# Complex Systems with Boundary
# and Non-Euclidean Geometry
# Lecture 2 Notes for CSSS2010

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**Disclaimer:** Same as in *Lecture 1: Complex Systems, Statistical Learning, and Pickles*. This lecture builds on that lecture

## 1 Systems with boundary

Often a system or network under consideration has a natural boundary. Sometimes this is in the simple sense, namely when collecting data on a system, the a network extracted that is is knowingly only a part of larger whole. For example, the boundary between the economy and a collection of stocks, or Web-pages linked to a particular subset oft he web extract during a particular web crawl. It is wise to have tools that allow one to deal with having data on particular piece of a system that respects that fact that this system has a natural boundary. In this section we look at what a boundary looks like from a Markov chain point of view.

## 2 The cross potential

In engineering there is an import tool for finding imperfections and cracks, called a *four point probe*. Somewhat simplistically, it works by using (say) a battery to create a charge of 1 at $a$ and of $-1$ at $b$, and then the probe measures the voltage drop between $c$ and $d$, see figure 1. Such a probe makes perfectly good sense on network, and can be used to explore the networks geometry, see figure 1. We call this voltage difference the *cross potential* and denote it as $[a,b;c,d]$. Using the Dirichlet norm from the previous lecture we see that

$$[a,b;c,d] = < V_a^b, V_c^d >_{Dir} = V_a^b(c) - V_a^b(d).$$

**Exercises:**

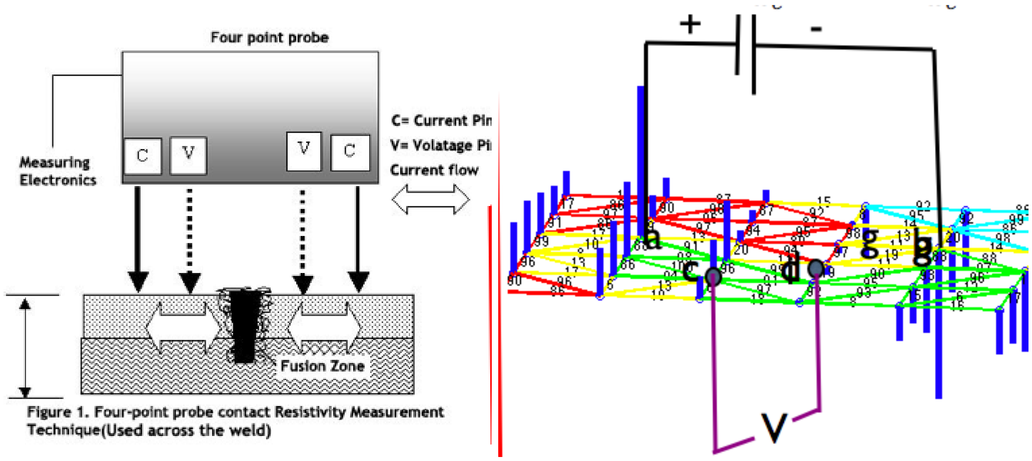


Figure 1: On the left we see a four point probe. The picture was taken from the American Acoustic Company's web site (a producer of four point probes). On the right we see the four point probe on a network. These four point probe measures the cross potential.

1. We may view $K$ a mapping from $\mathbb{R}^{|states|}/ < consts >$ to sum zero functions. Show the $[a,b;c,d]$ determines $K$ and hence $P$ (recall for us $p_{ii=0}$).

2. Prove $[a,b;c,d] = [b,a;d,c] = -[b,a;c,d] = -[a,b;d,c]$

## 2.1 Motivation for the name 'cross potential'

In the complex plane, if we put a charge of $+1$ at $a$, then the potential associated to the charge is

$$V_{a,S^2}(z) = \frac{1}{2\pi}\log|z-a|$$

Really this discussion should be taking place on the Riemann Sphere $S^2$ (the complex plane with the point at infinity adjoined), where the electric potential associated to a charge of $+1$ at $c$ and $-1$ at $d$ is given by

$$V^d_{c,S^2}(z) = \frac{1}{2\pi}\log\left|\frac{z-c}{z-d}\right|.$$

So we have

$$V^d_{c,S^2}(z) - V^d_{c,S^2}(w) = \frac{1}{2\pi}\log|\{z,w,c,d\}|$$

where

$$\{z,w,c,d\} = \frac{(z-c)(w-d)}{(z-d)(w-c)}$$

is the *cross ratio*, hence $[z,w;c,d]_{S^2} = V^d_{c,S^2}(z) - V^d_{c,S^2}(w)$ is called the cross potential. The cross ratio is **the** conformally invariant quantity on the Riemann Sphere, as is the cross potential on a network, see exercise 1.

