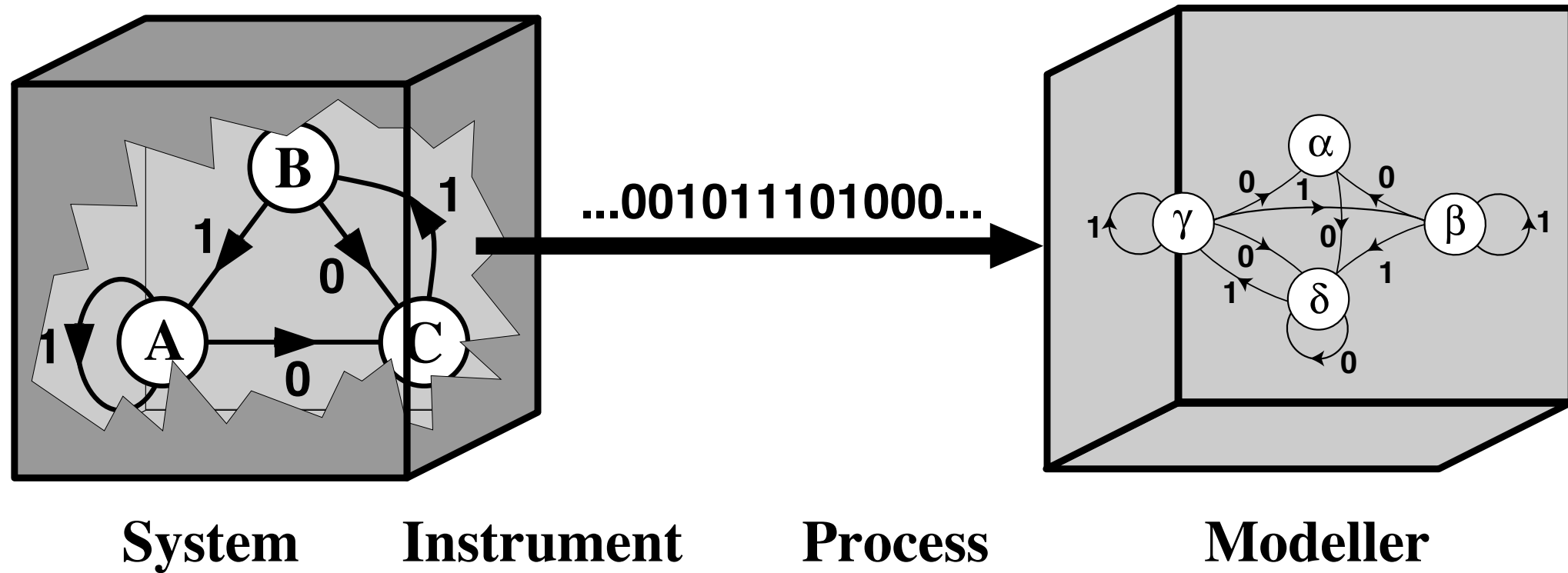


Complexity

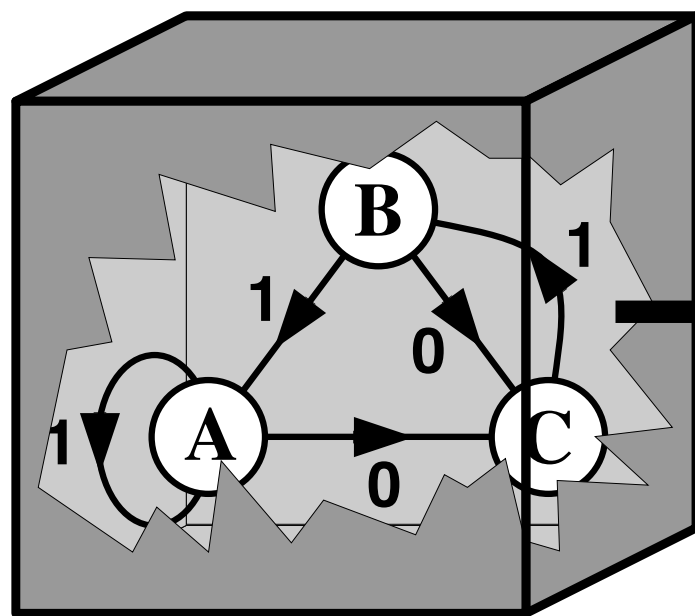
Jim Crutchfield
Complexity Sciences Center
Physics Department
University of California at Davis

Complex Systems Summer School
Santa Fe Institute
St. John's College, Santa Fe, NM
15 June 2016



The Learning Channel

Previous

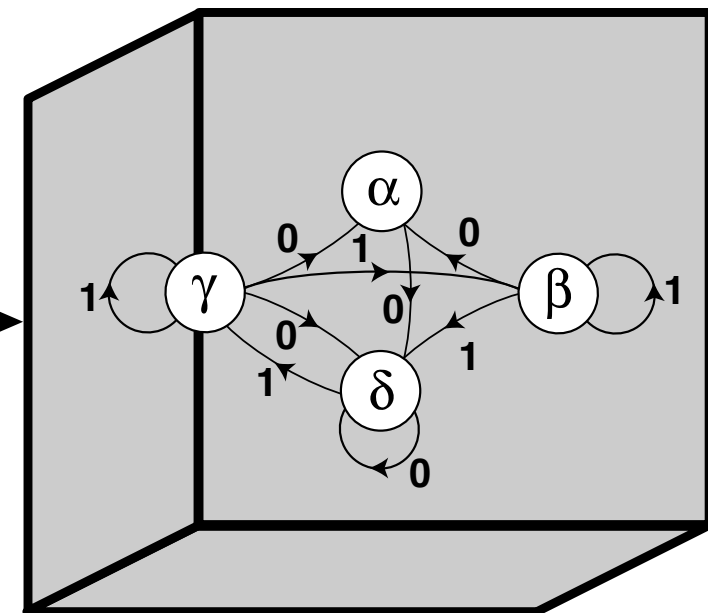


System

...001011101000...

Instrument

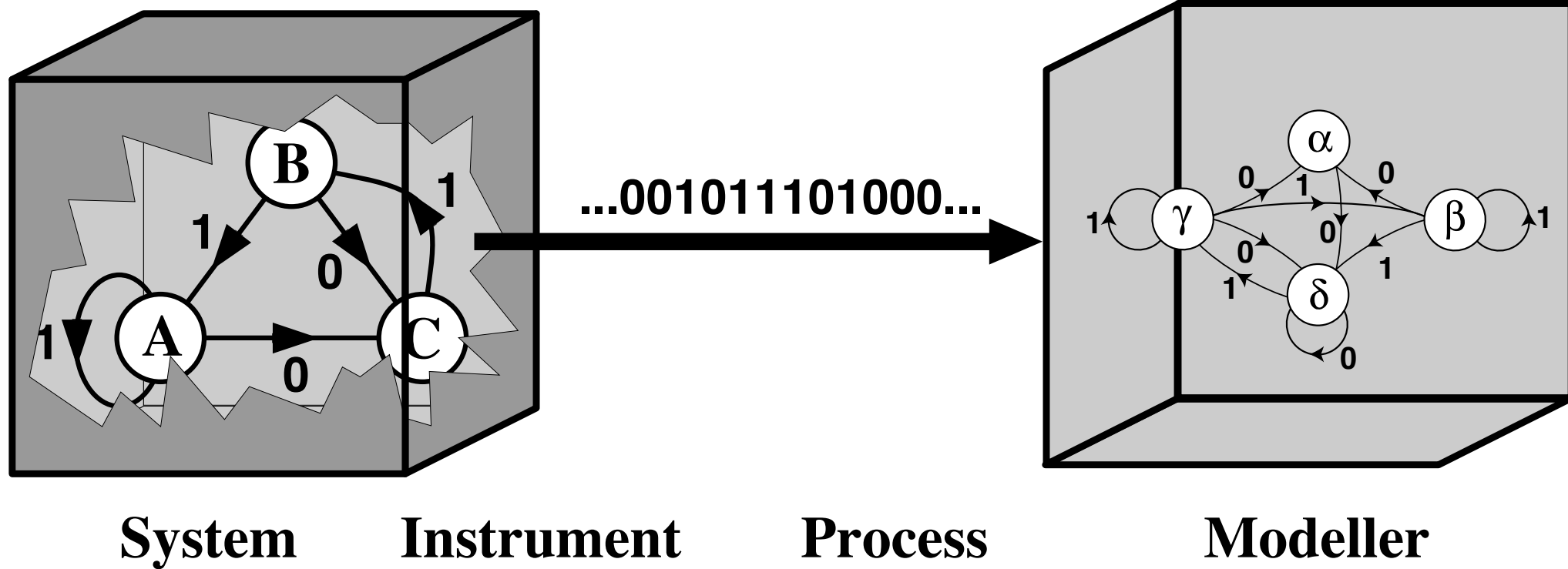
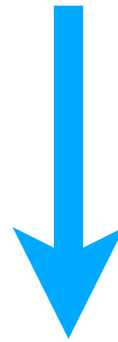
Process



Modeller

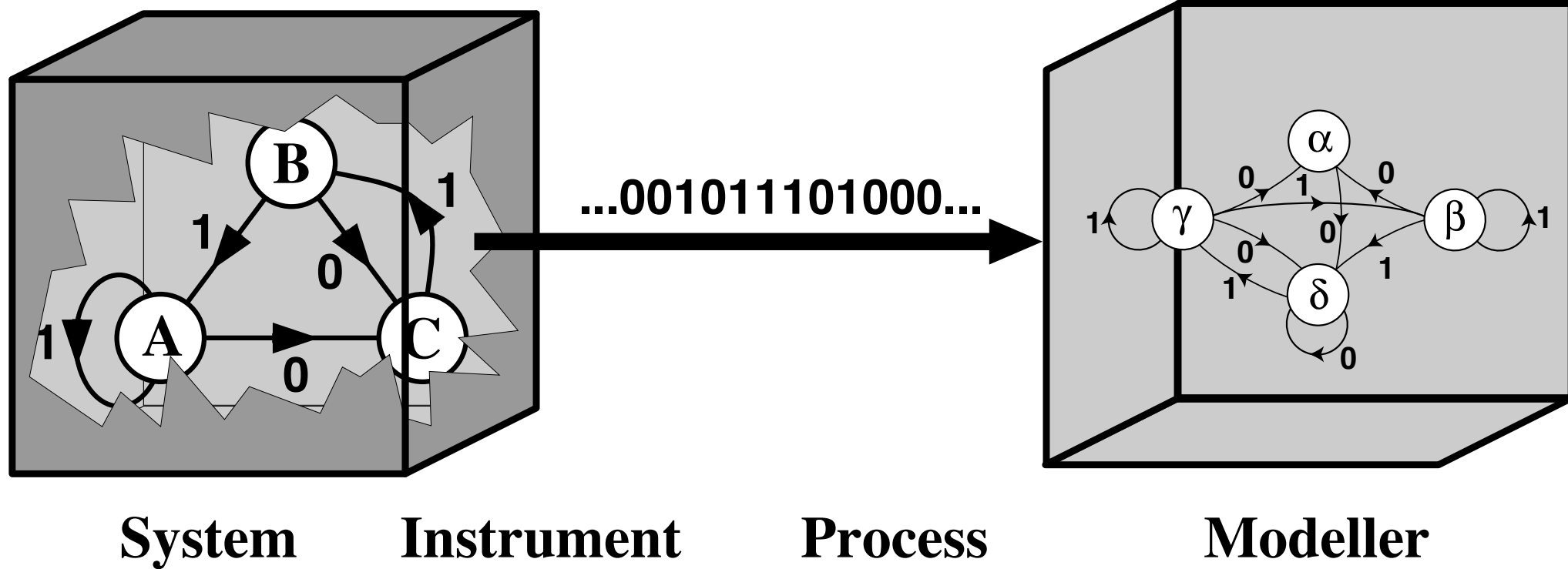
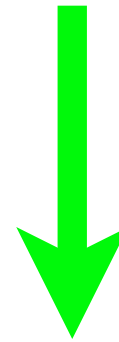
The Learning Channel

Now



The Learning Channel

Afternoon



The Learning Channel

Complexity

Information Theory for Complex Systems

Yesterday:

I. Information Theory

Information Measures

Algorithmic Basis

Now:

II. Information & Memory in Processes

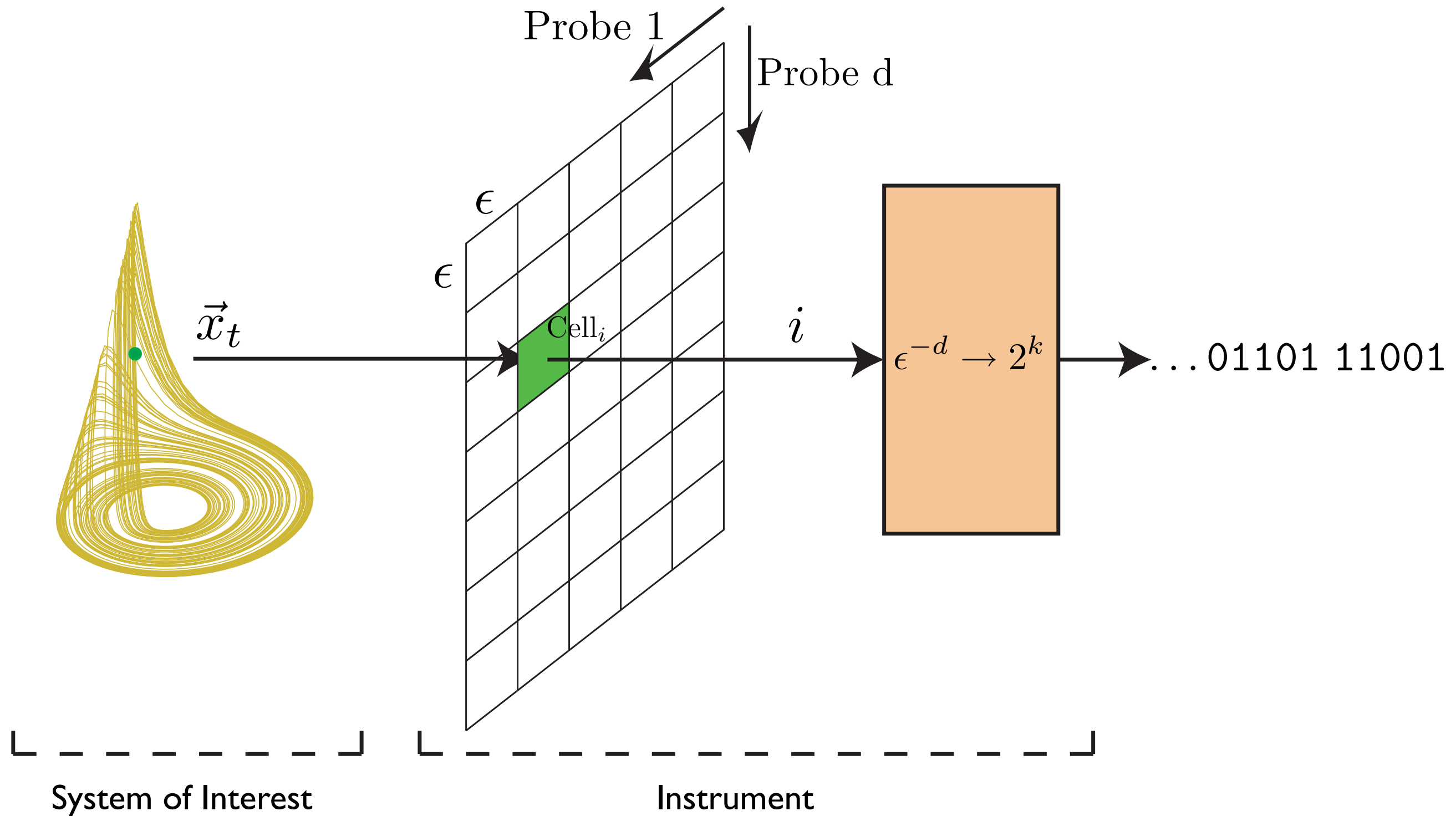
Intrinsic Computation

Measuring Structure

Intrinsic Computation

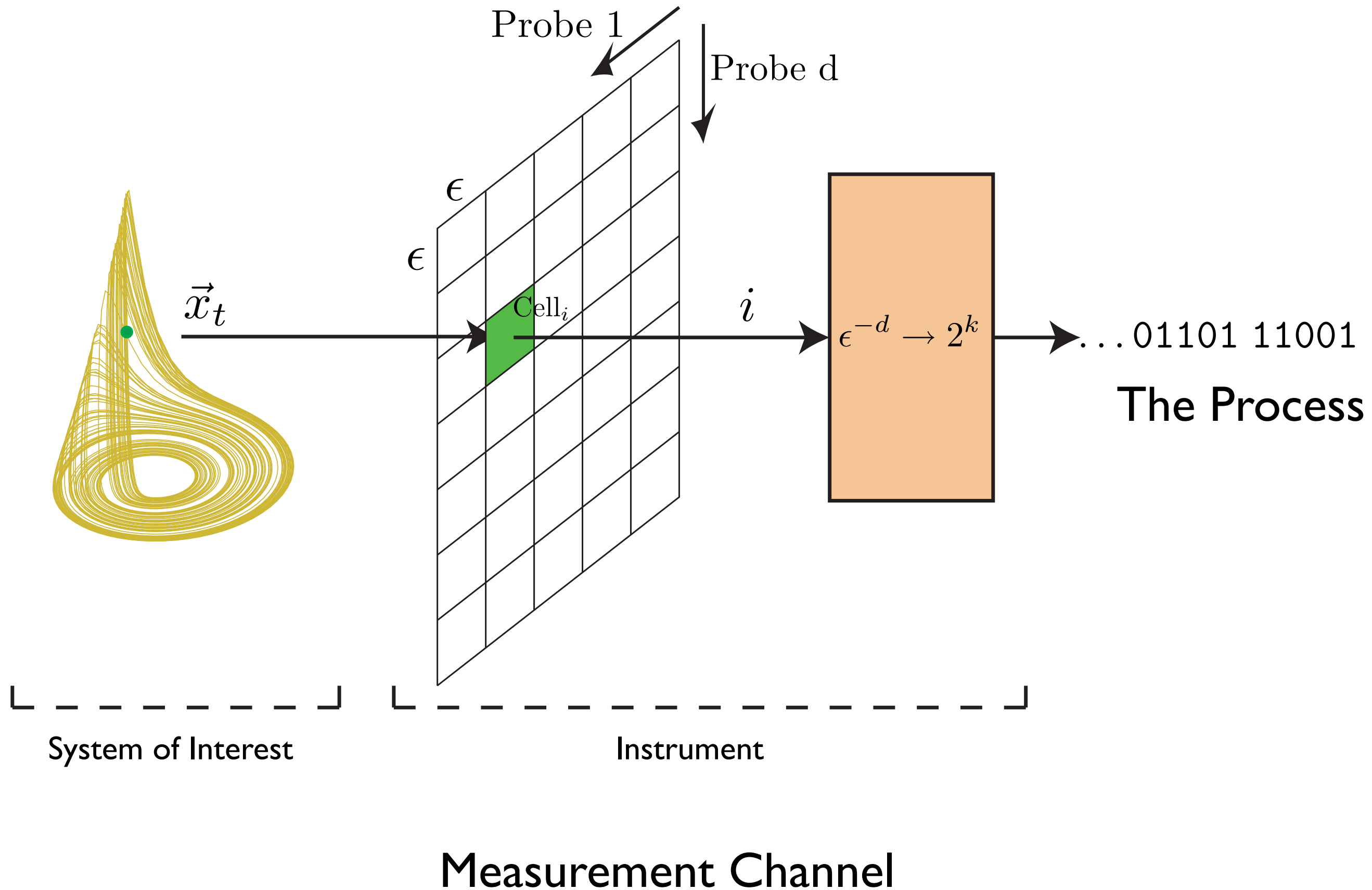
Optimal Models

Processes and Their Models



Measurement Channel

Processes and Their Models



Processes and Their Models ...

