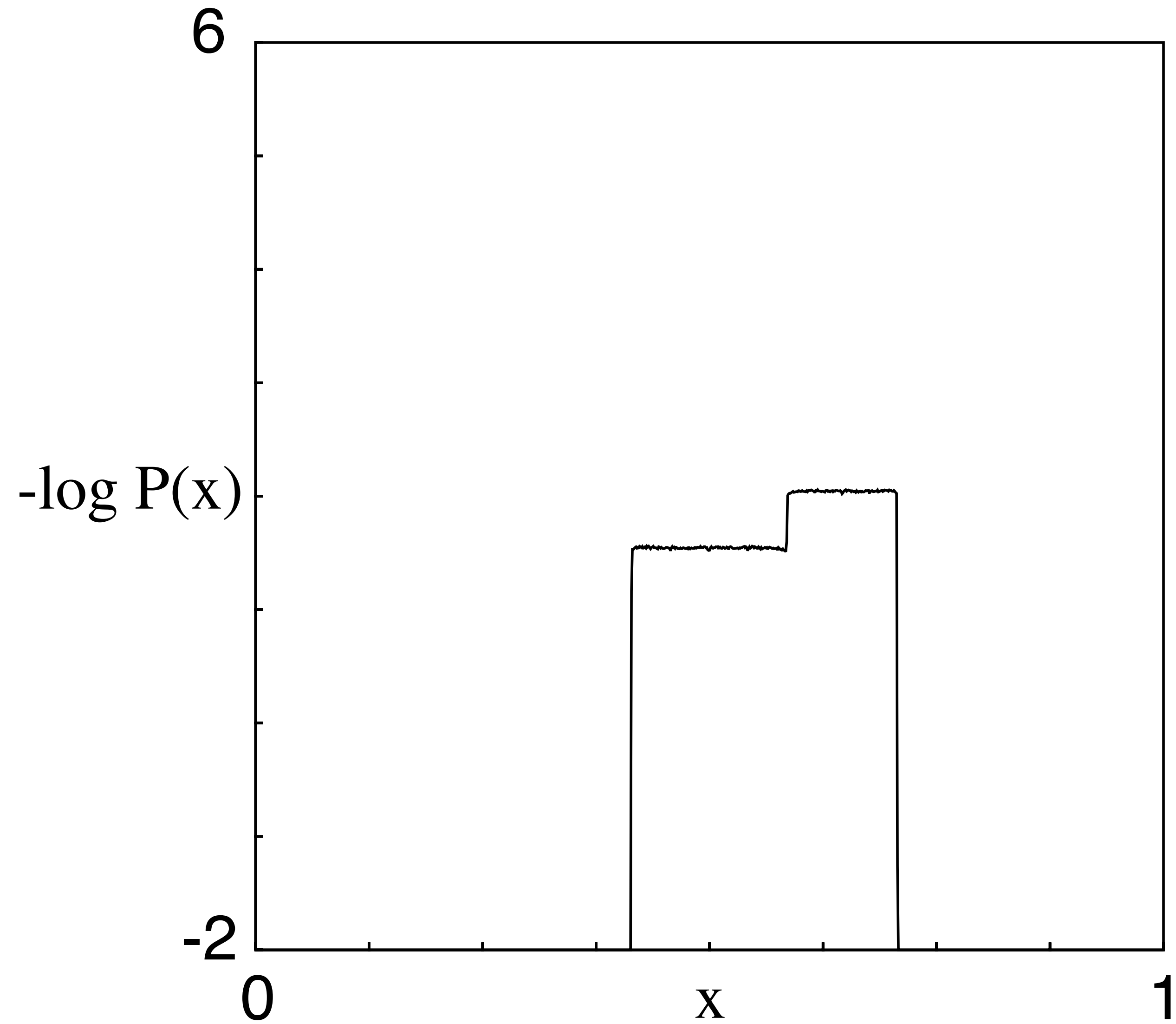
Typical chaotic parameter:

$$a = 1.75$$

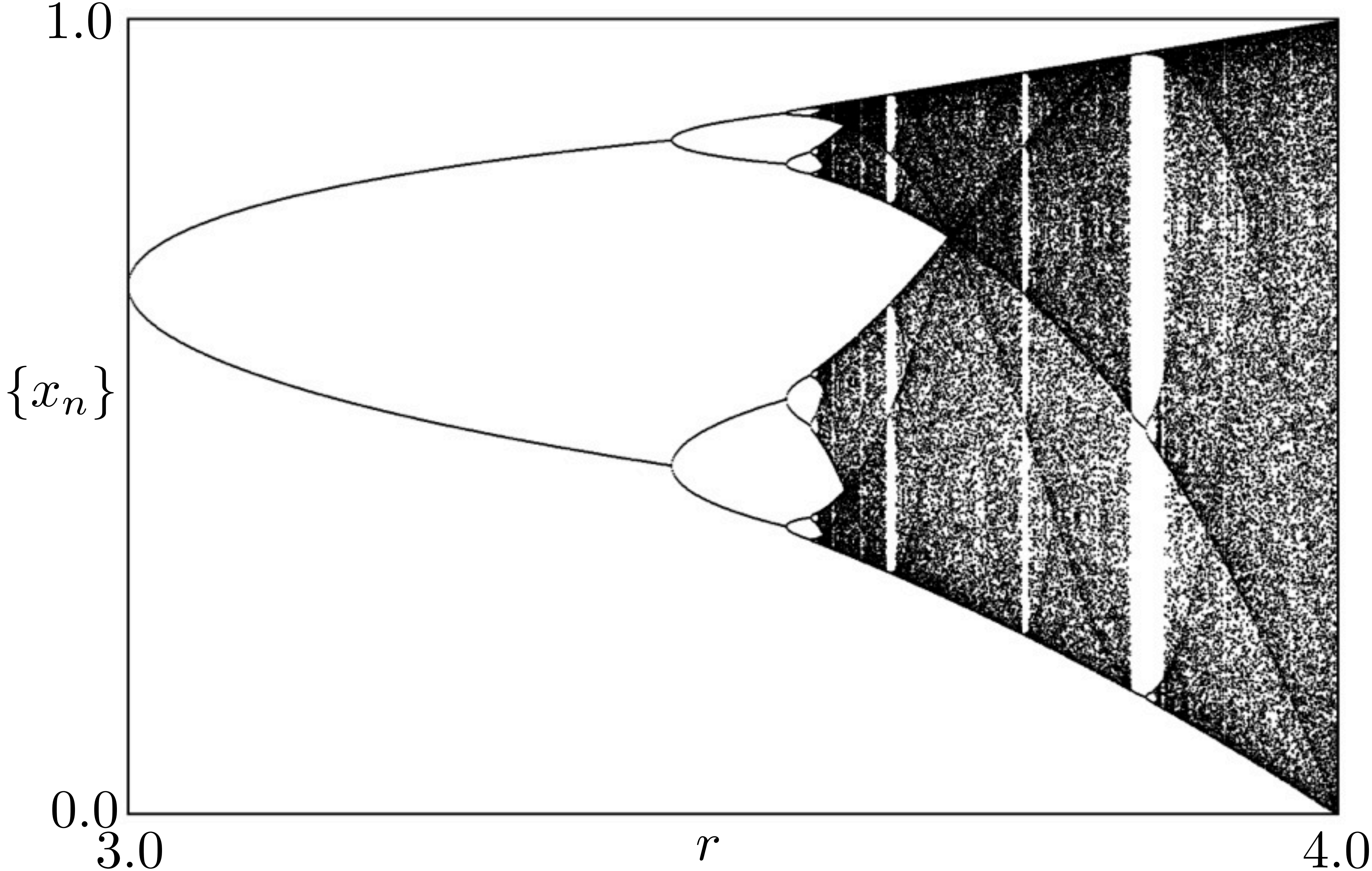


Two bands merge to one:

$$a = \sqrt{2}$$



From Determinism to Stochasticity ...
Probability Theory of Dynamical Systems ...
Numerical Example: Logistic map



From Determinism to Stochasticity ...

Probability Theory of Dynamical Systems ...