**Exercises:**

1. Cross ratio review.

   (a) Imagine $z$ as variable and fixing distinct points $\{b,c,d\} \in S^2$, prove that $\{z,a,b,c\}$ is an orientation preserving conformal homeomorphism of $S^2$

   (b) Prove every orientation preserving conformal homeomorphism of $S^2$ can be expressed at $\{z,a,b,c\}$ for some choice of $\{b,c,d\} \in S^2$.

   (c) Use this to prove there exist an orientation preserving conformal homeomorphism sending any four points on $S^2$ to any other.

   (d) Prove that the cross ratio is invariant under conformal homeomorphism of $S^2$.

   (e) Prove the cross ratio is real if and only if the four points lie on a circle (on $S^2$).

# 3 Boundary at $\infty$

Imagine we have an ergodic chain and special collection of states that we call the boundary or sphere at infinity, which we will denote as $\partial^\infty$. We denote $\partial^\infty$'s complement as $H$. In general, the boundary $\partial^\infty$ could be viewed at the boundary of the system one has collected or as a more natural boundary. Such a natural boundary might be much like the sphere at infinity in Euclidean space, namely the directions you can look, a sphere worth of directions. In the space of mathematics one might use the *list_of_* pages to captures the direction you can look. Hence we will let the *lists at infinity* be our $\partial^\infty$ in the Space of Mathematics. Here they are:

| Geodesic from ’central_limit_theorem’ to ’continuum_hypothesis’ | |
|---|---|
| Start at top | Continue to end |
| ’central_limit_theorem’ | ’effective_descriptive_set_theory’ |
| ’stretched_exponential_function’ | ’small_veblen_ordinal’ |
| ’monty_hall_problem’ | ’feferman%e2%80%93sch%c3%bctte_ordinal’ |
| ’spoof_(game)’ | ’ackermann_ordinal’ |
| ’probability_surveys’ | ’diamond_principle’ |
| ’studia_mathematica’ | ’grzegorczyk_hierarchy’ |
| ’montgomery%27s_pair_correlation_conjecture’ | ’codomain’ |
| ’entropy_power_inequality’ | ’conference_board_of_the_mathematical_sciences’ |
| ’president_of_the_institute_of_mathematical_statistics’ | ’finite’ |
| ’factorization_lemma’ | ’jensen%27s_covering_theorem’ |
| ’fuzzy_measure_theory’ | ’dynkin_system’ |
| ’pi_system’ | ’balls_and_vase_problem’ |
| ’shattering’ | ’large_veblen_ordinal’ |
| ’theory_of_conjoint_measurement’ | ’computable_real_function’ |
| ’linear_partial_information’ | ’church%e2%80%93kleene_ordinal’ |
| ’latin_square_property’ | ’ramified_forcing’ |
| ’m._riesz_extension_theorem’ | ’dynkin%27s_lemma’ |
| ’covering_theorem’ | ’hilbert%27s_paradox_of_the_grand_hotel’ |
| ’transitivity_(mathematics)’ | ’ordinal_notation’ |
| ’operation_(mathematics)’ | ’ontological_maximalism’ |
| | ’continuum_hypothesis’ |

