Symbolic Dynamics

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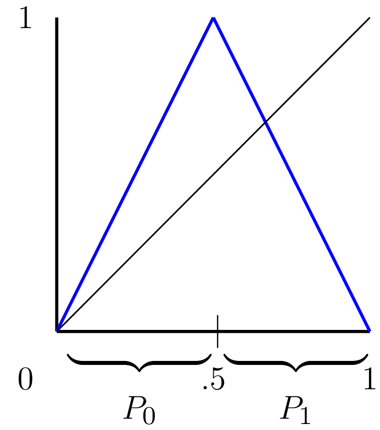
Symbolic Dynamics from a Dynamical System

Full tent map. $T : [0,1] \to [0,1]$ by

$$Tx = \begin{cases} 2x & \text{if } 0 \le x \le .5 \\ 2 - 2x & \text{if } .5 \le x \le 1 \end{cases}$$



Associate to each $x \in [0,1]$ the one-sided sequence $Px = x_0\, x_1\, x_2 \cdots$ where $x_i \in \{0,1\}$ and $T^i x \in P_{x_i}$.

$\frac{2}{5} \mapsto 01010101\cdots$

$\frac{1}{10} \mapsto 0001010101\cdots$

$P : [0,1] \to \{0,1\}^{\mathbb{N}}$ is called the itinerary map and Px is the $\mathcal{P}$-name of x.

Some Observations

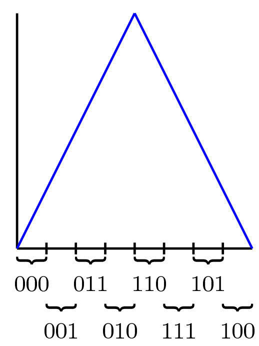
- $\frac{2}{5} = .4$ is periodic for T and $P(.4)$ is periodic for σ.
- $T^2(.1) = .4$ and $\sigma^2(P(.1)) = P(.4) = P(T^2(.1))$
- $x = .5$ has two $\mathcal{P}$-names since $.5 \in P_0 \cup P_1$: $0 1 \overline{0}$ and $1 1 \overline{0}$
- $T(P_0) = X = P_0 \cup P_1$ and $T(P_1) = X = P_0 \cup P_1$; this is a Markov condition on $\mathcal{P}$.
  Transition matrix: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow$ Every sequence $y \in \{0,1\}^{\mathbb{N}}$ is the $\mathcal{P}$-name of some point $x \in X$. ($P(X)$ is the full (one-sided) 2-shift.)

Applying T Refines the Partition

In the coding, $x_0 = \begin{cases} 0 & \text{if} \quad x \in [0, \frac{1}{2}] \\ 1 & \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$

$$x_1 = \begin{cases} 0 & \text{if} \quad Tx \in [0, \frac{1}{2}] & \text{iff} \quad x \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] \\ 1 & \text{if} \quad Tx \in [\frac{1}{2}, 1] & \text{iff} \quad x \in [\frac{1}{4}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \end{cases}$$



More Observations

- For all $n \geq 1$, the partition $\mathcal{P} \cap T^{-1}(\mathcal{P}) \cap \cdots \cap T^{-n+1}(\mathcal{P})$ consists of 2^n closed intervals of width 2^{-n} and the interiors are disjoint.

$\Rightarrow$ No two distinct points in X have the same $\mathcal{P}$-name.

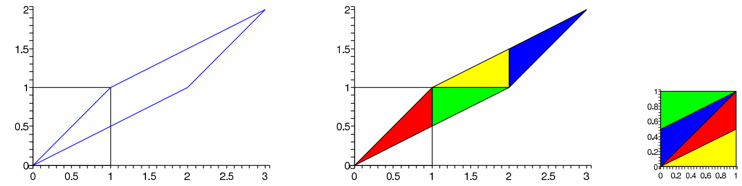
Theorem:

- $\pi : Px \to x$ is continuous
- π is onto
- $T \circ \pi = \pi \circ \sigma$
- π is at most 2-to-1, and is 1-to-1 on all transitive points (anything that does not eventually map to $.5$)

Another Example

$T_A : \mathbb{R}^2 \to \mathbb{R}^2$ by multiplication by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