Numerical Example: Logistic map $x_{n+1} = r x_n (1 - x_n)$

Typical chaotic parameter:
$r = 3.7$

Two bands merge to one:
$r = 3.678573510428321 9$

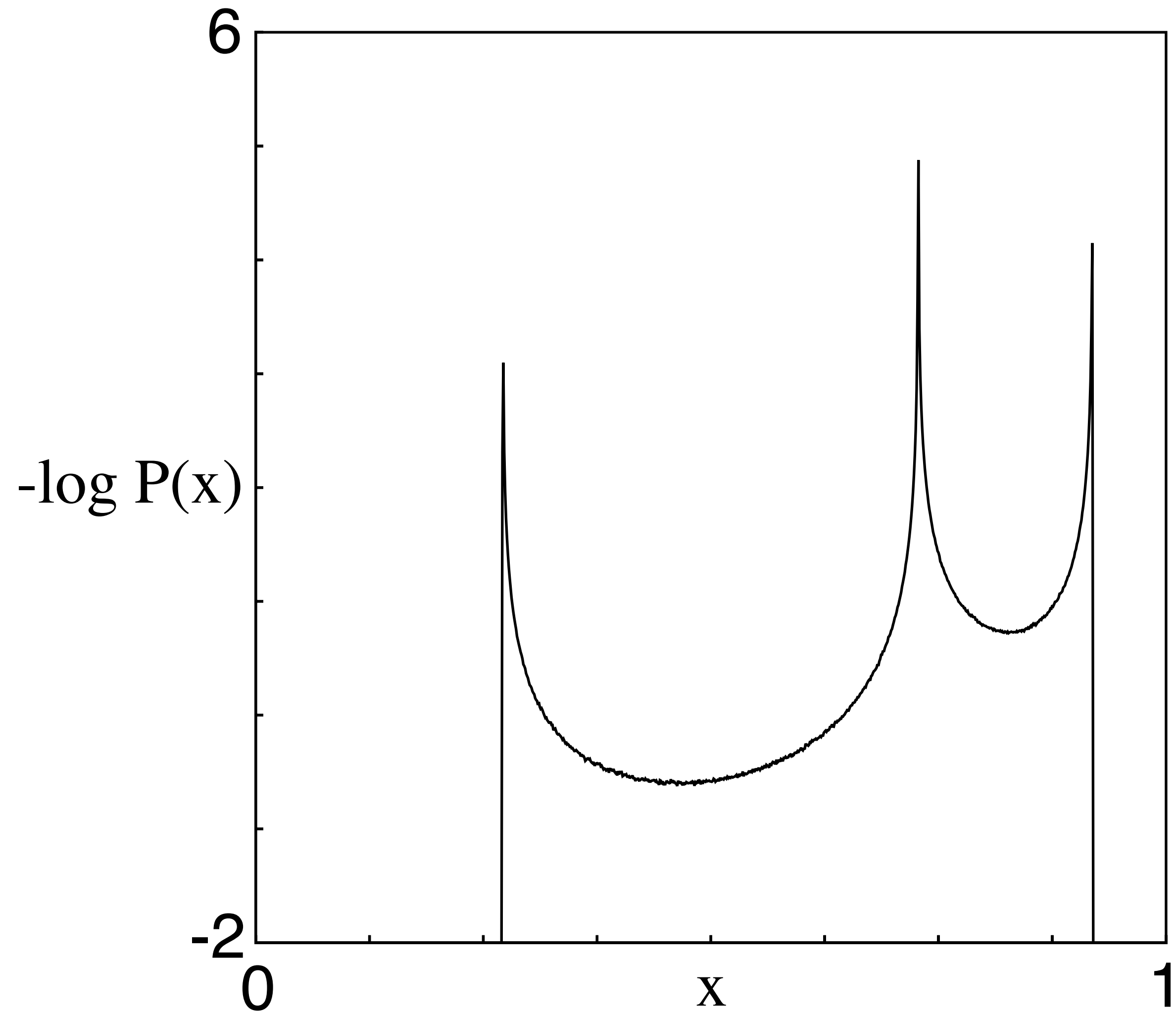
From Determinism to Stochasticity ...

Probability Theory of Dynamical Systems ...

Numerical Example: Logistic map $x_{n+1} = r x_n (1 - x_n)$

Typical chaotic parameter:
$r = 3.7$

Two bands merge to one:
$r = 3.6785735104283219$



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Probability Theory of Dynamical Systems ...

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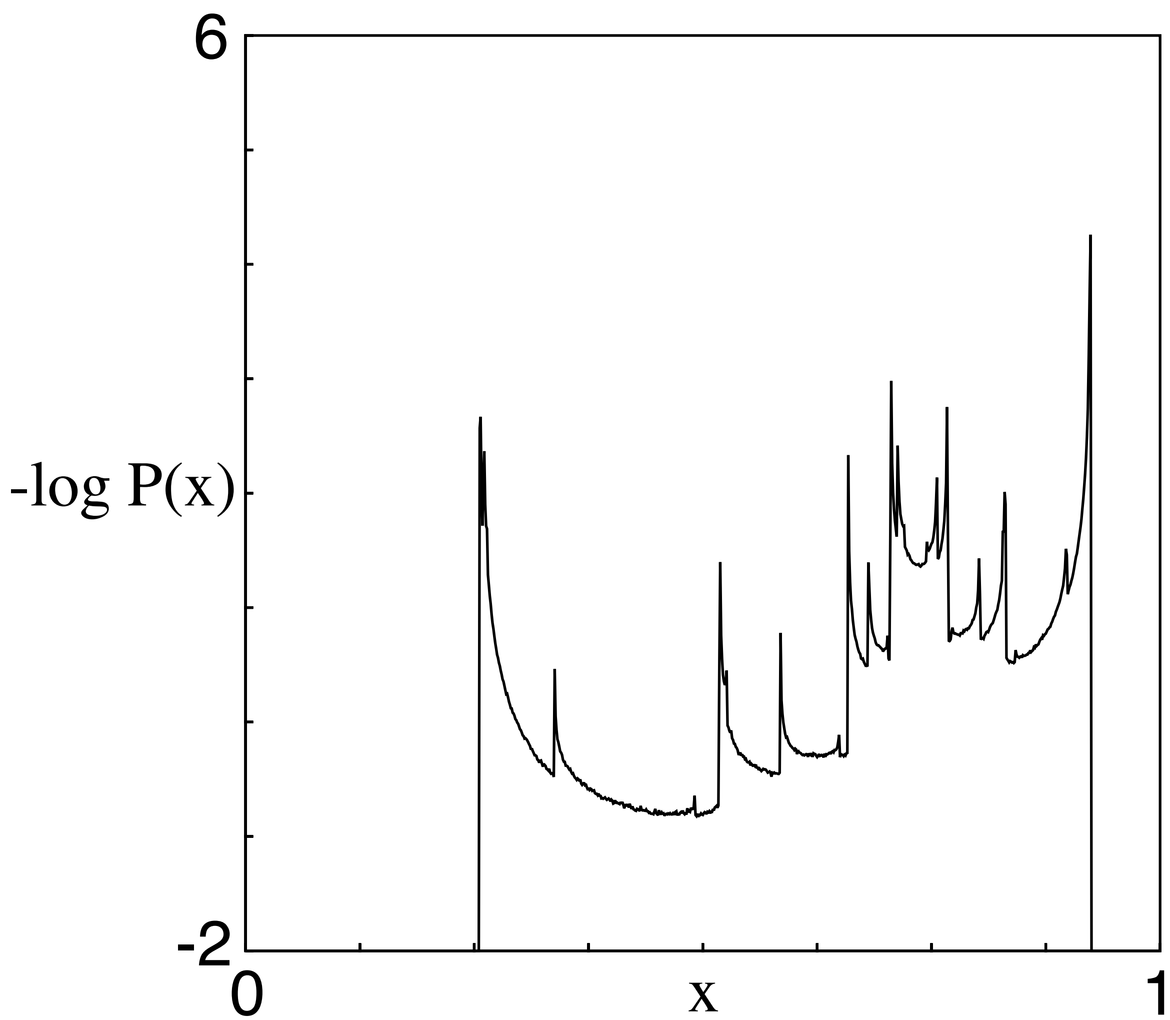
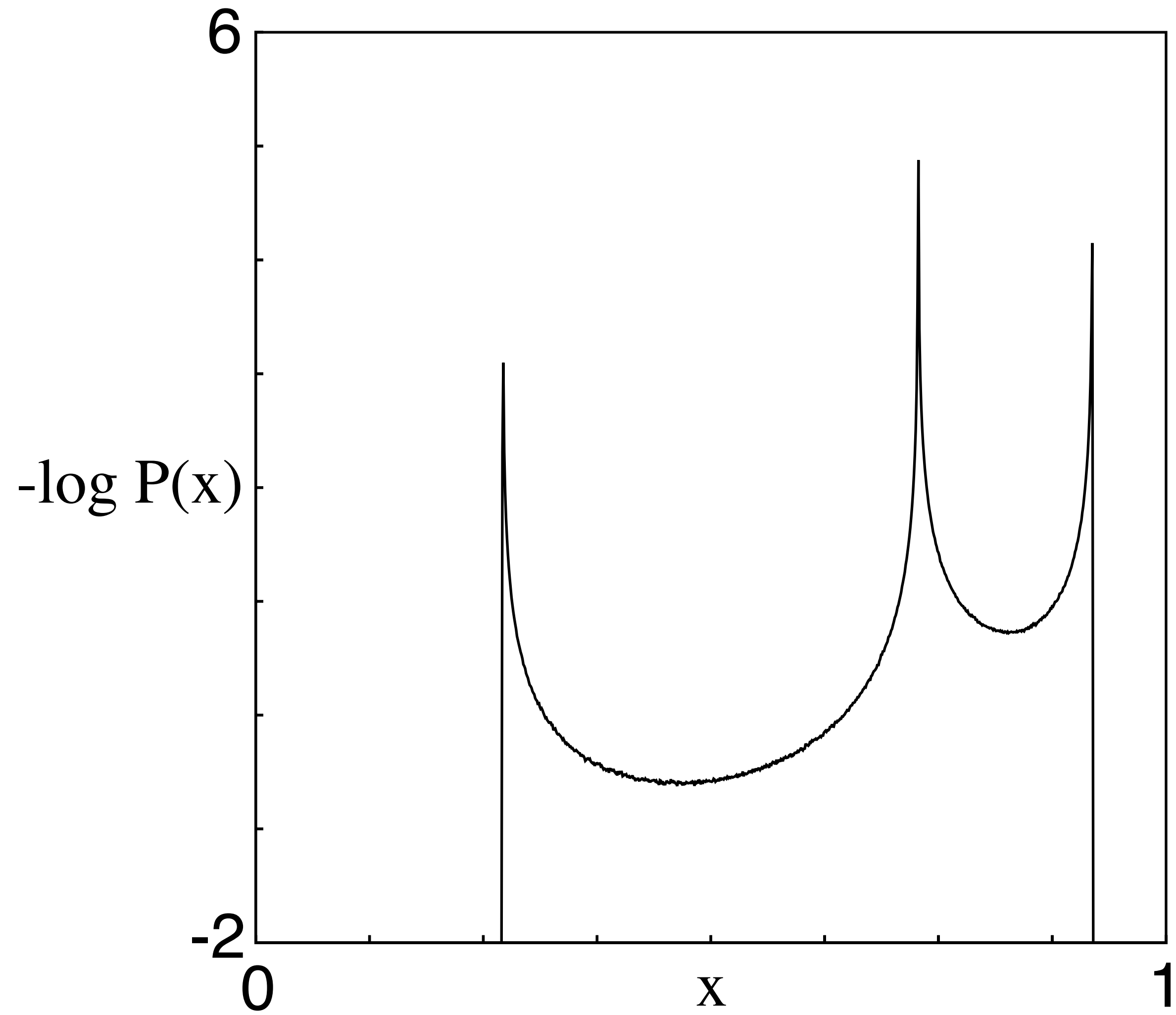