## 4 The hyperbolic metric

Given a $\partial^\infty$ there is a very natural pseudometric on $H$ which we will call the network's *hyperbolic metric*, it is denoted as as $d_{hyp}(a,b)$ and defined as

$$d_{hyp}(a,b) = max_{p,q\in\partial^\infty}[p,q;a,b].$$

**Hyperbolic Metric Lemma:** $d_{hyp}$ is a pseudometric.

**proof:**

1. $d_{hyp}$ is non-negative since $[p,q;a,b] = -[q,p;a,b]$, so the max is non-negative.
2. $d_{hyp}$ is symmetric since $[p;q;b,a] = [q,p;a,b]$, so $max_{p,q\in\partial^\infty}[p,q;a,b]$ and $max_{p,q\in\partial^\infty}[p,q;b,a]$ will agree.
3. To see $d_{hyp}$ satisfies the triangle inequality first notice that $[p,q;a,c] = [p,q;a,b] + [p,q;b,c]$ since

   $$[a,b;c,d] = V_p^q(a) - V_p^q(c) = V_p^q(c) - V_p^q(a) + (V_p^q(b) - V_p^q(b)) = [p,q;a,b] + [p,q;b,c].$$

   So using the $p$ and $q$ that maximize $[p,q;a,c]$, we have

   $$d_{hyp}(a,c) = [p,q;a,c] = [p,q;a,b] + [p,q;b,c] <= d_{hyp}(a,b) + d_{hyp}(b,c).$$

**Q.E.D**

$d_{hyp}$ behaves in many ways as one would expects the hyperbolic metric to behave. The *square grid chain with boundary* is a simple example to illustrate this claim. Take the square grid as on the left in figure 2 and view it as a Markov chain by using the adjacency matrix of the graph as a weight matrix $W$ and letting $P = [W\mathbf{1}]^{-1}W$. Let $\partial^\infty$ be the states connected to fewer than 4 other states (the obvious boundary). On the right in figure 2 we see the three dimensional MDS of this chain using $d_{hyp}(x,y)$. Notice we do get a nice negative curvature saddle surface as we had hoped for.

**Exercises:**

1. In practice it is rare for $x \neq y$ to satisfy $d_{hyp}(x,y) = 0$. But if this occurs, then we might as well identify the states $x$ and $y$ which satisfy $d_{hyp}(x,y) = 0$ forming the quotient space $H^*$.
   (a) Prove that $d^*_{hyp}$ is a metric on $H^*$.
   (b) By making a nice symmetric chain with only two states at infinity, construct an example where there are a pair of points where $d_{hyp}(x,y) = 0$.
2. For $x \in [0,1]$ let

   $$< V_a^b, V_c^d >_{Dir,x} = (1-x) < V_a^b, V_c^d >_{Dir} + x < V_c^d, V_a^b >_{Dir}$$

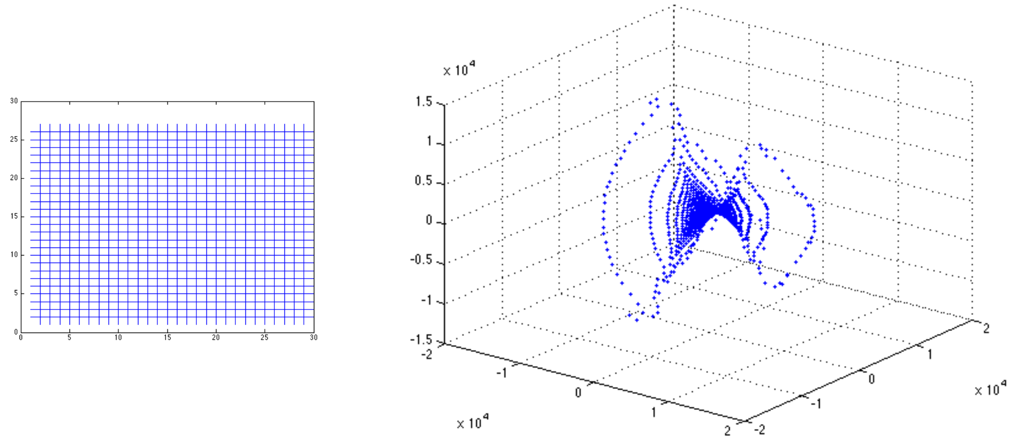


Figure 2: Here we see our square grid and the 3 dimensional MDS of the hyperbolic metric on the chain it determines (with $\partial^\infty$ the states connected to fewer than 4 other states (the obvious boundary).

(a) Show this is allows one to form a metric.