Main questions now:

How do we characterize the resulting process?

Measure degrees of unpredictability & randomness.

What correlational structure is there?

How do we build a model from the process itself?

How much can we reconstruct about the
hidden internal dynamics?

Processes and Their Models ...

Stochastic Processes:

Chain of random variables: $\overleftrightarrow{S} \equiv \dots S_{-2}S_{-1}S_0S_1S_2 \dots$

Random variable: S_t Alphabet: $\mathcal{A}$

Past: $\overleftarrow{S}_t = \dots S_{t-3}S_{t-2}S_{t-1}$

Future: $\overrightarrow{S}_t = S_tS_{t+1}S_{t+2} \dots$

L-Block: $S_t^L \equiv S_tS_{t+1} \dots S_{t+L-1}$

Word: $s_t^L \equiv s_t s_{t+1} \dots s_{t+L-1} \in \mathcal{A}^L$

Processes and Their Models ...

Stochastic Processes ...

Process:

$$\Pr(\vec{S}) = \Pr(\dots S_{-2}S_{-1}S_0S_1S_2\dots)$$

Sequence (or word) distributions:

$$\{\Pr(S_t^L) = \Pr(S_t S_{t+1} \dots S_{t+L-1}) : S_t \in \mathcal{A}\}$$

Process:

$$\{\Pr(S_t^L) : \forall t, L\}$$

Consistency conditions:

$$\Pr(S_t^{L-1}) = \sum_{S_{t+L-1}} \Pr(S_t^L) \qquad \Pr(S_{t+1}^{L-1}) = \sum_{S_t} \Pr(S_t^L)$$

Processes and Their Models ...

Types of Stochastic Process:

Stationary process:

$$\Pr(S_t S_{t+1} \dots S_{t+L-1}) = \Pr(S_0 S_1 \dots S_{L-1})$$

Assume stationarity, unless otherwise noted.

Notation: Drop time indices.

Processes and Their Models ...

Models of Stochastic Processes:

Markov chain model of a Markov process:

States: $v \in \mathcal{A} = \{1, \dots, k\}$

$$\overleftrightarrow{V} = \dots V_{-2} V_{-1} V_0 V_1 \dots$$

Transition matrix: $T_{ij} = \Pr(v_{t+1} | v_t) \equiv p_{vv'}$

$$T = \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix}$$

Stochastic matrix: $\sum_{j=1}^k T_{ij} = 1$

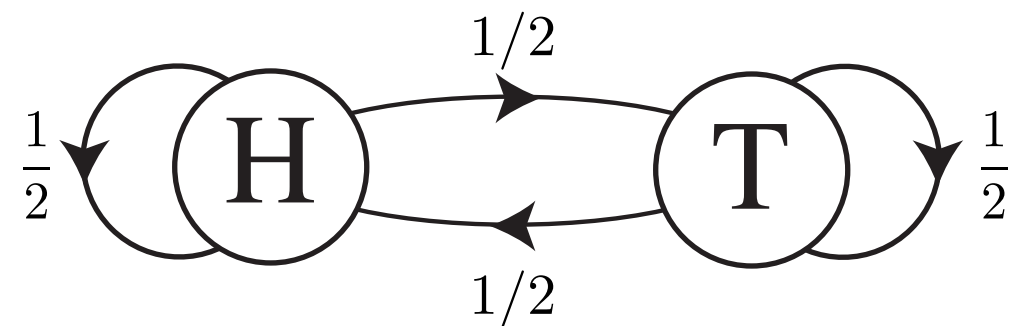
Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Fair Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



$$\Pr(H) = \Pr(T) = 1/2$$

Asymptotic invariant distribution: $\pi \equiv \Pr(H, T)$

$$\pi = \pi T$$

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Fair Coin ...

Sequence Distribution: $\Pr(v^L) = 2^{-L}$

Word as binary fraction:

$$s^L = s_1 s_2 \dots s_L$$

$$“s^L” = \sum_{i=1}^L \frac{s_i}{2^i}$$

$$s^L \in [0, 1]$$

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Fair Coin ...

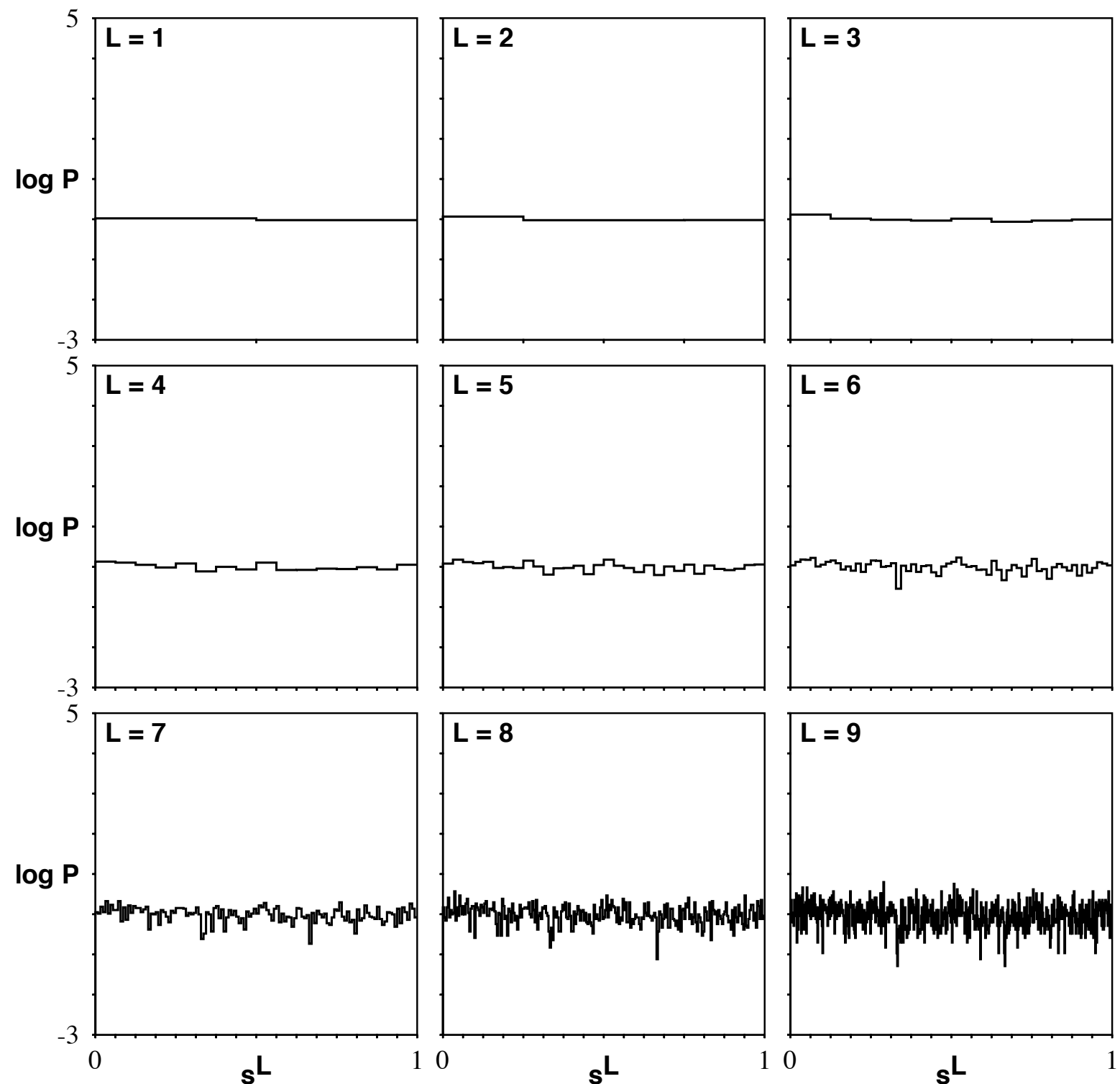
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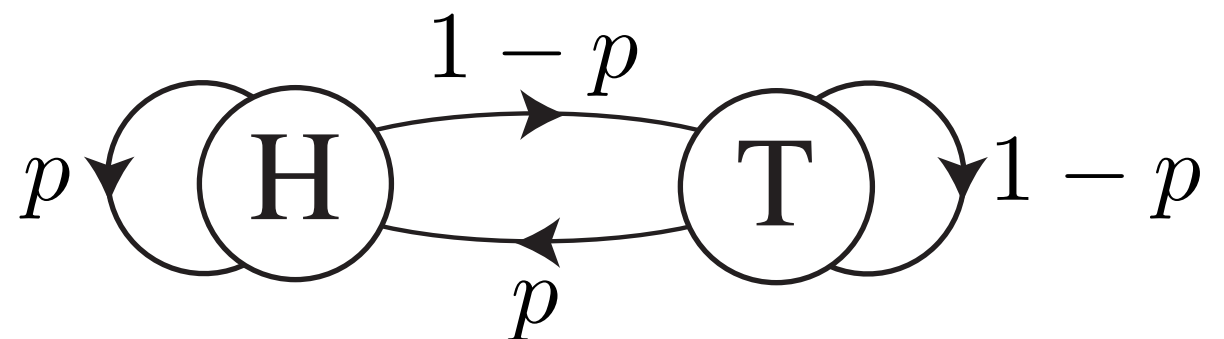
Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Biased Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} p & 1 - p \\ p & 1 - p \end{pmatrix}$$



$$\Pr(H) = p$$

$$\Pr(T) = 1 - p$$

$$\pi = \Pr(p, 1 - p)$$

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Biased Coin ...

Sequence Distribution:

$$\Pr(s^L) = p^n (1 - p)^{L-n},$$

$n = \text{Number } H\text{s in } s^L$

Processes and Their Models ...

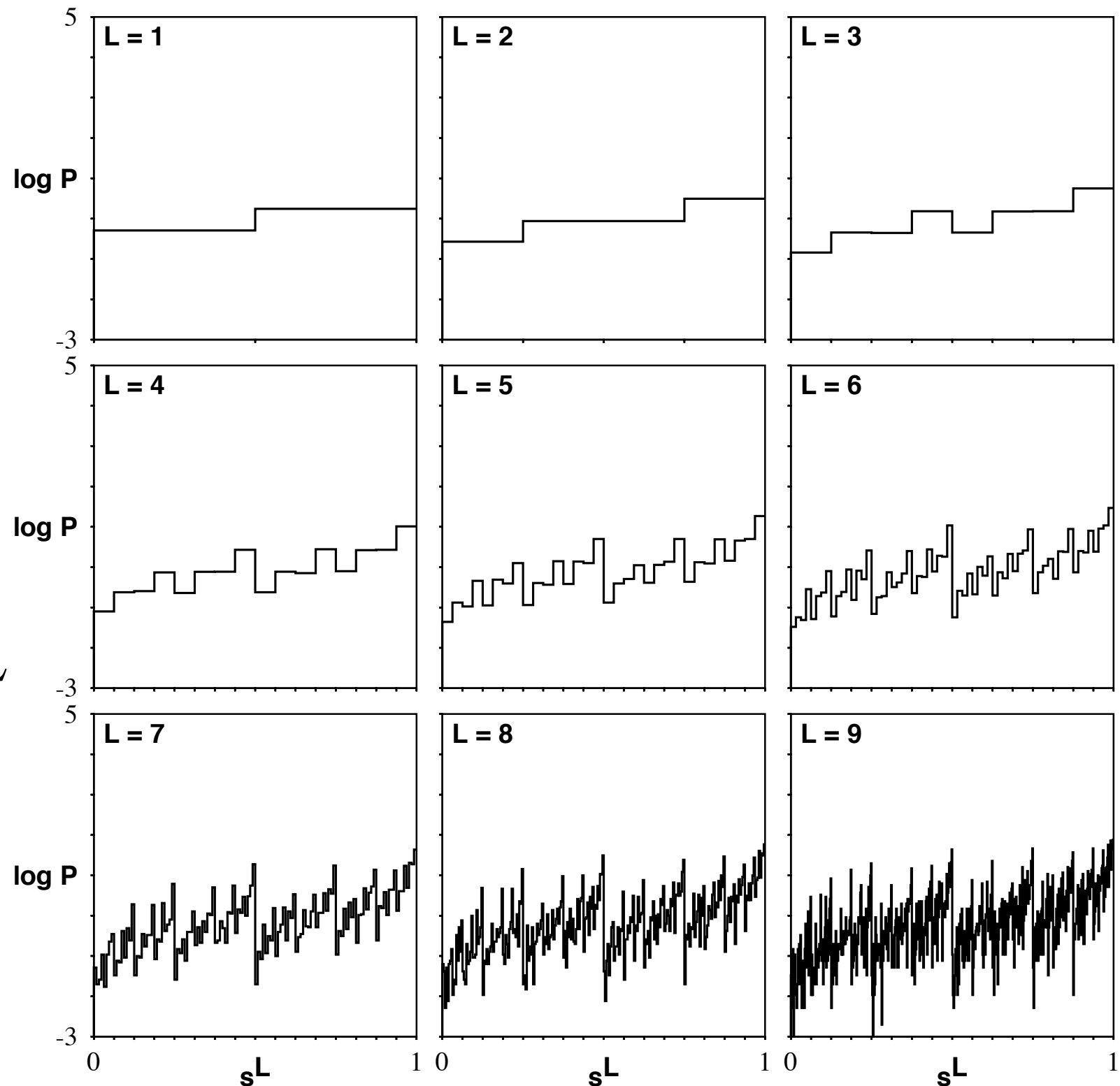
Models of Stochastic Processes ...

Example:
Biased Coin ...

Sequence Distribution:

$$\Pr(s^L) = p^n (1 - p)^{L-n},$$

n = Number H s in s^L



Processes and Their Models ...

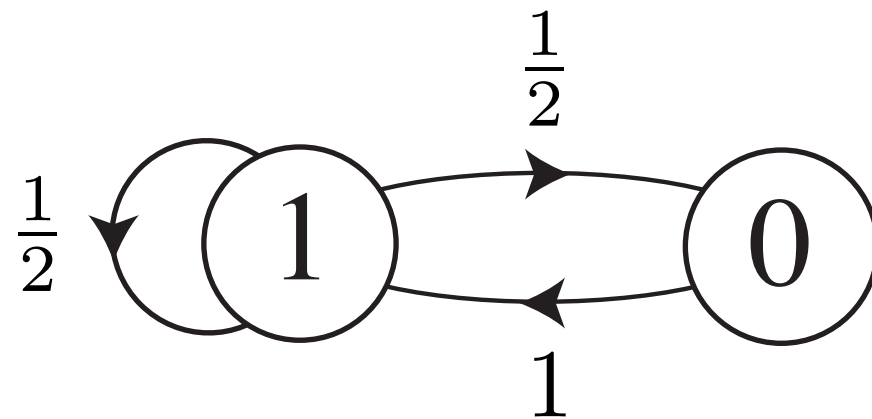
Models of Stochastic Processes ...

Example: Golden Mean Process = “No consecutive 0s”

Markov chain over 1-Blocks: $\mathcal{A} = \{0, 1\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \pi &= \Pr(V = 1, V = 0) \\ &= \left(\frac{2}{3}, \frac{1}{3}\right) \end{aligned}$$



As an order-1 Markov chain.

A minimal-order model of the GM Process.

Processes and Their Models ...

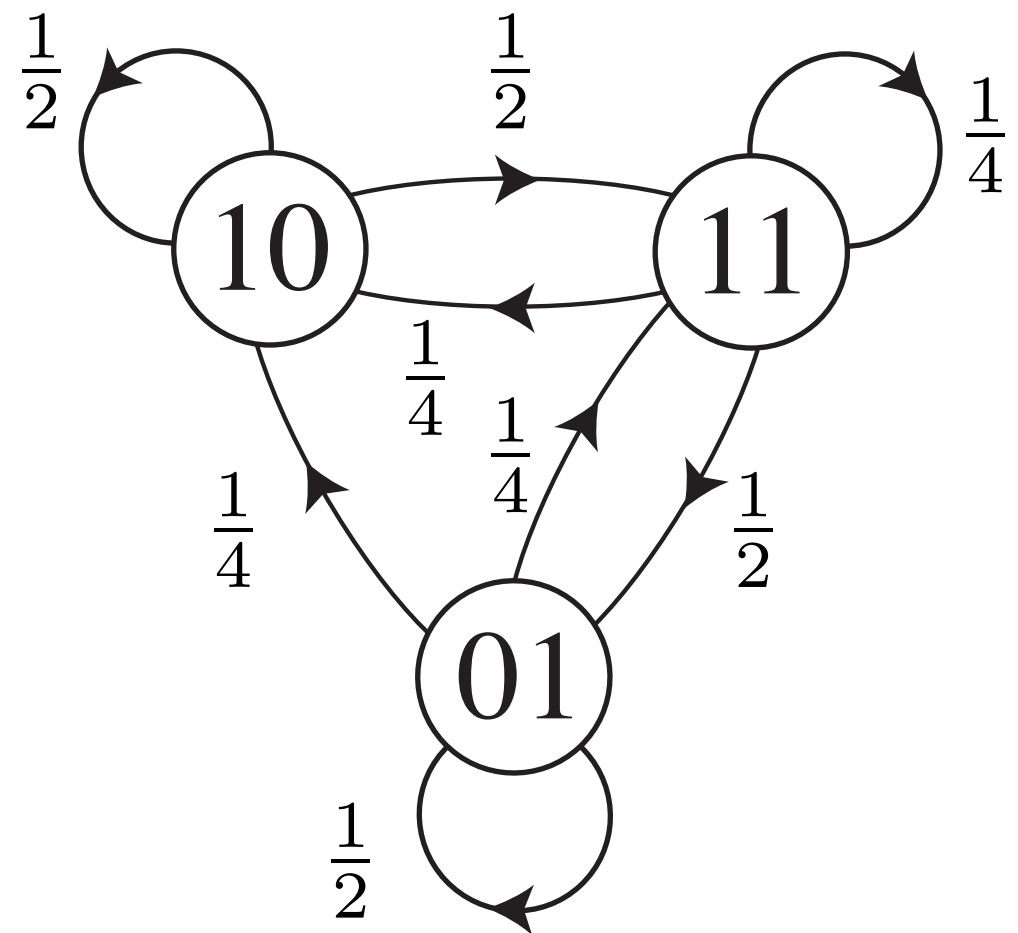
Models of Stochastic Processes ...