Two bands merge to one: $r = 3.678573510428321$9

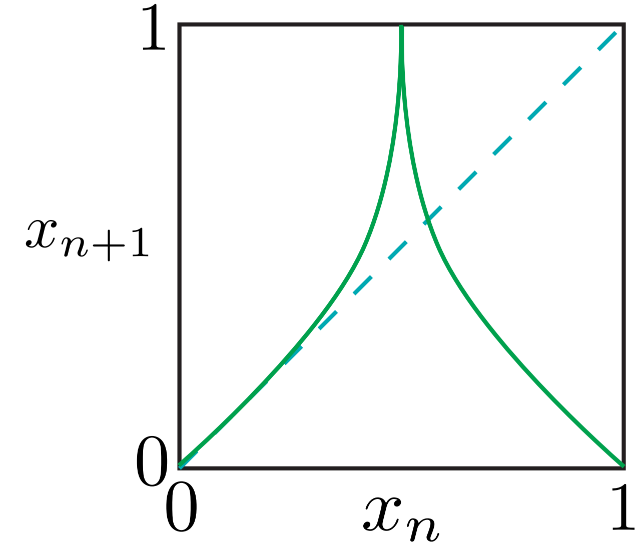


From Determinism to Stochasticity ...

Probability Theory of Dynamical Systems ...

Numerical Example: Cusp map $x_{n+1} = a(1 - |1 - 2x_n|^b)$

$(a, b) = (1, 1/2)$



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Probability Theory of Dynamical Systems ...

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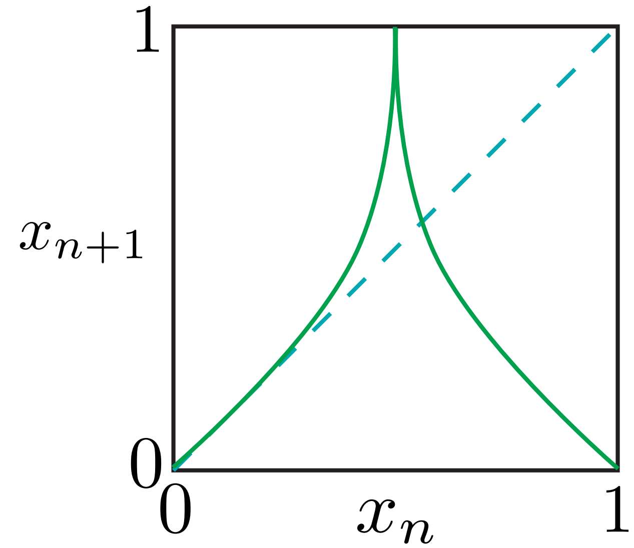

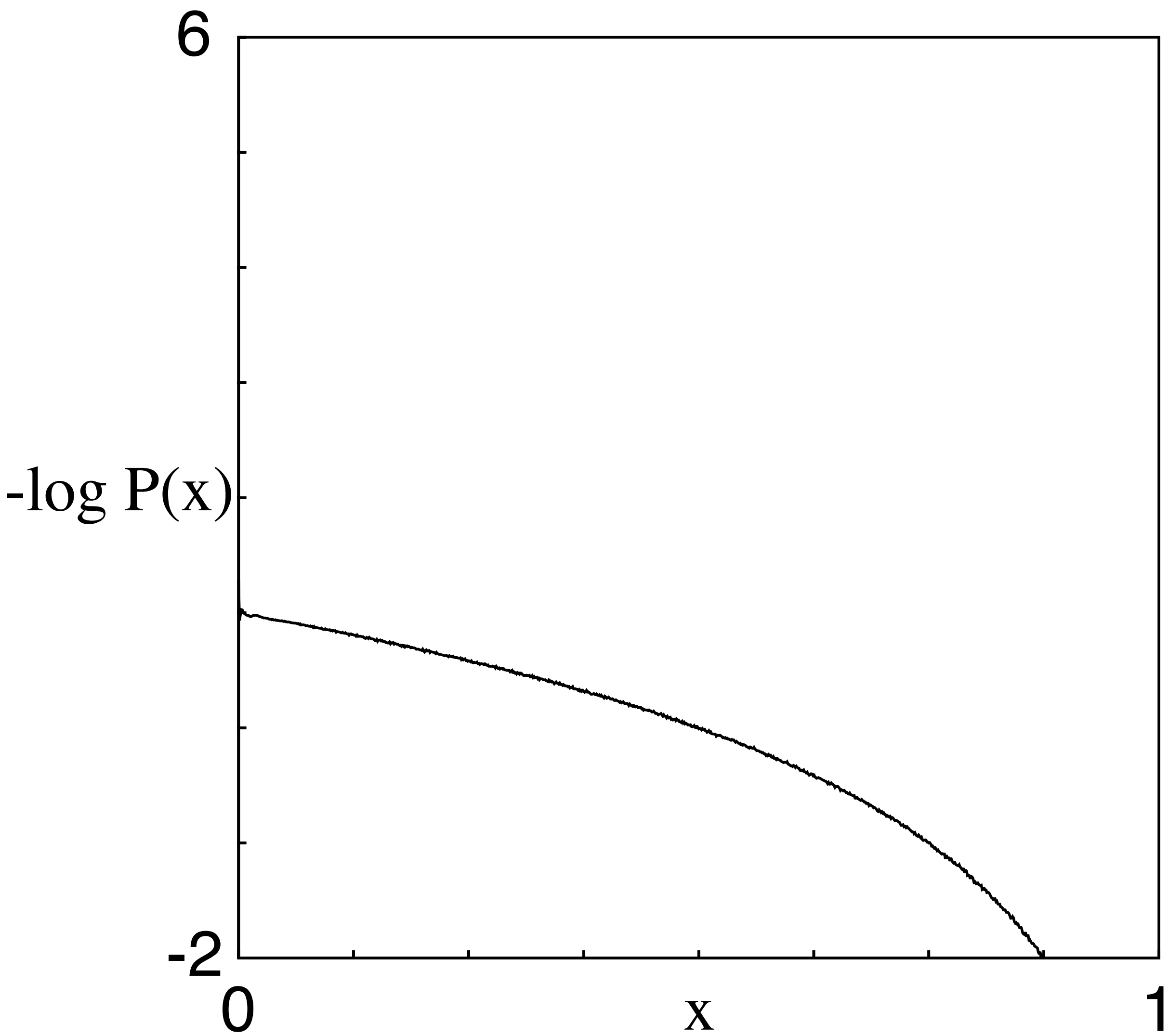

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End of Probability Theory of Dynamical Systems.

