Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Summer School

9-10 June 2011
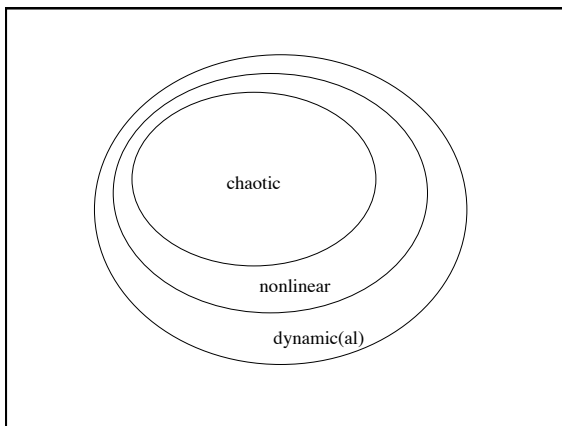
Liz Bradley

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Liz Bradley

lizb@cs.colorado.edu

Chaos

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

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chaotic

nonlinear

dynamic(al)

Chaos

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and "well-understood"

Where chaos turns up

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
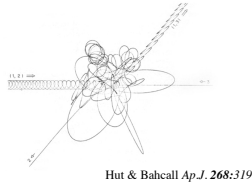 - Air/fuel flow in combustion chambers

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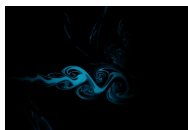
Where chaos turns up

- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies

  - and lots of other electronic, chemical, & biological systems

Where chaos turns up

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….
- Protein folding
- Population biology
- And many, many other fields (including yours)

Hut & Bahcall *Ap.J. 268:319*

- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differential equations

- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: difference equation

UH—OH. MY AUDIENCE HAS FALLEN INTO A POWERPOINT COMA.

THE ONLY THING I CAN DO NOW IS PUT THEM IN FUNNY POSES AND LEAVE.

IT LOOKS LIKE HIS FINGER HIT BRAIN.

A useful graphical solution technique

- "cobweb" diagram
- *aka* return map
- *aka* correlation plot
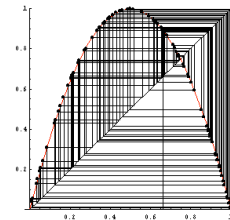
Image from Doug Ravenel's website at URochester
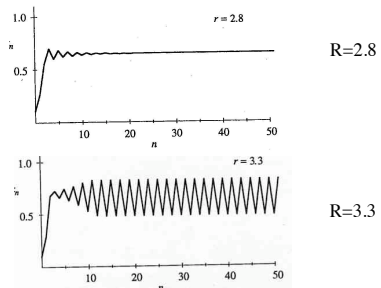

Bifurcations

Qualitative changes in the dynamics caused by changes in *parameters*

Bifurcations

Qualitative changes in the dynamics caused by changes in *parameters:*
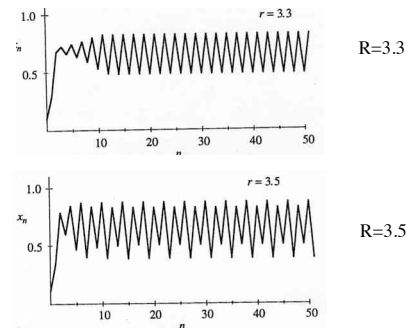
- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- etc. etc.

Bifurcations in the logistic map

R=2.8

R=3.3

Note: in discrete time plots, it makes no sense to connect dots!!

Plots from *Strogatz*

R=3.3

R=3.5

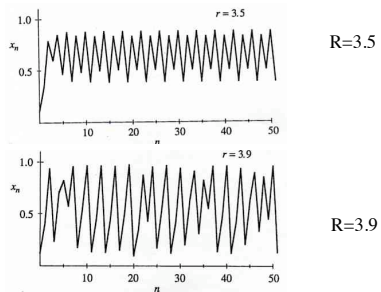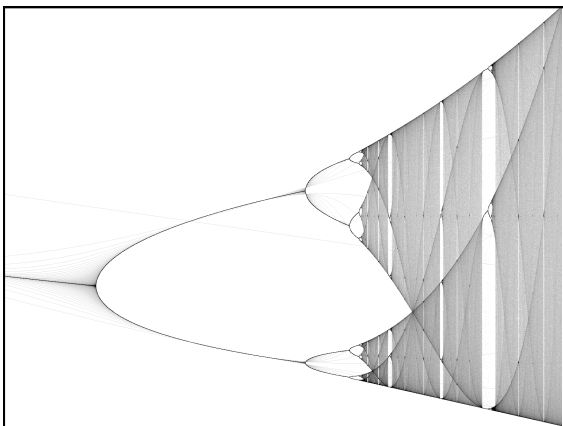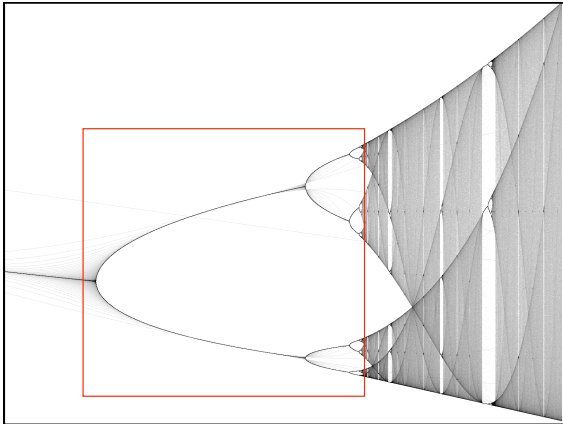Plots from *Strogatz*

R=3.5

R=3.9
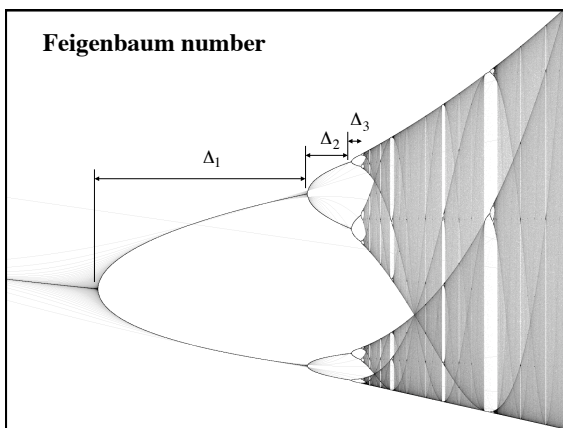
Figure 10.2.5

Plots from *Strogatz*

Courtesy of Allison Brown;
Chaos, in preparation.

- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)

3

- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- *period-doubling cascade @ low R*

Feigenbaum number

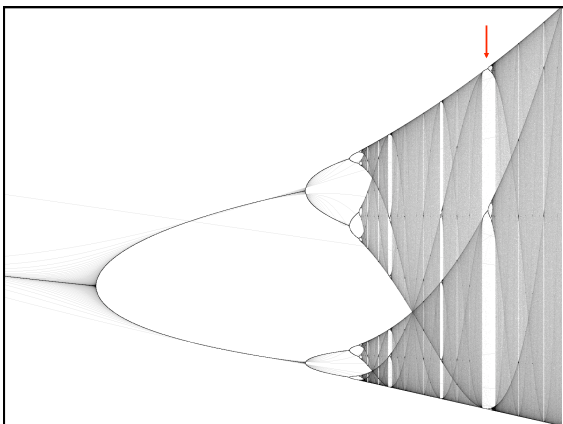Δ_1 Δ_2 Δ_3

Universality!

Feigenbaum number and many other interesting chaotic/dynamical properties hold for any 1D map with a quadratic maximum.

Proof: renormalizations. See Strogatz §10.7

Don't take this too far, though…

- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- *windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)*
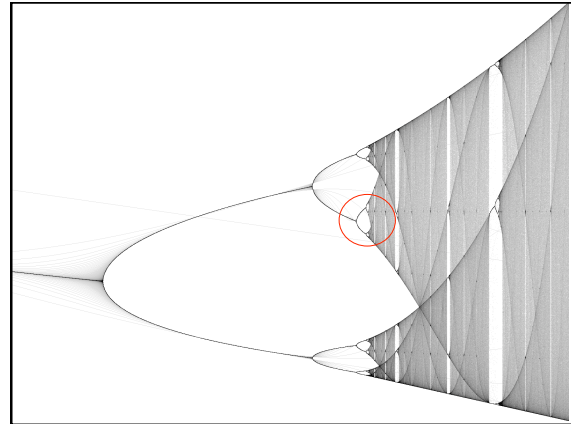
A bit more lore on periods and chaos

• Sarkovskii (1964)

  3, 5, 7, …3x2, 5x2, …3x2², 5x2², … 2², 2, 1

  Yorke (1975)

• Metropolis *et al.* (1973)

• chaos

• veils/bands: places where chaotic attractor is dense (UPOs)

• period-doubling cascade @ low R

• windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)

• *small copies of object embedded in it (fractal)*

Fractals

• non-integer Hausdorff dimension
• self-similar

Images from Gleick
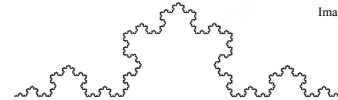
Examples: Cantor set, coastlines, trees, lungs, clouds, drainage basins, …

The Mandelbrot set
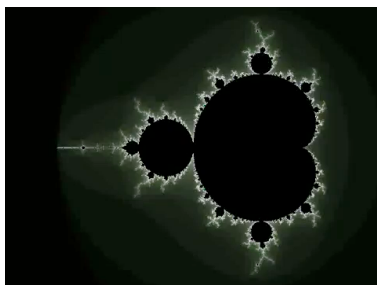
www.youtube.com/watch?v=G_GBwuYuOOs

Fractals in computer graphics
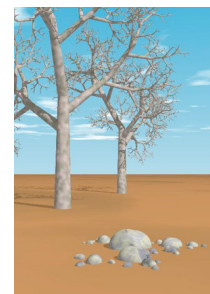
Matthew Ward, WPI
davis.wpi.edu/~matt/courses/fractals/trees.html

Fractals in maps

Newton's method on $x^4 - 1 = 0$

From Strogatz

Fractals and chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.
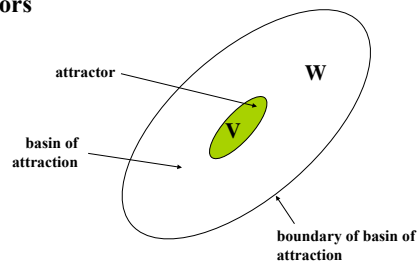
But **not** "*all*."

So far: mostly about *maps*.

- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: difference equation

Next up: *flows*

- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differential equations

Attractors

attractor

W

basin of attraction

V

boundary of basin of attraction

- Attractors exist only in dissipative systems!
- Dissipation $\Longleftrightarrow$ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

Conditions for chaos in continuous-time systems

Necessary:
- Nonlinear
- At least three state-space dimensions   (NB: only one needed in maps)

Necessary and sufficient:
- "Nonintegrable"
  - i.e., cannot be solved in closed form

Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter
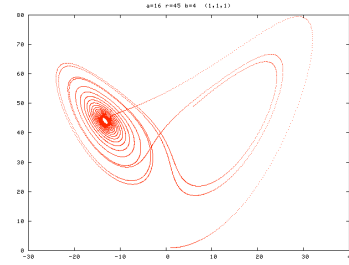
(Note: by Jim Crutchfield!)

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.
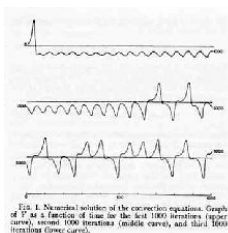
Fig. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).
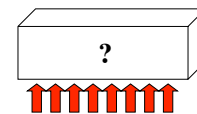
Fig. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

- Equations:

$$x' = a(y\text{-}x)$$
$$y' = rx - y - xz$$
$$z' = xy - bz$$

?