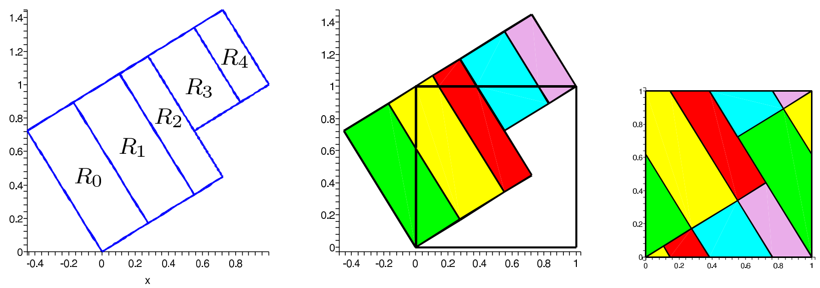
A induces a map f_A on $\mathbb{T}^2 = \mathbb{R}^2/_{\mathbb{Z}^2}$



A has two eigenvalues, $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ and $0 < \lambda_2 = \frac{3-\sqrt{5}}{2} < 1$; T_A and f_A expand in the direction of v_1 and contract in the direction of v_2.

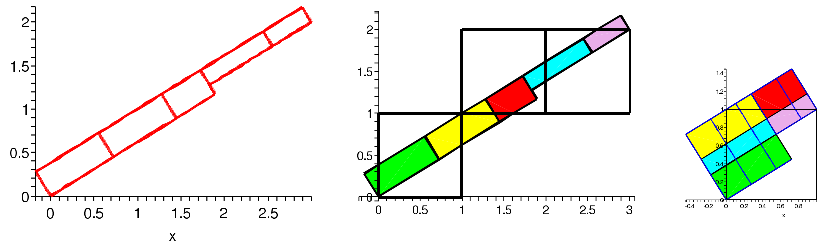
Markov Partition for f_A

Partition a fundamental region into rectangles (in $\mathbb{R}^2$) with boundaries parallel to v_1 and v_2:



Markov Partition for f_A

Apply f_A to these rectangles: boundaries expand in the direction of v_1 and contract in the direction of v_2:



Markov Partition for f_A

Markov condition: the contracting boundaries of $f_A(R_i)$ are contained in the contracting boundaries of R_i, and the expanding boundaries of $f_A^{-1}(R_i)$ are contained in the expanding boundaries of R_i

Transition matrix: $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

So $P(\mathbb{T}^2) \subsetneq \{0,1,2,3,4\}^{\mathbb{Z}}$.

$P(\mathbb{T}^2)$ is an example of a subshift, a closed σ-invariant subset of a full shift.

Near Conjugacy Again!

Let $\pi : P(\mathbb{T}^2) \to \mathbb{T}^2$ be the map which assigns a bi-infinite sequence over $\{0,1,2,3,4\}$ to the unique point in $\mathbb{T}^2$ having that sequence as its $\mathcal{R}$-name.

Theorem:

- π is continuous,
- π is onto,
- $f_A \circ \pi = \pi \circ \sigma$,
- π is uniformly bounded-to-one, and
- π is one-to-one on doubly transitive points

Recall the Shift Spaces

- $A_d = \{0, 1, \cdots, d-1\}$ is a finite alphabet
- $\Sigma_d = A_d^{\mathbb{Z}}$ is the full (two-sided) d-shift
- $\Sigma_d^+ = A_d^{\mathbb{N}}$ is the full one-sided d-shift
  That is, $x \in \Sigma_d^{(+)}$ is a sequence $\{x_i \in A_d\}_{i \in \mathbb{Z}(\text{or } \mathbb{N})}$.
- Metric: $d(x,y) = 2^{-k}$, where $k = \min\{|i| : x_i \neq y_i\}$
- Shift map: $\sigma : \Sigma_m^{(+)} \to \Sigma_m^{(+)}$ by $\sigma(x)_i = x_{i+1}$

Subshift Spaces

- A subshift is a closed subspace $X \subseteq \Sigma_m^{(+)}$ with the property $\sigma(X) \subseteq X$.
- Some examples:
  1. $X_{GM} = \{x \in \{0,1\}^{\mathbb{Z}} : x_i\, x_{i+1} \neq 11$ for all $i \in \mathbb{Z}\}$
  2. $X_{\text{even}} = \{x \in \{0,1\}^{\mathbb{Z}} :$
     between any pair of 1's is an even number of 0's$\}$
  3. A non-example:
     $X = \{x \in \{0,1\}^{\mathbb{Z}} : x_i = 1$ for exactly one $i \in \mathbb{Z}\}$

  Why not?
- The sequence of iterates $\sigma^n(\cdots 0\,0\,0\,.1\,0\,0\,0 \cdots)$ converges (as $n \to \infty$) to the point $\overline{0} \notin X$.