(b) Notice $[a,b;c,d]_{1/2} =< V_a^b, V_c^d >_{Dir,1/2}$ satisfies the symmetry $[a,b;c,d]_{1/2} = [c,d;a,b]_{1/2}$

(c) Examine the differences in the geometry as you very $x$.

### 4.1 Motivation for the name 'hyperbolic metric'

The key motivation for the name hyperbolic metric is the following result:

**The Cowgirl Hall of Fame Lemma:** The Poincare metric $d_{hyp,D^2}(z,w)$ on the interior of the unit disk $D^2$ satisfies

$$d_{hyp,D^2}(z,w) = max_{p,q \in \partial^\infty}[p,q;z,w]_{S^2}$$

where $\partial^\infty = \partial D^2$.

**proof:** In the Poincare disk model, the geodesic through $a$ and $b$ is a circular arc that intersects the $\partial D^2$ (the unit circle) at right angles. Furthermore, from exercise 1, $d_{hyp,D^2}(z,w) = \log|\{p,q,z,w\}|$ where $p$ and $q$ are the points where this geodesic intersects $\partial D^2$. So all we need to show is that among all the $p$ and $q$ on disk's boundary that these $p$ and $q$ maximize the $\log|\{p,q,z,w\}|$. To do this we note that the cross ratio is invariant under orientation preserving conformal homeomorphisms and that the orientation preserving conformal self maps of the disk are a subset of these homeomorphisms that form the hyperbolic isometries. Furthermore, the hyperbolic isometries are transitive on the unit tangent bundle; so we can send our points to $a = 0$, $b = x$ with $0 < x < 1$ and hence force the geodesic to be $[-1,1]$. Other choices of $p$ and $q$ could be written as $p = e^{i\theta}$ and $q = e^{i\phi}$, and so we need to show that

$$\log|\{p,q,0,x\}| = \log\left|\frac{(e^{i\theta}-0)(e^{i\phi}-x)}{(e^{i\theta}-x)(e^{i\phi}-0)}\right| = \log\left|e^{i\phi}-x\right| - \log\left|(e^{i\theta}-x)\right|$$

is maximized at $p=-1$ and $q=1$. But since $|z-x|$ is the Euclidean distance from $x$ to $z$ and log monotonically increases, we see that is indeed true. **Q.E.D**

**Exercises:**

1. In this exercise we prove that $d_{hyp,D^2}(a,b) = \log|\{p,q,a,b\}|$ where $p$ and $q$ are the points where the geodesic through $a$ and $b$ intersects $\partial D^2$.
   (a) Move to upper half space model.
   (b) Prove isometries are transitive on the unit tangent bundle, and hence we may assume our points are $p=0$, $q=\infty$, $a=i$, $b=iy$.
   (c) Use the fact that the metric is $\langle v,w\rangle_{hyp} = \frac{1}{y}\langle v,w\rangle_{euc}$ to prove the distance from $i$ to $iy$ is $\log(y)$.
   (d) Prove $y = \{0,\infty,i,iy\}$.
   (e) Prove isometries are conformal self maps and hence by exercise **??** in section **??**, the cross ration is invariant. The formula follows.

# 5 Geodesics currents

Let $\{p(a,b),q(a,b)\} = argmax(max_{p,q\in A}[p,q;a,b])$ associated to $p,q$, and defined as

$$g_{p,q} = \{(a,b) \mid (p(a,b),q(a,b)) = (p,q)\}.$$

Notice, $g_{q,p}$ is the current with the opposite orientation as $g_{pq}$, in other words $(a,b)\in g_{p,q}$ then $(b,a)\in g_{q,p}$. In figure 3, on the left we see a conformal mapping of the interior of the disk to the interior of the square. The lines drawn are hyperbolic geodesics in the canonical hyperbolic metric on he square. On the right we see the states on three geodesic currents in our square grid chain with boundary. These geodesics closely resemble each other as anticipated. In figure 4, we see an ideal triangle drawn in our MDS approximation to the actual geometry of the square grid chain with boundary. In hyperbolic geometry, ideal triangles are triangles with all three vertexes at infinity, they play a very special role in hyperbolic geometry (see exercise **??**), and a special role in the hyperbolic geometry on a chain with boundary as well (see exercise 3).