Now:

How is a chaotic dynamical system like a stochastic process?

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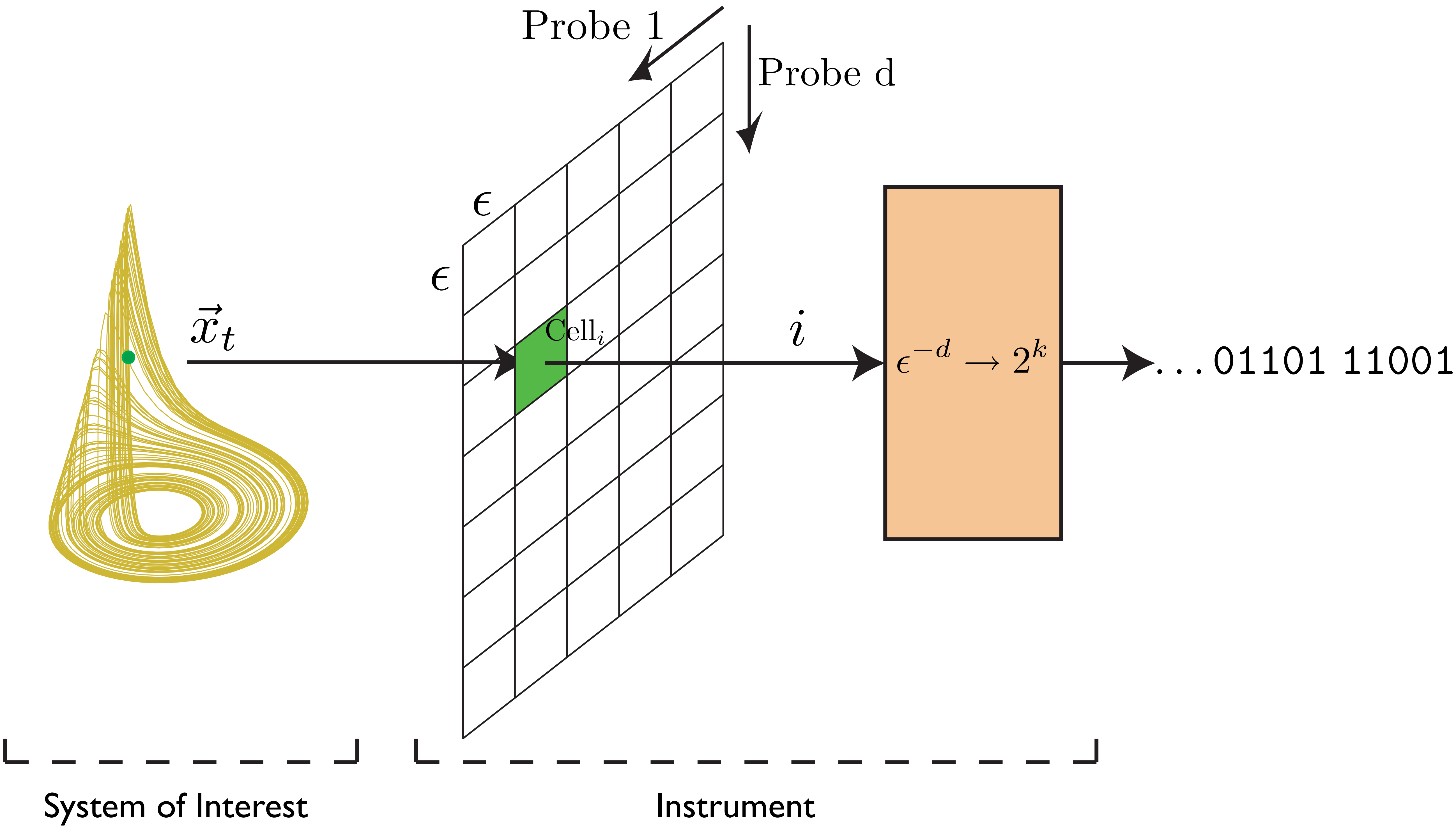


Measurement Channel

Measurement Theory: Making the connection

Hidden Dynamical System:

What can we learn from discrete time series?

Know how to evolve:

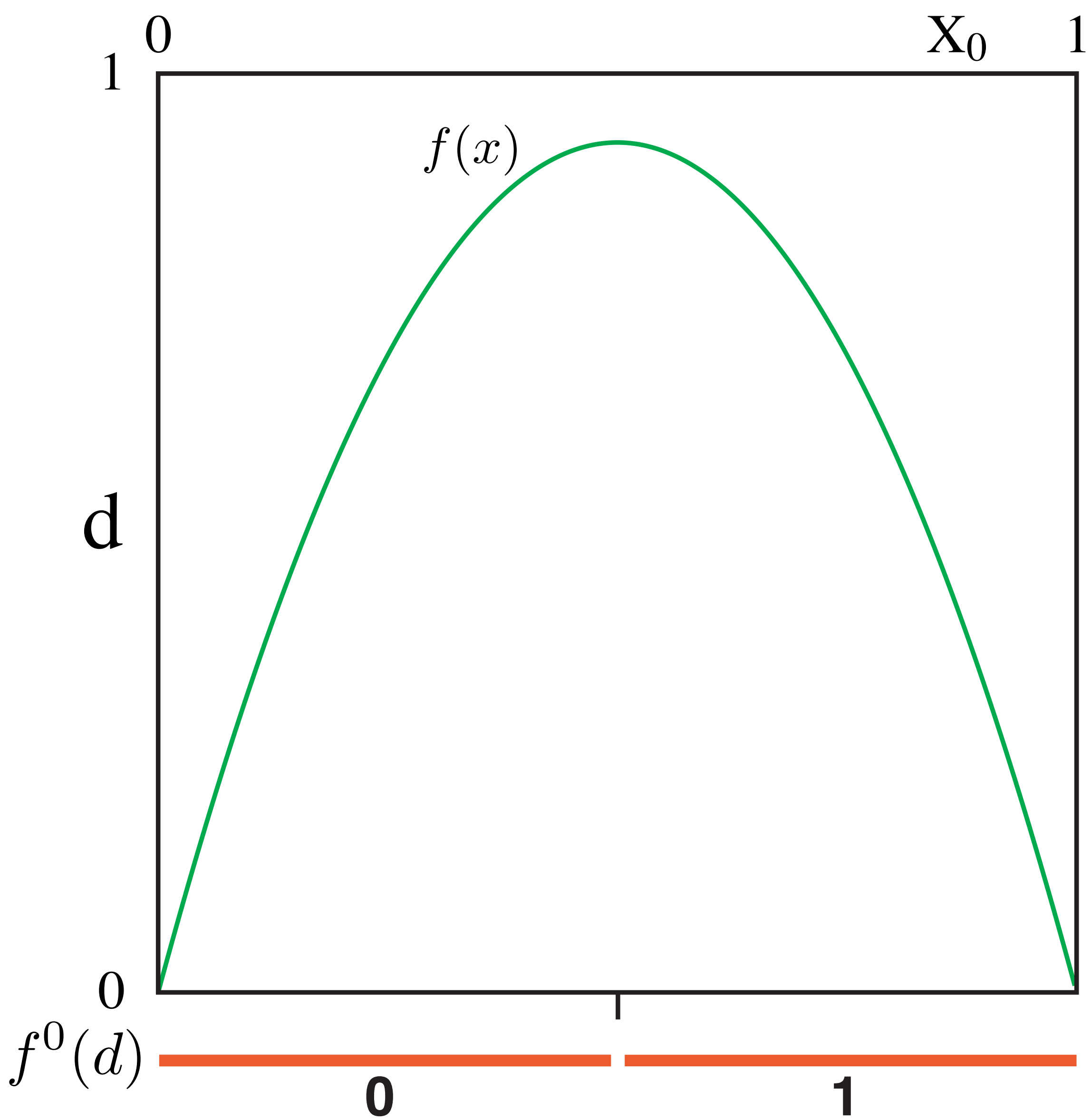
$$\mathcal{T} : \vec{x}_0 \rightarrow \vec{x}_1$$

$$\mathcal{T} : p_0(x) \rightarrow p_1(x)$$

How to evolve boundaries?