Language of a Subshift

A subshift, X, can be characterized by its language, $\mathcal{L}(X)$ (which is factorizable and extendible), or equivalently by a set of forbidden words, $\mathcal{F}$; $X = X_{\mathcal{F}}$.

- The full shift has $\mathcal{F} = \emptyset$
- X_{GM} has many options:
  $\mathcal{F}_1 = \{11\}$
  $\mathcal{F}_2 = \{110, 111\}$
  $\mathcal{F}_3 = \{1100, 1101, 1110, 1111\}$
  $\vdots$
- X_{even} has $\mathcal{F} = \{10^{2k+1}1 : k \geq 0\}$

If there exists a finite collection $\mathcal{F}$ describing the language of a subshift X, then $X = X_{\mathcal{F}}$ is a subshift of finite type (SFT).

SFT's and Walks on Directed Graphs

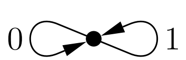
We can represent $(X_{\mathcal{F}}, \sigma)$ by bi-infinite walks on two kinds of graph shifts; edge shifts and vertex shifts.

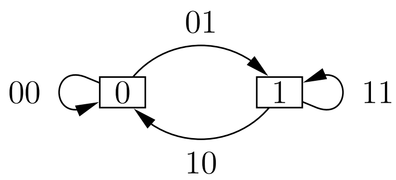
We may assume that all words in $\mathcal{F}$ have the same length, $m+1$, and say that $X_{\mathcal{F}}$ has memory m or is an m-step SFT.

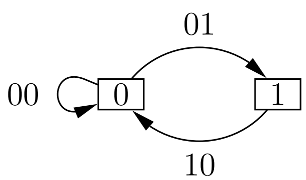
Construct a directed graph $G_{\mathcal{F}}$ with vertices $\mathcal{V}(G_{\mathcal{F}})$ and edges $\mathcal{E}(G_{\mathcal{F}})$ as follows:

- The vertices are (and are labeled by) the allowed m-blocks in $X_{\mathcal{F}}$.
- Put an edge from $B_1 \in A^m$ to $B_2 \in A^m$ if for $a, c \in A$ and $B \in A^{m-1}$, $B_1 = aB$, $B_2 = Bc$, and $aBc \notin \mathcal{F}$. Label the edge aBc.

Graph Shift Examples

- $\Sigma_d = X_{\emptyset}$: 

or 

- $X_{GM} = X_{\{11\}}$: 

Walks are labeled either by vertices of by edges, and words in $X_{\mathcal{F}}$ are represented by overlapping the middle symbols in the labels.

Adjacency Matrix for a Graph Shift

- A graph with walks labeled by vertices can be described by a $d \times d$ matrix of 0's and 1's, M: $M_{ij} = 1$ if there is an edge from vertex i to vertex j.
- Therefore, given such a matrix M, we obtain a graph G and the vertex shift, $\widehat{X}_M = \{x \in A_d^{\mathbb{Z}} : M_{x_i\, x_{i+1}} = 1$ for all $i \in \mathbb{Z}\}$.
- A graph with walks labeled by edges can be described by a square non-negative matrix M: M_{ij} is the number of edges from vertex i to vertex j.
- Therefore, given such a matrix, M, we obtain a graph, G, and the edge shift, $X_M = \{x \in \mathcal{E}(G)^{\mathbb{Z}}$: the terminal vertex of x_i is the initial vertex of x_{i+1} for all $i \in \mathbb{Z}\}$

Now to Linear Algebra

Theorem: Suppose G is a graph with adjacency matrix M. Then

1. The number of paths in G of length k from vertex i to vertex j
   $=$ number of words in $\widehat{X}_G$ of length $k+1$
   $= (M^k)_{ij}$.
2. The number of cycles in G of length k
   $=$ number of points in X_G with period k
   $=$ number of points in $\widehat{X}_G$ with period k
   $= \mathrm{tr}(A^k)$

So from an original dynamical system, we've passed through symbolic dynamics to graph theory and linear algebra!