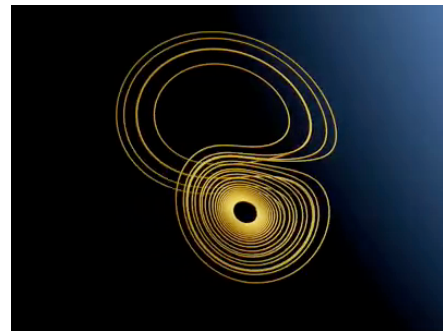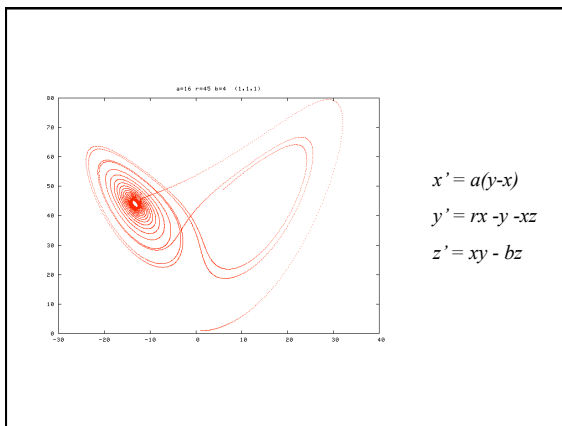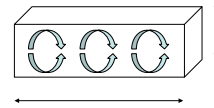(first three terms of a Fourier expansion of the Navier-Stokes eqns)

- State variables:
  - *x* convective intensity
  - *y* temperature
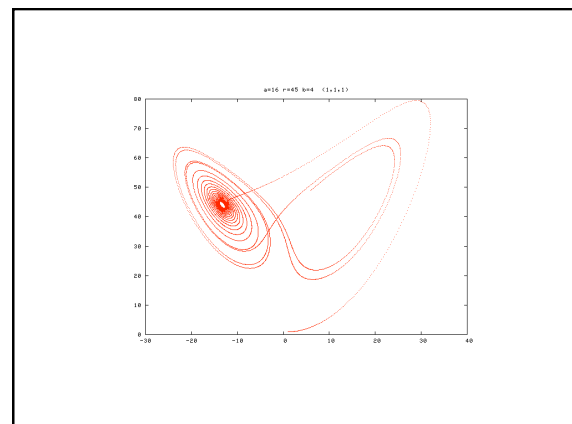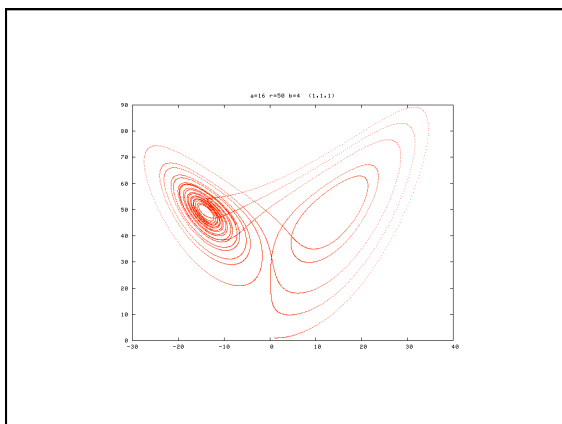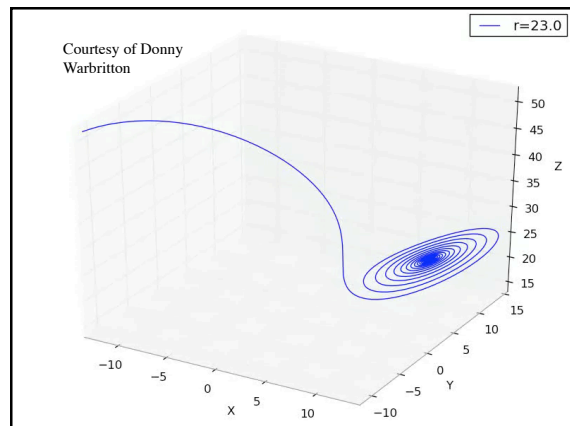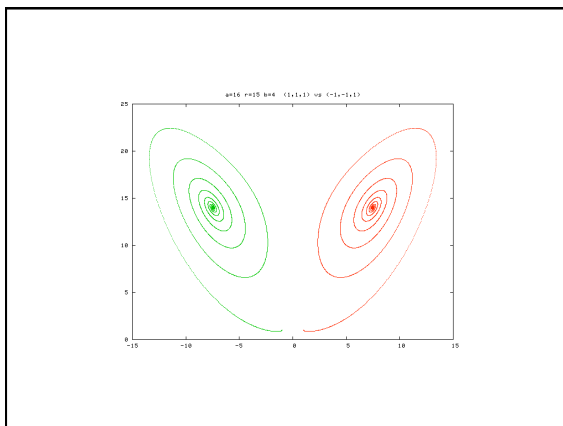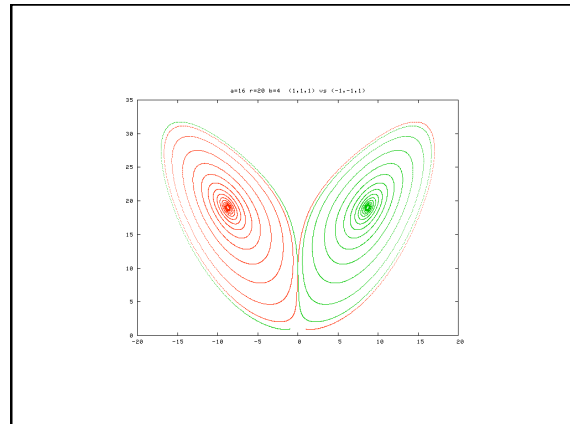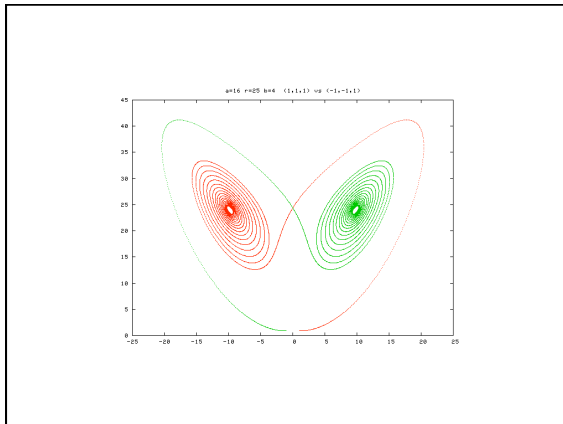 - *z* deviation from linearity in the vertical convection profile

- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to ΔT
  - *b* aspect ratio of the fluid sheet

$x' = a(y-x)$
$y' = rx - y - xz$
$z' = xy - bz$

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Courtesy of Donny Warbritton

Attractors

 Four types:

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction (plus the basin boundaries) *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc.

Attractors

- Fixed point
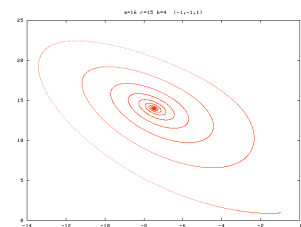
Attractors

• Limit cycle
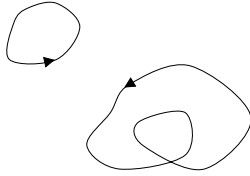
Attractors

• Quasi-periodic orbit…

"Strange" or chaotic attractors

• *often* fractal

• covered densely by trajectories

• exponential divergence of neighboring trajectories…
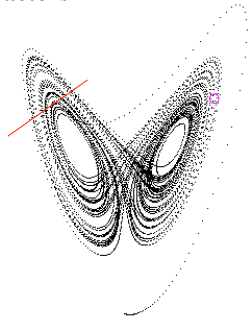
Lyapunov exponents

• nonlinear analogs of eigenvalues: one λ for each dimension
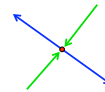
Lyapunov exponents: summary

• nonlinear analogs of eigenvalues: one λ for each dimension

• negative λ_i compress state space; positive λ_i stretch it

• $\Sigma\lambda_i < 0$ for dissipative systems

• long-term average in definition; biggest one dominates as $t \rightarrow$ infinity

• *positive λ is a signature of chaos*

• λ_i are same for all ICs in one basin

"Strange" or chaotic attractors:

• exponential divergence of neighboring trajectories

• *often* fractal

• covered densely by trajectories
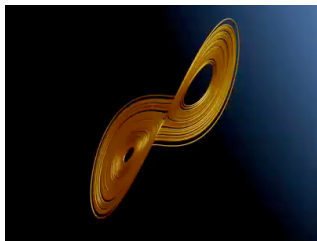
• contain an infinite number of "unstable periodic orbits"…
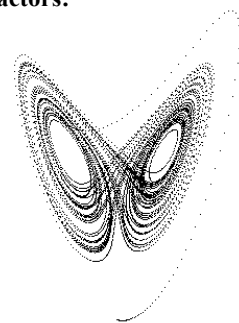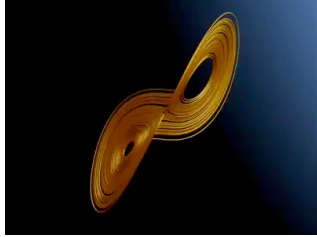
© 2006 Jos Leys and Etienne Ghys; www.josleys.com

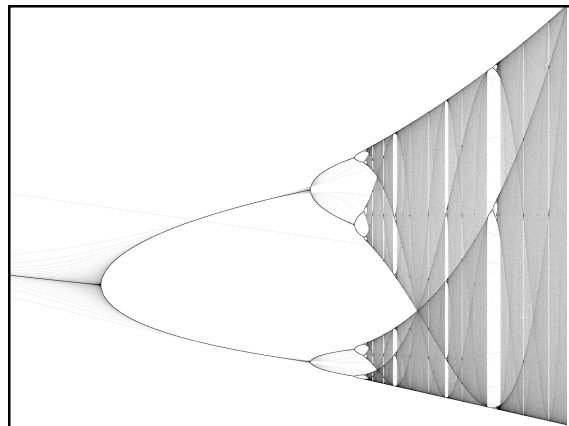**Unstable periodic orbits (UPOs)**
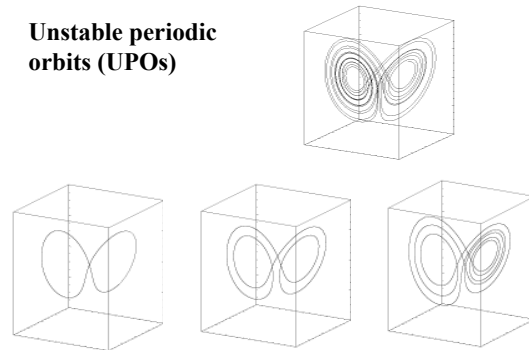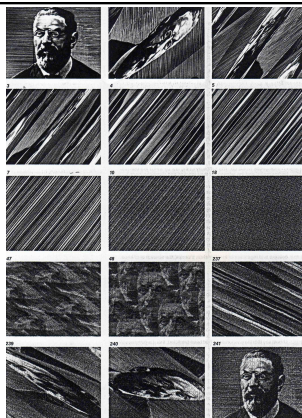

Bradley/Mantilla, *Chaos* **12**:596

Attractor "bones"…
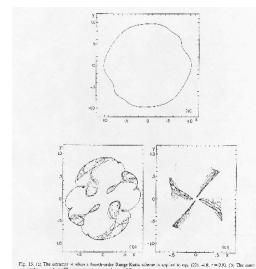

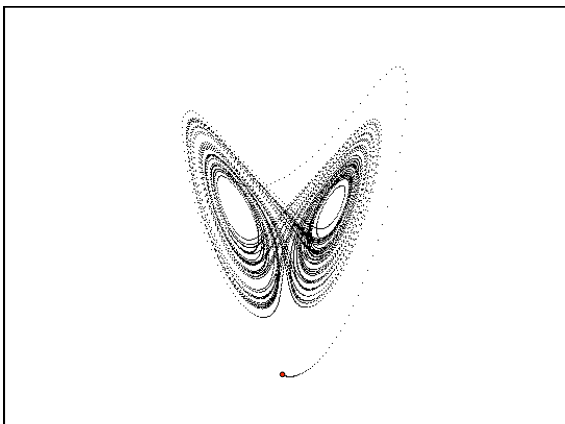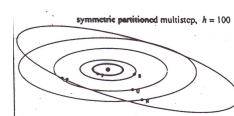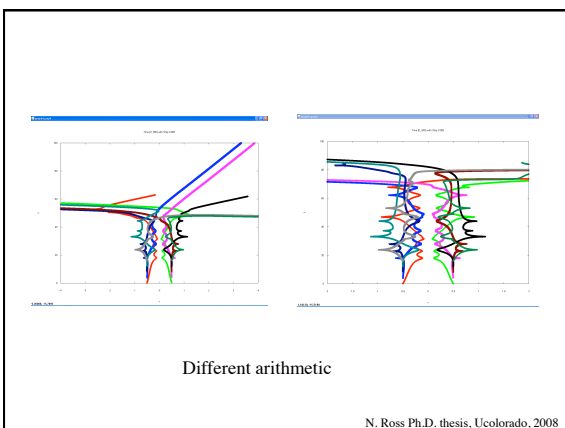© 2006 Jos Leys and Etienne Ghys; www.josleys.com

Poincare recurrence

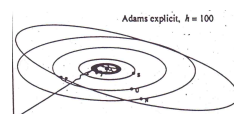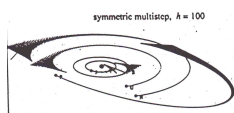Crutchfield *et al*.
Chaos **255:**46

The rest of today…

• Symbolics I: after lunch (here)
• Dynamics Lab I:
  • Meet here at 3pm
  • Bring your laptop, if you have one here
  • Lab handouts on the CSSS wiki

Different timestep

Lorenz, *Physica D* **35:**229

Different arithmetic

N. Ross Ph.D. thesis, Ucolorado, 2008

Different solver algorithm…

6/10/11

12

**Moral: numerical methods can run amok in "interesting" ways…**

- can cause distortions, bifurcations, etc.