From Determinism to Stochasticity ...

Measurement Theory of 1D Maps ...



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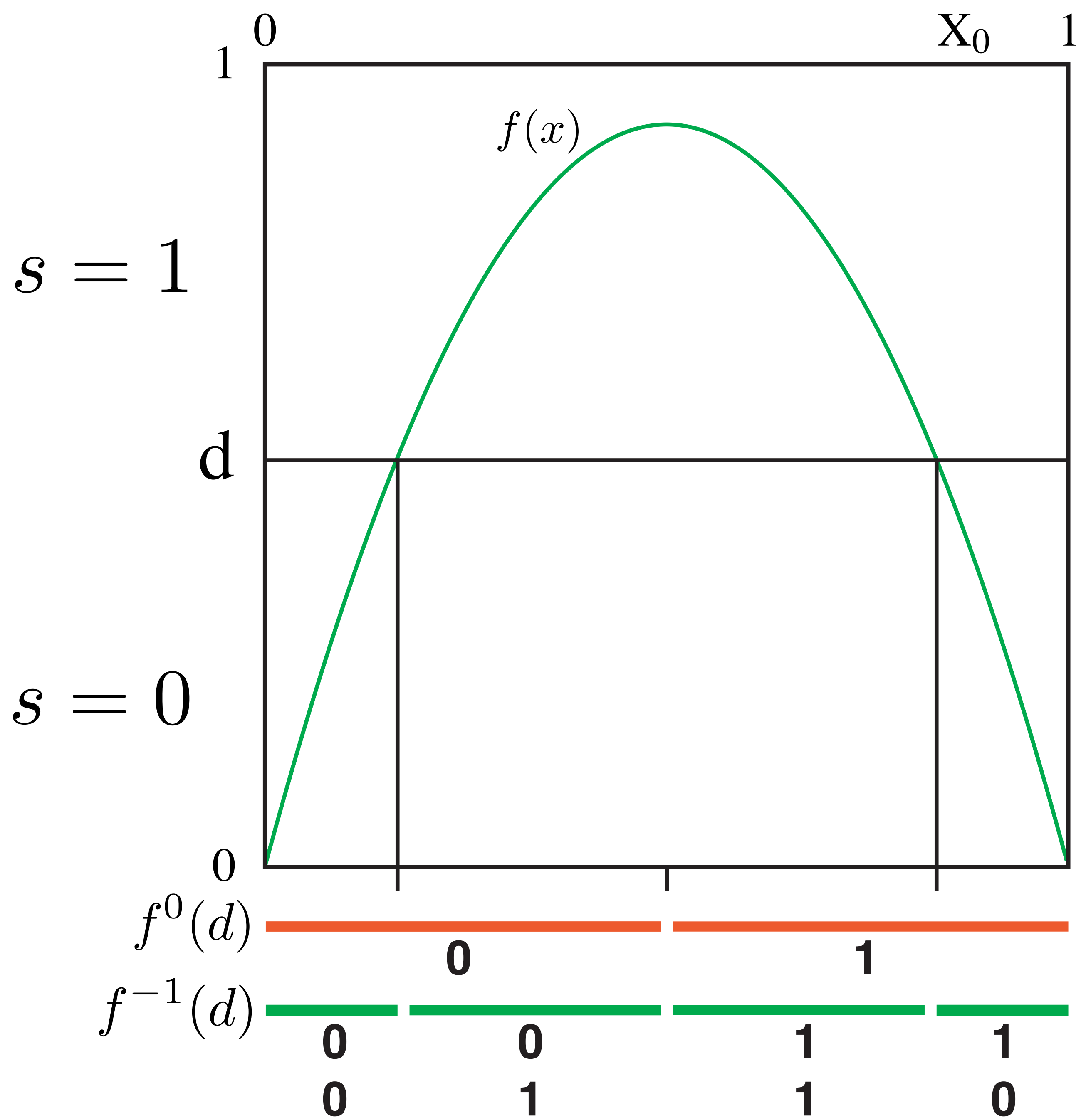
Measurement Theory of 1D Maps ...



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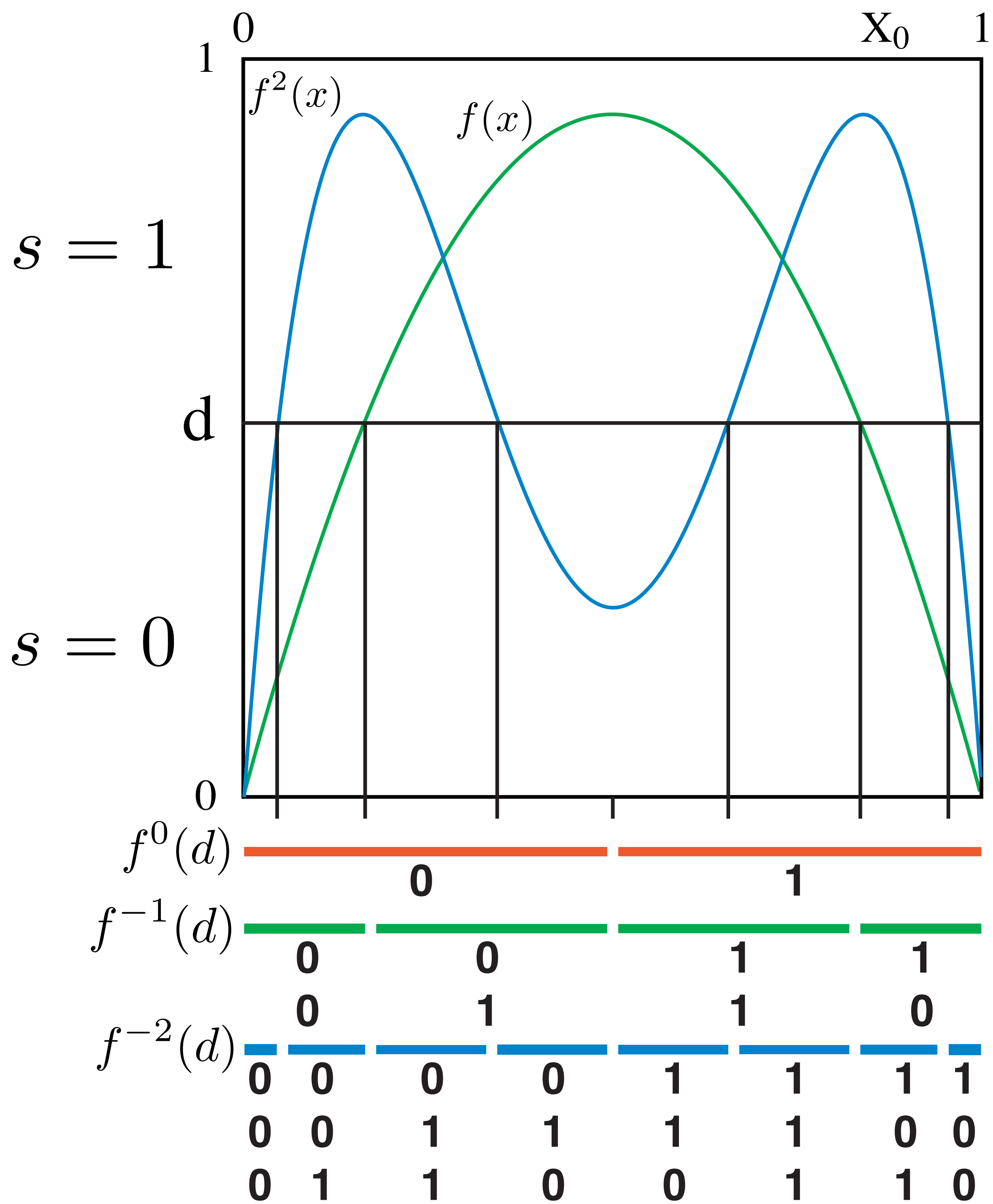
Measurement Theory of 1D Maps ...