In general, geodesic currents will not look quite so much like single geodesics. Often they will look and feel more like a collections of geodesics. This is largely because $p$ and $q$ will often behave more like *chunks* at infinity than like points at infinity. Hence the geodesic current $g_{pq}$ will contain all oriented pairs of states on geodesics in the union of all oriented geodesics heading from chunk $p$ to chunk $q$. In figure 5, we see schematic to help capture this intuition. One can prove the anticipated consistency relationship given this view point.

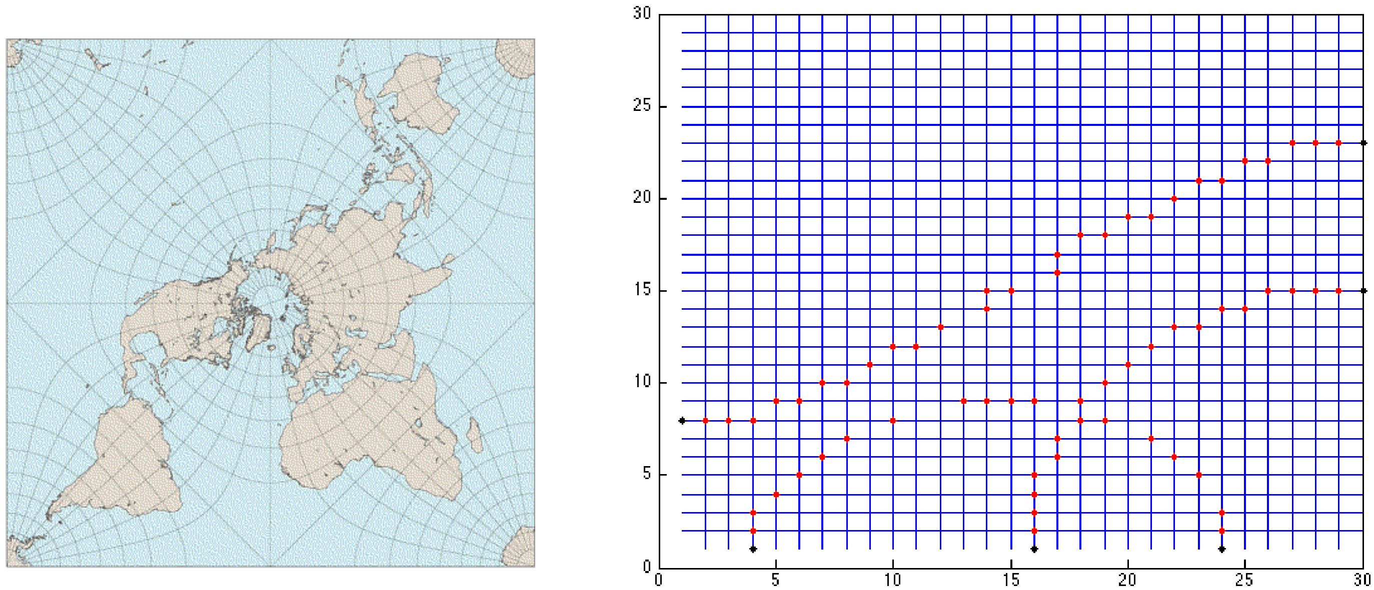


Figure 3: On the left we se a conformal mapping of the interior of the disk to the interior of the square. The lines drawn are the hyperbolic geodesics on the square. On the right we see three geodesics currents in our square grid chain with boundary. These geodesics closely resemble each other as anticipated.