- and these look a lot like *real, physical* dynamics…

- source: algorithms, arithmetic system, timestep, etc.

- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.

- and these look a lot like *real, physical* dynamics…

- source: algorithms, arithmetic system, timestep, etc.

- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

  - *change the timestep*

  - *change the method*

  - *change the arithmetic*

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….
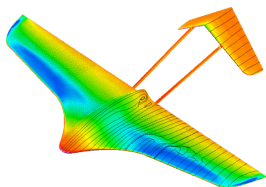

…??!?
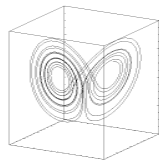
Shadowing lemma

Every* noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

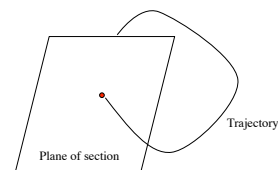Important: this is for *state* noise, not *parameter* noise.

(*) Caveat: not if the noise bumps the trajectory out of the basin
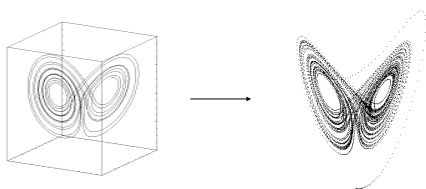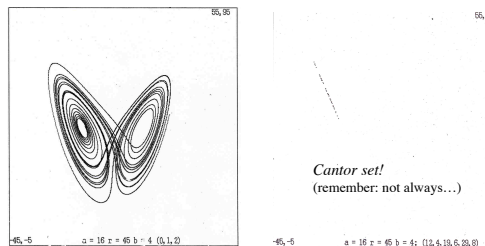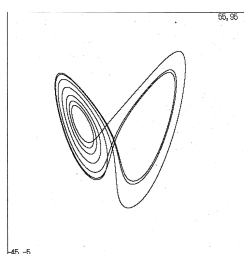

Solving *PDEs*

www.tecplot.com

Section

Trajectory

Plane of section

Not the same thing as a projection!

Section of a chaotic attractor

Cantor set!
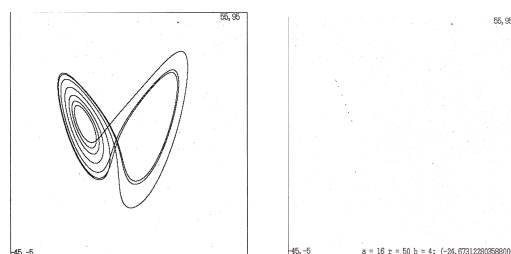(remember: not always…)

**Section of a UPO**

?

Aside: finding UPOs

- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers
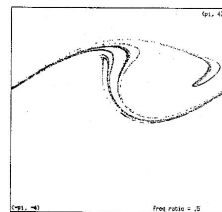
Back to sections…*time-slice* ones now.

Time-slice sections of periodic orbits: some thought experiments

- pendulum rotating @ 1 Hz and strobe @ 1 Hz?

- pendulum rotating @ 1 Hz and strobe @ 2 Hz?

- pendulum rotating @ 1 Hz and strobe @ 3 Hz?

- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?

- pendulum rotating @ 1 Hz and strobe @ π Hz? (or some other irrational)
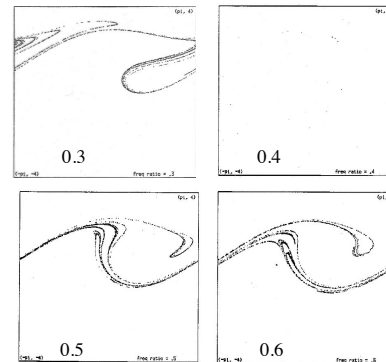

When this becomes really useful…

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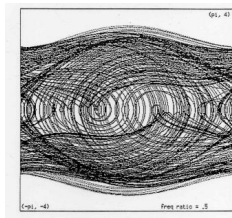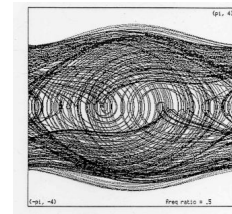
trajectory   Poincare section

What bifurcations look like on a Poincare section

0.3

0.4

0.5

0.6

Computing sections

- Space-slice: use the "inside-outside" function

- Time-slice: use modulo on the timestamp


Stability, λ, and the un/stable manifolds

Lyapunov exponents:

- one λ for each dimension; $\Sigma\lambda < 0$ for dissipative systems

- λ are same for all ICs in one basin

- negative λ compress state space along *stable manifolds*

- positive λ stretch it along *unstable manifolds*

- biggest one λ dominates as t $\longrightarrow$ infinity

- *positive λ_1 is a signature of chaos*

- calculating them:
  - From equations: eigenvalues of the variational matrix (see variational system notes on CSCI5446 homepage)
  - From data: various algorithms that are hideously sensitive to numerics, noise, data length, & algorithmic parameters...

Calculating λ (& other invariants) *from data*

- A good reference: Kantz & Schreiber, *Nonlinear Time Series Analysis* (Abarbanel's book is also very good)

- Associated software: TISEAN
  `www.mpipks-dresden.mpg.de/~tisean`

- *Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!*

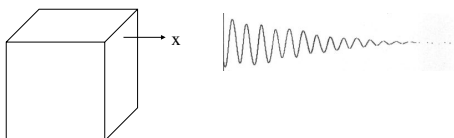- Use your dynamics knowledge to understand & use those knobs intelligently

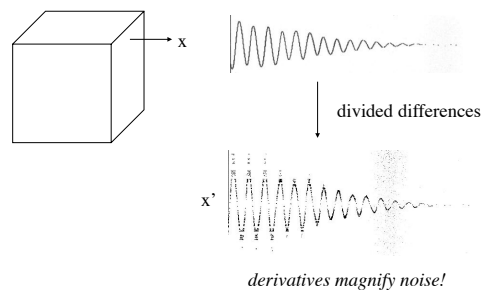## These λ & manifolds play a role in control of chaos...
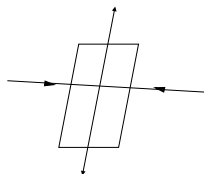
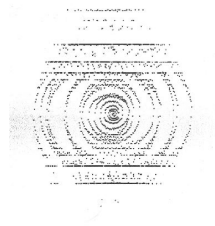## We've been assuming that we can measure all the state variables...
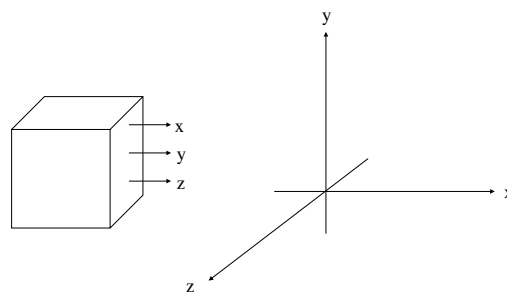
But often you can't.
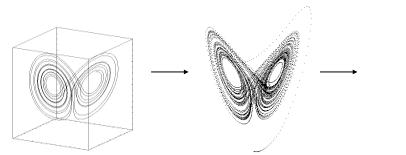

How to reconstruct the other state vars?

divided differences

x'

derivatives magnify noise!

What this looks like in the state space
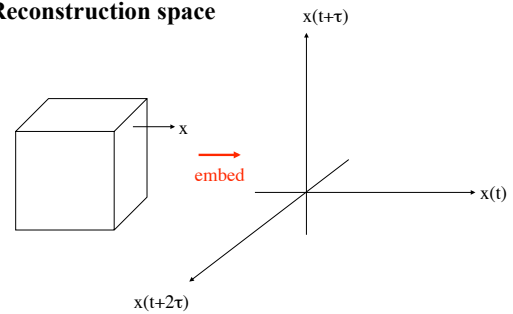

This is not useful.

What we want here is to undo a projection

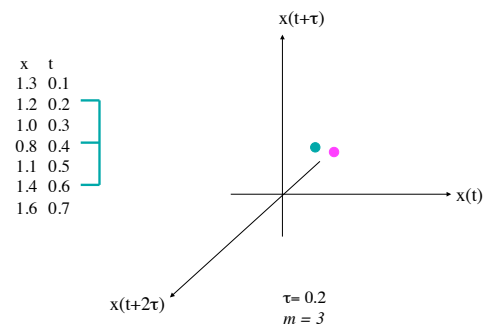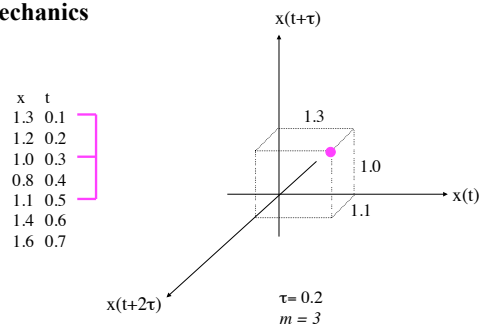**Delay-coordinate embedding**

"reinflate" that squashed data to get a *topologically identical* copy of the original thing.

Reconstruction space

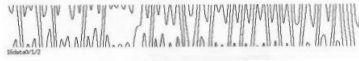
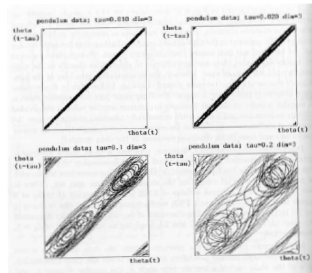
x(t+τ)

x

embed

x(t)

x(t+2τ)

Mechanics

x(t+τ)

x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7

1.3

1.0

1.1

x(t)

x(t+2τ)

τ = 0.2
m = 3

x(t+τ)

x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7

x(t)

x(t+2τ)

τ = 0.2
m = 3

TISEAN's `delay` command does this
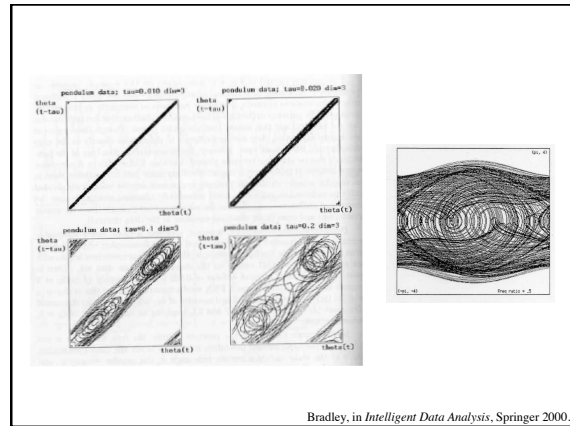
17

What this looks like

Scalar time-series data:

Reconstructed dynamics:

Bradley, in *Intelligent Data Analysis*, Springer 2000.