Measurement Theory ...

Symbolic dynamics:

1. Replace complicated dynamic (f) with trivial dynamic (shift)

2. Replace infinitely precise point $x \in M$
   with discrete infinite sequence $\mathbf{s} \in \Sigma_f$

3. If the partition is "good" then
   a. Study discrete sequences to learn about continuous system
   b. Can often calculate quantities directly

From Determinism to Stochasticity ...

Kinds of Instruments:

When are partitions good?

When symbol sequences encode orbits

$$\begin{tikzcd} M \arrow[r, "\mathcal{T}"] & M \\ \mathcal{A}^{\mathbb{Z}} \arrow[u, "\Delta"] \arrow[r, "\sigma"] & \mathcal{A}^{\mathbb{Z}} \arrow[u, "\Delta"'] \end{tikzcd}$$

Diagram commutes:

$$\mathcal{T}(x) = \Delta \circ \sigma \circ \Delta^{-1}(x)$$

Good kinds of instruments:

Markov partitions

Generating partitions

Measurement Theory ...

Markov Partitions for 1D Maps:

Discrete symbol sequences: $\overleftrightarrow{s} = \overleftarrow{s}\,\overrightarrow{s},\ s \in \mathcal{A}$

Markov = Given symbol, ignore history

$$\Pr(\overrightarrow{s} \,|\overleftarrow{s}) = \Pr(\overrightarrow{s} \,|s_1)$$

Maps of the interval: $f : I \to I,\ I = [0, 1]$

Partition: $\mathcal{P} = \{P_1, \ldots, P_p\}$

Open sets: $P_i = (d_{i-1}, d_i),\ 0 = d_0 < d_1 < \cdots < d_p = 1$

$$I = \bigcup_{i=1}^{p} \bar{P}_i$$

$\mathcal{P}$ is a Markov partition for f :

$$f(P_i) = \bigcup_j P_j \, , \forall i$$

$f(P_i)$ is 1-to-1 and onto (homeomorphism)

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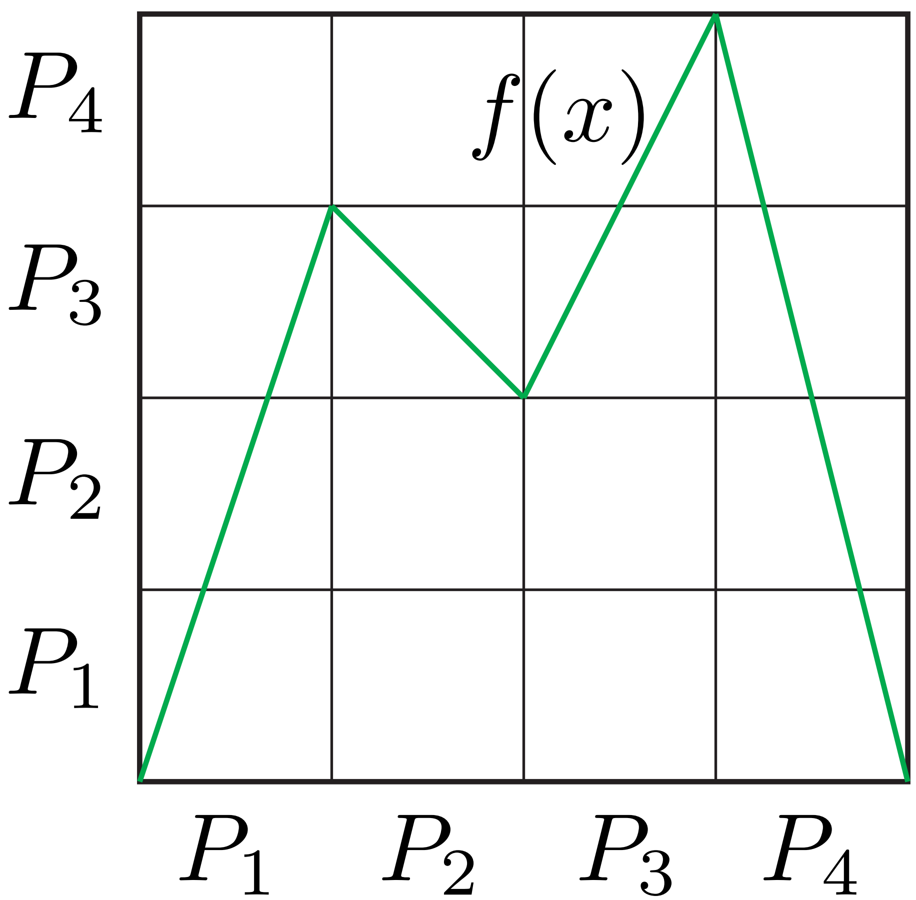
Measurement Theory ...
Markov Partitions for 1D Maps ...

$$s \in \mathcal{A} = \{1, 2, 3, 4\}$$



$$f(P_1) = P_1 \bigcup P_2 \bigcup P_3$$

$$f(P_2) = P_3$$

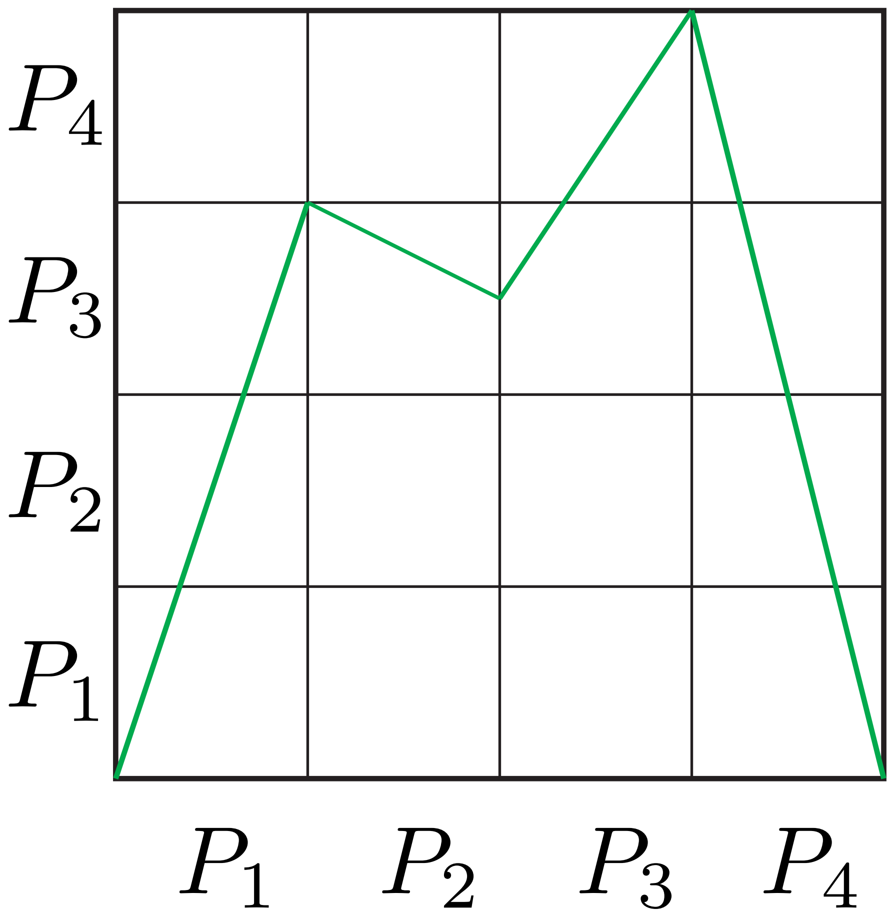
$$f(P_3) = P_3 \bigcup P_4$$

$$f(P_4) = P_1 \bigcup P_2 \bigcup P_3 \bigcup P_4$$

Markov!

$\Rightarrow \overrightarrow{s}, \ s \in \mathcal{A}$ Good coding

Markov Partitions for 1D Maps ...



$$f(P_2) \subset P_3$$

$$f(P_2) \neq \bigcup_i P_i$$

$$f(P_3) \subset P_3 \bigcup P_4$$

$$f(P_3) \neq \bigcup_i P_i$$

Not Markov!

$$\Rightarrow \overrightarrow{s}, \ s \in \mathcal{A}$$ Bad coding

From Determinism to Stochasticity ...