Entropy of a subshift

For a subshift space $X \subseteq A^{\mathbb{Z}}$,

$$h_{\text{top}}(\sigma|_X) = h_{\text{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\{w \in A^n : w \in \mathcal{L}(X)\}|.$$

Entropy of the Golden Mean Subshift, $X_{\{11\}}$:

$|\{w \in A^n : w \in \mathcal{L}(X)\}| =$ the number of paths of length n on the vertex shift representing $X_{\{11\}} = \sum_{i,j=1}^{2} (M^{n-1})_{ij}$

Diagonalize M in order to more easily count these entries: eigenvalues of M are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ with eigenvectors $v_1 = \langle \lambda_1, 1\rangle$ and $v_2 = \langle \lambda_2, 1\rangle$

So letting $P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$ gives $P^{-1}MP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Entropy of the Golden Mean Subshift

Thus $P^{-1}M^nP = (P^{-1}MP)^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$, and so

$M^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$, where

$f_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}}$ is the n^{th} Fibonacci number.

Now

$$\sum_{i,j=1}^{2} (M^{n-1})_{i,j} = f_n + f_{n-1} + f_{n-1} + f_{n-2} = f_{n+1} + f_n = f_{n+2}$$

Thus, $h_{\text{top}}(X_{\{11\}}) = \lim_{n\to\infty} \frac{1}{n} \log \left(\frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\sqrt{5}} \right) = \log \lambda_1.$

Entropy of Subshifts

Theorem:

- If G is an irreducible graph, M is the adjacency matrix for G, and λ_M is the largest eigenvalue of M, then $h_{\text{top}}(\widehat{X}_G) = \log \lambda_M$
- If X is an irreducible SFT with memory m and G is the essential graph with X_G representing X, then $h_{\text{top}}(X_G) = \log \lambda_M$, where M and λ_M are as above.

Measures on Shift Spaces

- Let p be any probability distribution on A such that $p(a) > 0\ \forall\ a \in A$. The Bernoulli measures μ_p on $A^{\mathbb{Z}}$ are defined by: $\mu_p(C^{0,\cdots,k_1}_{a_0\cdots a_{k-1}}) = p(a_0)\cdots p(a_{k-1})$.
- $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is measure preserving for such μ_p: for any cylinder set C, $\mu_p(\sigma^{-1}(C)) = \mu_p(C)$.
- Suppose X is an irreducible SFT and let G be a vertex graph for X with adjacency matrix M.
  Let P be any $|A| \times |A|$ stochastic matrix such that $P_{ij} > 0 \iff M_{ij} > 0$ and let $\vec{p}$ be a stationary probability vector for P; $\vec{p}P = \vec{p}$.
  Define the $(\vec{p}, P)$ Markov measures on X by $\mu([a_1 \cdots a_k]_i) = p_{a_1}P_{a_1a_2}\cdots P_{a_{k-1}a_k}$.

Measure-theoretic Entropy of Subshifts

- For an irreducible SFT, X, equipped with the $(\vec{p}, P)$-Markov measure, μ, the measure-theoretic entropy of the shift w.r.t. μ is $h_\mu(X) = -\sum_{i,j} p_i P_{ij} \log P_{ij}$.
- For an irreducible SFT, X, with associated vertex shift G and transition matrix M, there is a unique shift-invariant measure which has maximum entropy, $\log \lambda_M$.
- This measure is given by initial probability vector $\vec{p}$ given by $p_i = \ell_i r_i$, $i = 0, \cdots, d-1$, where $\vec{\ell}$ and $\vec{r}$ are the left- and right- positive eigenvectors of M corresponding to λ_M with $\sum_i \ell_i r_i = 1$, and stochastic matrix P given by $P_{ij} = M_{ij}\dfrac{1}{\lambda_M}\dfrac{r_j}{r_i}$.