Example: Golden Mean Process ...

as a Markov chain over 2-Blocks: $\mathcal{A} = \{10, 01, 11\}$

$$T = \begin{matrix} & \begin{matrix} 10 & 01 & 11 \end{matrix} \\ \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \end{matrix}$$

$$\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$



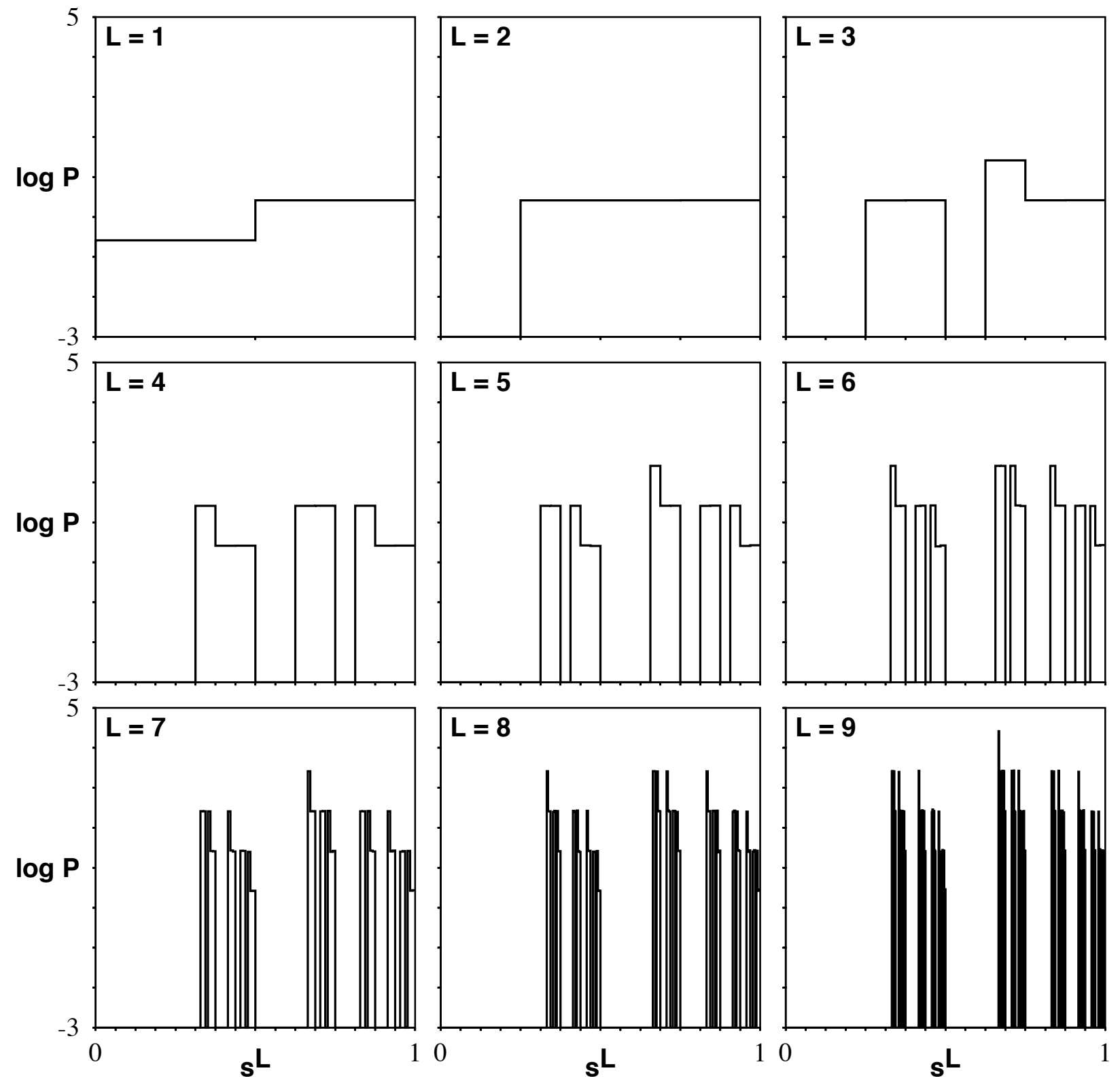
Previous model and this:

Different **presentations** of the same Golden Mean Process

Processes and Their Models ...

Models of Stochastic Processes ...

Example:
Golden Mean:



Processes and Their Models ...

Models of Stochastic Processes ...

Two Lessons:

Structure in the behavior: $\text{supp } \Pr(s^L)$

Structure in the distribution of behaviors: $\Pr(s^L)$

Processes and Their Models ...

Models of Stochastic Processes ...

Hidden Markov Models of Processes:

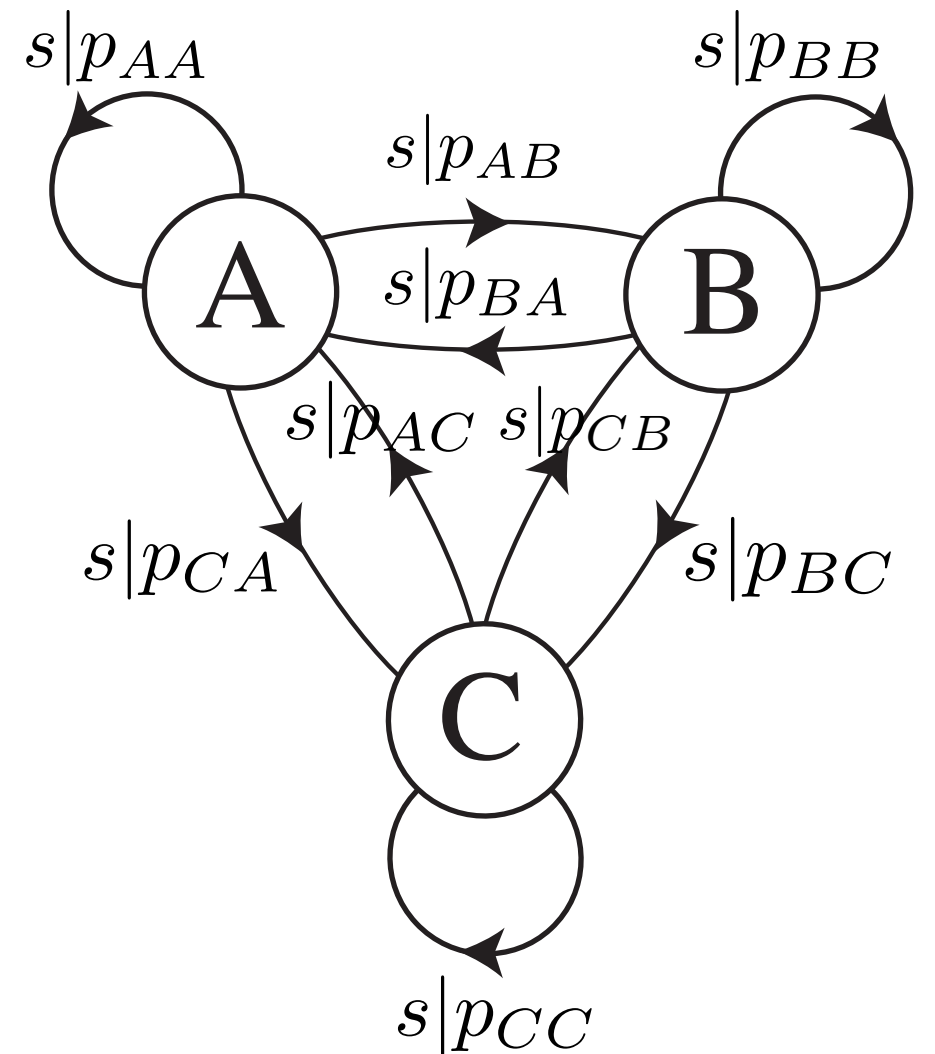
Internal: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(s)} = \begin{pmatrix} p_{AA;s} & p_{AB;s} & p_{AC;s} \\ p_{BA;s} & p_{BB;s} & p_{BC;s} \\ p_{CA;s} & p_{CB;s} & p_{CC;s} \end{pmatrix}$$

$$p_{AA} = \sum_{s \in \mathcal{B}} p_{AA;s}$$



symbol | transition probability

Processes and Their Models ...

Models of Stochastic Processes ...

Types of Hidden Markov Model:

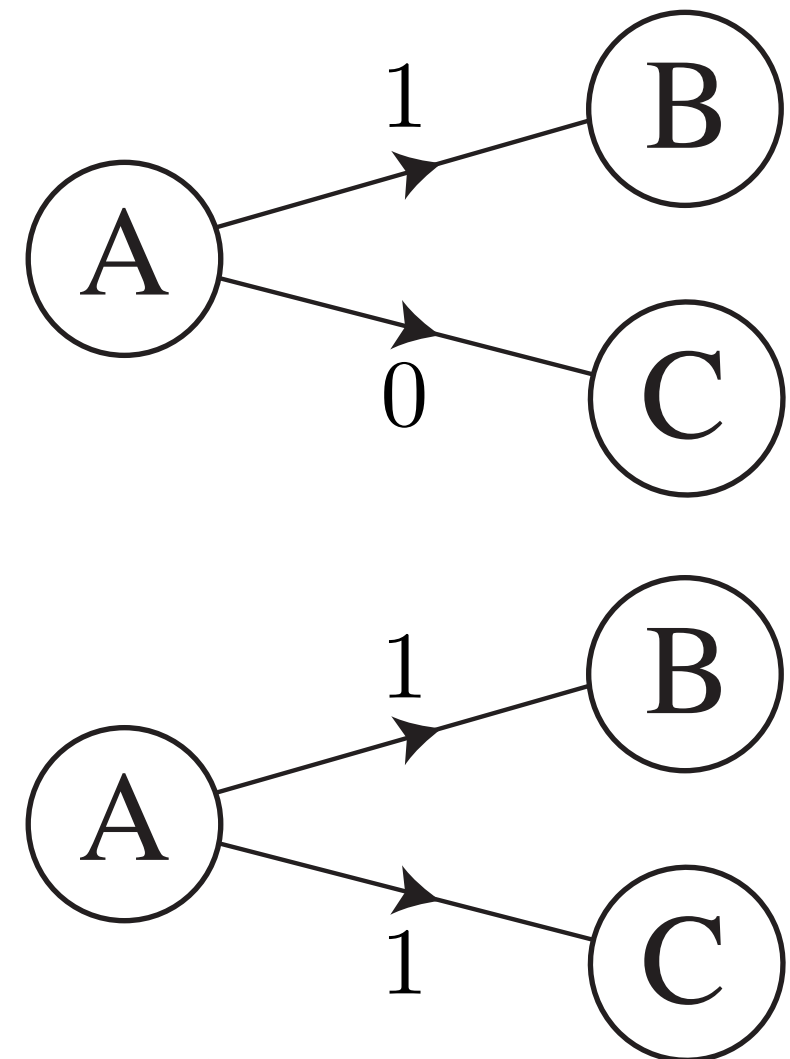
“**Unifilar**”: current state + symbol “determine” next state

$$\Pr(v'|v, s) = \begin{cases} 1 \\ 0 \end{cases}$$

$$\Pr(v', s|v) = p(s|v)$$

$$\Pr(v'|v) = \sum_{s \in \mathcal{A}} p(s|v)$$

“**Nonunifilar**”: no restriction



Multiple internal edge paths can generate same observed sequence.

Processes and Their Models ...

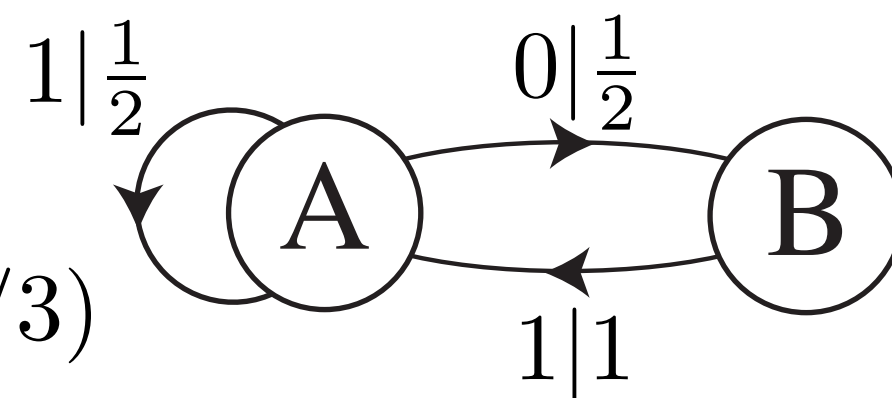
Models of Stochastic Processes ...

Example:

Golden Mean Process as a unifilar HMM:

Internal: $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$



Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

Initial ambiguity only: At most 2-to-1 mapping

$$BA^n = 1^n$$

$$AA^n = 1^n$$

$$\begin{aligned} \text{Sync'd: } s = 0 &\Rightarrow v = B \\ s = 1 &\Rightarrow v = A \end{aligned}$$

Irreducible forbidden words: $\mathcal{F} = \{00\}$

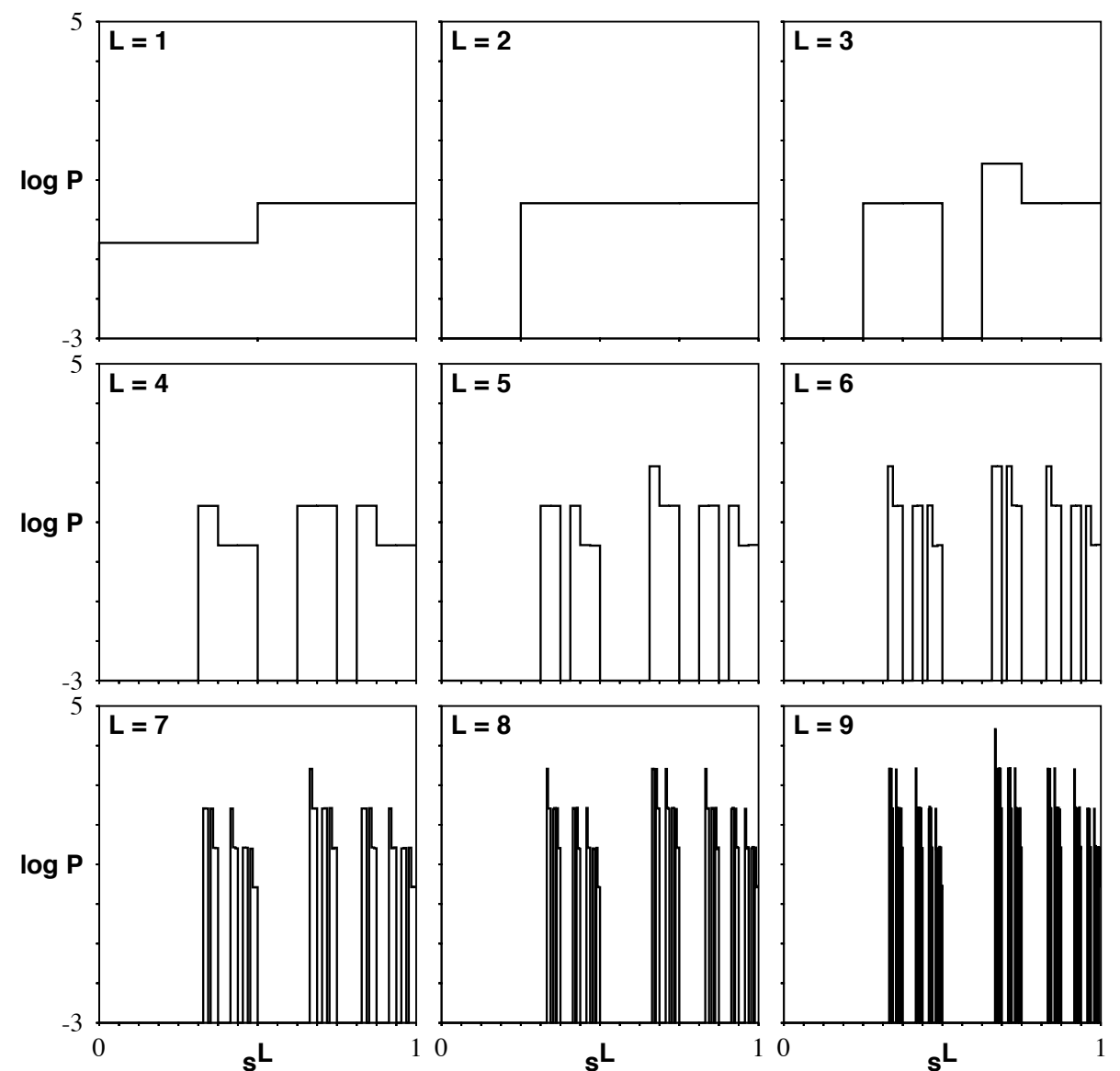
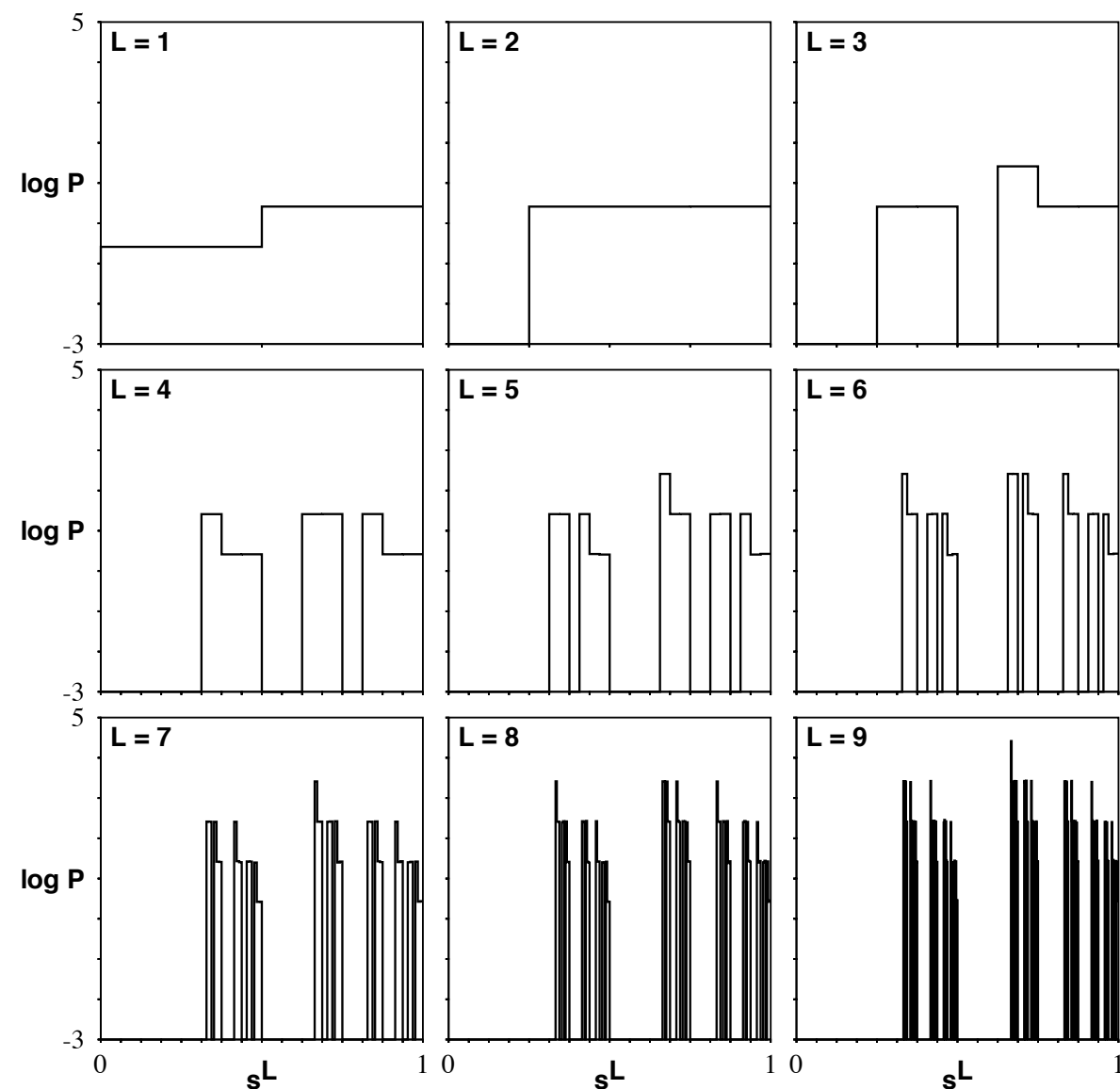
Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Golden Mean Process ... Sequence distributions:
Internal state sequences Observed sequences

($A = 1; B = 0$)



Same!