Measurement Theory ...

Why Markov Partition?

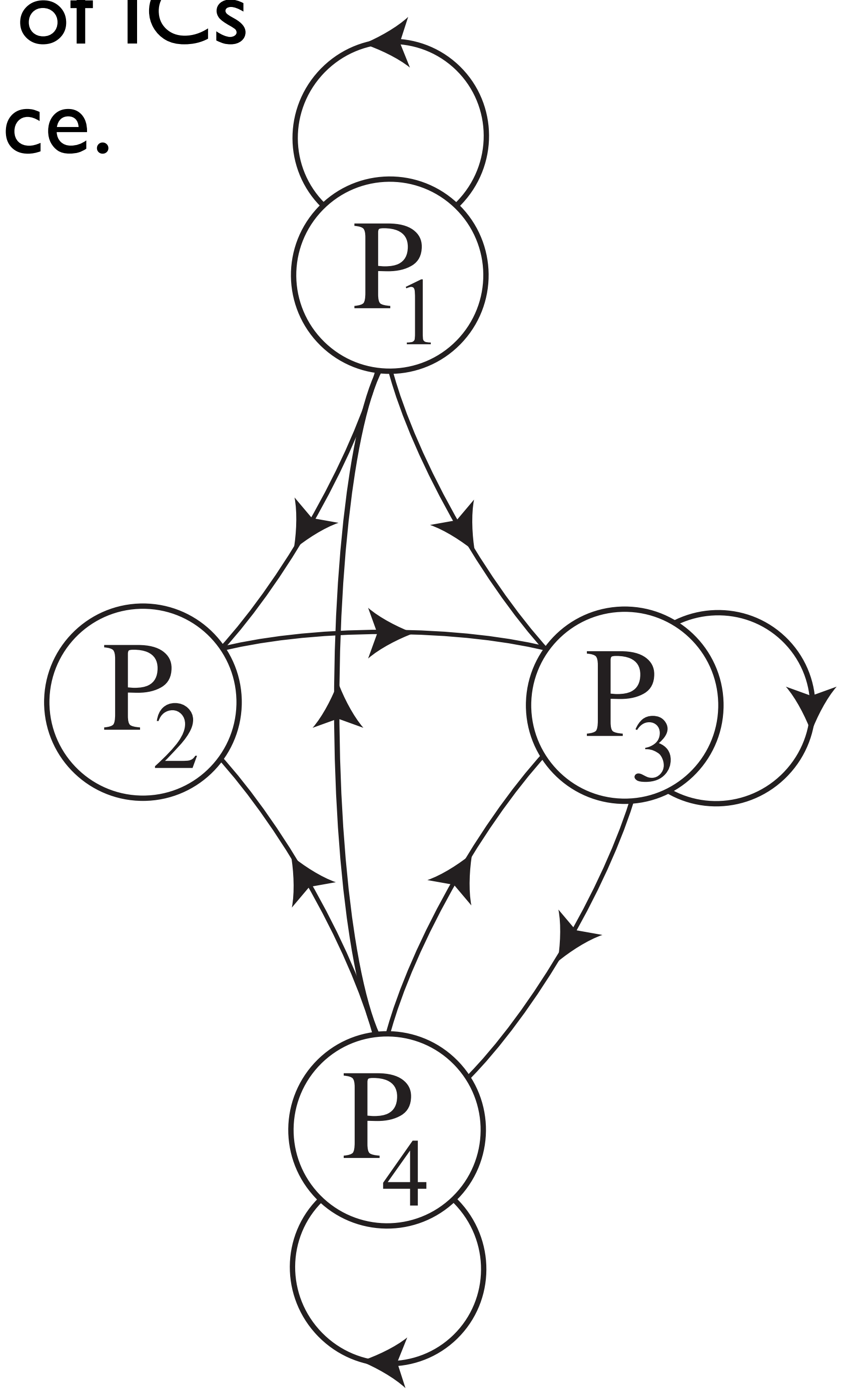
Symbol sequences track orbits:

Longer the sequence, the smaller the set of ICs that could have generated that sequence.

$$\lim_{L \to \infty} ||\Delta(\mathbf{s}^L)|| \to 0$$

Markov Partition is stronger:

Summarize map with a Markov chain over the partition elements.



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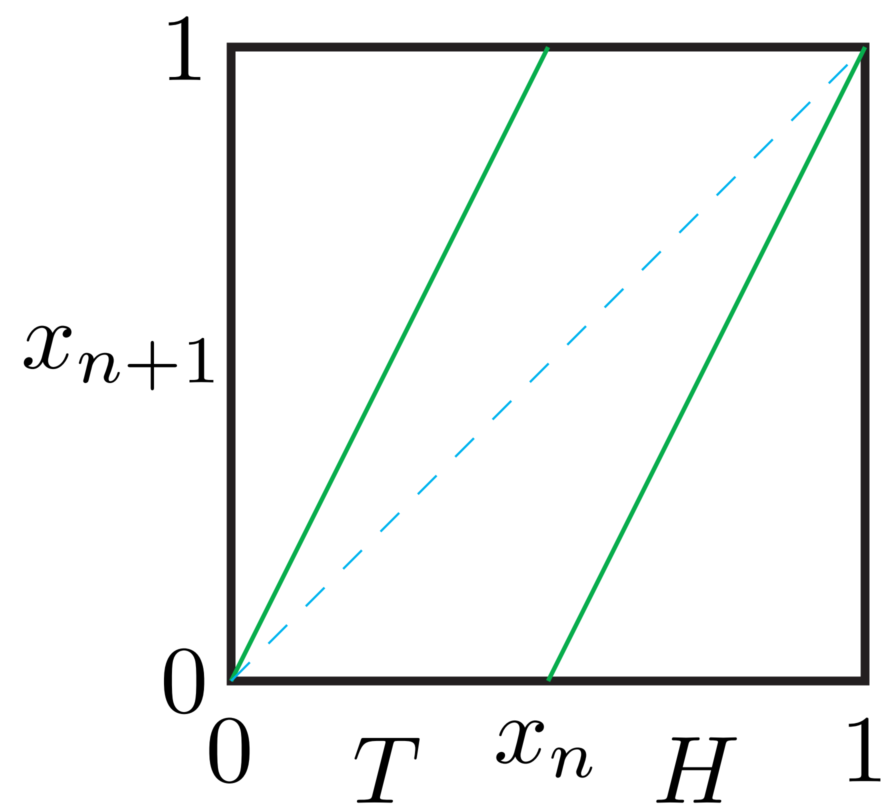
Measurement Theory ...

Markov partition for Shift map:

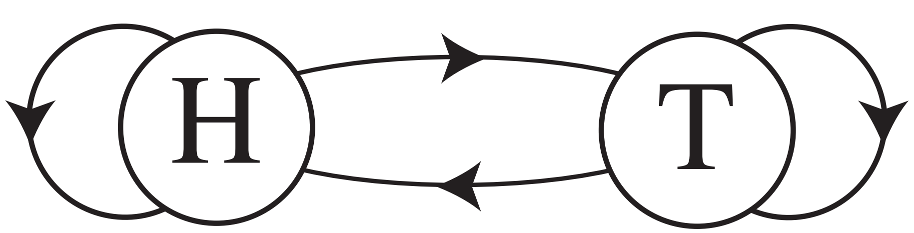


$$\mathcal{P} = \{T \sim (0, \tfrac{1}{2}), H \sim (\tfrac{1}{2}, 1)\}$$

$$f(P_T) = P_T \bigcup P_H \ \& \ f|_{P_T} \text{ is monotone}$$

$$f(P_H) = P_T \bigcup P_H \ \& \ f|_{P_H} \text{ is monotone}$$

Associated (topological) Markov chain:

