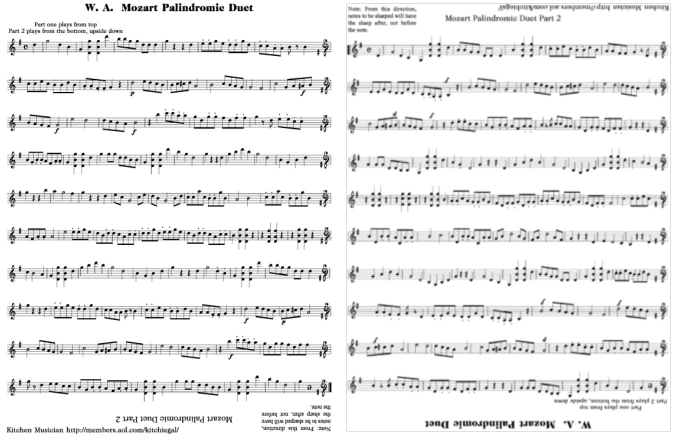
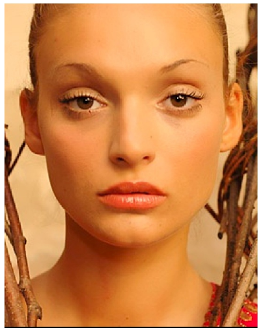
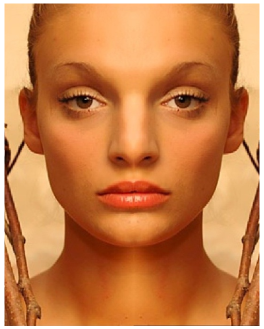
no lemon

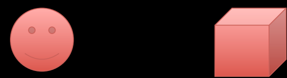
no melon

Mozarts: “no lemon, no melon”









Symmetry and its breaking

the quest for universals in nature

Sander Bais
Institute for Theoretical Physics
University of Amsterdam

Santa Fe Institute

A Preamble

Symmetrie

Symmetry, as wide or as narrow as you may define it, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection

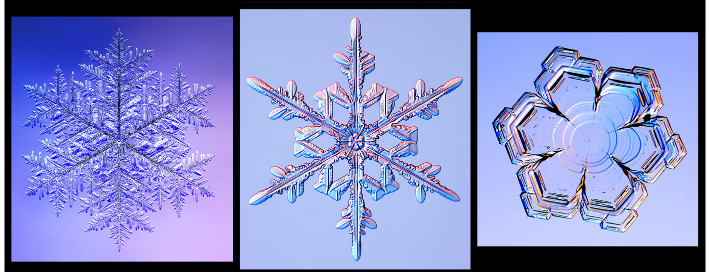
Hermann Weyl

The power of symmetry

Symmetry as a natural language for structure:

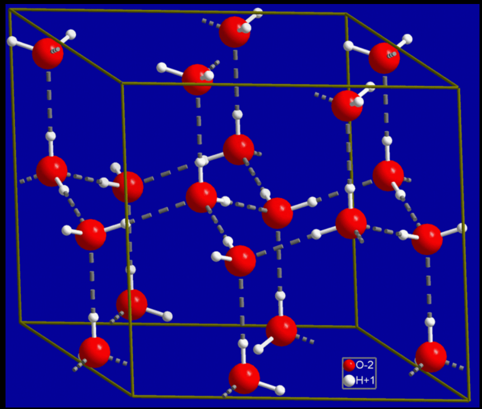
- Symmetries ⇔ groups/algebra's
- Symmetries of things
- Symmetries of spaces (and time)
- Symmetries of (sets of) equations
- Symmetry as guiding principle in the search for hidden order
- Symmetry breaking and its generic features
- Symmetry breaking and phase transitions.

Snowflakes

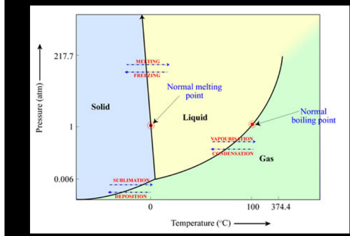


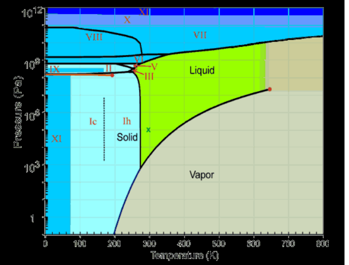
Splendid (im)perfection / snowflake evolves / path dependence

Basic hexagonal ice crystal

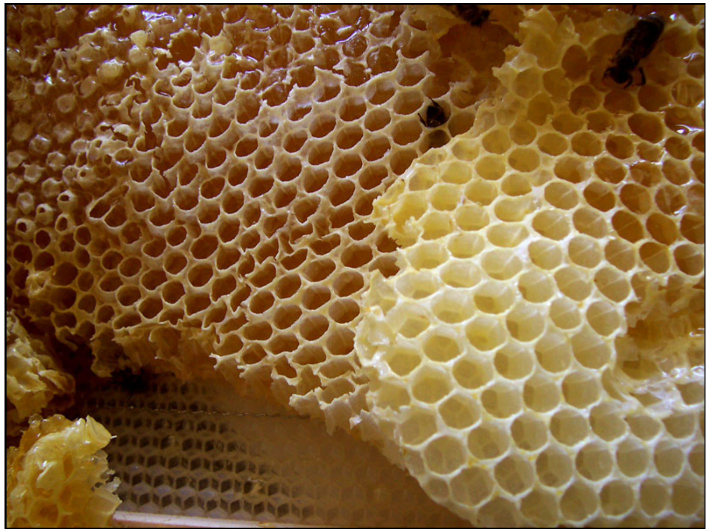


The many faces (phases) of water

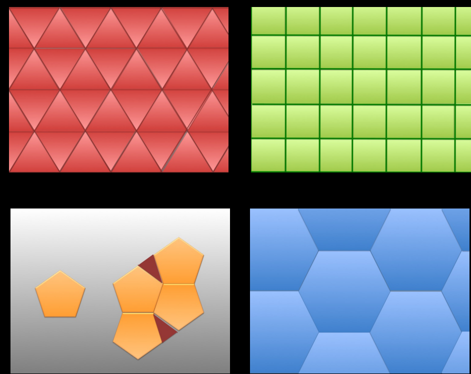


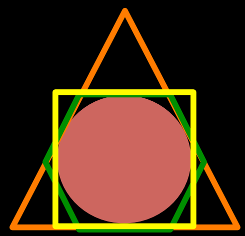


Ice-eight	ordered	Tetragonal
Ice-nine	ordered	Tetragonal
Ice-ten	symmetric	Cubic
Ice-eleven	ordered	Orthorhombic
Ice-eleven	symmetric	Hexagonal
Ice-twelve	disordered	Tetragonal