Bradley, in *Intelligent Data Analysis*, Springer 2000.

Takens(*) theorem

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (have same topology as) the original state-space dynamics.

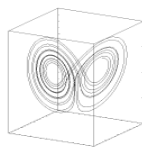Note: the measured quantity must be a smooth, generic function of at least one state variable, and must be uniformly sampled in time.

* Whitney, Mane, …

Diffeomorphisms and topology

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape

- have same dynamical invariants (e.g., λ)

Measure just the *x* coordinate…

…and then embed:
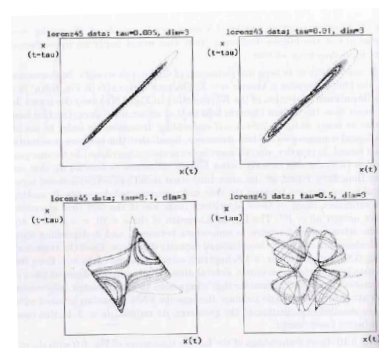
Takens* theorem

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (have same topology as) the original state-space dynamics.

Picking τ
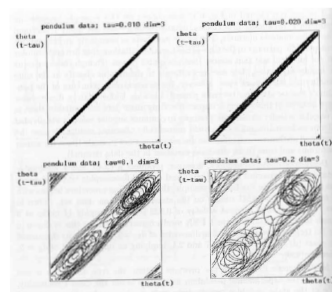
TISEAN contains tools that help you do this (e.g., `mutual`)

Picking *m*

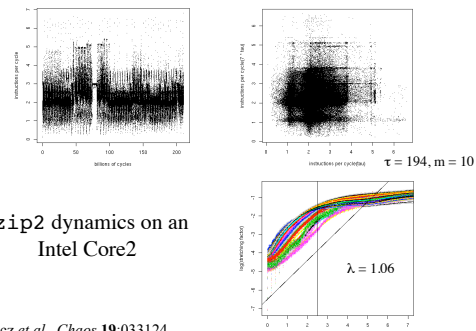$m > 2d$: sufficient to ensure no crossings in reconstruction space:

…may be overkill.
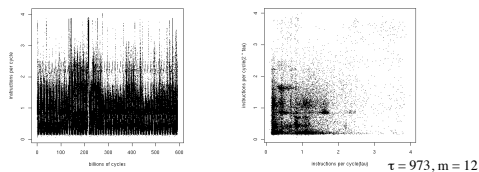
"Embedology" paper: $m > 2$ d_box
(box-counting dimension)

TISEAN contains tools that help you do this (e.g., `false_nearest`)

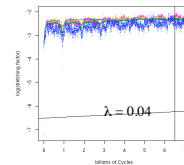**NLTSA* of computer performance dynamics**   * nonlinear time-series analysis

τ = 194, m = 10

`bzip2` dynamics on an Intel Core2

λ = 1.06

Mytkowicz *et al.*, *Chaos* **19**:033124

τ = 973, m = 12

`bzip2` dynamics on an Intel Pentium 4

λ = 0.07

Mytkowicz *et al.*, *Chaos* **19**:033124

τ = 930, m = 8
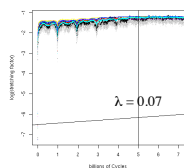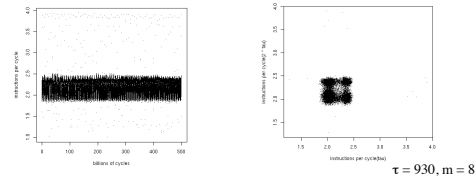
`povray` dynamics on an Intel Core2

λ = 0.04

Mytkowicz *et al.*, *Chaos* **19**:033124

If Δt is not uniform

Theorem (Takens): for τ>0 and m ≥ 2d,
   reconstructed trajectory is diffeomorphic to
   the true trajectory

Conditions: evenly sampled in time

Interspike interval embedding

idea: lots of systems generate spikes —
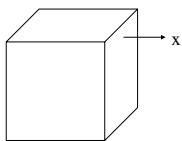   hearts, nerves, etc.

if you assume that the spikes are the result of
   an integrate-and-fire system, then the Δt
   has a one-to-one correspondence to some
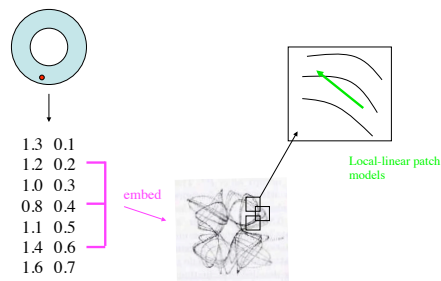   state variable's *integrated* value…

in which case the Takens theorem still holds.

(with the Δts as state variables)

Sauer *Chaos* **5**:127

What if we measured time-series data from a roulette wheel?

x

The Eudaemonic Pie
(or The Newtonian Casino)

1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7

embed

Local-linear patch models

The Santa Fe competition

- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past,* Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)
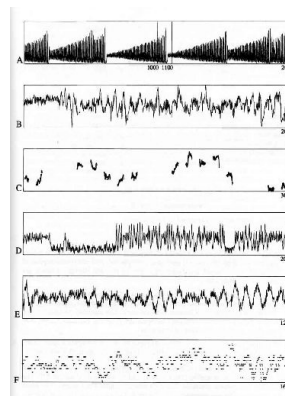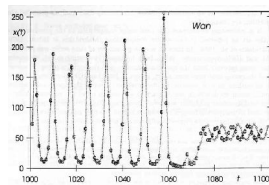

The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
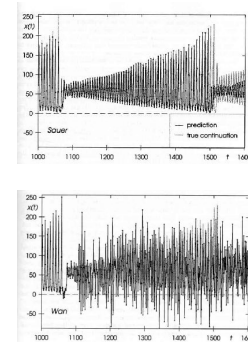- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

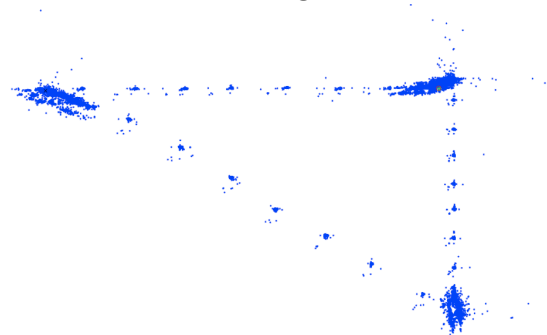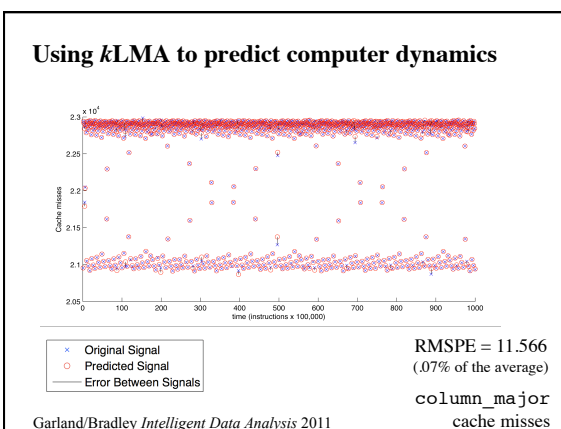**Embedding + patch models: (Sauer)**
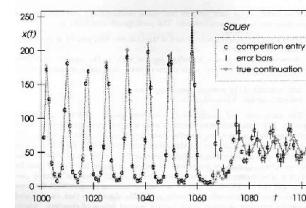
Neural net: (Wan)
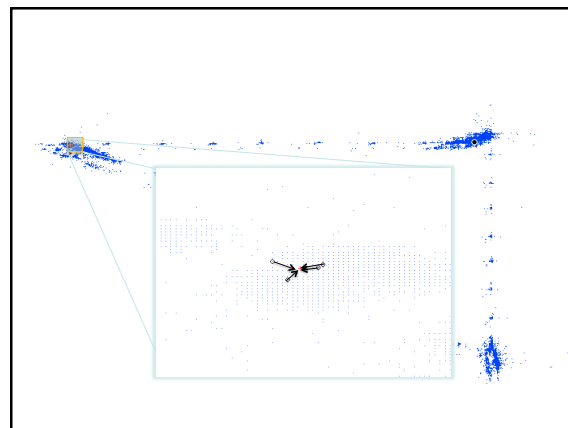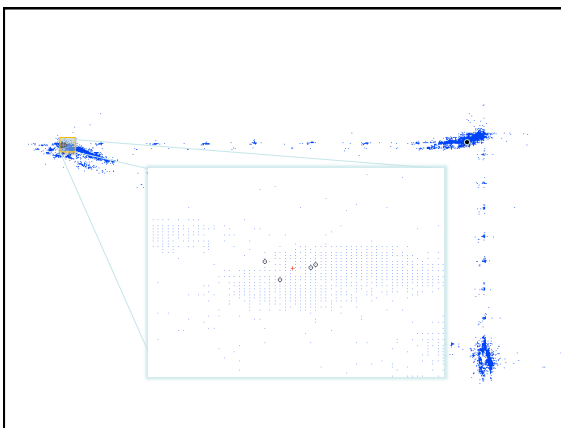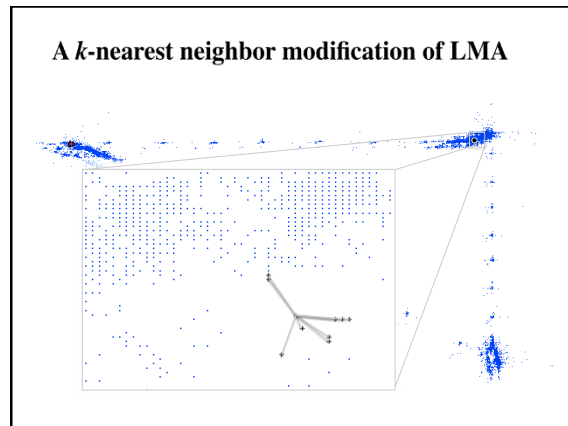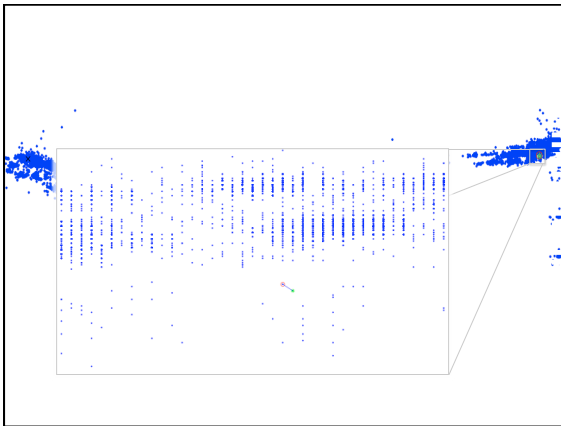

Further out:

Sauer

Wan

Sauer's algorithm:

In his competition entry, shown in Figure 3, Sauer used a careful implementation of local-linear fitting that had five steps:

1. Low-pass embed the data to help remove measurement and quantization noise. This low-pass filtering produces a smoothed version of the original series. (We explained such filtered embedding at the end of Section 4.1.)
2. Generate more points in embedding space by (Fourier-) interpolating between the points obtained from Step 1. This is to increase the coverage in embedding space.
3. Find the k nearest neighbors to the point of prediction (the choice of k tries to balance the increasing bias and decreasing variance that come from using a larger neighborhood).
4. Use a local SVD to project (possibly very noisy) points onto the local surface. (Even if a point is very far away from the surface, this step forces the dynamics back on the reconstructed solution manifold.)
5. Regress a linear model for the neighborhood and use it to generate the forecast.