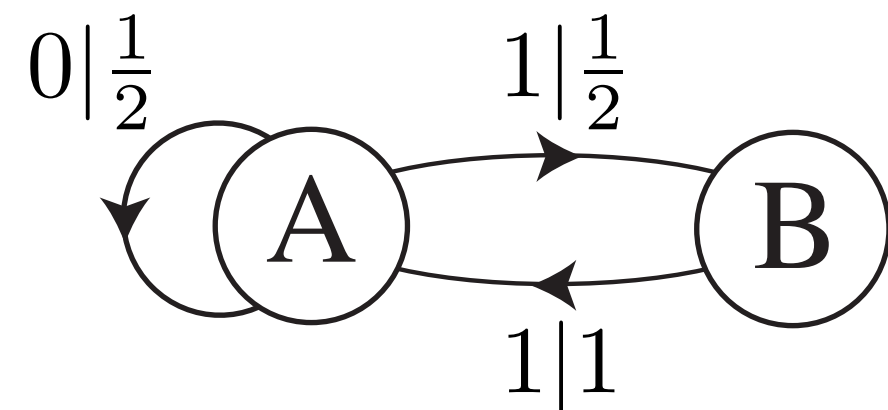
Processes and Their Models ...

Models of Stochastic Processes ...

Example: Even Process = Even #Is

As a unifilar HMM:

Internal (= GMP): $\mathcal{A} = \{A, B\}$



$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$v^L = \dots AAB AAB ABA A \dots$$

$$s^L = \dots 011011110 \dots \quad s^L = \{\dots 01^{2n}0 \dots\}$$

Irreducible forbidden words: $\mathcal{F} = \{010, 01110, 0111110, \dots\}$

No finite-order Markov process can model the Even process!

Lesson: Finite Markov Chains are a subset of HMMs.

Processes and Their Models ...

Models of Stochastic Processes ...

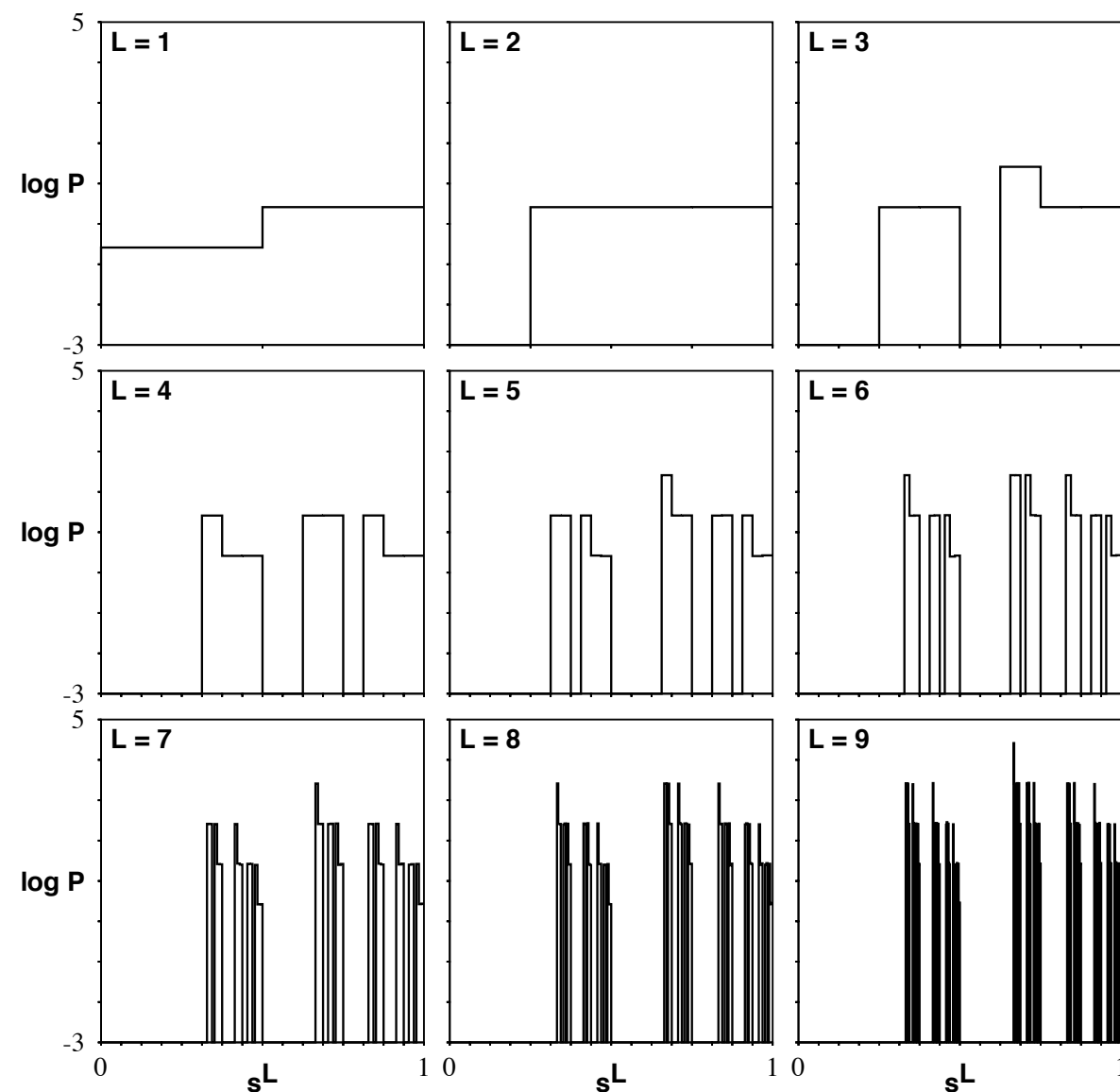
Example:

Even Process ... Sequence distributions:

Internal states (= GMP)

Observed sequences

($A = 1; B = 0$)



Rather different!

Processes and Their Models ...

Models of Stochastic Processes ...

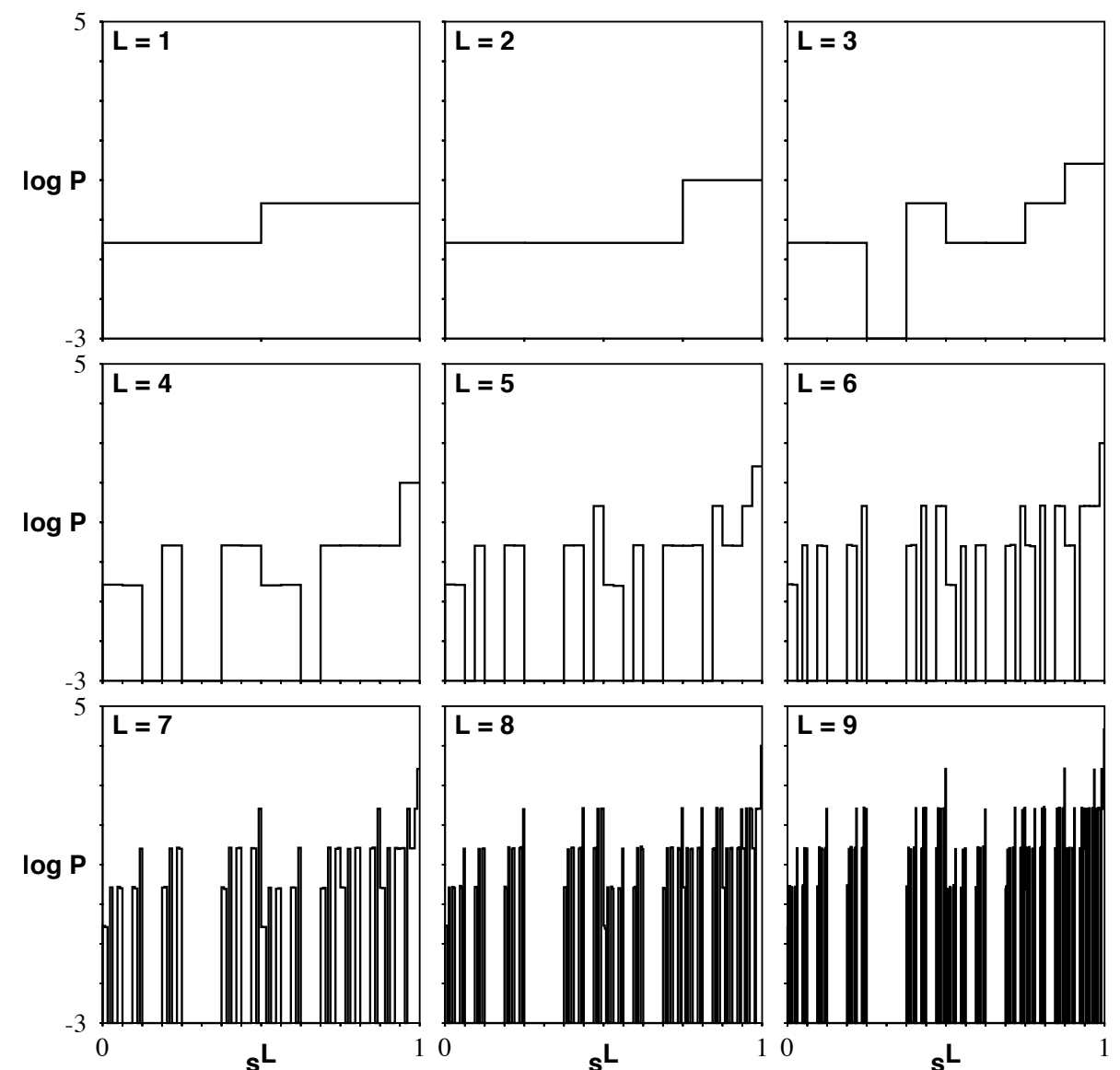
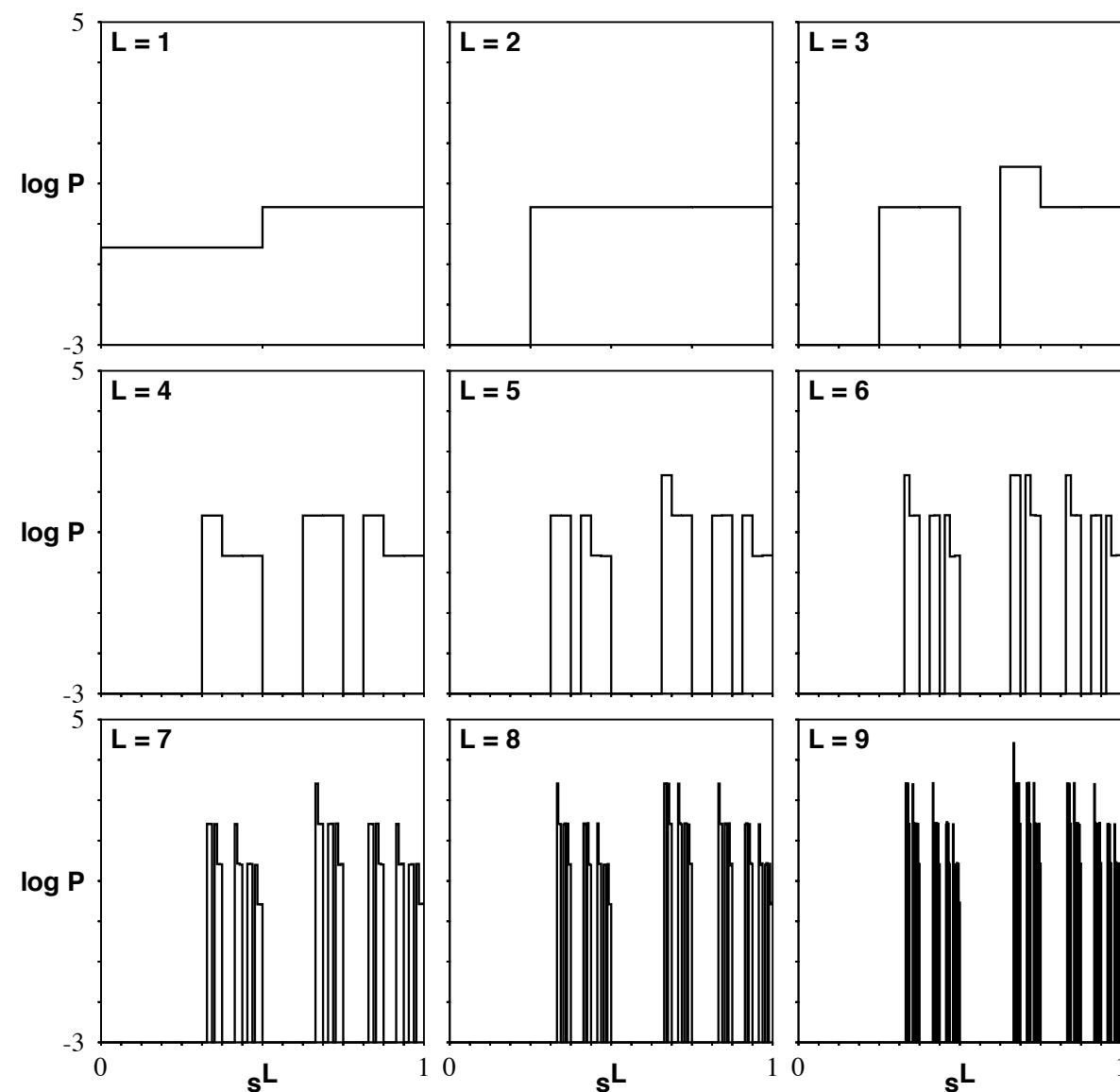
Example:

Even Process ... Sequence distributions:

Internal states (= GMP)

($A = 1; B = 0$)

Observed sequences



Rather different!

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

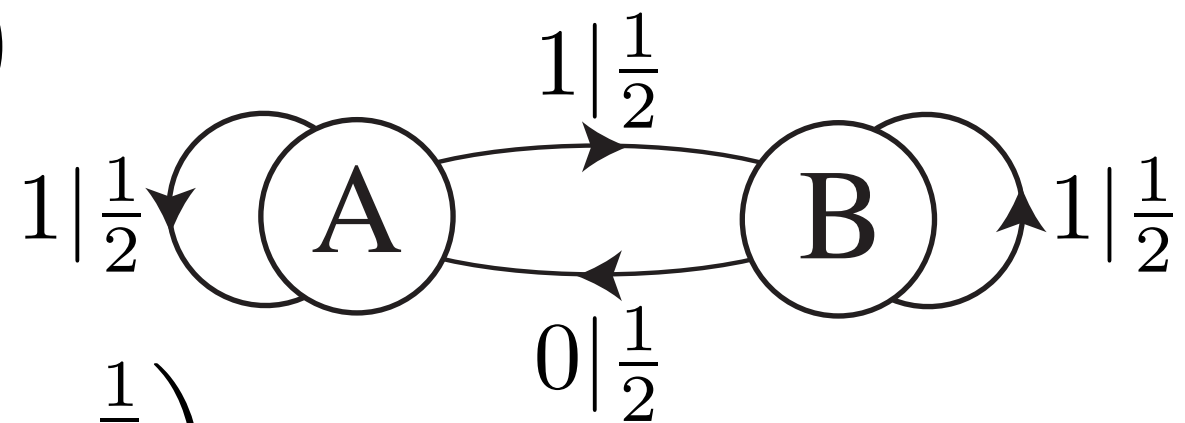
Simple Nonunifilar Source:

Internal (= Fair Coin): $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \pi_V = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$



Many to one: $1111111 \Leftarrow \begin{cases} AAAAAAAAAA\dots \\ AB BBB BBB\dots \\ AAB BBB BBB\dots \\ AAAB BBB BBB\dots \\ \dots \\ BBB BBB BBB\dots \end{cases}$

Is there a unifilar HMM presentation of the observed process?

Processes and Their Models ...

Models of Stochastic Processes ...

Example:

Simple Nonuniform Process ...

Internal states (= Fair coin)
(A = 1; B = 0)

Observed sequences

Processes and Their Models ...

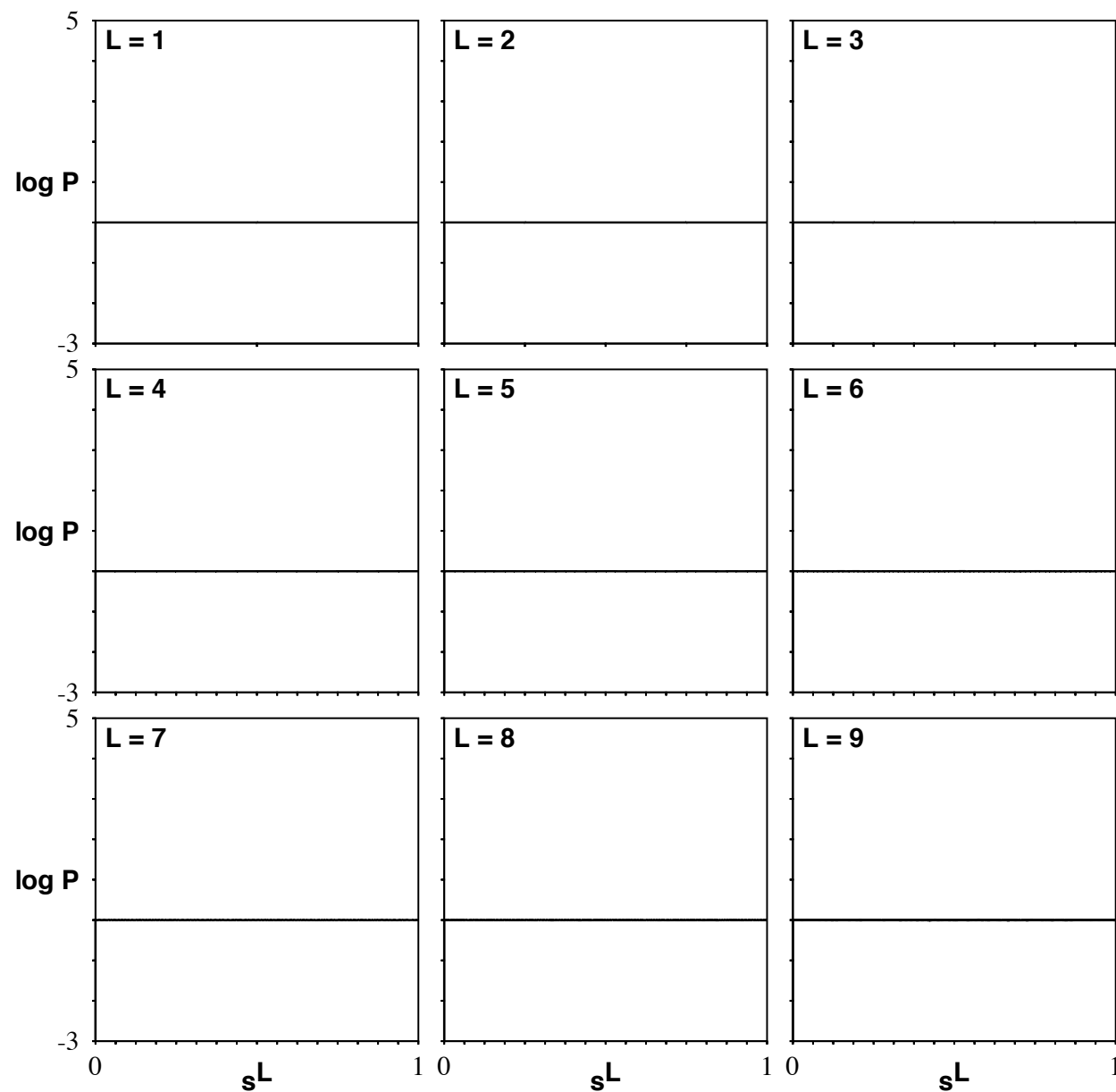
Models of Stochastic Processes ...

Example:

Simple Nonuniform Process ...

Internal states (= Fair coin)
($A = 1; B = 0$)

Observed sequences



Processes and Their Models ...