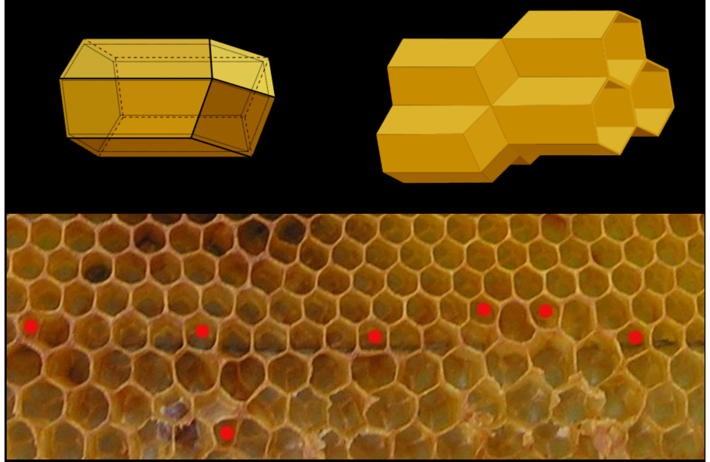


Space filling lattices

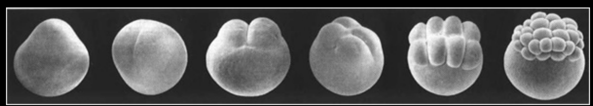




Optimal solutions often have the highest symmetry

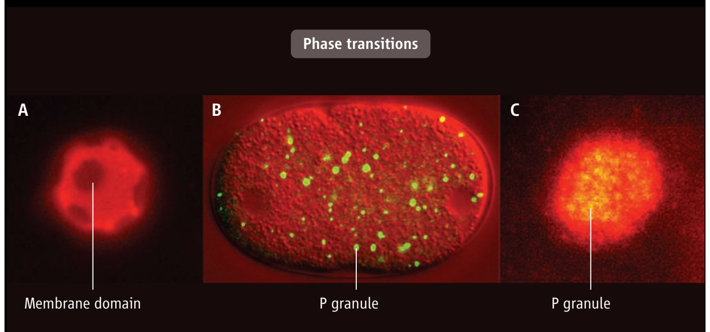


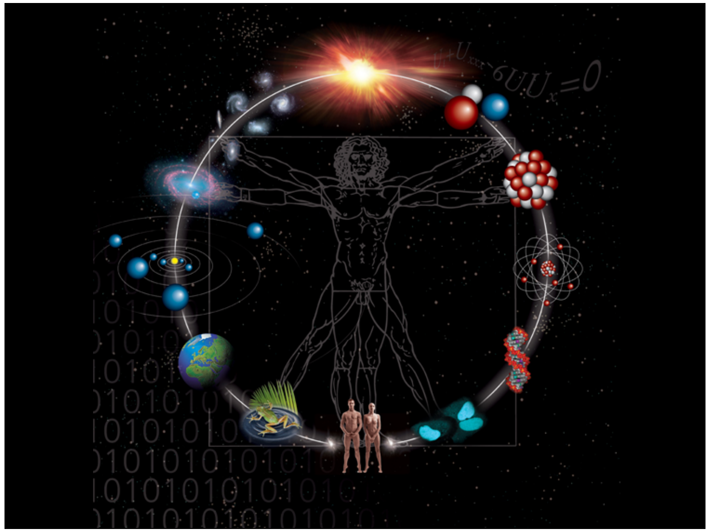
Symmetry breaking in cell division

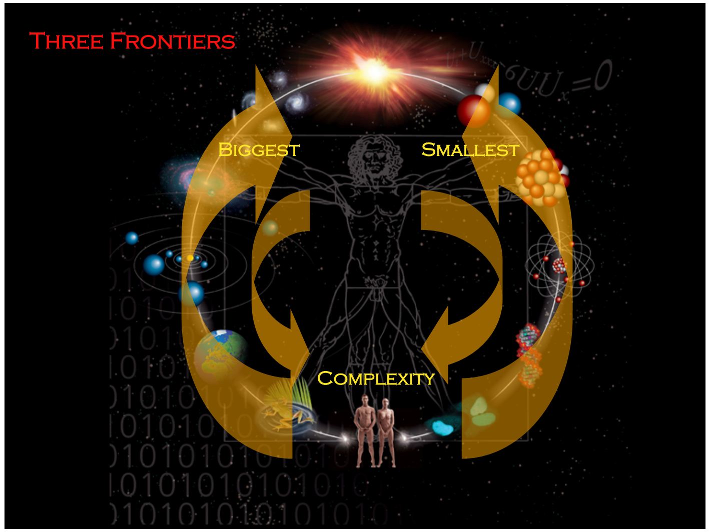


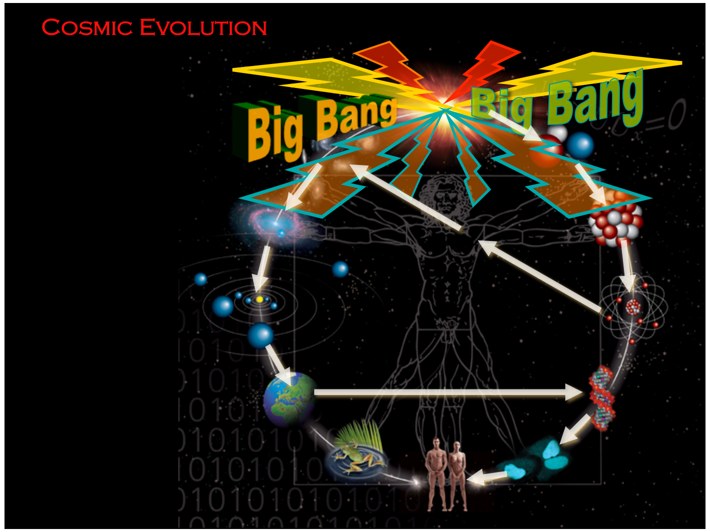
The quest for universals in biology

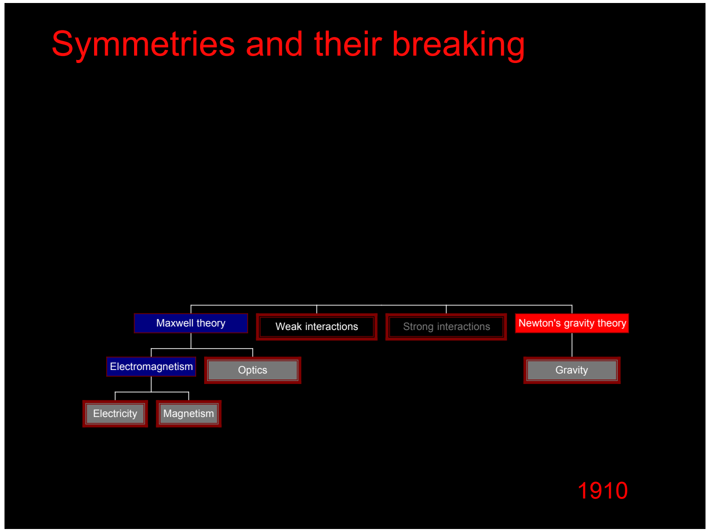
Phase transitons in membrane and cytoplasm