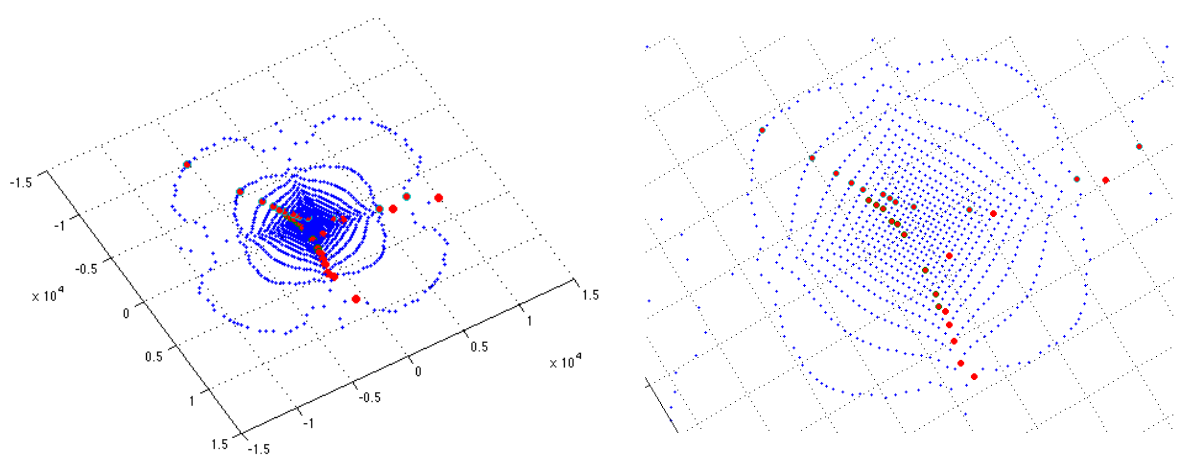


Figure 4: We see an ideal triangle in our square grid chain with boundary.

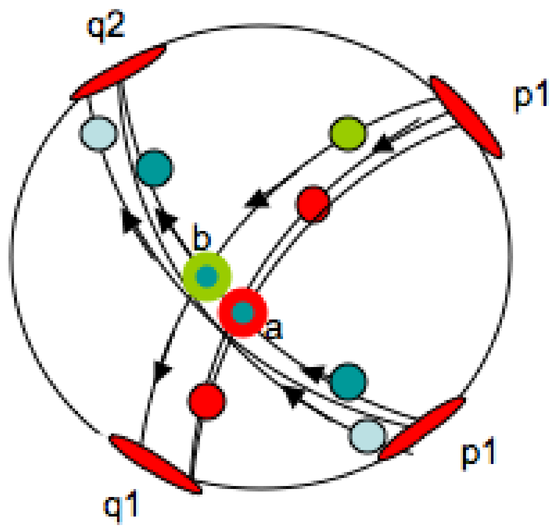


Figure 5: Here we witness a schematic of a pair of geodesic currents.

**Geodesic Current Consistency Lemma** If $(a,b) \in g_{p,q}$ and $(b,c) \in g_{p,q}$, then $(a,c) \in g_{p,q}$.

**proof:** If $(a,c)$ is not in $g_{p,q}$, then $(p,q)$ is not in $argmax(max_{p_1,q_1 \in A}[p_1,q_1;a,c])$ and there exist $(p_0,q_0)$ such that

$$[p,q;a,c] < [p_0,q_0;a,c].$$

As in the proof of the hyperbolic metric lemma, we have

$$[p_0,q_0;a,c] = [p_0,q_0;a,b] + [p_0,q_0;b,c]$$

which by the definition of the distance must satisfy

$$[p_0,q_0;a,b] + [p_0,q_0;b,c] <= d(a,b) + d(b,c) = [p,q;a,b] + [p,q;b;c],$$

which, in turn, satisfies

$$[p,q;a,b] + [p,q;b;c] = [p,q;a,c].$$

So in the end assuming $(a,c)$ is not in $g_{p,q}$ results in $[p,q;a,c] < [p,q;a,c]$, a contradiction. **Q.E.D**

## 5.1 The coalescence of geodesic currents

Let us take a look at the behavior of geodesics in the $d_{hyp}$ metric in the space of mathematics using our lists at infinity as our $\partial^{\infty}$. On the right in figure 6 we see an example of triangle in this space. This triangle was formed by randomly selecting points on the geodesic computing the $d_{hyp}$ between all pairs of points in this collection and then using MDS view it in 3 dimensional Euclidean space. These theorems where chosen to be far apart in the world of mathematics, so the directions we see when looking at the Central Limit Theorem from both the vantage point of the Continuum Hypothesis and the Guass-Bonnet Theorem agree, and it happens to be the *list_of_basic_probability_topics*. In fact the states on the geodesics begin to agree as we approach the *list_of_basic_probability_topics*. This property that distinct geodesics begin to agree as we approach infinity we call *geodesic current coalescence*. From exercise 1, we should expect this. Now when topics are closer together the direction we look will be much more variable. In figure 7, we experience this where we se an example of some theorems from two dimensional geometry and the hyperbolic triangle that they live on.