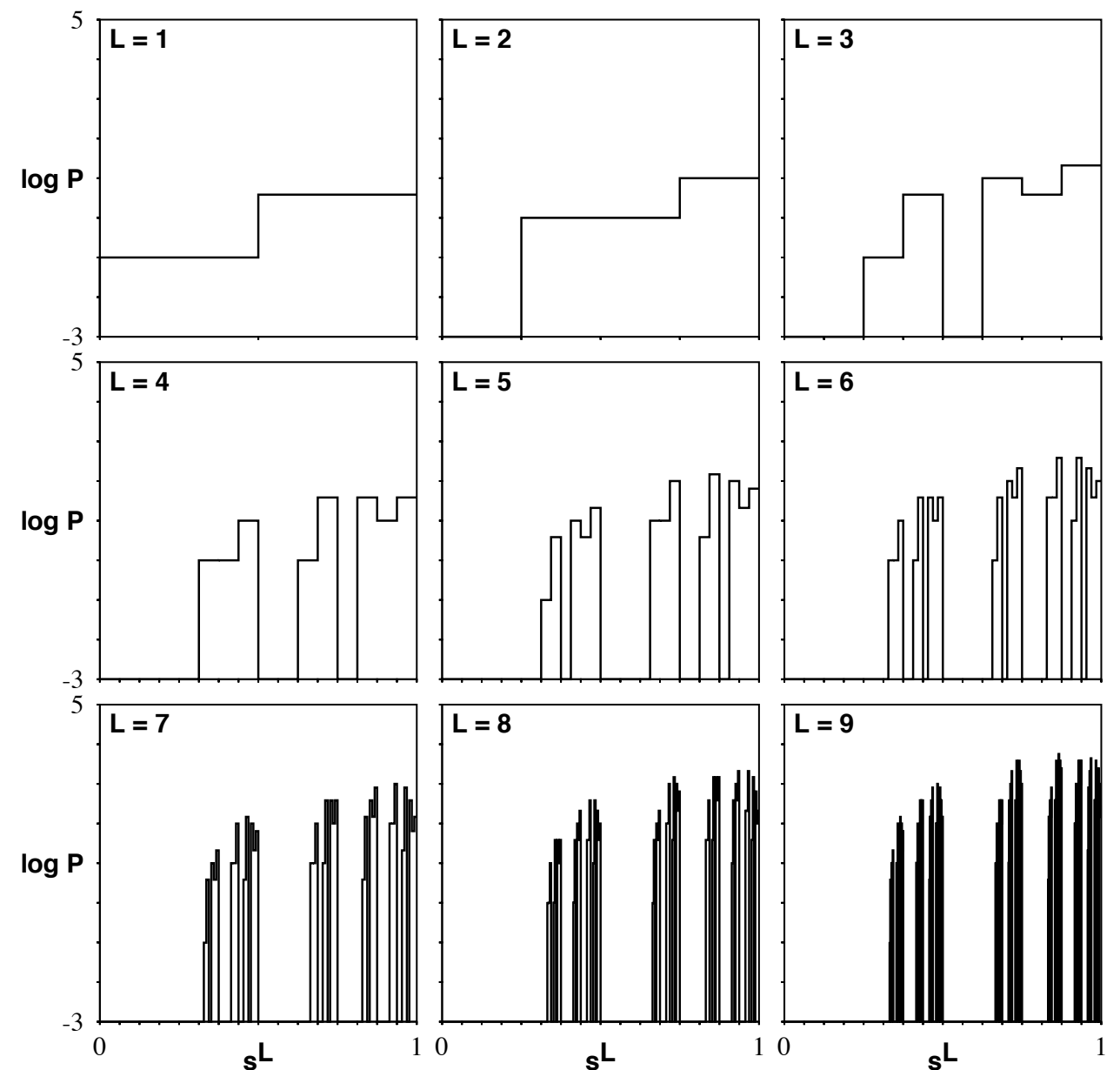
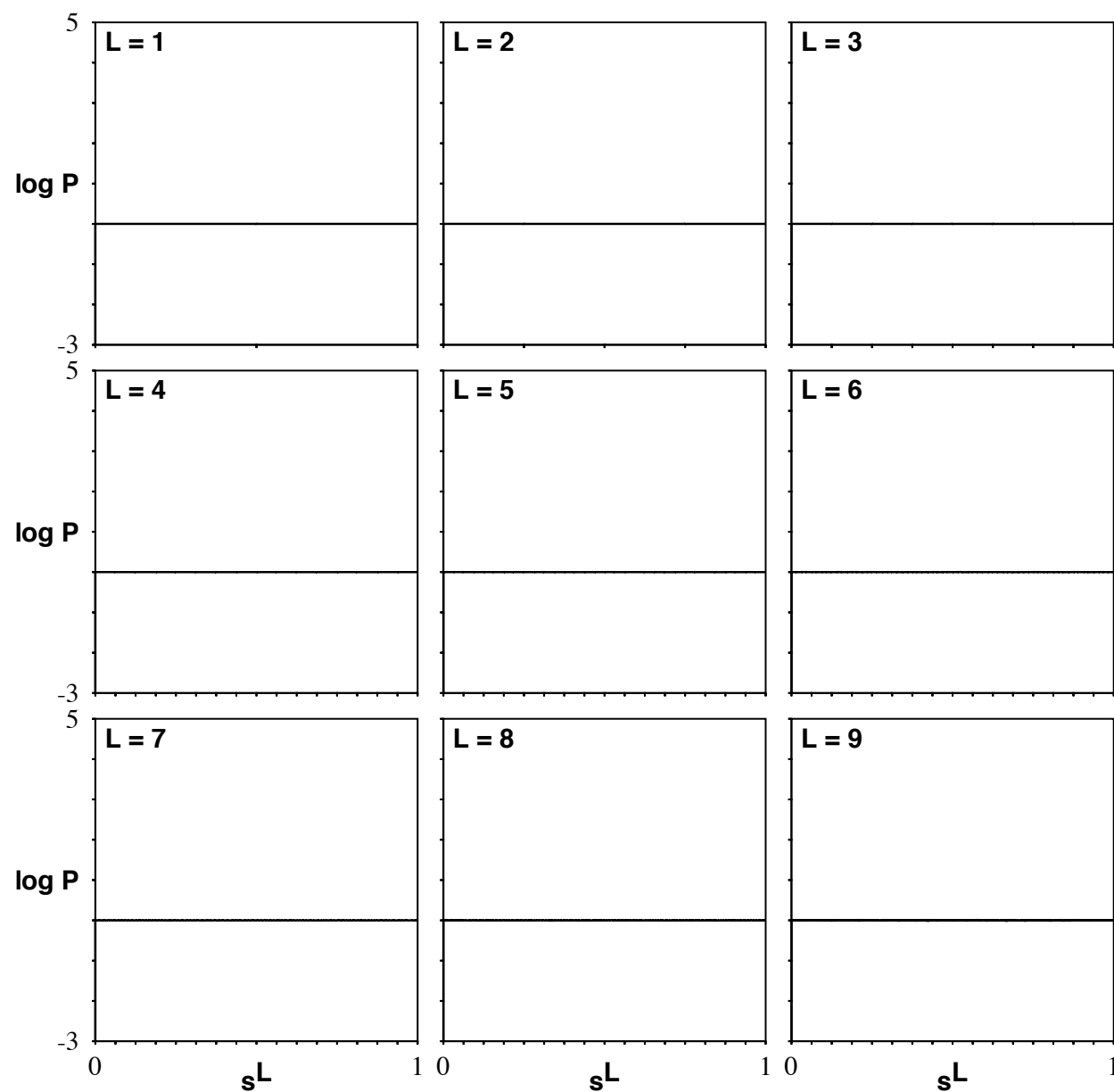
Models of Stochastic Processes ...

Example:

Simple Nonuniform Process ...

Internal states (= Fair coin)
($A = 1; B = 0$)

Observed sequences



Processes and Their Models ...

What to do with all of this complicatedness?

1. Information theory for complex processes
2. Measures of complexity
3. Optimal models and how to build them

Information in Processes

Information in Processes ...

Entropy Growth for Stationary Stochastic Processes: $\Pr(\vec{S})$

Block Entropy:

$$H(L) = H(\Pr(s^L)) = - \sum_{s^L \in \mathcal{A}} \Pr(s^L) \log_2 \Pr(s^L)$$

Monotonic increasing: $H(L) \geq H(L - 1)$

Adding a random variable cannot decrease entropy:

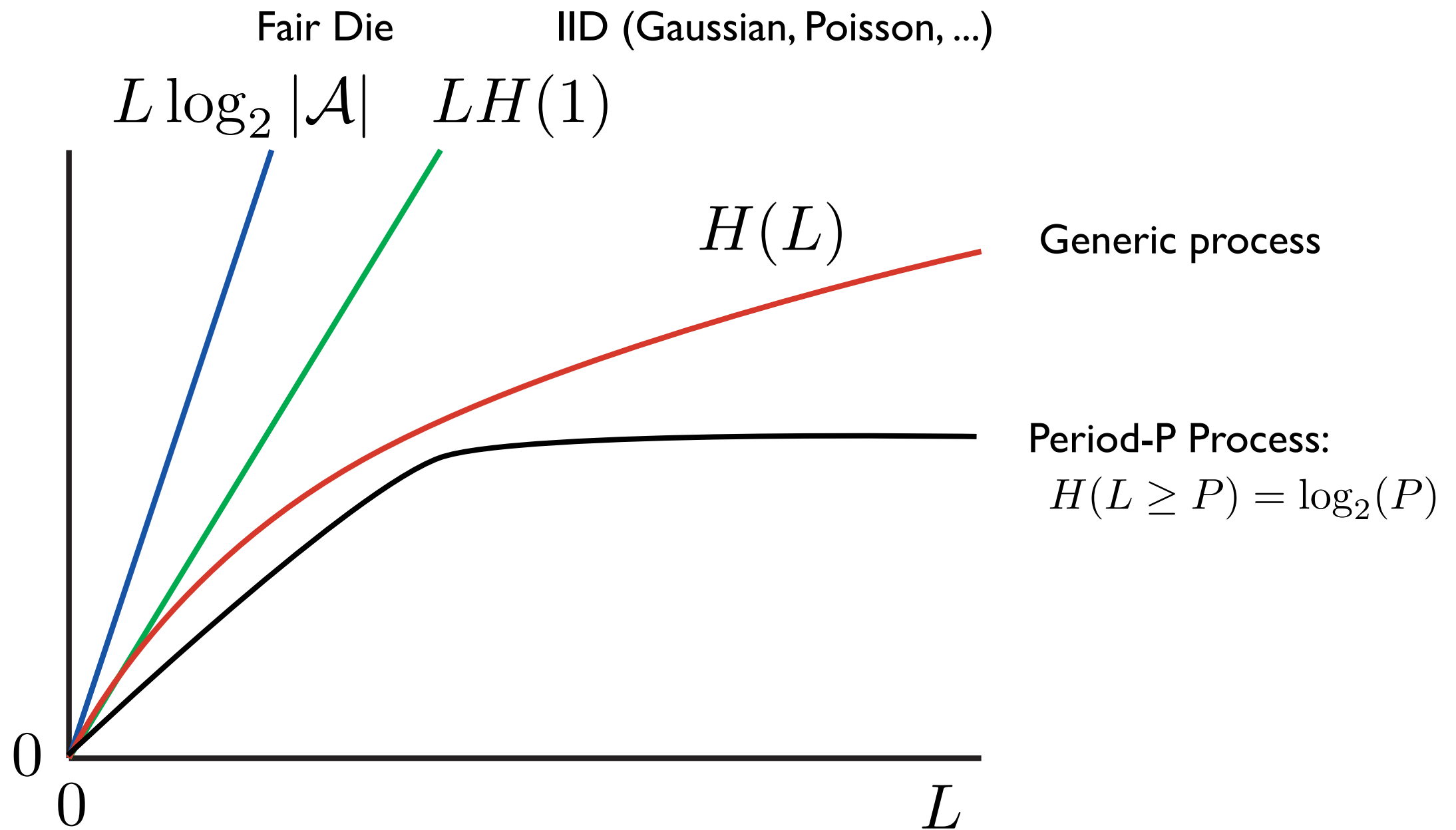
$$H(S_1, S_2, \dots, S_L) \leq H(S_1, S_2, \dots, S_L, S_{L+1})$$

No measurements, no information: $H(0) = 0$

Information in Processes ...

Entropy Growth for Stationary Stochastic Processes ...

Block Entropy ...



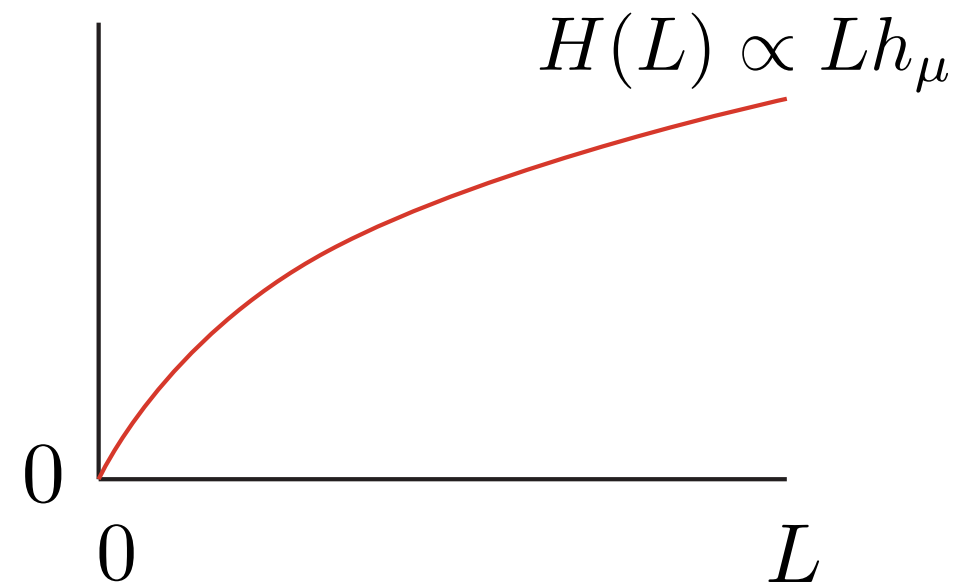
Information in Processes ...

Entropy Rates for Stationary Stochastic Processes:

Entropy per symbol is given by the **Source Entropy Rate**:

$$h_\mu = \lim_{L \rightarrow \infty} \frac{H(L)}{L}$$

(When limits exists.)



Interpretations:

Asymptotic growth rate of entropy

Irreducible randomness of process

Average description length (per symbol) of process

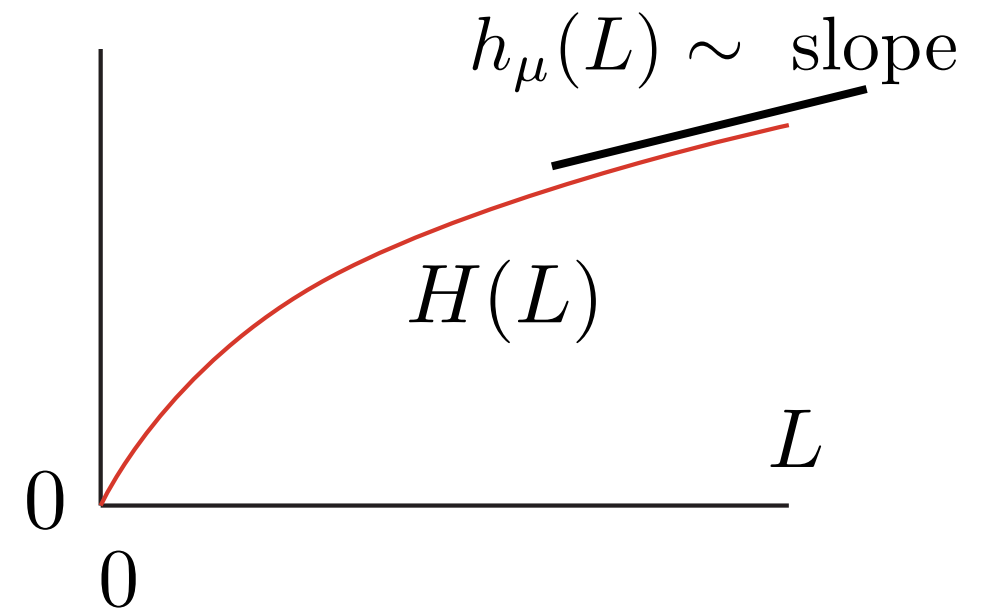
Information in Processes ...

Entropy Rates for Stationary Stochastic Processes ...

Length-L Estimate of Entropy Rate:

$$\hat{h}_\mu(L) = H(L) - H(L-1)$$

$$\hat{h}_\mu(L) = H(s_L | s_1 \cdots s_{L-1})$$



Monotonic decreasing: $\hat{h}_\mu(L) \leq \hat{h}_\mu(L-1)$

Conditioning cannot increase entropy:

$$H(s_L | s_1 \cdots s_{L-1}) \leq H(s_L | s_2 \cdots s_{L-1}) = H(s_{L-1} | s_1 \cdots s_{L-2})$$

Information in Processes ...

Entropy Rates for Stationary Stochastic Processes:

Entropy rate ...

$$\hat{h}_\mu = \lim_{L \rightarrow \infty} \hat{h}_\mu(L) = \lim_{L \rightarrow \infty} H(s_0 | \overleftarrow{s}^L) = H(s_0 | \overleftarrow{s})$$

Interpretations:

Uncertainty in next measurement, given past

A measure of unpredictability

Asymptotic slope of block entropy

Alternate entropy rate definitions agree:

$$\hat{h}_\mu = h_\mu$$

Information in Processes ...

Entropy Rate for a Markov chain: $\{V, T\}$

$$\begin{aligned} h_\mu &= \lim_{L \rightarrow \infty} h_\mu(L) \\ &= \lim_{L \rightarrow \infty} H(v_L | v_1 \cdots v_{L-1}) \\ &= \lim_{L \rightarrow \infty} H(v_L | v_{L-1}) \end{aligned}$$

Assuming asymptotic state distribution:

Process in statistical equilibrium

Process running for a long time

Forgotten it's initial distribution

Closed-form:

$$h_\mu = - \sum_{v \in V} p_v(\infty) \sum_{v' \in V} T_{vv'} \log_2 T_{vv'}$$

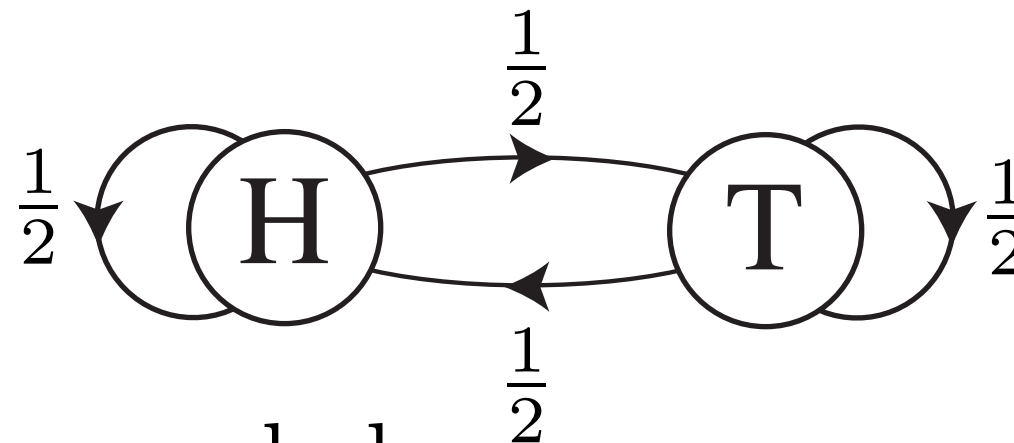
$$\begin{aligned} \vec{p}(n) &= \vec{p}(0) T^n \\ \vec{p}(\infty) &= \vec{p}(\infty) T^n \end{aligned}$$

Information in Processes ...

Entropy Rate for Markov chains ...