Lorenz's method of analogues

A *k*-nearest neighbor modification of LMA

Using *k*LMA to predict computer dynamics

- × Original Signal
- ○ Predicted Signal
- — Error Between Signals

RMSPE = 11.566
(.07% of the average)

column_major
cache misses

Garland/Bradley *Intelligent Data Analysis* 2011

Noise…

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:

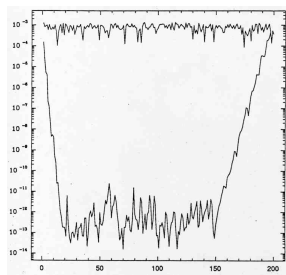- use the stable and unstable manifold structure on a chaotic attractor…
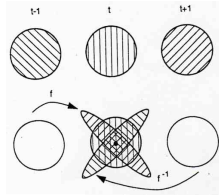
Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time

- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls

- Since all three versions of that data should be identical at the middle time, can average them

- ⇒ noise reduction!

- Works best if manifolds are perpendicular, but requires only transversality

Results:

Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Noise…

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:

- use the stable and unstable manifold structure on a chaotic attractor

- use the *topology* of the attractor…

Computational Topology

Why: this is the fundamental mathematics of shape. complements geometry.

What: compute topological properties *from finite data*

How:
- introduce resolution parameter
- count components and holes at different resolutions
- deduce topology from patterns therein

V. Robins Ph.D. thesis, UColorado, 1999

Connectedness: definitions

- how many "lumps" in a data set:

- ε-connectedness (after Cantor)

- ε-connected components

- ε-isolated points:

Connectedness: examples

If the data points are samples of a disconnected fractal like this:

The number of connected components looks like this:
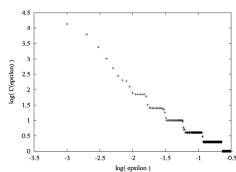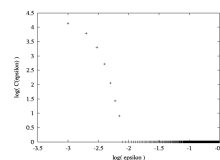

(note obvious tie-in to fractal dimension…)

Robins *et al.*, *Physica D* **139**:276, *Nonlinearity* **11**:913

Connectedness: examples

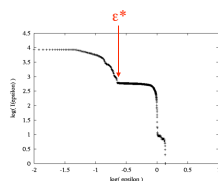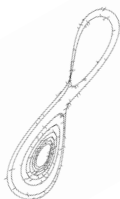If the data points are samples of a connected set like this:
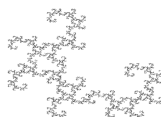
The number of connected components looks like this:

Robins *et al.*, *Physica D* **139**:276, *Nonlinearity* **11**:913

Connectedness and filtering

The effect of noise is to add isolated points to the set and a shoulder to the C(ε) curve:

ε*

So if you know that the object is connected, you can reasonably assume that any isolated points are noisy, and remove them by pruning with ε = ε*

Robins *et al.*, *Intelligent Data Analysis* **8**:505, *Chaos* **14**:305

Continuity and filtering
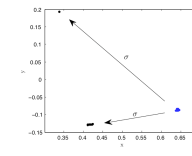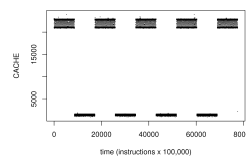
Idea:
  • deterministic dynamics, if embedded correctly, are continuous
Conjecture:
  • if the image of a connected set is not connected, more than one dynamics is at work
Approach:
  • track connectedness over time
Applications:
  • pulling apart interleaved dynamics, removing noise…

Alexander *et al.*, *PhysRevLett.*, in preparation

Chaos and control…
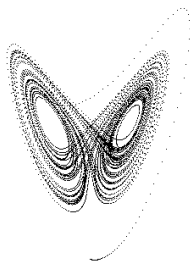
key concepts:

• dense attractor coverage

• exponential trajectory separation
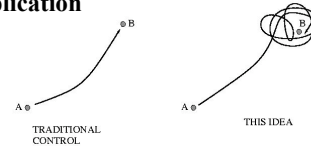
• un/stable manifold structure

• local-linear control

Control:

getting from A to B, minimizing some cost functional…

Lorenz System:

denseness, reachability, and control

55,95

-45,-5

R = 50
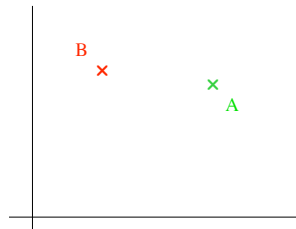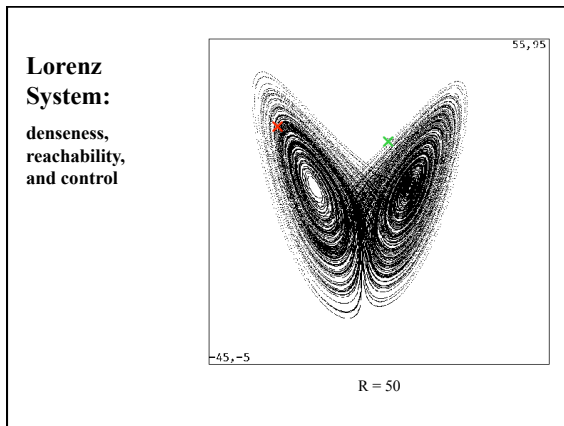
Denseness & reachability in a real engineering application

B

B

A

A

TRADITIONAL CONTROL

THIS IDEA

• **can control position/volume/density of attractor –within limits**

• **possibly not reachable any other way**

• **nondeterministic – not for time-critical applications**

IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—II: FUNDAMENTAL THEORY AND APPLICATIONS, VOL. 40, NO. 11, NOVEMBER 1993

Using Chaos to Broaden the Capture
Range of a Phase-Locked Loop

Elizabeth Bradley, *Member, IEEE*

OGY control: taking advantage of the unique properties of chaos

• dense attractor coverage → reachability (*)

• un/stable manifold structure → controllability

* eventually…

Ott *et al.*, *PRL* **64**:1196

• dense attractor coverage → reachability

• un/stable manifold structure → controllability

• exploit sensitive dependence, too???

⟹ "targeting"

Lorenz System:

SDOIC-based targeting

OGY & co. have been used in *tons* of systems; see Shinbrot review paper.

Also see Hubler's work on open-loop control.

55,95

-45,-5

Four R switches; 240X faster

Bradley, *Cybernetics & Systems* **26**:299

Program in
Applied
Mathematics

Erik Bollt

*University of Colorado at Boulder
Boulder CO 80309-0526
(303) 492-4668*
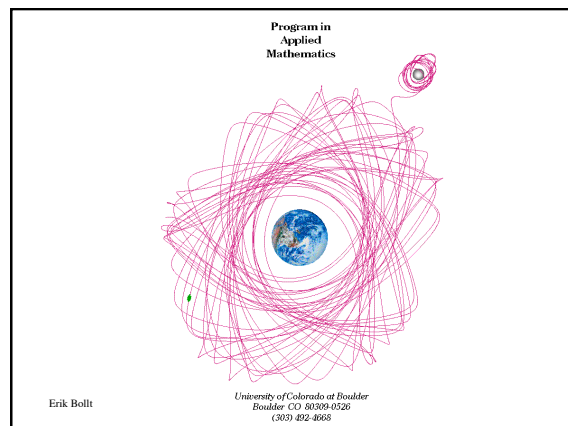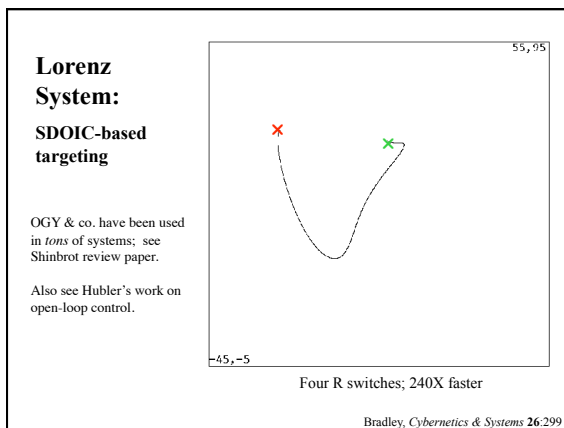
Other cool ways to use invariant manifolds

Want to get a spacecraft onto a "halo orbit," which is a UPO of the dynamics.

W^S

Halo Orbit

Unstable Periodic Orbits (UPOs) have invariant manifolds:

W^U

- Stable Invariant Manifold (W^S)
  - *The set of all trajectories a particle could use to arrive onto the UPO.*
- Unstable Invariant Manifold (W^U)
  - *The set of all trajectories a particle could take after a small perturbation from the UPO.*
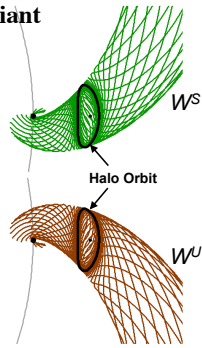
Jeff Parker, PhD thesis, UColorado 2008

Low-energy (cheap) orbit transfers

- Depart along W^U_{L1} & arrive on W^S_{L2}

2D Lyapunov Orbit about Lunar L_1

2D Lyapunov Orbit about Lunar L_2

Unstable Manifold

Stable Manifold

L_1 Moon L_2
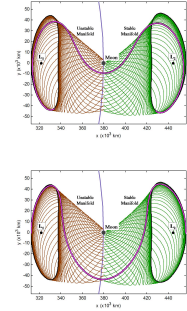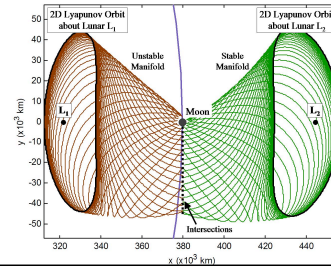
Intersections

Jeff Parker, PhD thesis, UColorado 2008

Homoclinic orbits - The best case

- If a trajectory in Stable and Unstable intersect ("homoclinic connection")
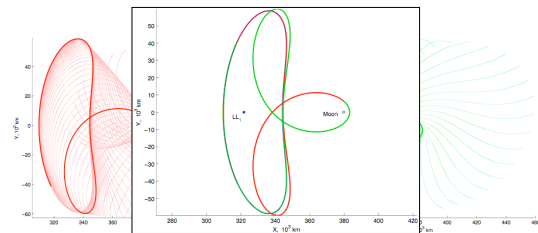
Unstable Manifold of an LL₁ Lyapunov Orbit   Stable Manifold of an LL₁ Lyapunov Orbit

Jeff Parker, PhD thesis, UColorado 2008

Can we do any of that in spatially extended systems?

(i.e. harness the butterfly effect, exploit un/stable manifold geometry?)

Sensitive flames (1856 – 1930s)

I repeat a passage from Spenser:

" Her ivory forehead full of bounty brave,
    Like a broad table did itself dispread ;
For love his lofty triumphs to engrave,
    And write the battles of his great godhead.
All truth and goodness might therein be read,
    For there their dwelling was, and when she spake,
Sweet words, like dropping honey she did shed ;
    And through the pearls and rubies softly brake
A silver sound, which heavenly music seemed to make."

The flame selects from the sounds those to which it can respond. It notices some by the slightest nod, to others it bows more distinctly, to some its obeisance is very profound, while to many sounds it turns an entirely deaf ear.