**An Example Geodesic:** Here we see an example of the geodesic Going from the Central Limit Theorem to the Continuum Hypothesis:

| The Lists at $\infty$ | |
|---|---|
| 'list_of_abstract_algebra_topics' | 'list_of_curve_topics' |
| 'list_of_triangle_topics' | 'list_of_mathematical_topics_in_quantum_theory' |
| 'list_of_lie_group_topics' | 'list_of_algebraic_coding_theory_topics' |
| 'list_of_complex_analysis_topics' | 'list_of_set_theory_topics' |
| 'list_of_basic_probability_topics' | 'list_of_fourier_analysis_topics' |
| 'list_of_general_topology_topics' | 'list_of_algorithm_general_topics' |
| 'list_of_geometry_topics' | 'list_of_partial_differential_equation_topics' |
| 'list_of_numerical_computational_geometry_topics' | 'list_of_topology_topics' |
| 'list_of_geometric_topology_topics' | 'list_of_group_theory_topics' |
| 'list_of_computer_graphics_and_descriptive_geometry_topics' | 'list_of_multivariable_calculus_topics' |
| 'list_of_partition_topics' | 'list_of_differential_geometry_topics' |
| 'list_of_statistical_topics' | 'list_of_variational_topics' |
| 'list_of_stochastic_processes_topics' | 'list_of_permutation_topics' |
| 'list_of_linear_algebra_topics' | 'list_of_algebraic_topology_topics' |
| 'list_of_calculus_topics' | 'list_of_homological_algebra_topics' |
| 'list_of_exponential_topics' | 'list_of_number_theory_topics' |
| 'list_of_commutative_algebra_topics' | 'list_of_recreational_number_theory_topics' |
| 'list_of_computability_and_complexity_topics' | 'list_of_basic_algebra_topics' |
| 'list_of_boolean_algebra_topics' | 'list_of_mathematical_logic_topics' |
| 'list_of_representation_theory_topics' | 'list_of_integration_and_measure_theory_topics' |
| 'list_of_factorial_and_binomial_topics' | 'list_of_string_theory_topics' |
| 'list_of_numerical_analysis_topics' | 'list_of_topics_related_to_%cf%80' |
| 'list_of_real_analysis_topics' | 'list_of_mathematical_topics_in_relativity' |
| 'list_of_knot_theory_topics' | 'list_of_trigonometry_topics' |
| 'list_of_convexity_topics' | 'list_of_algebraic_number_theory_topics' |
| 'list_of_functional_analysis_topics' | 'list_of_numeral_system_topics' |
| 'list_of_probability_topics' | 'list_of_combinatorial_computational_geometry_topics' |
| 'list_of_dynamical_systems_and_differential_equations_topics' | 'list_of_polynomial_topics' |
| 'list_of_graph_theory_topics' | 'list_of_order_theory_topics' |
| 'list_of_mathematical_topics_in_classical_mechanics' | 'list_of_circle_topics' |
| 'list_of_harmonic_analysis_topics' | 'list_of_algebraic_geometry_topics' |

**Exercises:**

1. Ideal triangle in $D^2$ with the $d_{hyp,D^2}$ metric are triangles with all three vertexes at infinity. We see a schematic of one in figure 6.

   (a) Prove all ideal triangle are congruent.

   (b) **Geodesic coalescence:** Show as move along an edge of our triangle towards infinity that the distance to the other edge sharing this point at infinity is $< 1/(y+C)$ for some constant $C$.