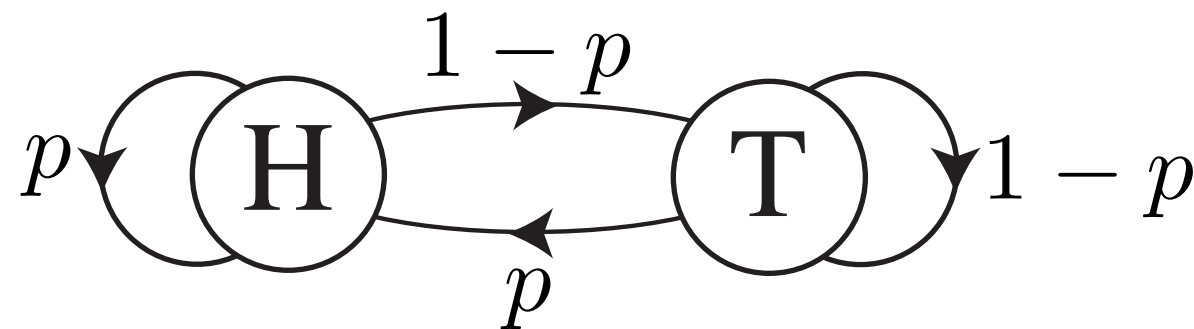
Examples:

(1) Fair Coin:



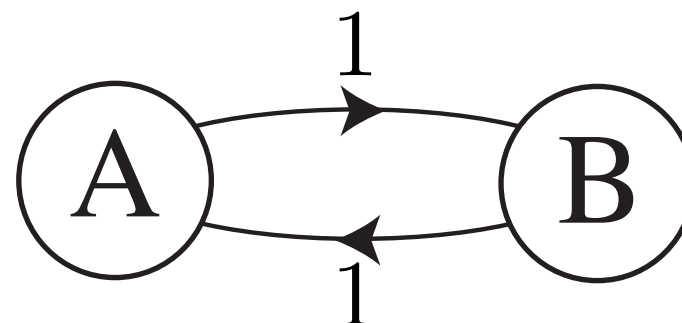
$$h_\mu = 1 \text{ bit per symbol}$$

(2) Biased Coin:



$$h_\mu = H(p) \text{ bits per symbol}$$

(3) Period-2 Process:



$$h_\mu = 0 \text{ bits per symbol}$$

Information in Processes ...

Entropy Rate for **Unifilar** Hidden Markov Chain:

Internal: $\{V, T\}$

Observed: $\{T^{(s)} : s \in \mathcal{A}\}$

Closed-form for entropy rate:

$$h_\mu = - \sum_{v \in V} p_v(\infty) \sum_{s \in \mathcal{A}} \sum_{v' \in V} T_{vv'}^{(s)} \log_2 T_{vv'}^{(s)}$$

Due to unifilarity:

Observed sequences are (effectively) unique state paths

Information in Processes ...

Entropy Rate for **Nonunifilar** Hidden Markov Chain:

Internal: $\{V, T\}$

Observed: $\{T^{(s)} : s \in \mathcal{A}\}$

Entropy rate: **No closed-form!**

$$h_\mu \neq - \sum_{v \in V} \pi_v \sum_{s \in \mathcal{A}} \sum_{v' \in V} T_{vv'}^{(s)} \log_2 T_{vv'}^{(s)} \quad \pi_v = p_v(\infty)$$

Upper and Lower Bounds:

$$H(S_L | V_1 S_1 \cdots S_{L-1}) \leq h_\mu(L) \leq H(S_L | S_1 \cdots S_{L-1})$$

Unrealistic for inference: Must know about internal states.

Unrealistic for analysis: Simulate chain, do empirical estimate.

Entropy rate? But there exists a way ... stay tuned!

Information in Processes ...

Entropy Convergence:

Length- L entropy rate estimate:

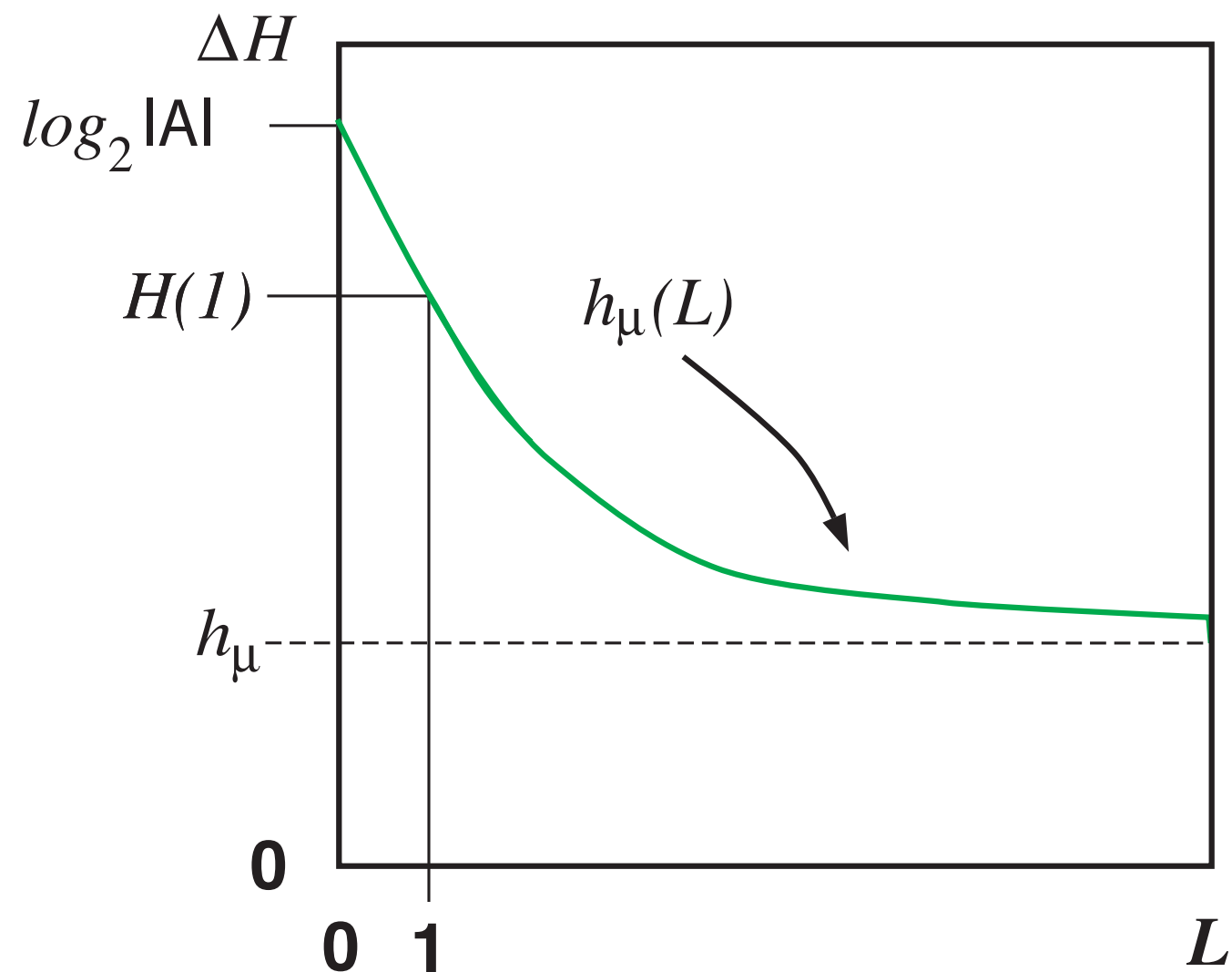
$$h_{\mu}(L) = H(L) - H(L - 1)$$

$$h_{\mu}(L) = \Delta H(L)$$

Monotonic decreasing:

$$h_{\mu}(L) \leq h_{\mu}(L - 1)$$

Process appears less random
as account for longer correlations



Memory in Processes

Information in Processes ...

Motivation:

Previous: Measures of randomness of information source

Block entropy $H(L)$

Entropy rate h_μ

Current target point:

Measures of memory & information storage

Big Picture:

Complementary.

Information in Processes ...

Motivation:

Previous: Measures of randomness of information source

Block entropy $H(L)$

Entropy rate h_μ

Current target point:

Measures of memory & information storage

Big Picture:

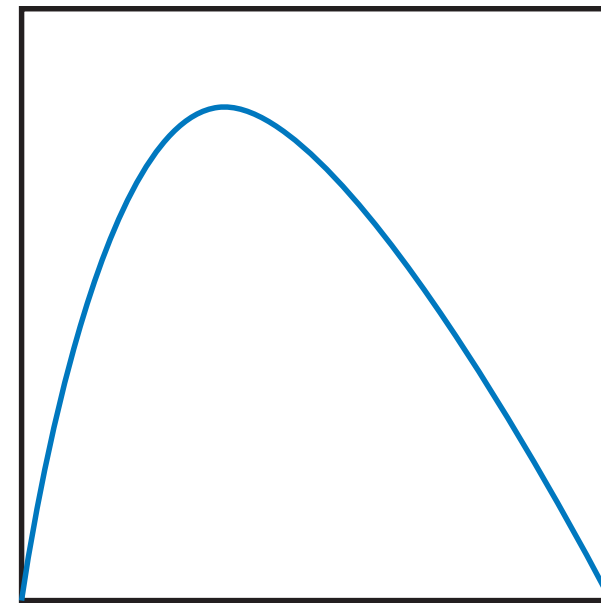
Complementary.

Structurally
Complex

Memory

Simple

Predictable



Randomness

Unpredictable

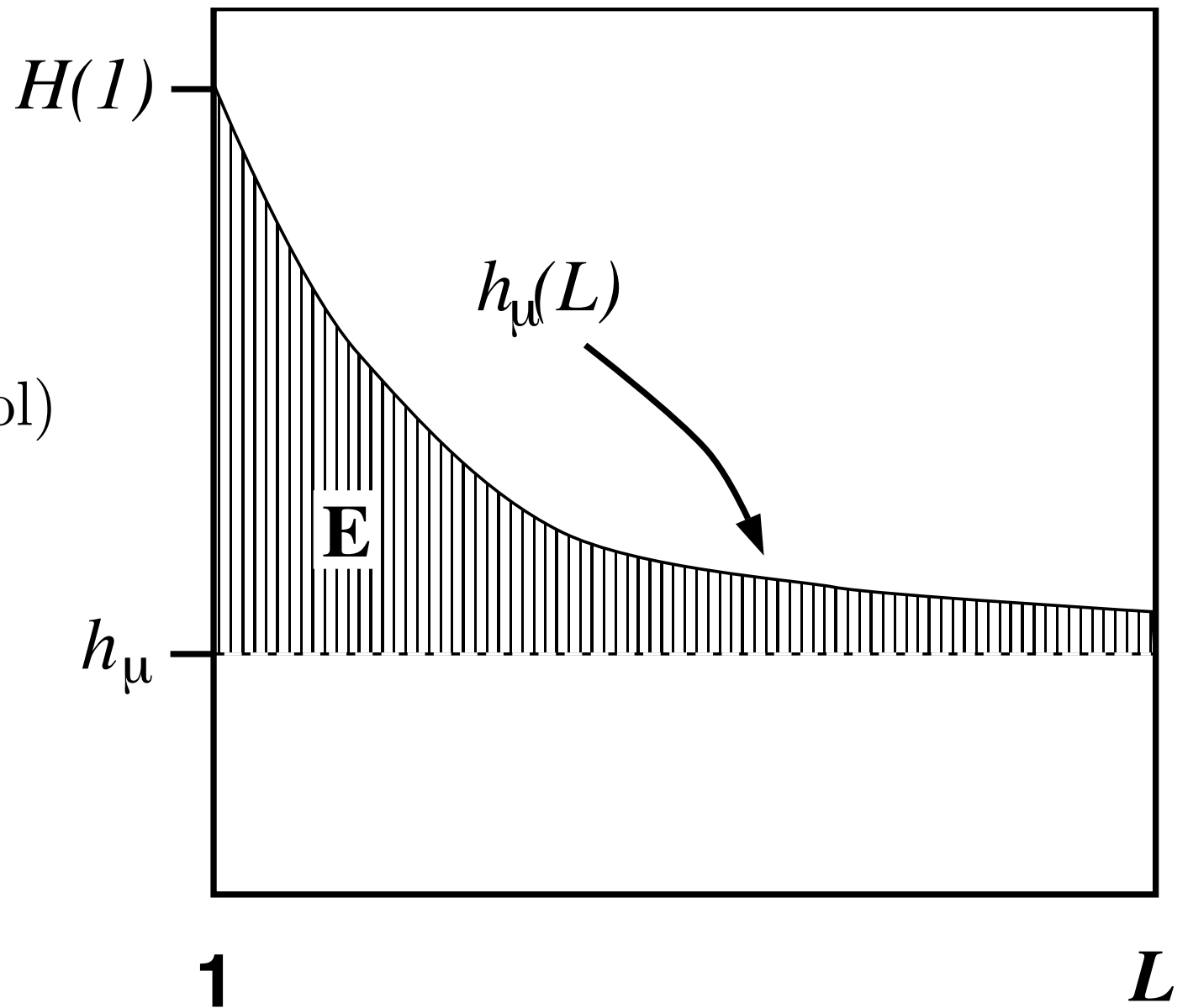
Memory in Processes ...

Excess Entropy:

As entropy convergence:

$$\mathbf{E} = \sum_{L=1}^{\infty} [h_{\mu}(L) - h_{\mu}]$$

($\Delta L = 1$ symbol)



Properties:

- (1) Units: $\mathbf{E} = [\text{bits}]$
- (2) Positive: $\mathbf{E} \geq 0$
- (3) Controls convergence to actual randomness.
- (4) Slow convergence $\Leftrightarrow$ Correlations at longer words.
- (5) Complementary to entropy rate.

Memory in Processes ...

Excess Entropy ...

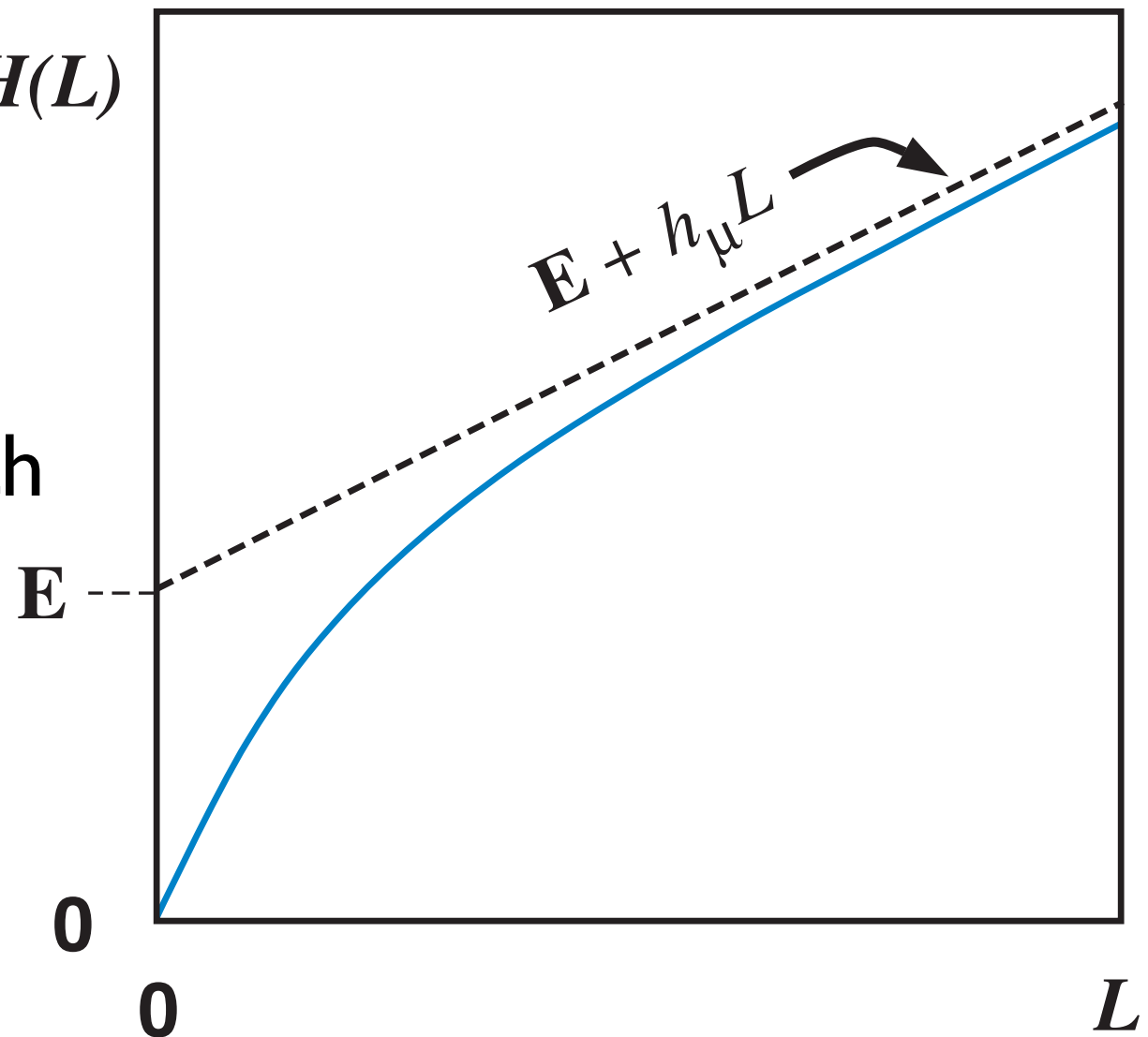
Asymptote of entropy growth:

$$\mathbf{E} = \lim_{L \rightarrow \infty} [H(L) - h_\mu L]$$

That is,

$$H(L) \propto \mathbf{E} + h_\mu L$$

Y-Intercept of entropy growth

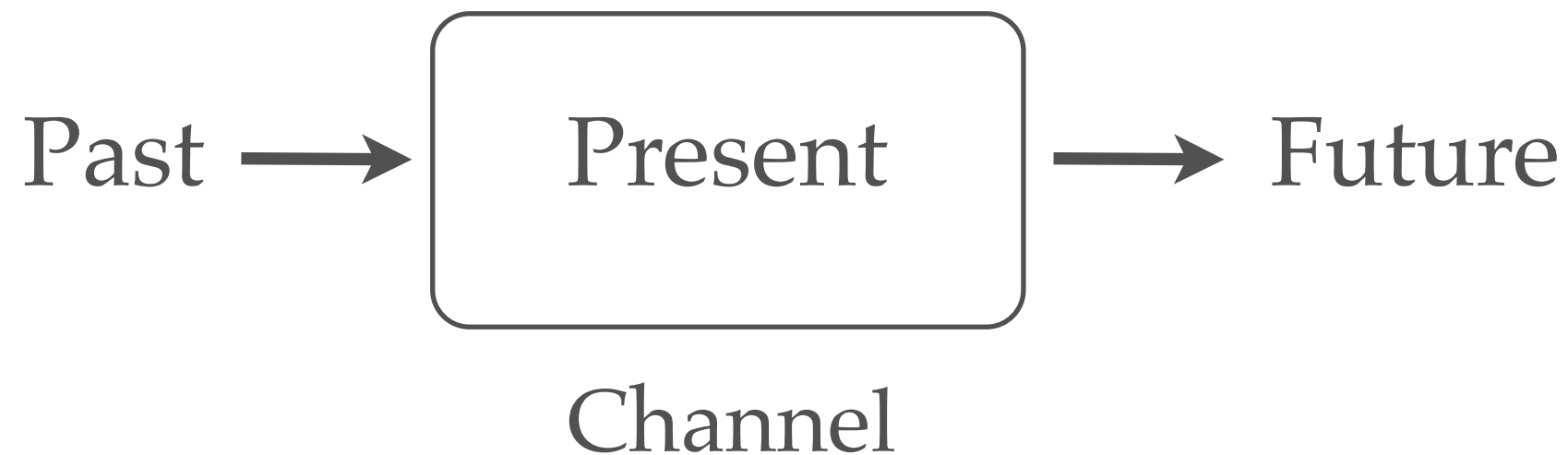


Memory in Processes ...