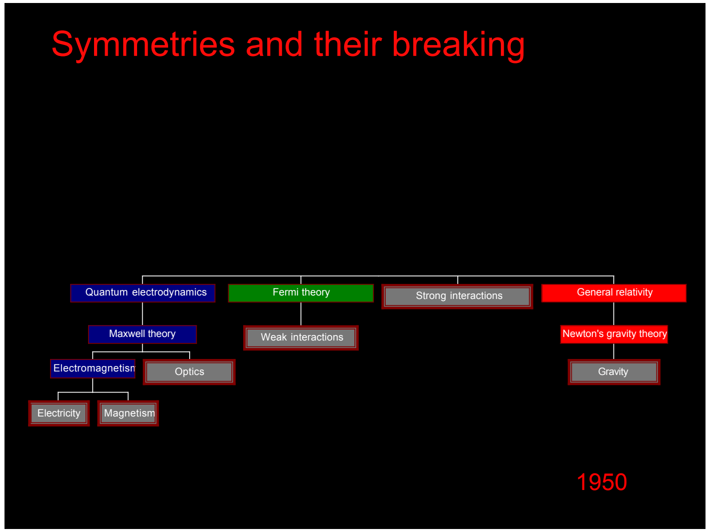


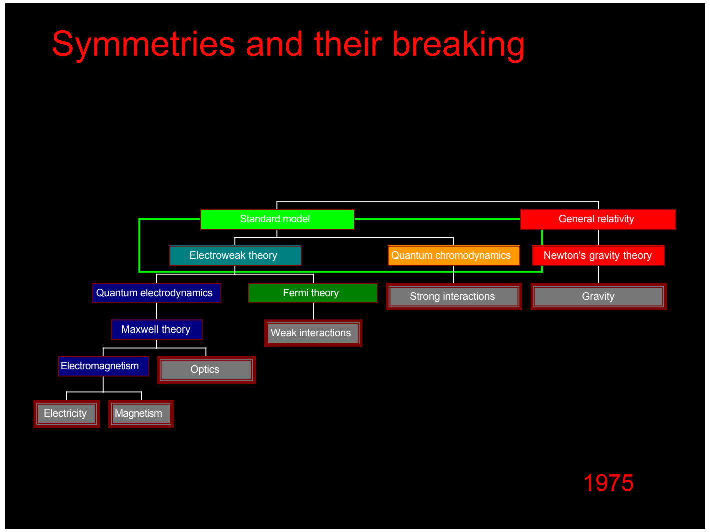


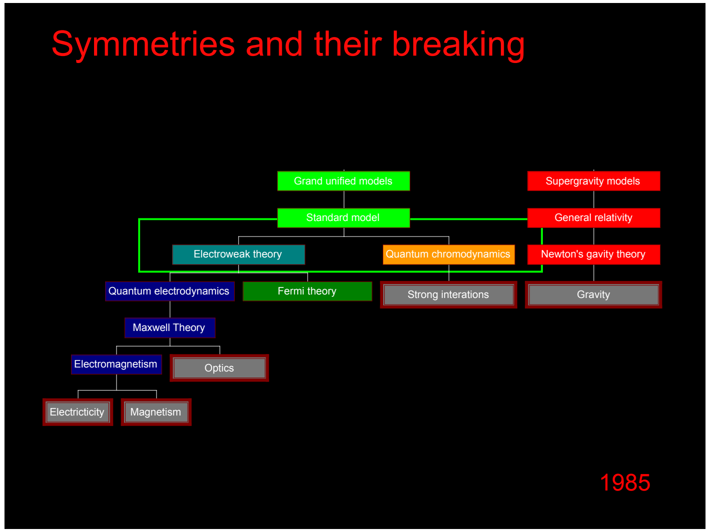

THREE FRONTIERS
BIGGEST
SMALLEST
COMPLEXITY

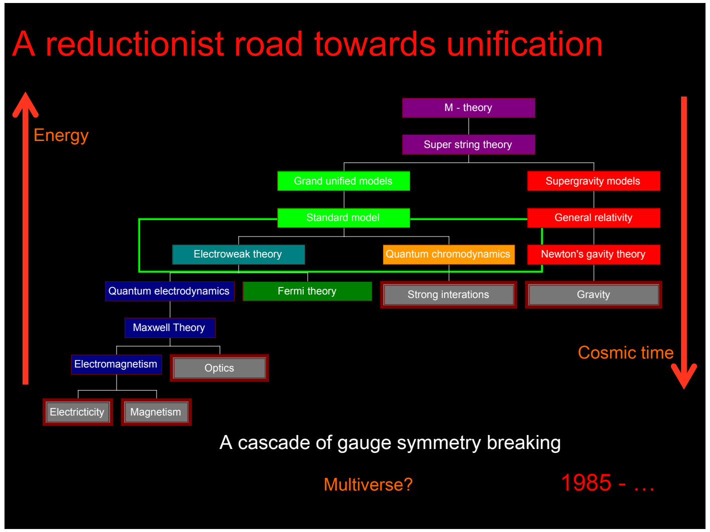

COSMIC EVOLUTION
Big Bang
Big Bang


Symmetries and their breaking
Maxwell theory
Weak interactions
Strong interactions
Newton's gravity theory
Electromagnetism
Optics
Gravity
Electricity
Magnetism
1910


Symmetries and their breaking
Quantum electrodynamics
Fermi theory
Strong interactions
General relativity
Maxwell theory
Weak interactions
Newton's gravity theory
Electromagnetism
Optics
Gravity
Electricity
Magnetism
1950

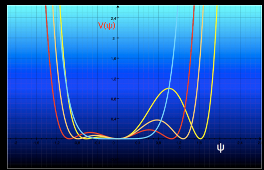

Symmetries and their breaking
Standard model
General relativity
Electroweak theory
Quantum chromodynamics
Newton's gravity theory
Quantum electrodynamics
Fermi theory
Strong interactions
Gravity
Maxwell theory
Weak interactions
Electromagnetism
Optics
Electricity
Magnetism
1975


Symmetries and their breaking
Grand unified models
Supergravity models
Standard model
General relativity
Electroweak theory
Quantum chromodynamics
Newton's gavity theory
Quantum electrodynamics
Fermi theory
Strong interations
Gravity
Maxwell Theory
Electromagnetism
Optics
Electricticity
Magnetism
1985


A reductionist road towards unification
Energy
M - theory
Super string theory
Grand unified models
Supergravity models
Standard model
General relativity
Electroweak theory
Quantum chromodynamics
Newton's gavity theory
Quantum electrodynamics
Fermi theory
Strong interations
Gravity
Maxwell Theory
Cosmic time
Electromagnetism
Optics
Electricticity
Magnetism
A cascade of gauge symmetry breaking
Multiverse?
1985 - …

Reaction diffusion

Reaction diffusion equations



$$\dot{\psi} = \varepsilon\psi - \left(\nabla^2 + 1\right)^2 \psi + g_1 \psi^2 - \psi^3$$

Symmetry breaking in Biology?

- Symmetry breaking, which may occur at multiple levels, is a prevalent process in biology, because organismal survival depends critically on well-defined structures and patterns at both microscopic and macroscopic scales — indeed, patterns like those seen on the fearsome tiger are consequences of broken symmetry.
- P.W. Anderson, speculated that increasing levels of broken symmetry in many-body systems (systems of many interacting components) correlates with increasing complexity and functional specialization (Anderson 1972). This is certainly true in biology, as symmetry breaking along well- defined axes is intimately linked to functional diversification on every scale, from molecular assemblies, to subcellular structures, to cell types themselves, tissue architecture, and embryonic body axes.

Li and Bowerman, Cold Spring harbor perspectives in Biology (2010)

2. Symmetries and groups

What's in the name?

Symmetry:

discrete / continuous

global / local

spacetime / internal

broken / unbroken

abelian / non-abelian

super / ordinary

finite / infinite

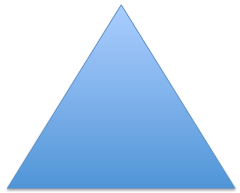
classical / quantum

Eskimo's…

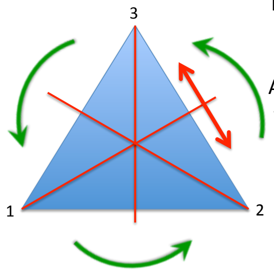
The physics of symmetry

- Systems (building blocks, constituents, cells)
- Interactions => Dynamics => collective behavior. Conservation laws.
- Geometric perspective
- Algebraic perspective
- Breaking of symmetries and their characteristics
- Phase transitions and symmetrybreaking

Symmetries of an object



Symmetries of an object for a Group



The set of operations (transformations) $G=\{g_i\}$, that conserve the shape of an obje (leave the object invariant) forms a group. Also two successive transformations leave the object invariant so we have that:

$g_1 \cdot g_2 = g_3$ and e, g^{-1}

Rotations over 60: e, r, r^2
Mirror symmetry: s_{12}, s_{13}, s_{23}

The group G of an equilateral triangle is discrete and consists of 6 element

(123)=> (312)=>(213)=>...