A 2D jet

contraction   air jet exits

exit slit

plenum

1.0 metre

compressed air enters
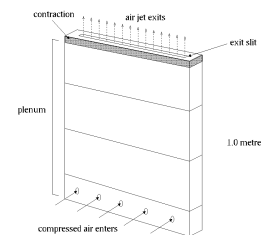
Peacock *et al.*, *Exp. Fluids* **37**:22

End view
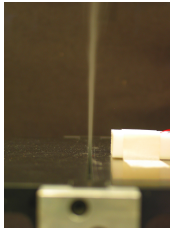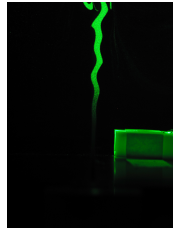
room lighting   stop-action laser "slice"

aerosolized canola oil

Forcing the jet flow
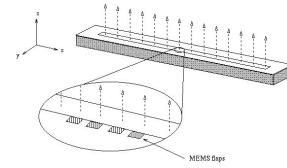
MEMS flaps

Slit: 2.5 X 400 mm

MEMS actuators

poly 2   poly 1

solder

ceramic   air jet
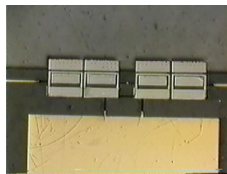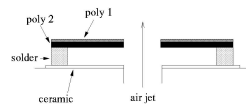
Video : overhead view at 2Hz, 10V

Area of individual flap is 1.0 x 0.25mm

Ma *et al., IEEE Trans. Adv. Packaging* **26**:268

The Butterfly effect in action…

no forcing   *6Hz forcing*
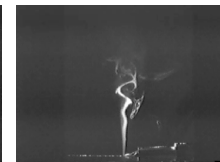
MEMS flap

Forcing generates coherent structures that enhance entrainment and mixing

Peacock *et al., Exp. Fluids* **37**:22

Does this have anything to do with reality?
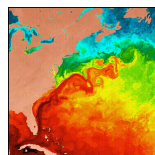
Measurement & isolation:
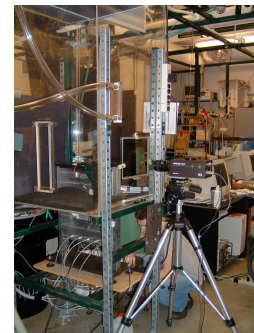
Communication and chaos:

- Two coupled Lorenz systems will synchronize
- Robust w.r.t. a small amount of noise
- Use this to transmit & receive information

$x' = a(y-x)$
$y' = rx -y -xz$
$z' = xy - bz$

$\rightarrow$

$x' = a(y-x)$
$y' = r(x+\varepsilon x) -y -xz$
$z' = xy - bz$

- Chaotic carrier wave, so hard to intercept or jam

Pecora & Carroll *Phys. Rev. Lett* **64**:821

Another interesting application:
chaos in the solar system

- orbits of Pluto, Mars
- Kirkwood gaps
- rotation of Hyperion & other satellites
- …

Solar system stability:

- recall: two-body problem not chaotic
- but three (or more) can be…

Hut & Bahcall
Ap J. **268:319**
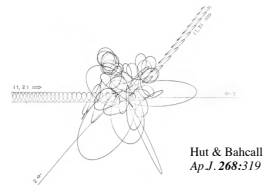
Exploring that issue, circa 1880:

- an *orrery*
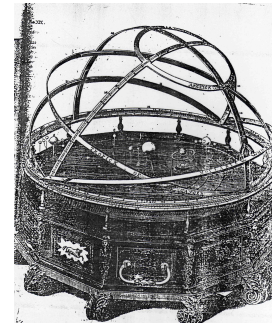
Exploring that issue, circa 1980:

- write the *n*-body equations for the solar system
- solve them on a special-purpose computer

The *digital orrery*
(Wisdom & Sussman)

ONR N00014-86-K-0180
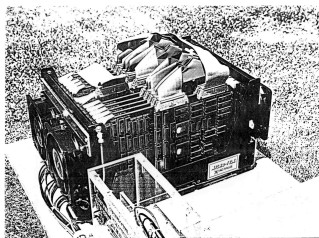
Numerical Evidence That
the Motion of Pluto Is Chaotic

GERALD JAY SUSSMAN AND JACK WISDOM

Science **241**:433

Should we worry?

- No.

Kirkwood gaps:
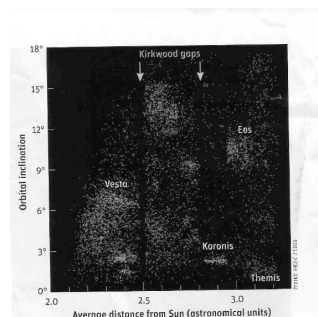
From *Sky & Telescope*

Chaos and the Kirkwood gaps
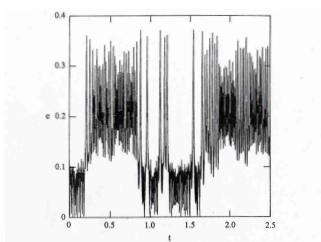
FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. the time is now measured in millions of years. Bursts of high eccentricity behavior are interspersed with intervals of irregular low eccentricity behavior, broken by occasional spikes.

Wisdom, *Nuclear Phys. B* **2:**391
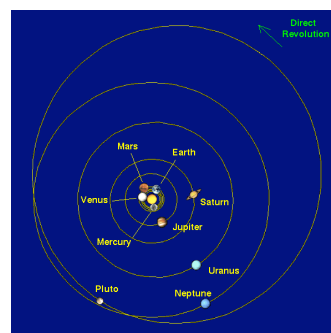
csep10.phys.utk.edu/astr161/lect/solarsys/revolution.html

Evidence in favor of the conjecture:
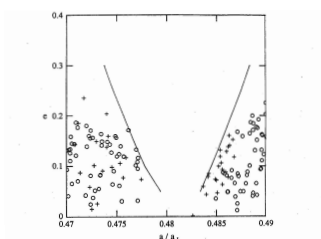
FIGURE 9. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and quasiperiodic region in the gap, but trajectories of both types are planet crossing.

Wisdom, *Nuclear Phys. B* **2:**391

Chaotic tumbling of satellites:

Voyager and Galileo saw this...

From *Sky & Telescope*

Ap. J. **97**:570
Ap. J. **98**:1855

www.nasa.gov/mission_pages/cassini/multimedia/pia06243.html
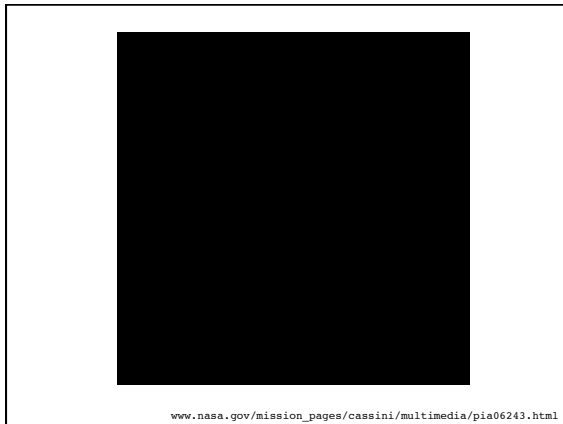
Chaotic tumbling of satellites:

This happens for **all** satellites at some point in their history, unless they are perfectly spherical and in perfectly circular orbits (pf: KAM theorem; see Wisdom paper on syllabus)

Some of them are still tumbling chaotically because of their geometry, but most (like the earth and its moon) have settled down into equilibria

More chaos in the solar system:

• obliquity of Mars (Touma & Wisdom, *Science* **259:**1294)

www.solarviews.com

• etc.

Musical Variations from a Chaotic Mapping

Dabby *Chaos* **6**:95

Pitch sequence:
C, E, G, C, E, G, C, E…

C symbol dynamics

variation!

Also fun: http://www.youtube.com/watch?v=B2XtE9Yy1CA
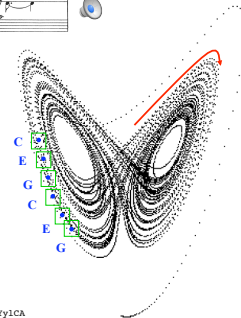
Chaotic variations on movement sequences

original piece
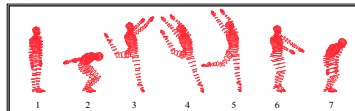
chaotic   mapping
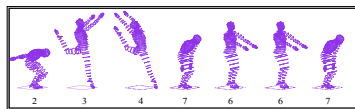
chaotic variation

Bradley & Stuart, *Chaos* **8:**800

Lorenz

Rossler

original cell
itinerary →

variation
cell
itinerary →

medley

Rossler medley

random medley

original cell
itinerary →

variation
cell
itinerary →

abrupt transition

Interpolation

Stuart & Bradley, *ICML* 1998

Corpus-based approach

• **graph captures motions of one joint**
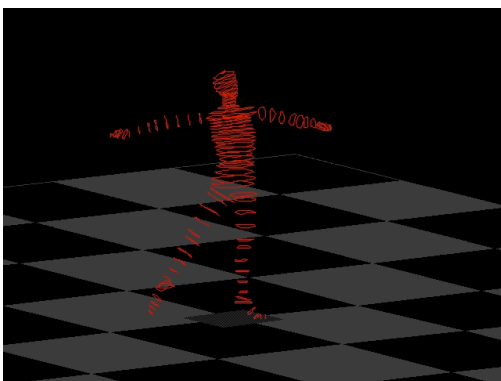• **note: specific to the genre of the corpus!**