Excess Entropy ...

Mutual information between past and future: Process as channel

Process $\Pr(\overleftarrow{X}, \overrightarrow{X})$ communicates past $\overleftarrow{X}$ to future $\overrightarrow{X}$:



Excess Entropy as Channel Utilization:

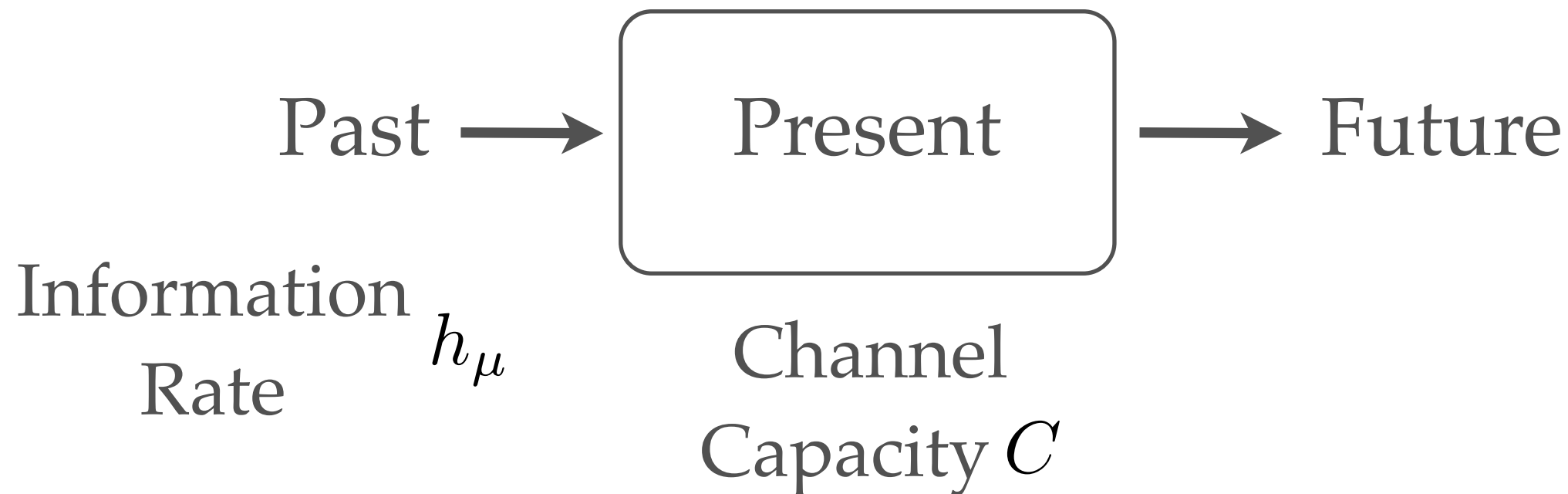
$$\mathbf{E} = I[\overleftarrow{X}; \overrightarrow{X}]$$

Memory in Processes ...

Excess Entropy ...

Mutual information between past and future: Process as channel

Process $\Pr(\overleftarrow{X}, \overrightarrow{X})$ communicates past $\overleftarrow{X}$ to future $\overrightarrow{X}$:



Excess Entropy as Channel Utilization:

$$\mathbf{E} = I[\overleftarrow{X}; \overrightarrow{X}]$$

Memory in Processes ...

Examples of Excess Entropy:

Fair Coin:

$h_\mu = 1$ bit per symbol

$\mathbf{E} = 0$ bits

Biased Coin:

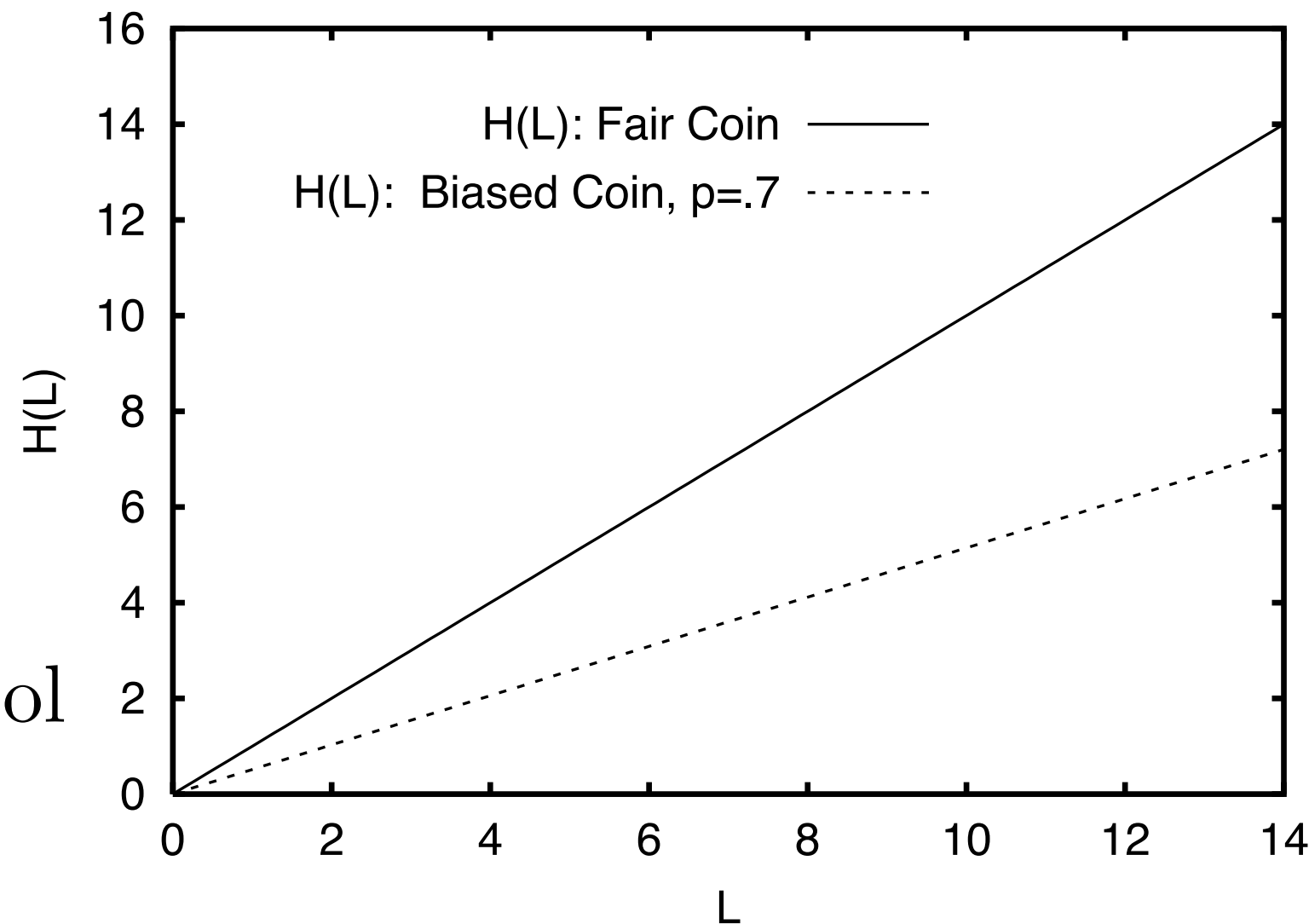
$h_\mu = H(p)$ bits per symbol

$\mathbf{E} = 0$ bits

Any IID Process:

$h_\mu = H(X)$ bits per symbol

$\mathbf{E} = 0$ bits



Memory in Processes ...

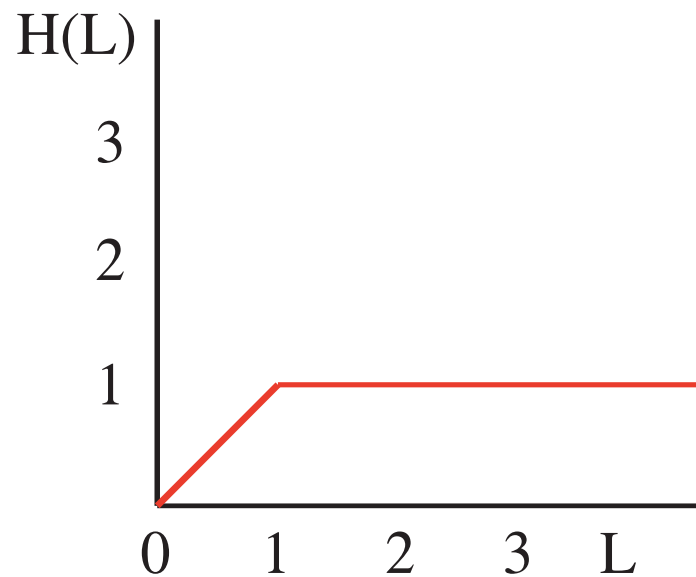
Examples of Excess Entropy ...

Period-2 Process: 0101010101

$$H(1) = 1$$

$$H(2) = 1$$

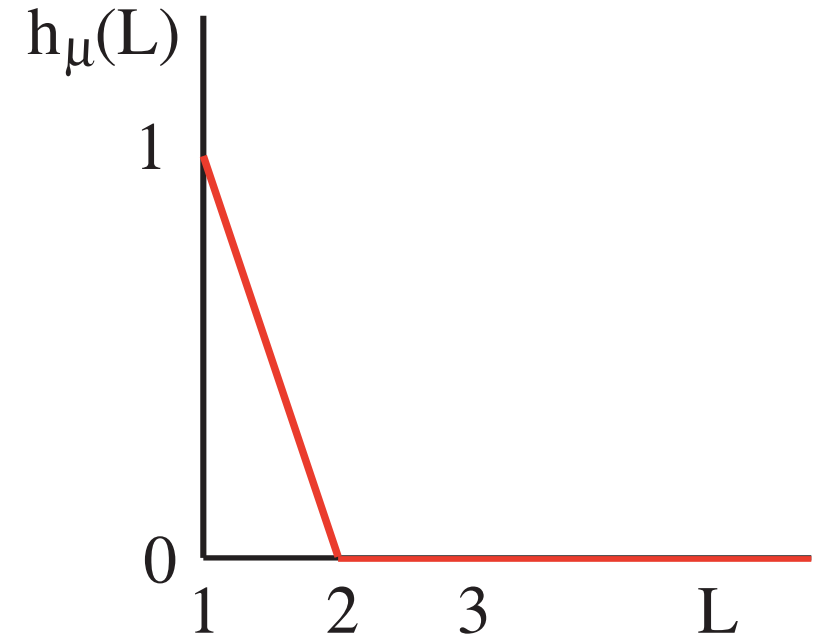
$$H(3) = 1$$



$$h_{\mu}(1) = 1$$

$$h_{\mu}(2) = 0$$

$$h_{\mu}(3) = 0$$



$$h_{\mu} = 0 \text{ bits per symbol}$$

$$\mathbf{E} = 1 \text{ bit}$$

Meaning:

1 bit of phase information

0-phase or 1-phase?

Memory in Processes ...

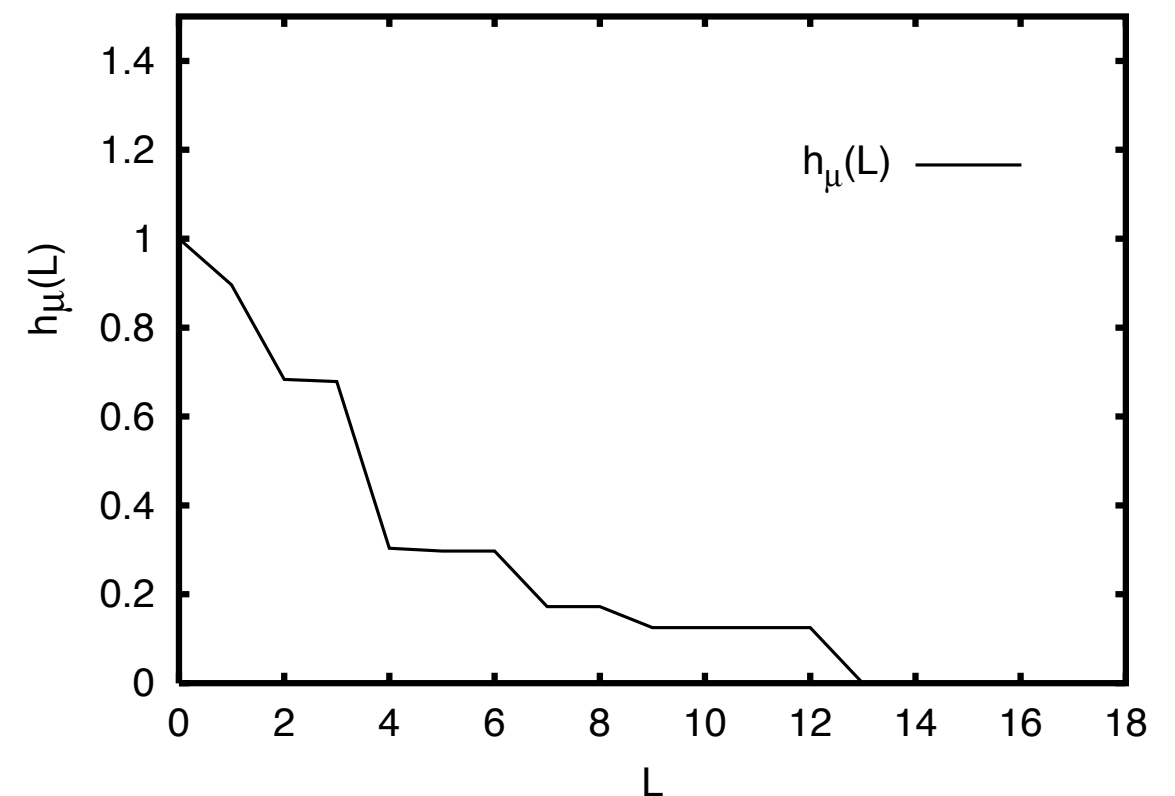
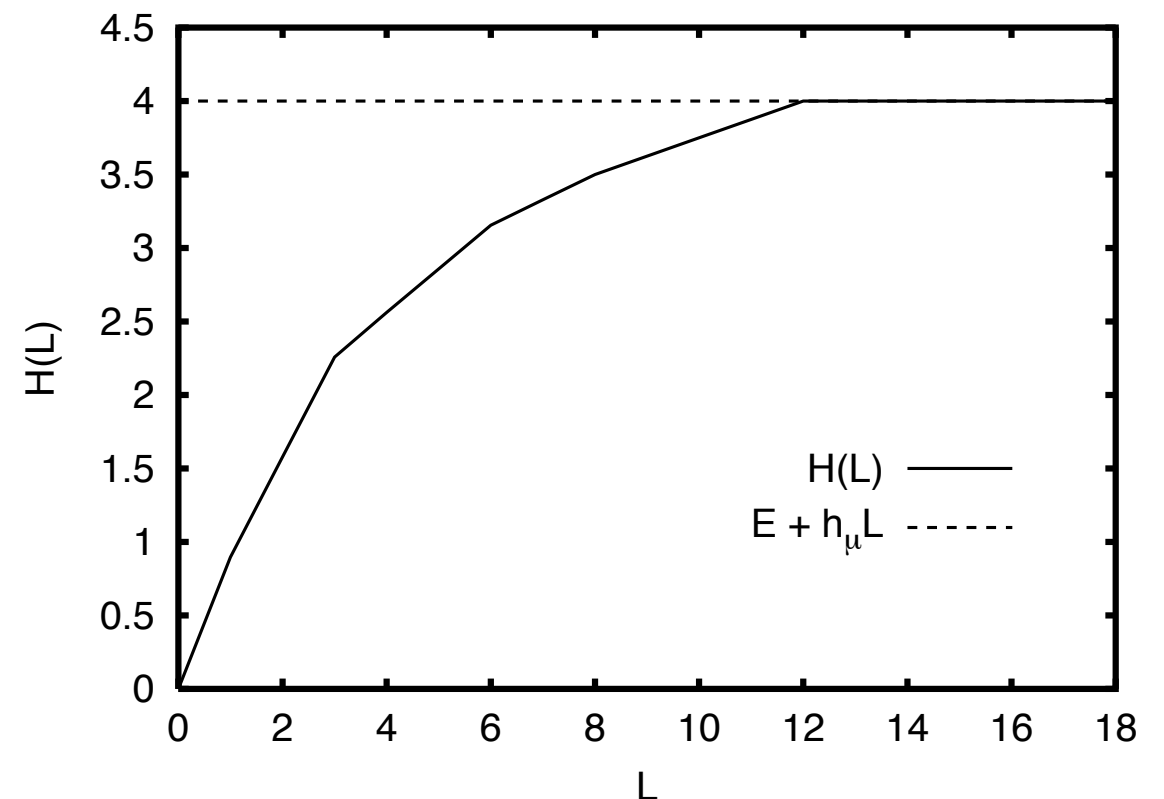
Examples of Excess Entropy ...

Period-16 Process:

$$(1010111011101110)^\infty$$

$$h_\mu = 0 \text{ bits per symbol}$$

$$\mathbf{E} = 4 \text{ bits}$$



Cf., entropy rate does not distinguish periodic processes!

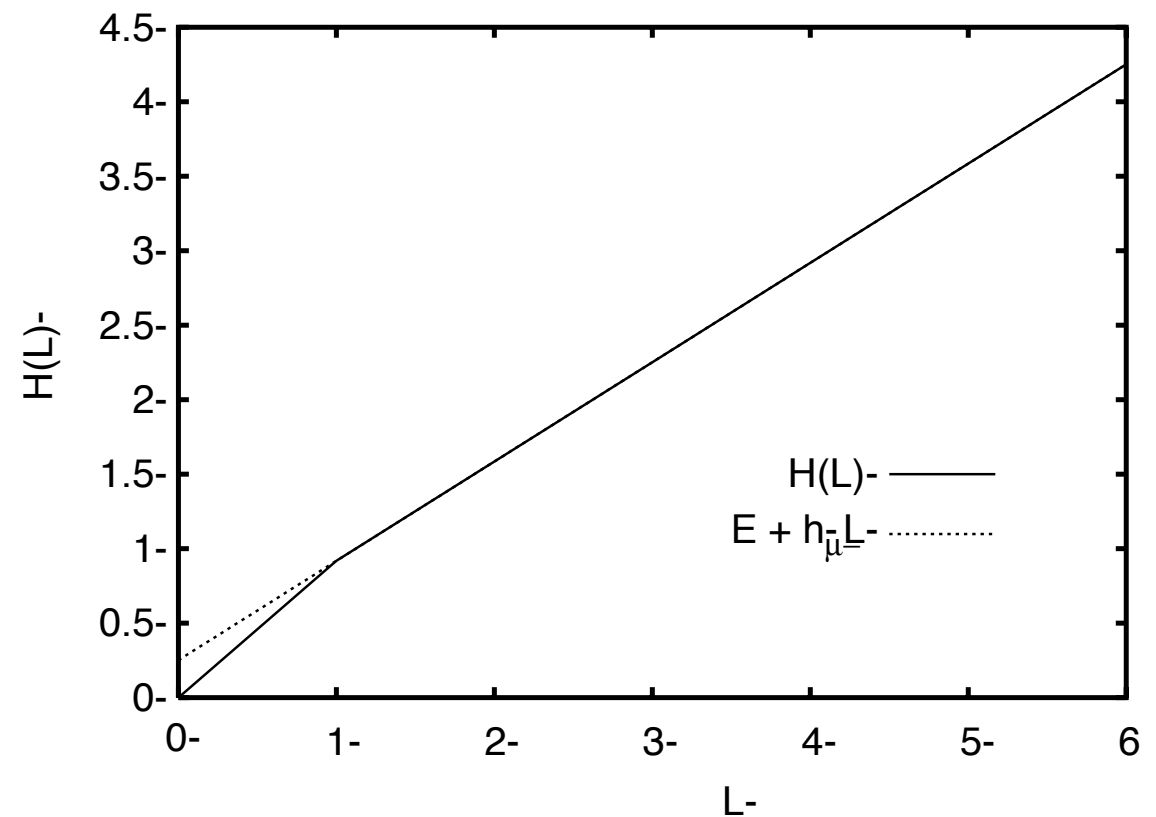
Memory in Processes ...