This group is D_3 or S_3 the permutation group of three objects.

Representation of group on a vector space

G ={g} ←→ { n x n matrices} ←→ n-dim vectors

e=

1	0	0
0	1	0
0	0	1

r=

0	0	1
1	0	0
0	1	0

r^2=

0	1	0
0	0	1
1	0	0

s_{23}=

1	0	0
0	0	1
0	1	0

s_{13}=

0	0	1
0	1	0
1	0	0

s_{12}=

0	1	0
1	0	0
0	0	1

<1> =

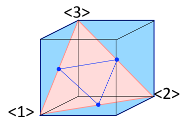
1
0
0

<2> =

0
1
0

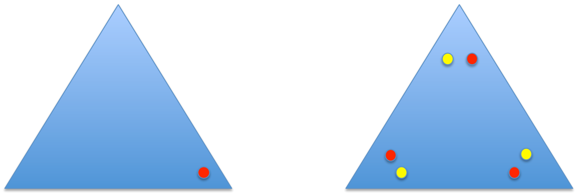
<3> =

1
0
0



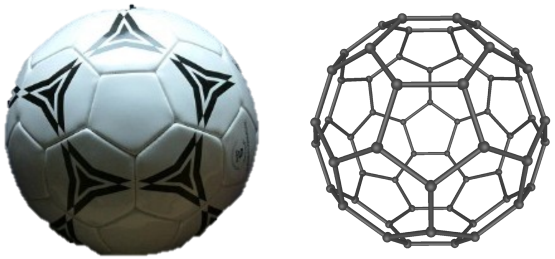
Breaking of the symmetry

G => H



$D_3 \Rightarrow Z_2$ $D_3 \Rightarrow Z_3$

Other objects: today's Bucky ball special



The Fisher-Griess Monstergroup F or M1

Largest finite group $G=\{g_i \,;\, g_i\, g_j = g_k\}$

\# of elements g_i equals:

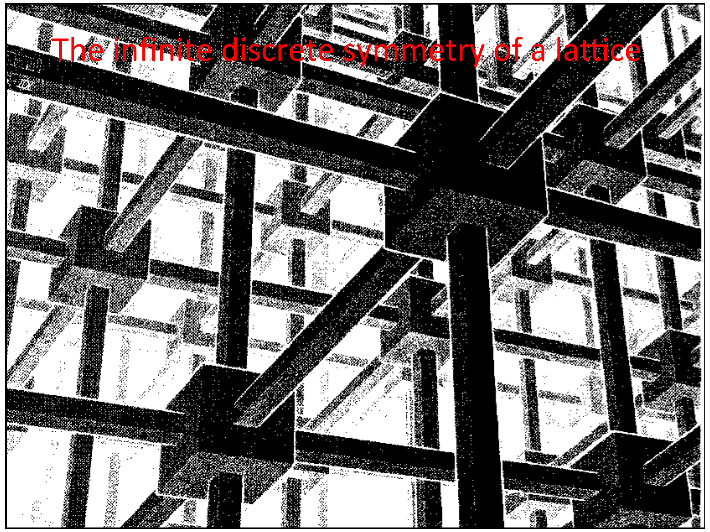
$= 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

=
808.017.424.794.512.875.886.459.904.961.710.757.005.754.368.000.000.000

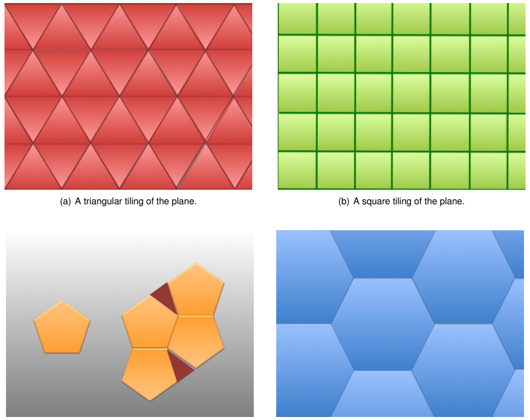
$\approx 8 \times 10^{53}$.

Smallest faithful representation: 196882×196882 matrices

The infinite discrete symmetry of a lattice



Lattices



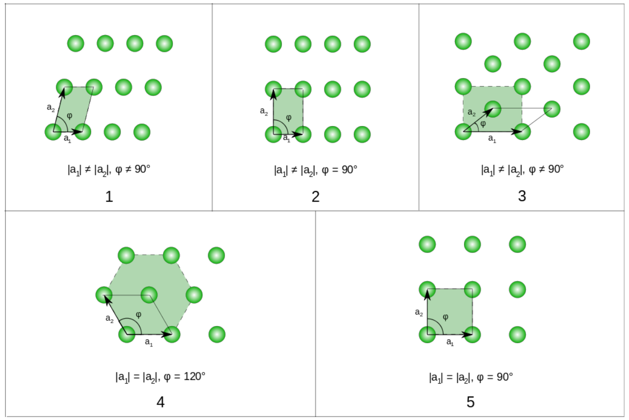
(a) A triangular tiling of the plane.

(b) A square tiling of the plane.

(c) The plane cannot be filled with pentagons.

(d) A hexagonal tiling of the plane. Adding the centers would make it a triangular lattice like (a) again.

Lattices (space filling)

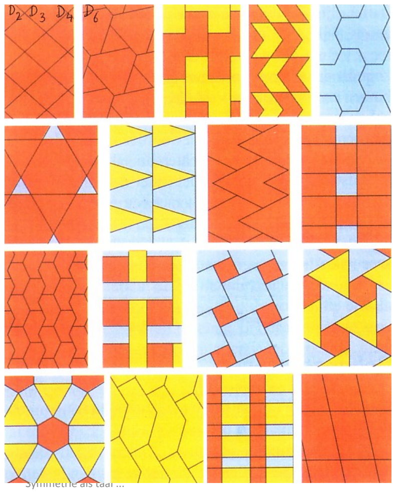


Space groups

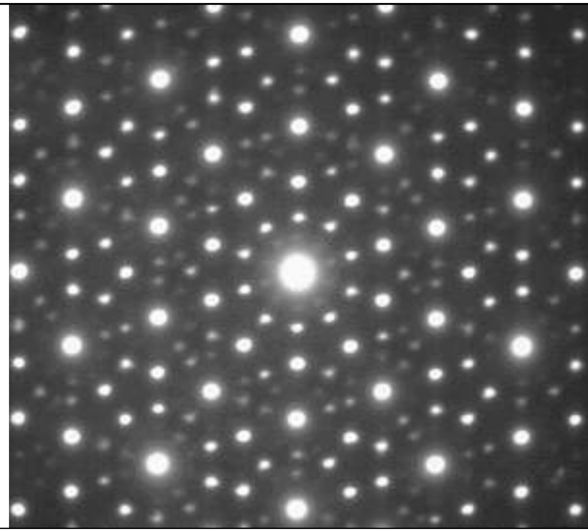
In 2 dimensions:
17 space groups
10 point groups

No five fold symmetries!

In 3 dimensions:
240 space groups
32 point groups

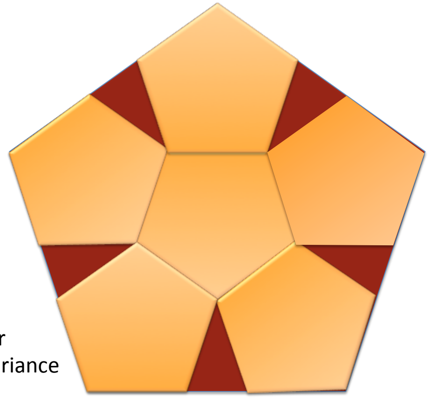


10-06-14 Symmetrie als taal ...