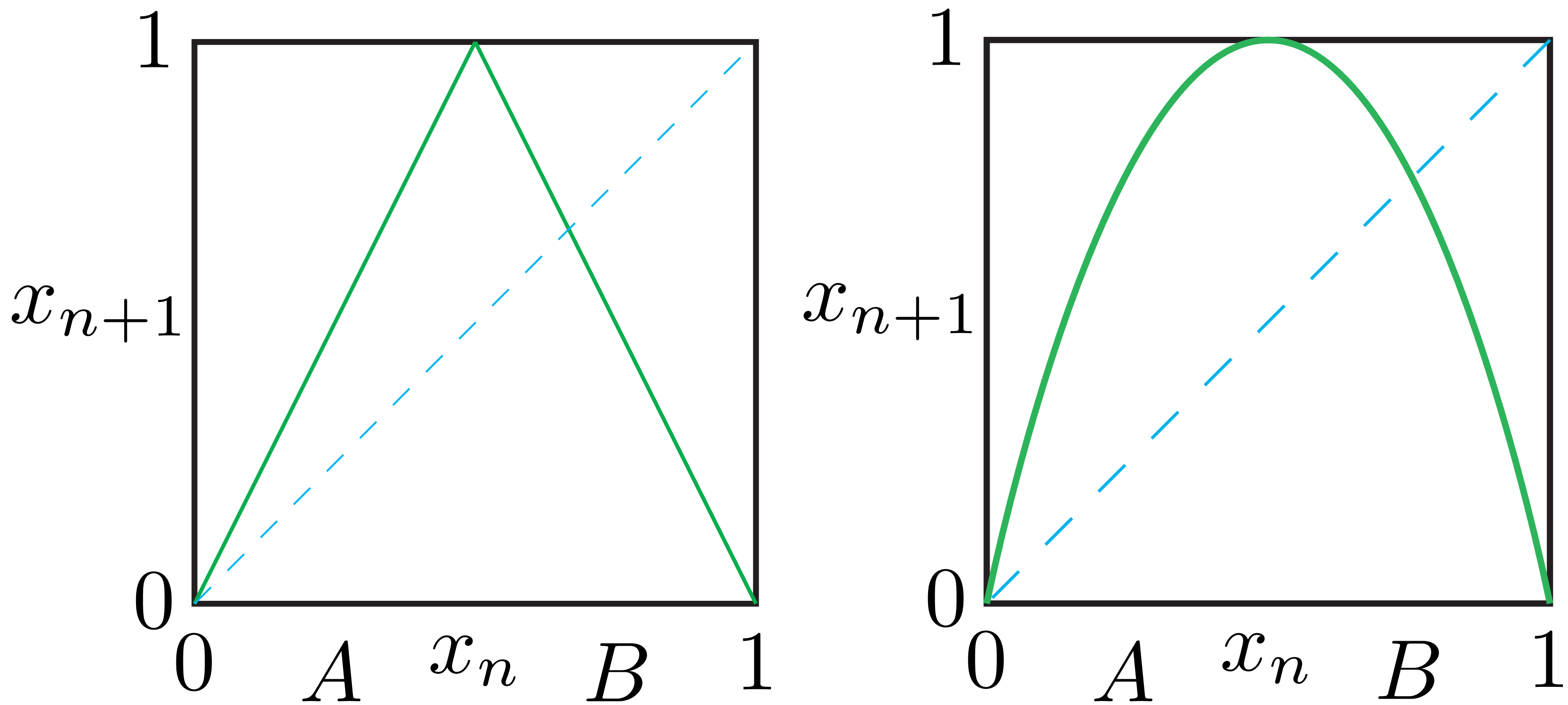


From Determinism to Stochasticity ...

Measurement Theory ...

Markov partition for Tent and Logistic maps (Two-onto-One):

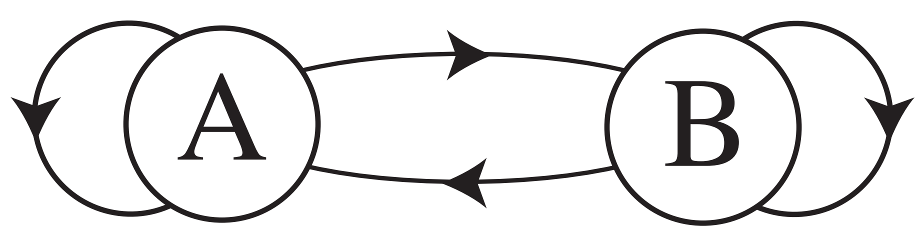
$$\mathcal{P} = \{A \sim (0, \tfrac{1}{2}), B \sim (\tfrac{1}{2}, 1)\}$$



$$f(P_A) = P_A \bigcup P_B \;\&\; f|_{P_A} \text{ is monotone}$$

$$f(P_B) = P_A \bigcup P_B \;\&\; f|_{P_B} \text{ is monotone}$$

Associated Markov chain:



Notice what is thrown away

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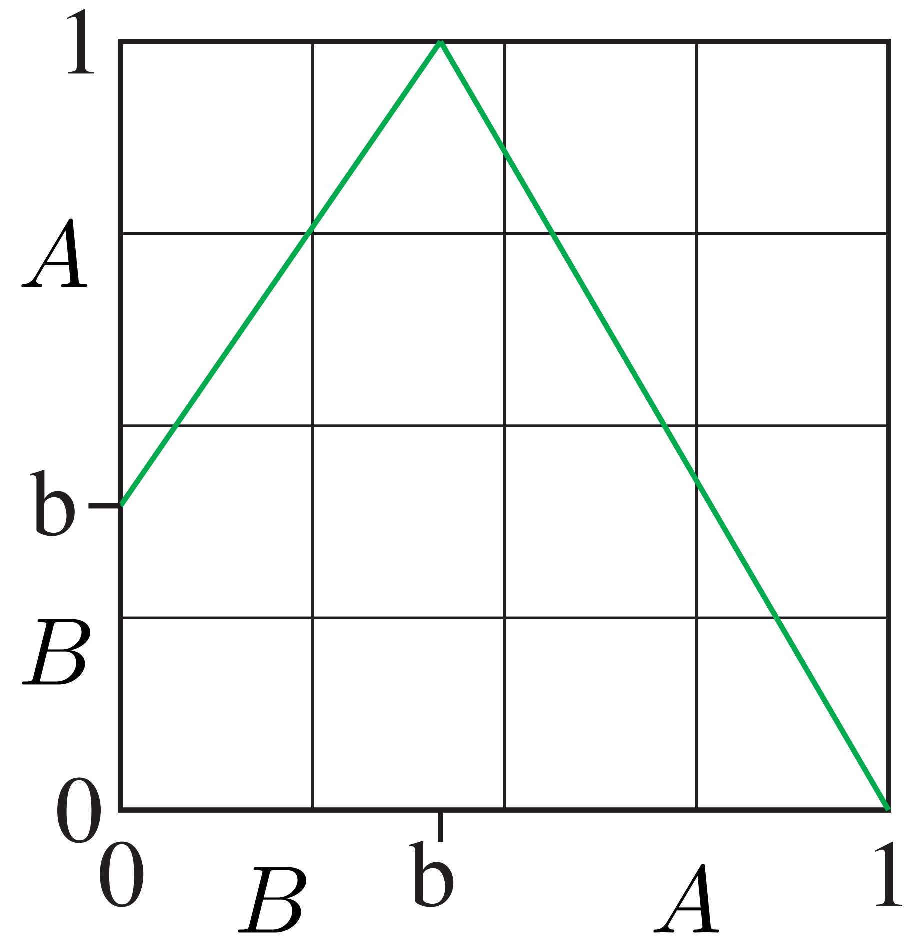
Measurement Theory ...

Markov partition for Golden Mean map:

$$x_{n+1} = \begin{cases} \phi x_n + b & 0 \leq x_n \leq b \\ (x_n - 1)/(b-1) & b < x_n \leq 1 \end{cases}$$



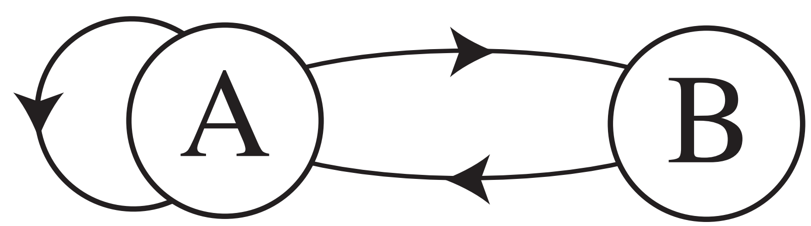
$$\phi = \frac{1+\sqrt{5}}{2} \qquad b = \frac{1}{1+\phi}$$

$$\mathcal{P} = \{B \sim (0,b), A \sim (b,1)\}$$

$$f(P_B) = P_A \ \& \ f|_{P_B} \text{ is monotone}$$

$$f(P_A) = P_B \bigcup P_A \ \& \ f|_{P_A} \text{ is monotone}$$

Markov chain is Golden Mean Process:



From Determinism to Stochasticity ...

End

(Faithful-) Measurement Theory:

One way to see how a chaotic dynamical system produces a stochastic process.

Next lecture:

1. More flexible instruments (than Markov partitions)
2. When measurements mislead
2. General stochastic processes