Initial state

Target state

33
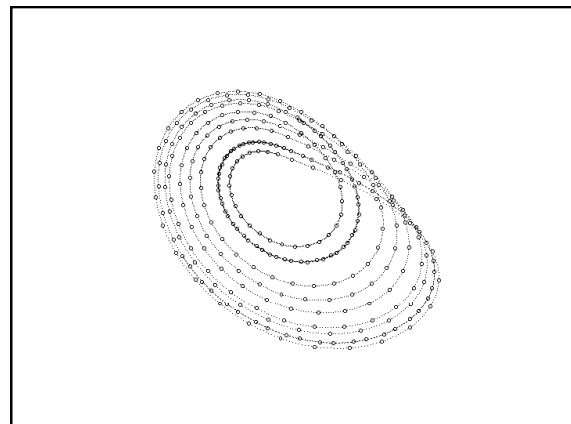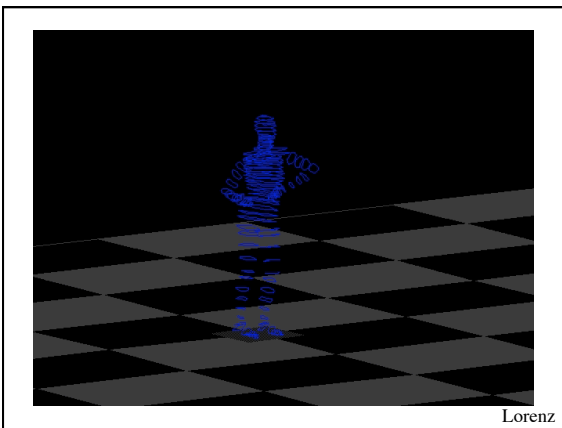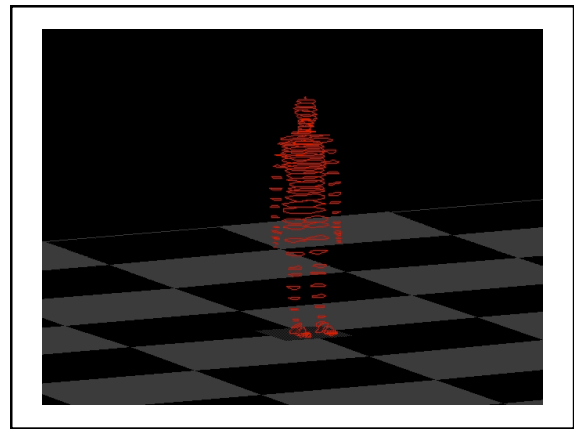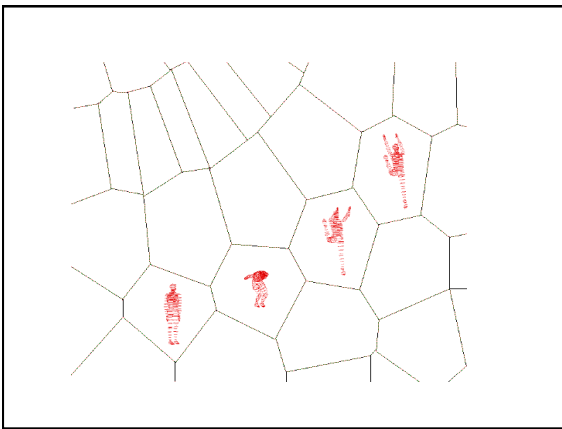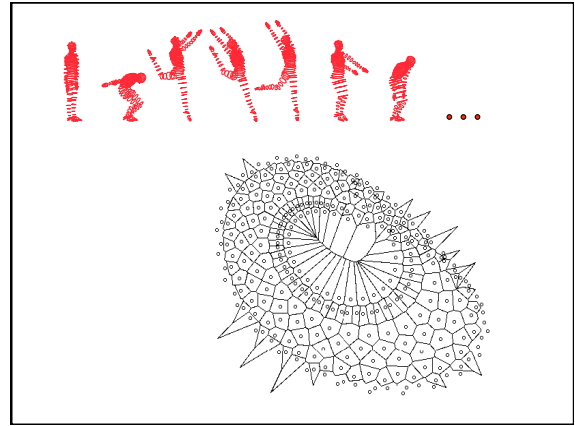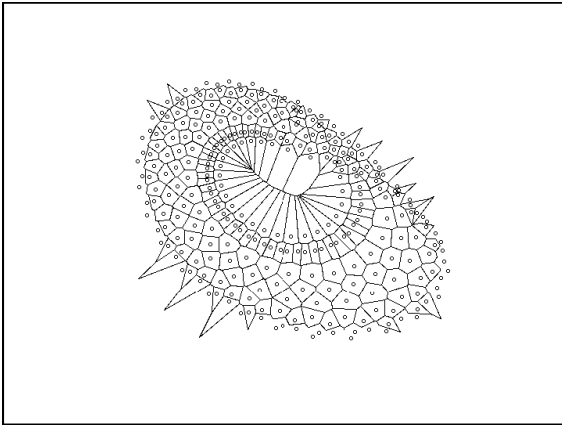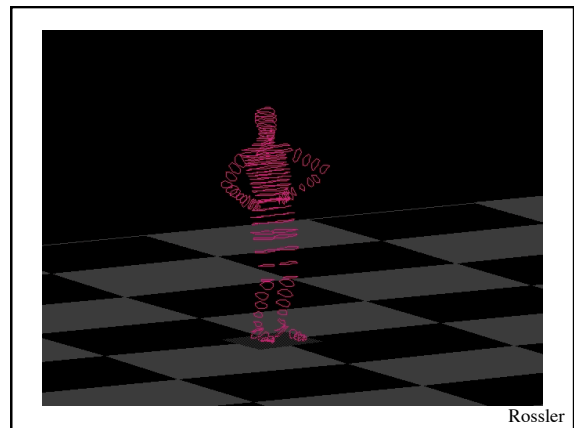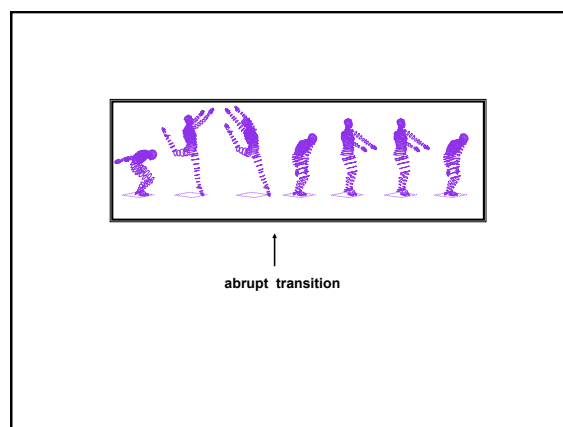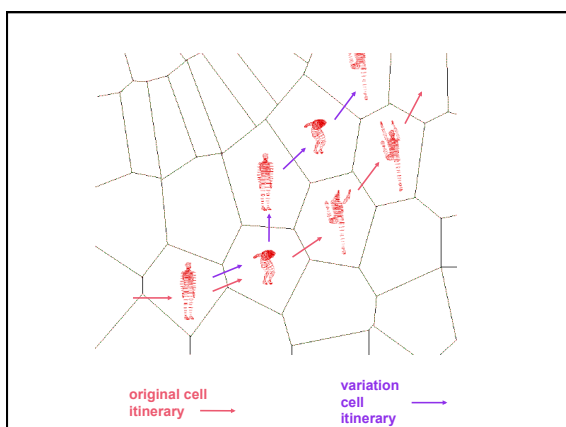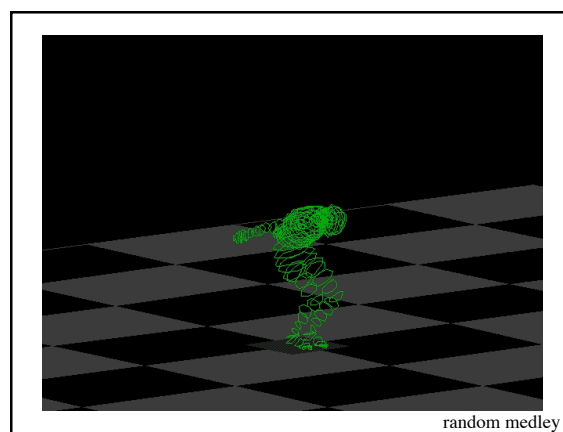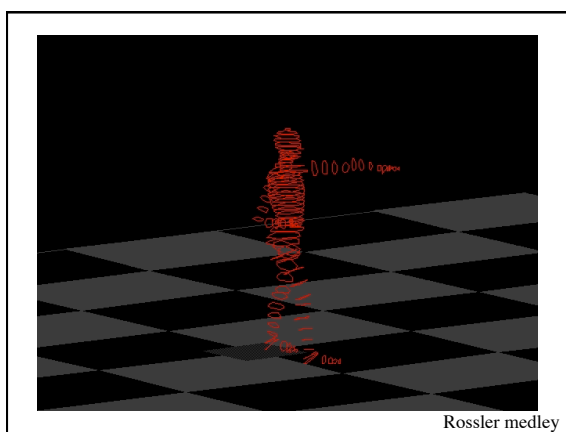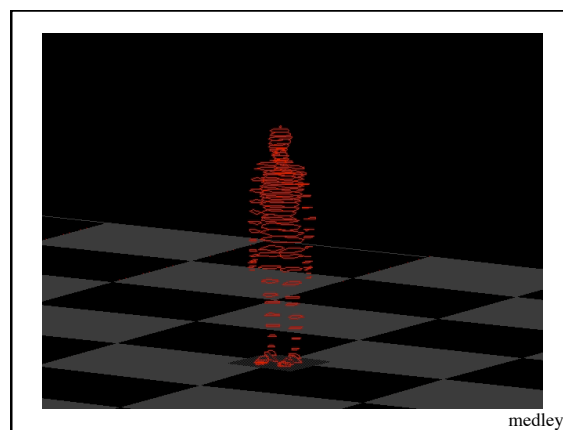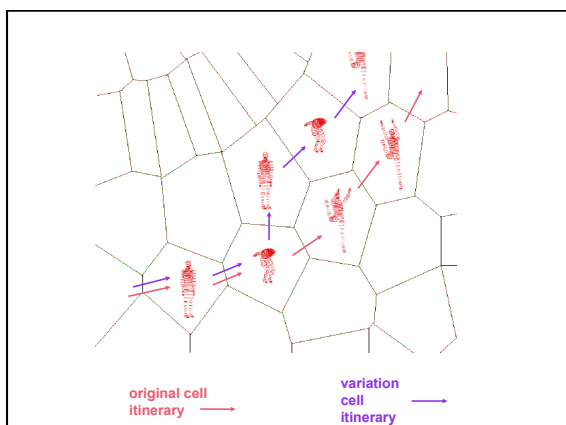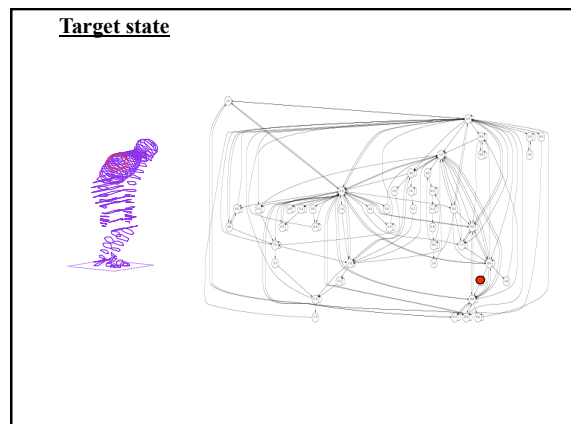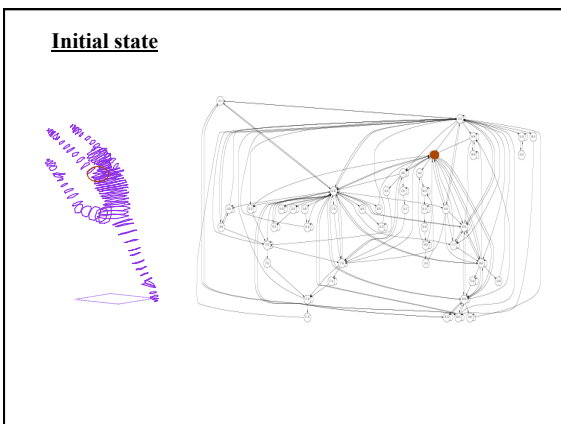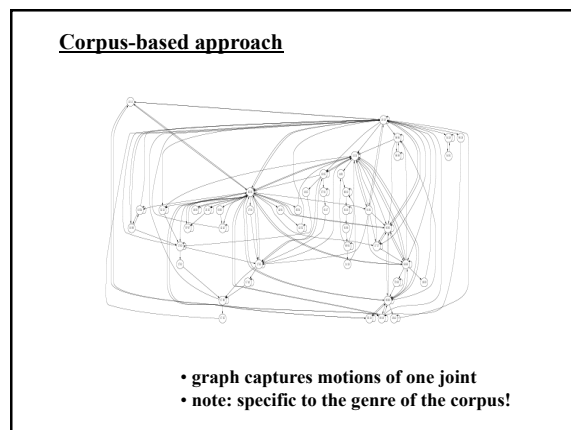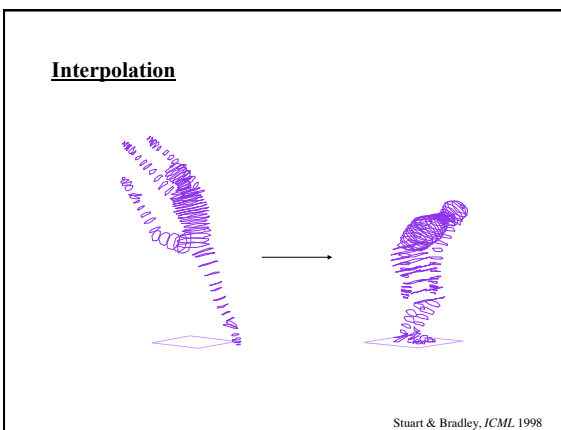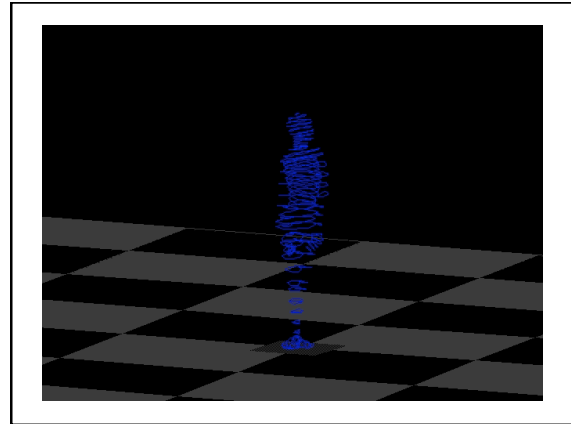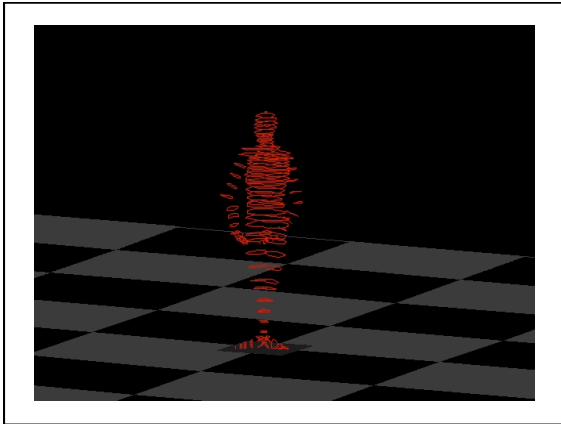
Graph search
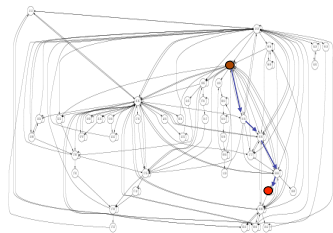
…for 44 joints in parallel!
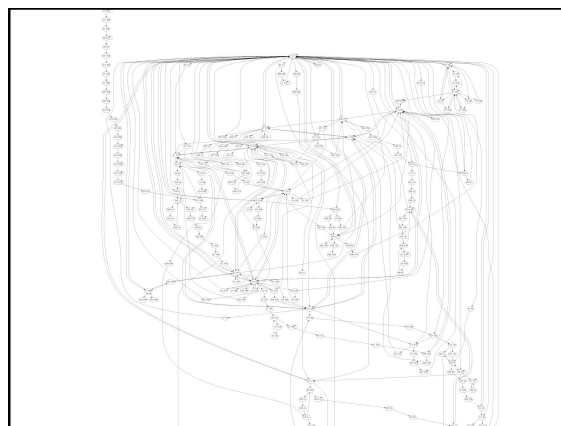
initial
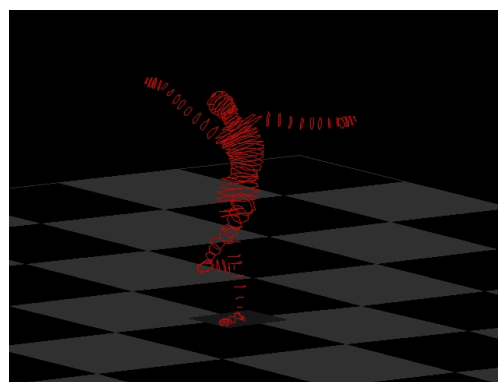
target