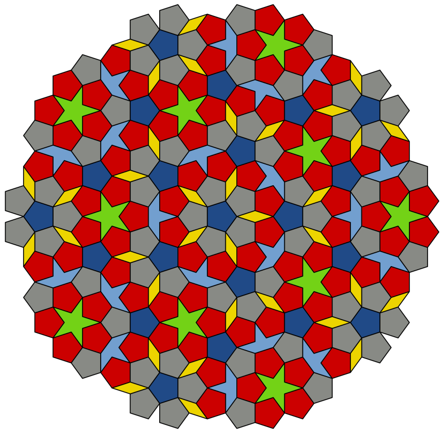
The pentagon

Stacking pentagons

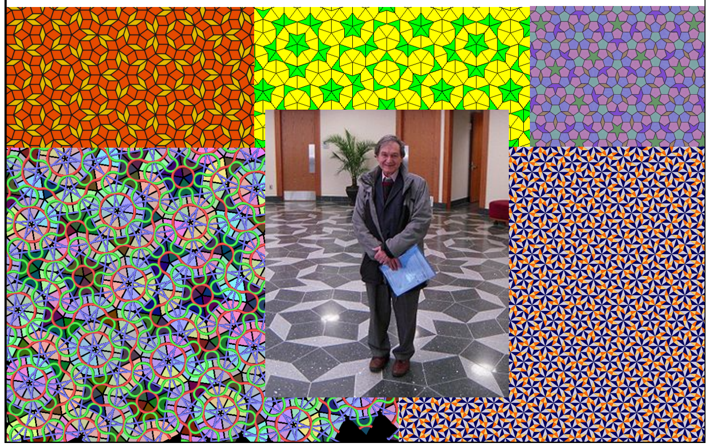


Selfsimilar
Scale invariance
Fractal

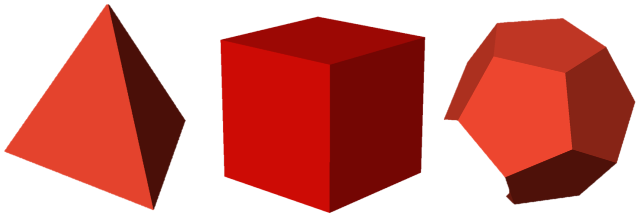
Penrose tilings



Penrose tilings

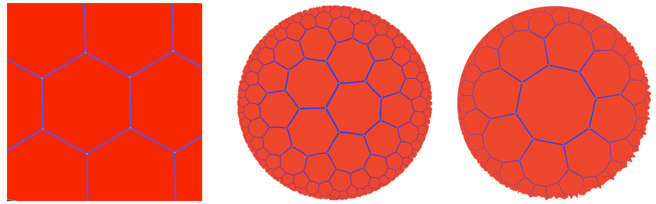


Regular polygons



{3,3} {4,3} {5,3}

Relation with tilings



{6,3} {7,3} {8,3}

Symmetry of a space

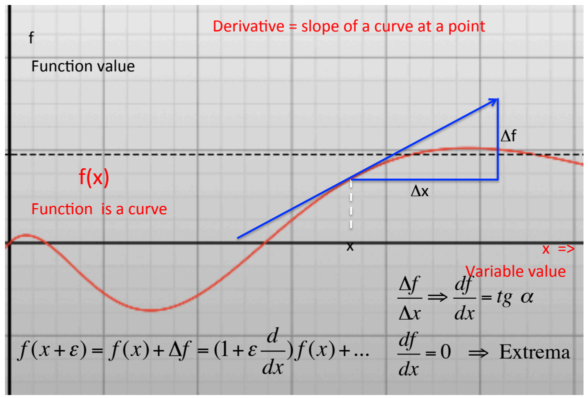
- An infinite line R:

x

- what are the symmetries?
- translations x => x+a
- scale transformations x => λ x
- reflection in a point x => -x

- These symmetries are mostly continuous

The slope (derivative) of a function



Translations and their effect

- Transformation $x \Rightarrow x' = x+a$
- $G = R$ is a continous group with $T_a\, T_b = T_{a+b}$
- $f(x) \Rightarrow f'(x) = f(x') = f(x+a)$
- Effect of a small translation $a=\varepsilon$ of a function:
  $f(x+\varepsilon) = f(x) + \varepsilon df/dx + \varepsilon^2 \ldots + \ldots$
- The “derivative” $p = d/dx$ “generates” an infinitesimal translation of the function at x
- Knowing df/dx in a point, tells you something about f in the neighborhood of that point
- *Eigenfunctions* $f_k(x)$ of $p=d/dx \Rightarrow p\, f_k = k\, f_k$ then $f_k = e^{kx}$
- The f_k form (irreducible) representations of R
- Tensor “multiplication” of irreps $f_a\, f_b = f_{a+b}$
- Invariant function requires : k= 0 so f= const.

Scale transformations

- $x \Rightarrow \lambda x$
- Generator is $D = x\, d/dx$
- $D\, f_b = x\, \lambda^b\, b x^{b-1} = b\, f_b$
- Eigenfunctions: $f_b(x) = x^b \Rightarrow$
  $f'(x) = f(\lambda x) = \lambda^b\, x^b = \lambda^b\, f_b(x)$
- The f_b form representations with ”scaling dimension” b
- Invariant function: f = const
- (Lie) Algebra of commutators of generators of T and D:
  $[D,p] = x(d/dx)^2 - d/dx - x((d/dx)^2 = -p$
  $[p,p]=0$ and $[D,D]=0$
- Non-commuting <=> non-abelian algebra (group)
- No common eigenfunctions of p and D except f = const

Continuous symmetry

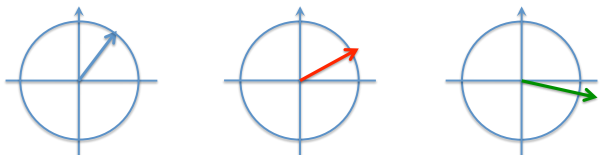
Group: Rotations in the plane: SO(2)

$G = \{R_\alpha\}$ $(0 < \alpha < 360°)$

Circle $G = S^1 = R/Z$

$R_{\alpha 1} \cdot R_{\alpha 2} = R_{\alpha 1+\alpha 2} = R_{\alpha 2} \cdot R_{\alpha 1}$

Representations of the group: r_n $f_n = e^{in\alpha}$ $n = 0, \pm 1, \ldots$



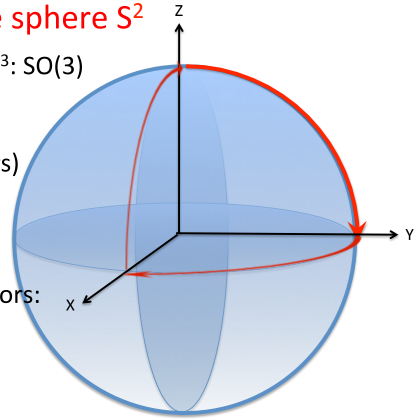
Symmetry of the sphere S^2

Group: Rotations in R^3: SO(3)
$G = \{R_{\check{n},\alpha}\}$

Generators (operators)
$L_x = y\, d/dz - z\, d/dy$
…

Algebra of commutators:
$[L_x, L_y] = L_z$
…

Representations:
r_l with $l = 0,1,2\ldots$
dimension $2l+1$ also r_s $s=1/2, 3/2, \ldots$