Examples of Excess Entropy ...

Golden Mean Process:

$$h_{\mu} = \frac{2}{3} \text{ bits per symbol}$$

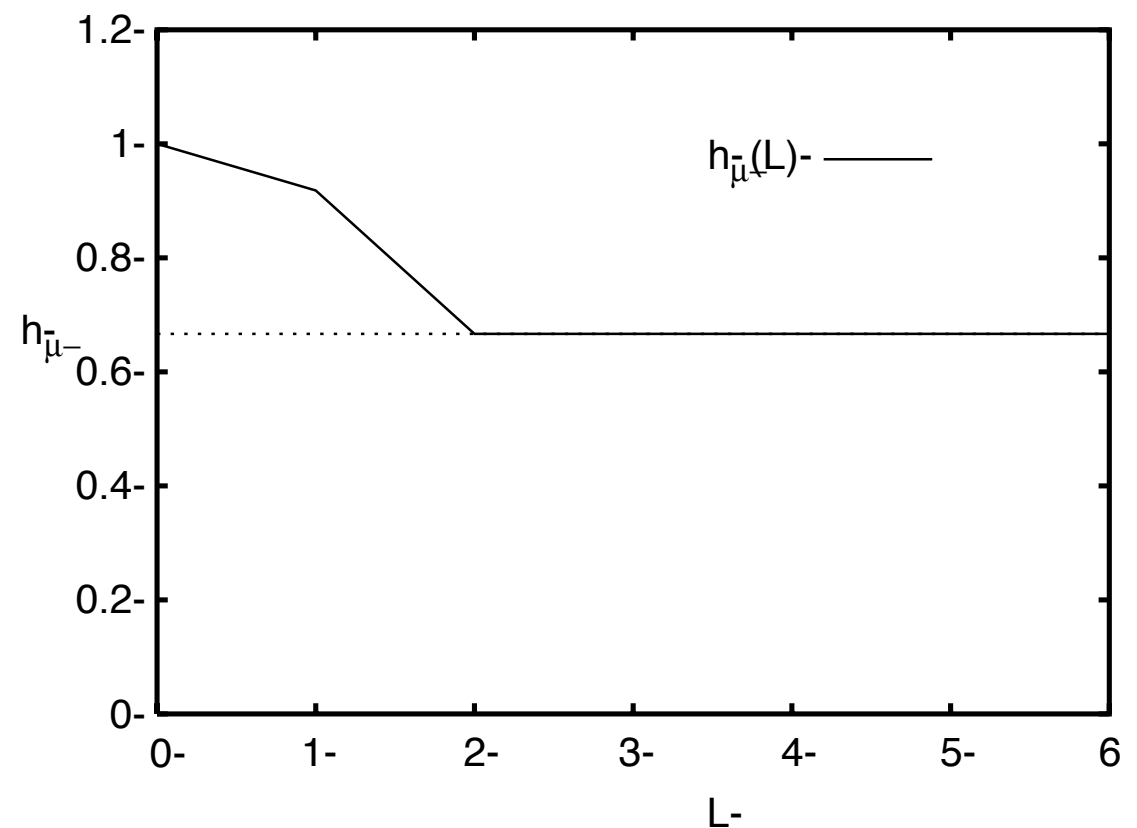
$$\mathbf{E} \approx 0.2516 \text{ bits}$$



R-Block Markov Chain:

$$\mathbf{E} = H(R) - R \cdot h_{\mu}$$

(E.g., 1D Ising Spin System)



Memory in Processes ...

Examples of Excess Entropy:

Finitary Processes: Exponential entropy convergence

Random-Random

XOR (RRXOR) Process:

$$S_t = S_{t-1} \text{ XOR } S_{t-2}$$

$$h_\mu = \frac{2}{3} \text{ bits per symbol}$$

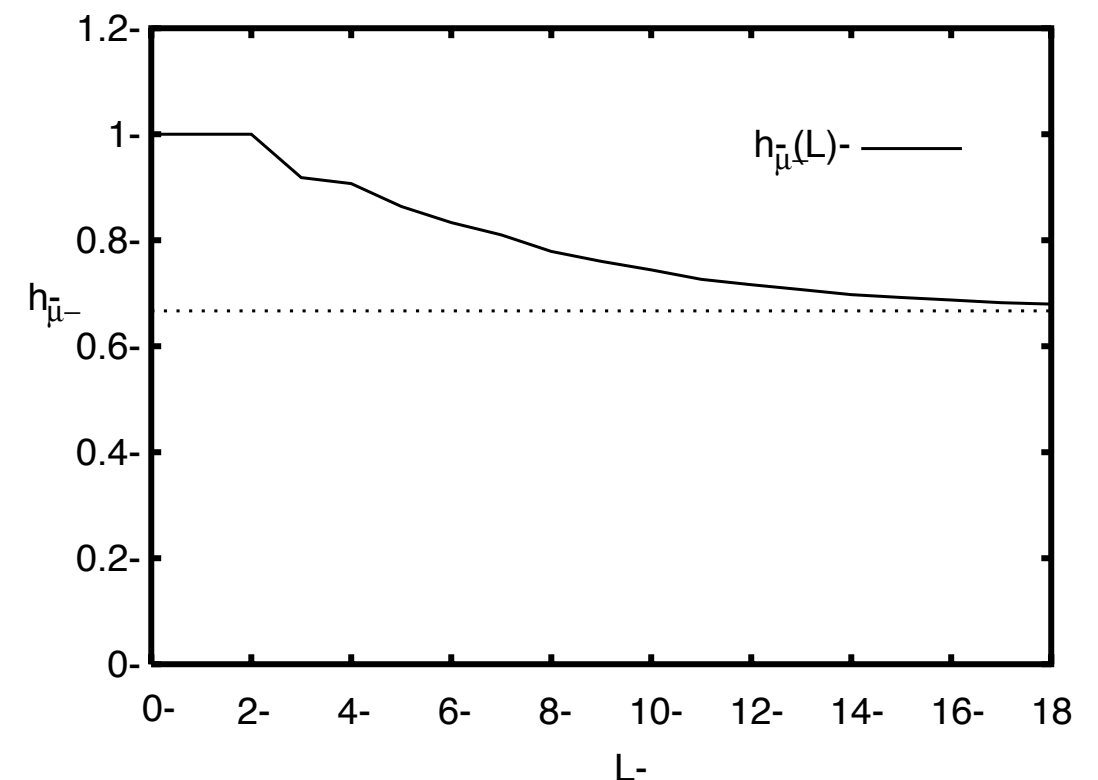
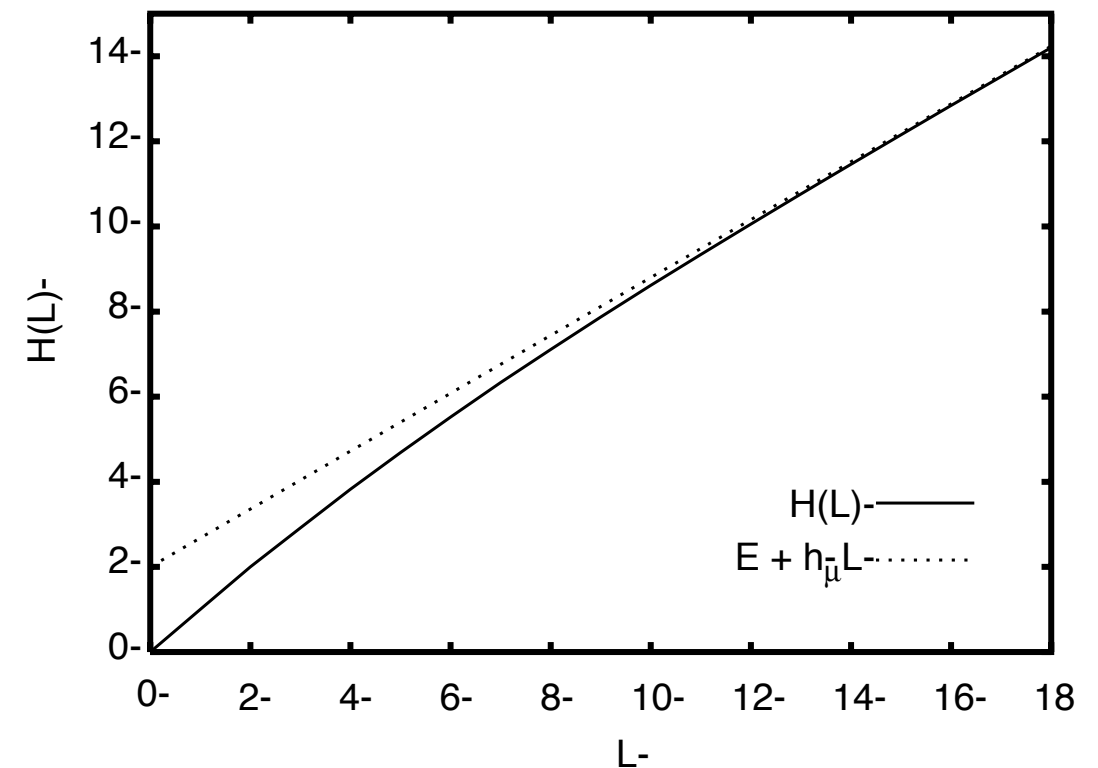
$$\mathbf{E} \approx 2.252 \text{ bits}$$

Finitary processes:

Exponential convergence:

$$h_\mu(L) - h_\mu \approx 2^{-\gamma L}$$

$$\mathbf{E} = \frac{H(1) - h_\mu}{1 - 2^{-\gamma}} \quad \gamma \approx 0.30$$



Memory in Processes ...

Examples of Excess Entropy:

Infinitary Processes:

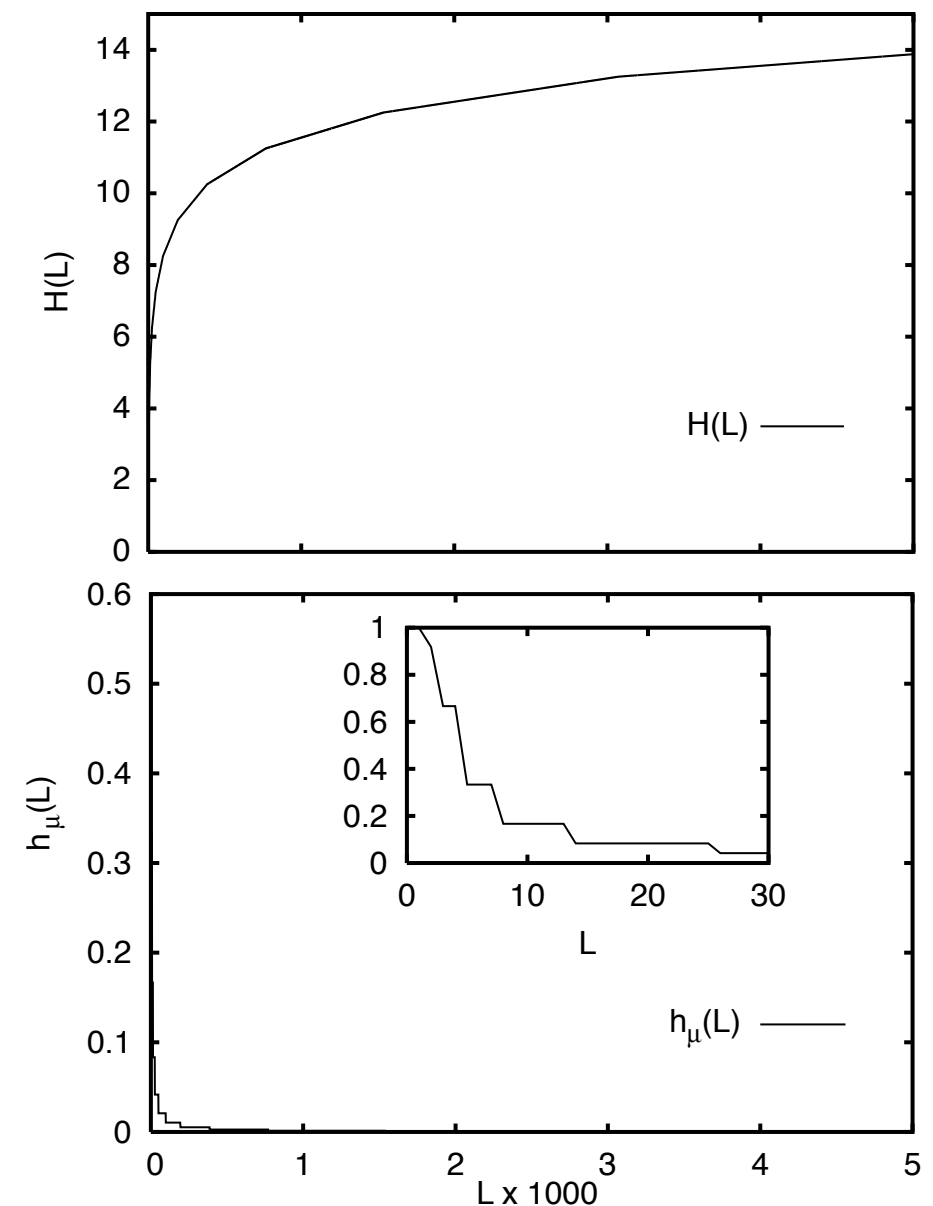
$$\mathbf{E} \rightarrow \infty$$

Excess entropy can diverge:
Slow entropy convergence
Long-range correlations
(e.g., at phase transitions)

Morse-Thue Process:

A context-free language

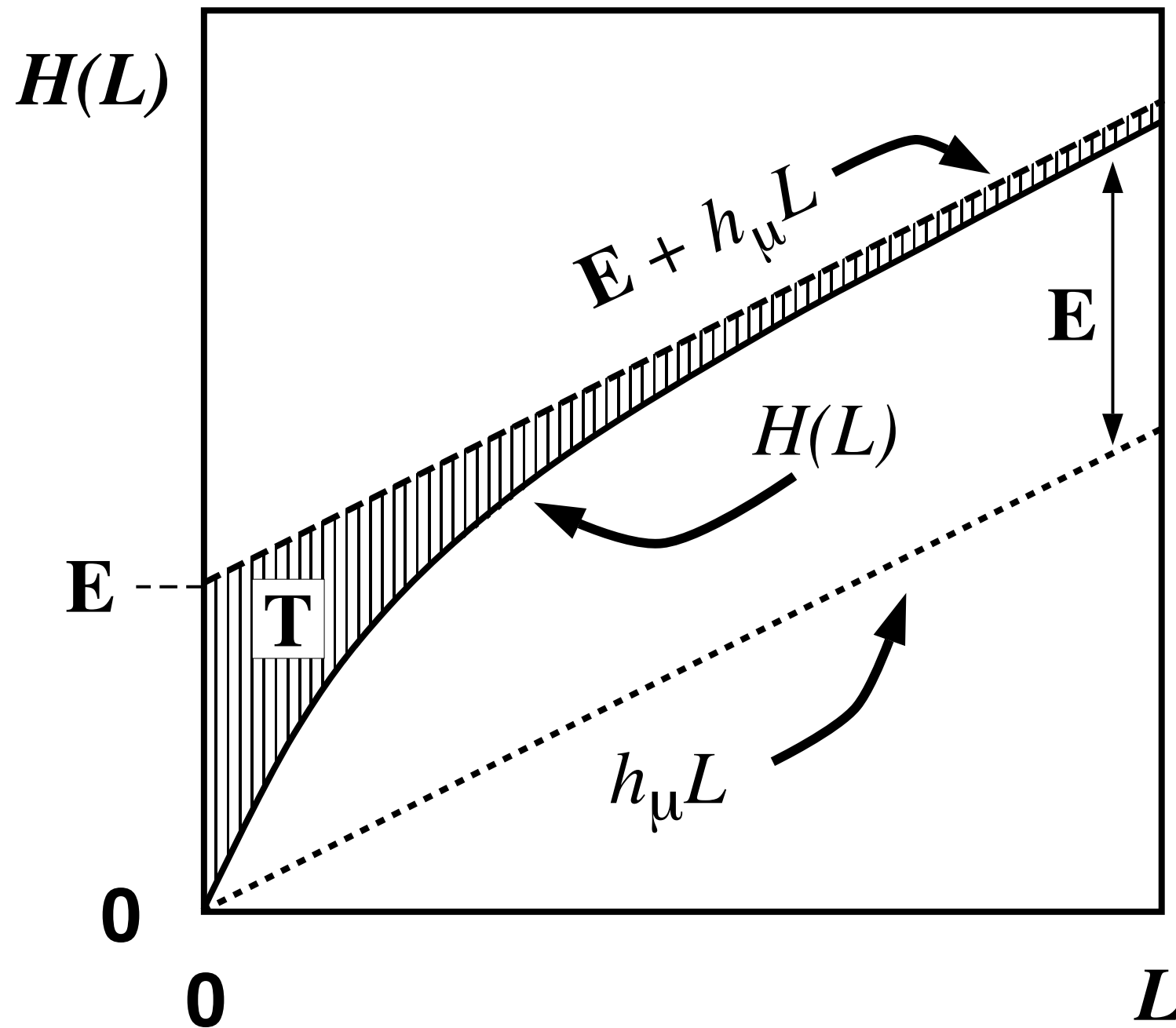
From Logistic map at onset of chaos



$$h_\mu = 0 \text{ bits per symbol}$$

Memory in Processes ...

Information-Entropy Roadmap for a Stochastic Process:



Memory in Processes ...

What is information?

Depends on the question!

Uncertainty, surprise, randomness,

Compressibility.

Transmission rate.

Memory, apparent stored information,

Synchronization.

...

Algorithmic Basis of Information

Kolmogorov-Chaitin Complexity versus Shannon Information

KC Complexity versus Shannon Information

Consider average KC Complexity of source:

$$K(\ell) \equiv \langle K(x_{0:\ell}) \rangle_{\text{realizations}}$$

Recall Block Entropy:

$$H(\ell) \equiv H[\text{Pr}(X_{0:\ell})]$$

Their growth rates equal the Shannon entropy rate:

$$h_\mu = \lim_{\ell \rightarrow \infty} \frac{H(\ell)}{\ell} = \lim_{\ell \rightarrow \infty} \frac{K(\ell)}{\ell}$$

KC Complexity of typical realizations from an information source grows proportional to the Shannon entropy rate [Brudno 1978].

KC Complexity versus Shannon Information

Again, KC Complexity is a measure of randomness, unpredictability, surprise, ...

As well as being a measure of the *deterministic* computing resources requires to *exactly* reproduce a given finite string.

KC Complexity and entropy rate maximized by IID processes.

KC Complexity versus Statistical Complexity

KC Complexity Theory:

Great mathematics.

Uncomputable.

Not quantitative: constants of proportionality unknown

Quantitative sciences use Information Theory instead.

Complexity

Information Theory for Complex Systems

Yesterday:

Complex Processes

Information in Processes

Just Finished:

Memory in Processes

Next:

Intrinsic Computation

Measuring Structure

Optimal Models

Structure = Computation

See online course:

<http://csc.ucdavis.edu/~chaos/courses/ncaso/>