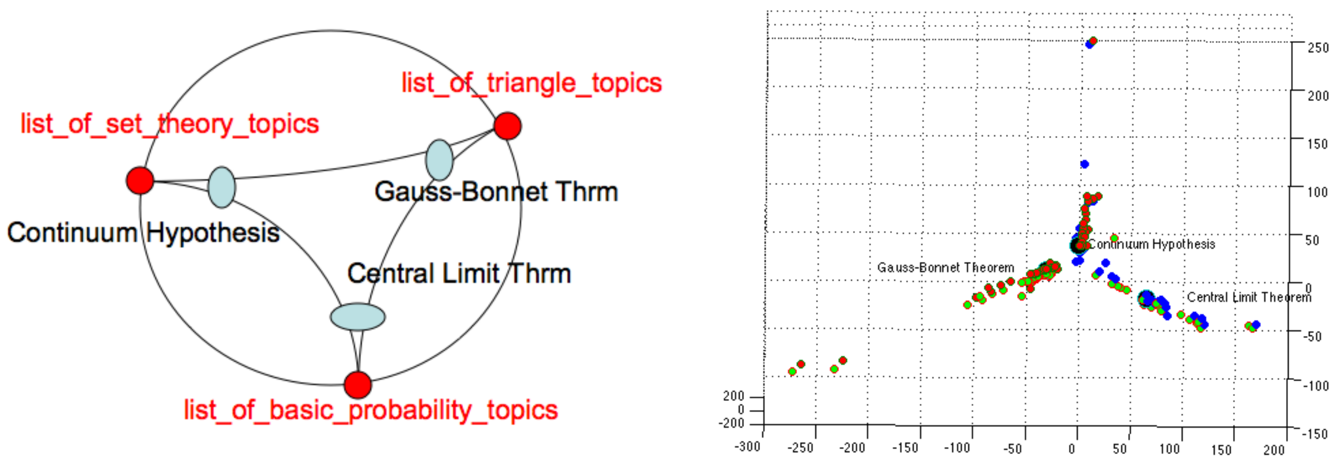


Figure 6: We see an the triangle connecting three relatively distant theorems, and a schematic showing us the points at infinity involved.

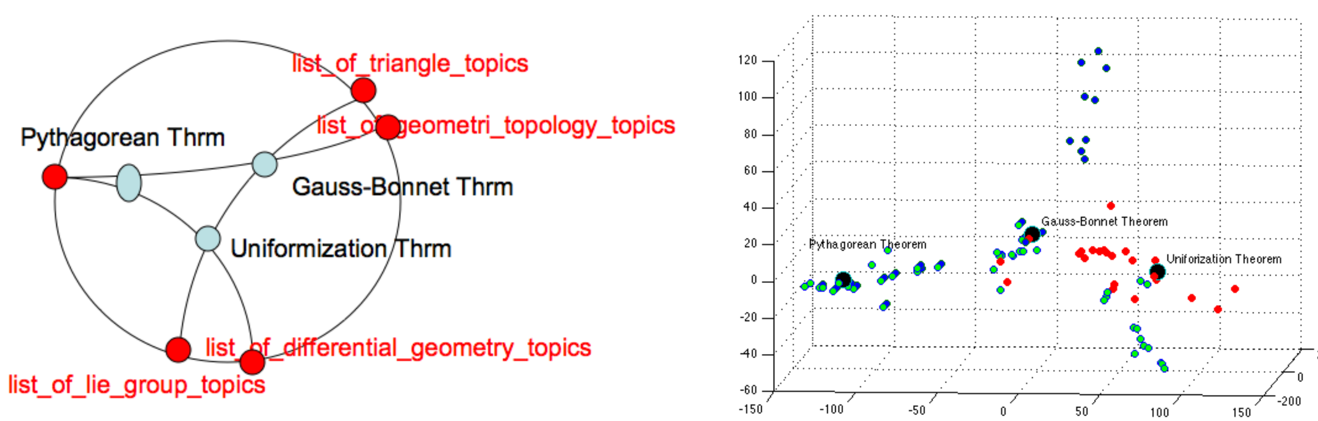


Figure 7: We see an the triangle connecting three relatively close theorems, and a schematic showing us the points at infinity involved.

2. We can define the angle between geodesics meeting at a state in analogy with hyperbolic plane. Do it.

3. Construct an example of a simple chain that illustrates Geodesic coalescence.