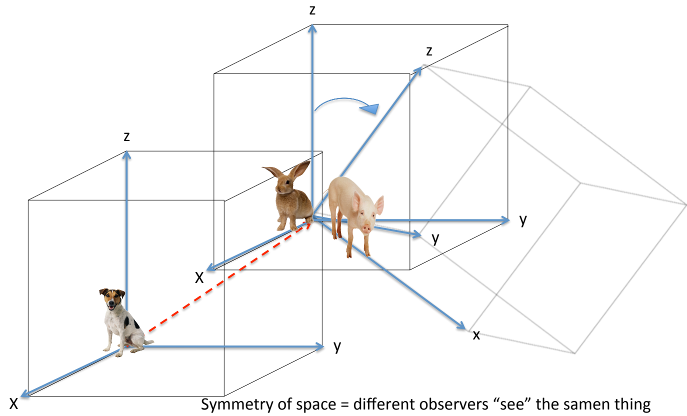
Euclidean space is homogeous en isotropic

Euclidean space R^3 : {x,y,z}
The empty space looks the same from any point {x,y,z} and in any direction

⇒ Empty space is *invariant* under *translations* and *rotations.* This group is the Euclidean group

$$E_3 \simeq R^3 \rtimes O(3)$$

Symmetries of empty space link the reference (rest) frames of different observers



Symmetry of space = different observers "see" the samen thing

Equations

The symmetries that are important in nature, are not the symmetries of things but the symmetries of equations

Steven Weinberg

3. Symmetries and equations

Cubic equation

Which values for x solve the following equation?

$$x^3 + 89x^2 + 2638x + 26040 = 0$$

$$x = 0 \Rightarrow 26040 = 0 \quad No!$$

$$x = 1 \Rightarrow 28768 = 0 \quad No!$$

$$\Rightarrow x = ?$$

Solution:

$$x_1 = 28 \quad x_2 = 30 \quad x_3 = 31$$

Help!

Cubic equation:

$$x^3 + bx^2 + cx + d = 0$$

and

$$(x - x_1)(x - x_2)(x - x_3) = 0$$

This eqn is invariant under permutations of x_1, x_2 en x_3, so the group is $D_3 \simeq S_3$.

Comparing the coefficients yields

$$\left.\begin{array}{l} b = -(x_1 + x_2 + x_3) \\ c = (x_1x_2 + x_2x_3 + x_3x_1) \\ d = x_1x_2x_3 \end{array}\right\} \begin{array}{c} Invariante\ polynomen \\ in\ x_1, x_2\ en\ x_3 \end{array}$$

Conclusions:
- Symmetry group helps solving equation
- Symmetry group transform solutions into each other
- "Space" of solutions is invariant

Energy function H(p,q) and Newton's equations

Energy of particle in potential depends on momenta and coordinates only

$$H(q,p) = H_{kin} + H_{pot}$$
$$= p^2/2m + V(q)$$

Newton's laws

$$\dot{q} = \frac{\partial H}{\partial p} \quad \rightarrow \quad \dot{q} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \quad \rightarrow \quad \dot{p} = -\frac{\partial V}{\partial q} = F$$

Time evolution of some dynamical variable f(p,q)

In general we may write

$$\begin{aligned} \frac{df}{dt} &= \\ &= \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} \\ &= \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} \end{aligned}$$

We define a Poisson Bracket:

$$\{f, g\} \equiv \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

This yields the following equation fot the time derivative of f:

$$\frac{df}{dt} = \{f, H\}$$

Conserved quantities generate symmetries

H generates time evolution through Poisson bracket implying that:

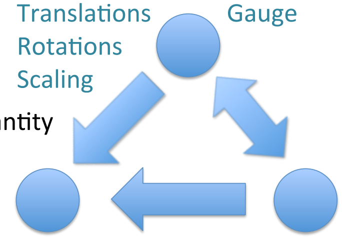
$$\{p, H\} = 0 \Rightarrow \frac{dp}{dt} = 0$$ p is conserved

$$\{f, p\} = \frac{df}{dq}$$ p generates translations

$$\{H, p\} = 0 \Rightarrow \frac{\partial H}{\partial q} = 0$$ H is translation invariant

Continuous space symmetries

Symmetry operators: group
Infinitesimal transformations:
Lie algebra of generators:
Translations
Rotations
Scaling
Gauge



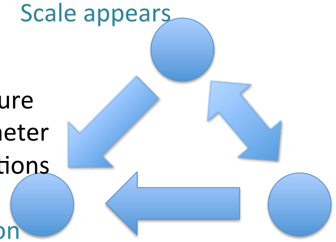
Physical quantity
Generator
Operator:
Energy
Momentum
Angular momentum

Conservation laws
Quantum numbers
Labeling of states:
Conservation of:
Energy,
Momentum
Charge

4. Symmetries and their breaking

Symmetry breaking

Symmetry Breaking
Translations => lattice
Rotations => anisotropy
Scale appears



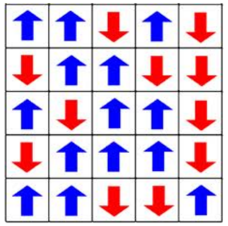
Phase structure
Order parameter
Phase transitions
Masses
Magnetisation
Superconductivity
Mode analysis

Defects
in broken phase:

Kinks, walls,
monopoles
vortices

Ising Model

$$H(\sigma) = -\sum_{<ij>} J_{ij}\sigma_i\sigma_j - \mu\sum_j h_j\sigma_j$$



$$\sigma_i = \pm 1$$

Global internal Z_2 symmetry: $\sigma_i \Rightarrow -\sigma_i$

$$P_\beta(\sigma) = \frac{e^{\beta H(\sigma)}}{Z_\beta}$$

Ising simulator

Partition sum

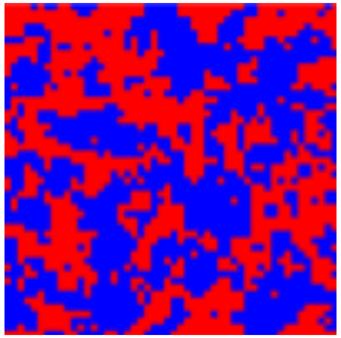
$$Z_\beta = \sum_\sigma e^{-\beta H(\sigma)}$$

Ising simulator

Magnetisation

$$M = < \sum \sigma_i >$$

Critical Ising



$T>T_c$

- Symmetric phase
- Disorder
- Short range correlations

$$< \sigma(x)\sigma(y) > \simeq e^{-|y-x|/\zeta} >$$

$T< T_c$

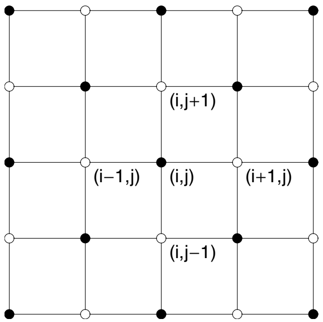
- Broken phase
- Order
- Correlations are constant

$$< \sigma(x)\sigma(y) > \simeq 1$$

$T= T_c$

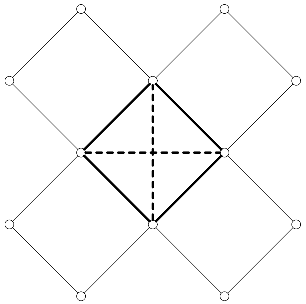
- Fluctuations on all scales; scale invariance
- Correlations: powerlaws: $< \sigma(x)\sigma(y) > \simeq 1/|y-x|^\alpha >$

2-d Ising; renormalization group



$$\mathcal{H} = -J\sum_{i,j}(s_{i,j}s_{i+1,j} + s_{i,j}s_{i,j+1})$$

Ising: Blockspinning



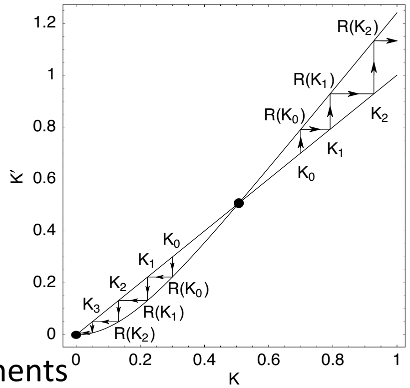
$$\mathcal{Z} = \sum_{s_{i,j}=\pm 1}\prod_{i,j} e^{K(s_{i,j}s_{i+1,j}+s_{i,j}s_{i,j+1})}$$