interpolation 1

initial

target

interpolation 2
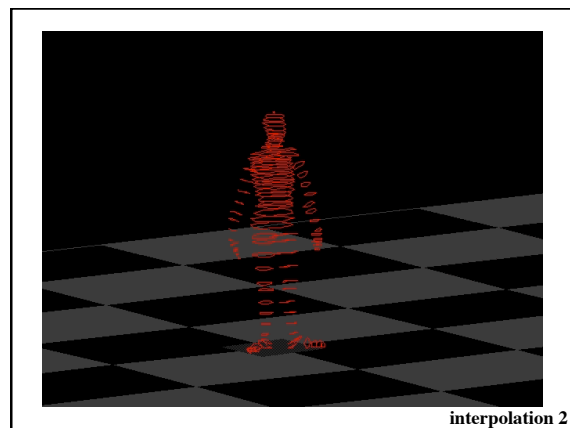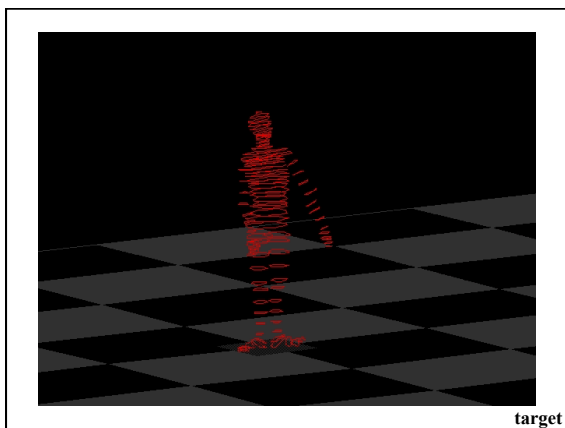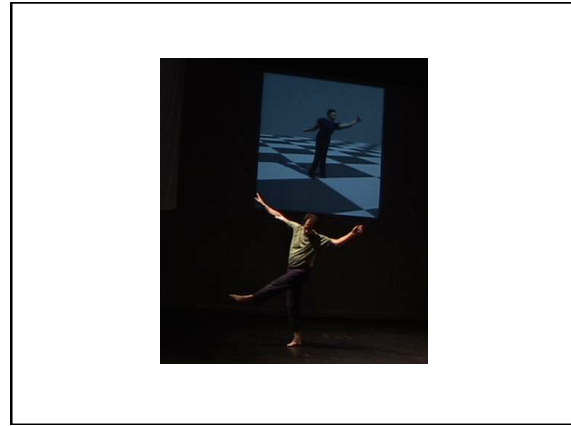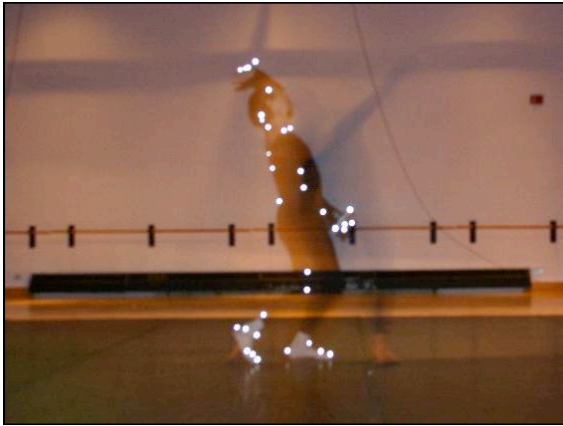


initial

target

interpolation 3
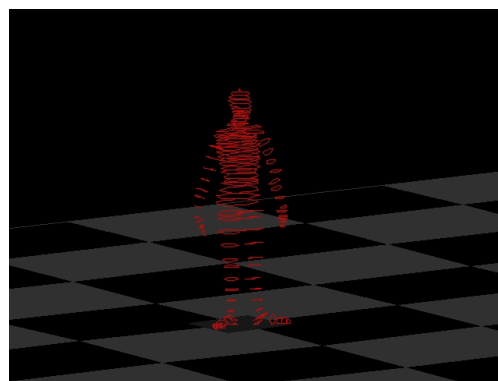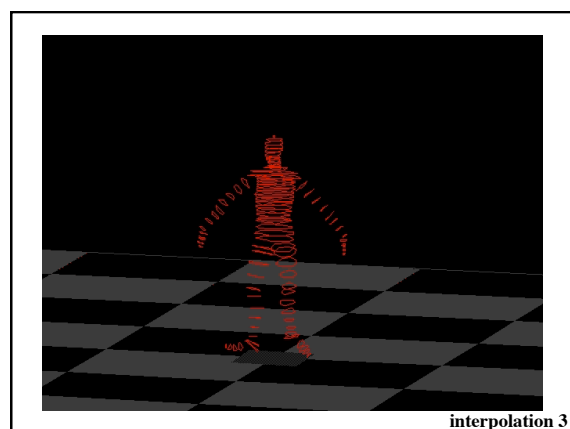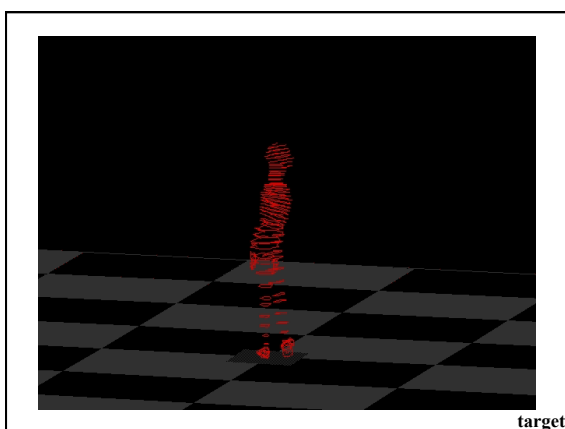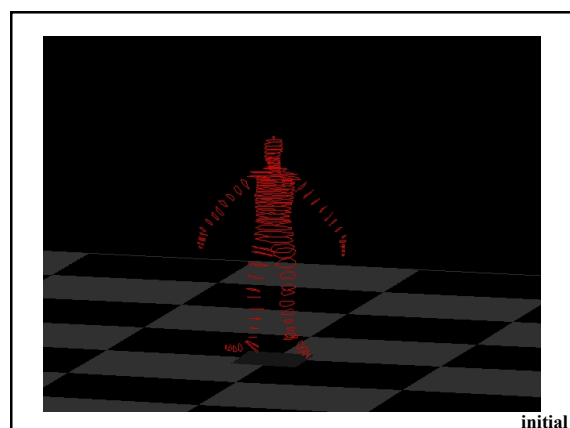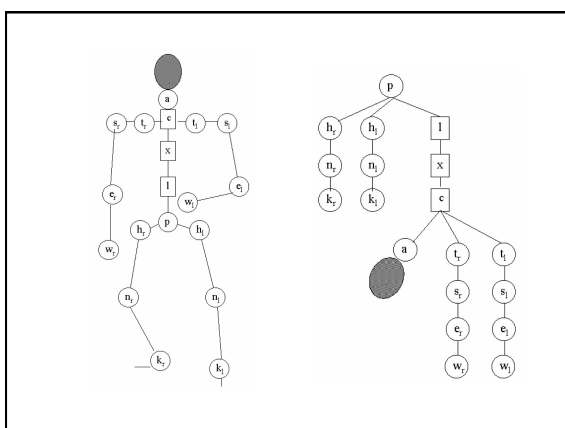
Con/cantation: (chaotic variations)
A computer-assisted theme and variations performance project

Created by David Capps and Liz Bradley
Video and layout: Angelika von Chamier

Ideas and algorithms: Josh Stuart
Motion capture and animation: Carnegie Mellon Graphics Laboratory
(Professor Jessica Hodgins, leader; Justin Macey, motion-capture
technician; Mo Mahler, animation and character design)
Code: David Trowbridge and Evan Sheehan
Inspiration: Diana Dabby

Made possible with support from the Radcliffe Institute for Advanced
Study, the National Science Foundation (IIS-0326322), the David and
Lucile Packard Foundation, and the Graduate Council on Arts and
Humanities at the University of Colorado.

Radcliffe Institute for Advanced Study
Tuesday, April 17th
5pm
Radcliffe Gym

Radcliffe Yard
10 Garden Street
Cambridge, MA 02138

Free Admission

synchronousobjects.osu.edu/content.html

synchronousobjects.osu.edu/content.html
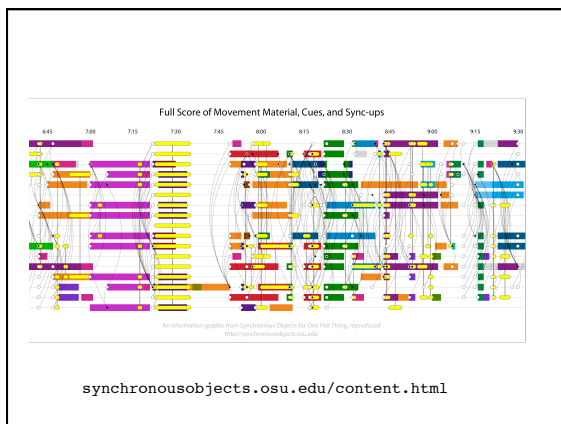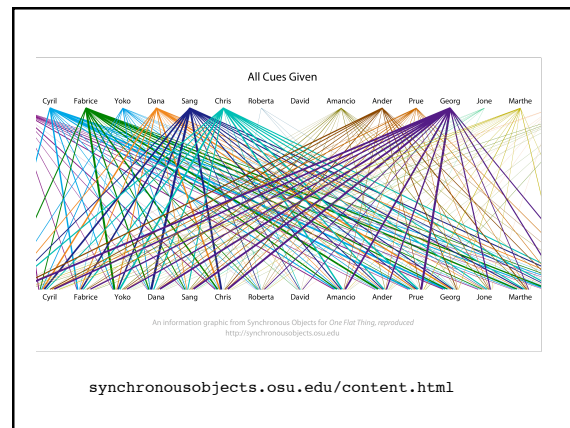
Invariant Manifolds

Halo Orbits: Manifolds

- Unstable Periodic Orbits (UPOs) have invariant manifolds.

- Stable Invariant Manifold (W^S)
  - *The set of all trajectories a particle could use to arrive onto the UPO.*

- Unstable Invariant Manifold (W^U)
  - *The set of all trajectories a particle could take after a small perturbation from the UPO.*