Ising: Blockspinning

Critical point

=>

Critical exponents



Conformal algebra in d=2

Symmetry algebra is infinite dimensional:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}cm(m^2-1)\delta_{m,-n}$$

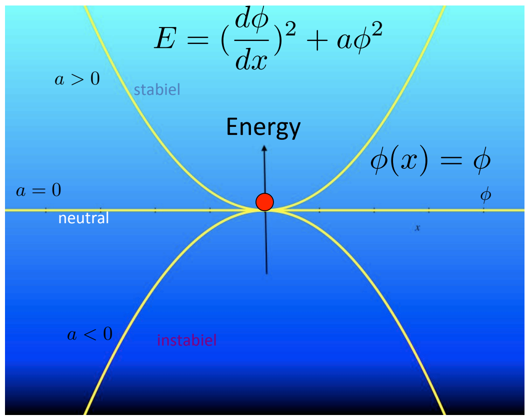
Fields are organized in representations of conformal algebra
And the lowest weight of such a representation corresponds to the Critical exponents (powers):

$$c = c^{(m)} = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, 5, \ldots .$$

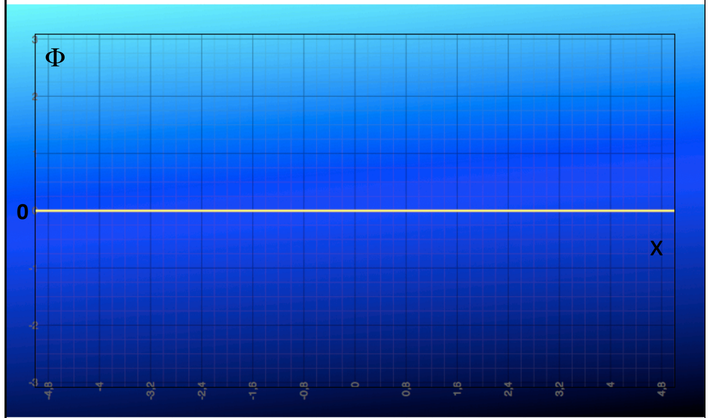
$$h_{p,q}^{(m)} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \qquad \begin{cases} 1 \leqslant p \leqslant m-1 \\ 1 \leqslant q \leqslant m \end{cases}$$

Mean field theory effective descriptions of phase transitions

Local order parameter $\phi(x) = \langle\sigma(x)\rangle$ and stability



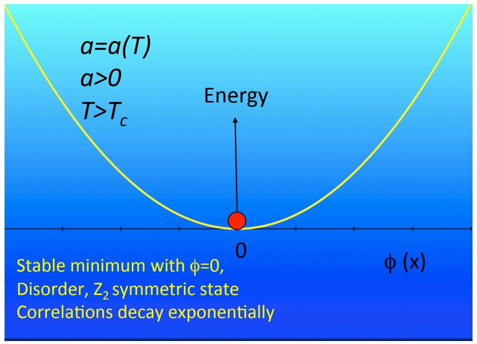
The symmetric groundstate (vacuum): $\phi(x)=0$ for all x



Second order transitions

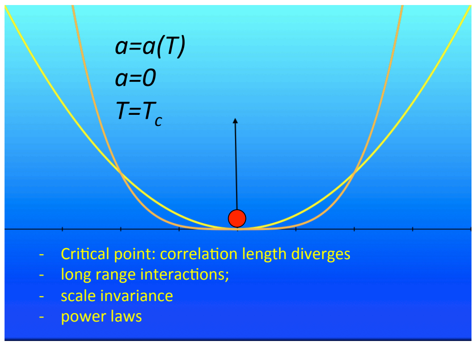
Phase transitions and symmetry breaking

$$E = (\frac{d\phi}{dx})^2 + a\phi^2 + b\phi^4$$



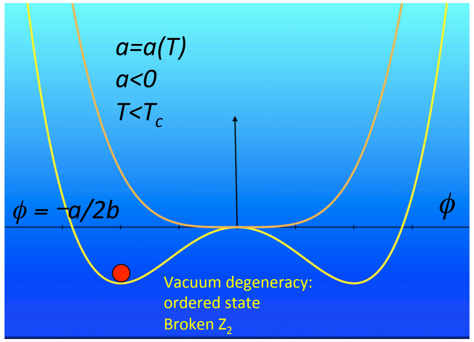
Phase transitions and symmetry breaking

$$E = (\frac{d\phi}{dx})^2 + a(\phi^2 - b)^2$$



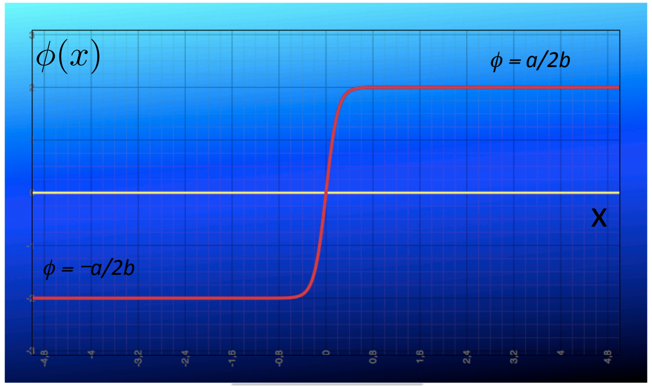
Symmetry breaking: ϕ^2=b

$$E = (\frac{d\phi}{dx})^2 + a\phi^2 + b\phi^4$$



The topological defect: Kink or Domain wall

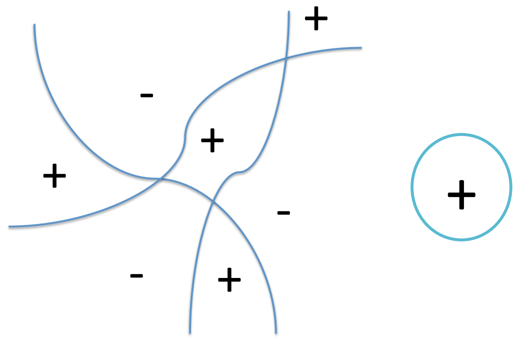
$$\phi(x) = 2\tanh(5x)$$



Defects

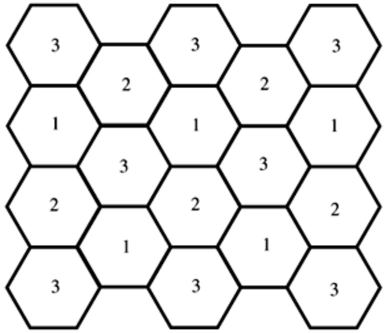
- Defects are emergent structures
- They are characteristic for phases with broken symmetries (with groundstate degeneracy)
- They are locally topologically stable
- They play an important role in the dynamics of the phase transition (duality)
- They can be quite generally classified.

Domain structure in d=2 => network of walls



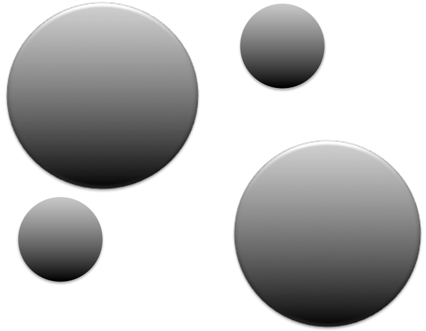
Domain structure => pattern formation

Three distinct groundstates:

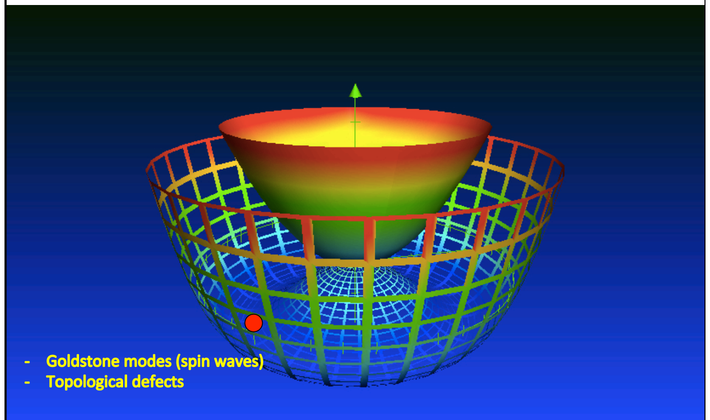


Domains in 3 dimensions

Domains are locally stable but not globally



Excitations in broken phase



Defect classification

- G breaks to H (subgroup of G)
- Orbit of orderparameter ϕ under G equals G/H
  This is the so called vacuum manifold of degenerate groundstates
- Spatial manifold M with boundary dM
- Study (homotopy) classes of maps dM => G/H
- These classes form a group and label the topological charges of the defect

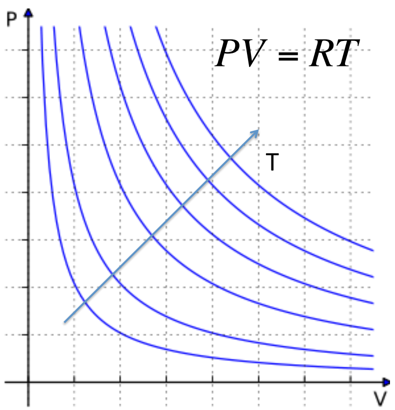
Defect Types

- D=1 dM = Z_2 π_0(G/H) Kinks or walls
- D=2 dM=S^1 π_1(G/H) Particles or line
- D=3 dM= S^2 π_2(G/H) = π_1(H) Monopoles

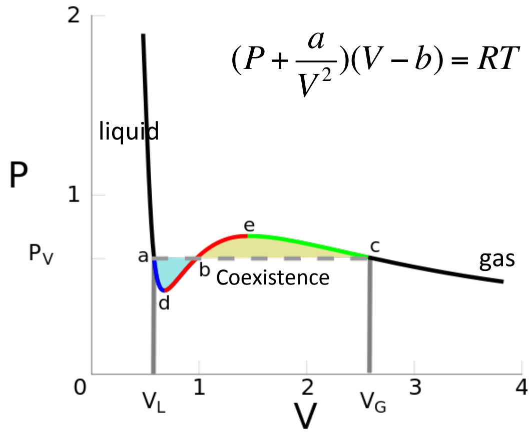
- Non-abelian line defects e.g. if G =SO(3) and H discrete => π_1 = H

First order transitions

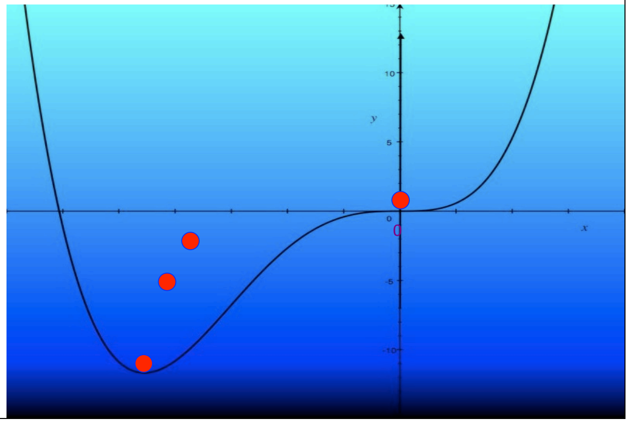
Equation of state: Ideal gas law



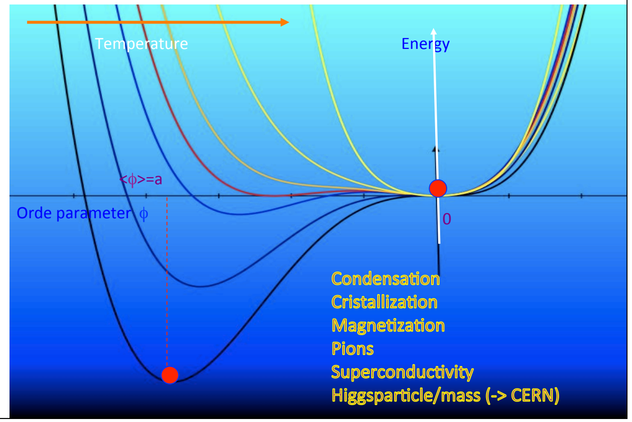
Van der Waals equation of state

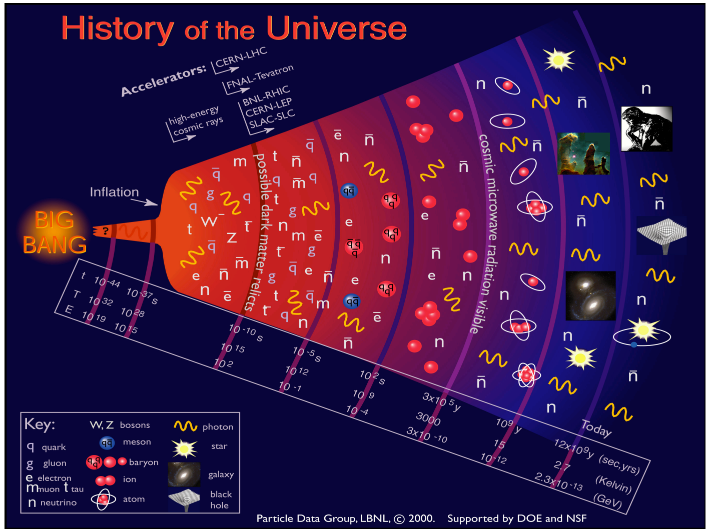


1st order transitions: bubble nucleation



Metastable state persist until transition through bubble formation





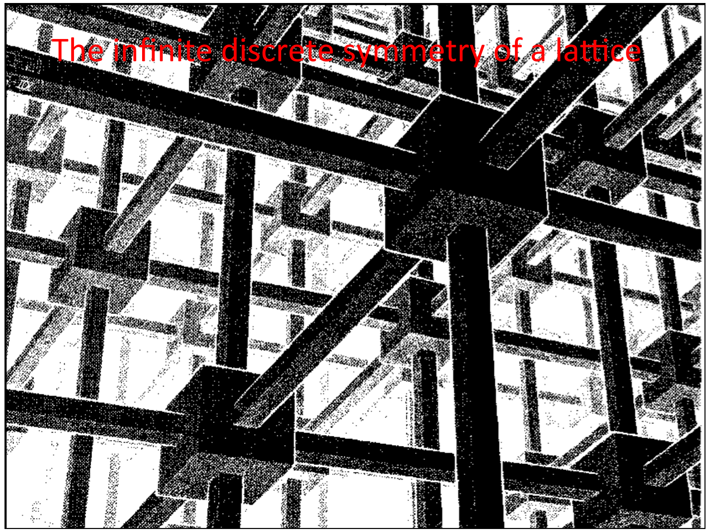
Euclidean space is homogeous en isotropic

Euclidean space R^3 : {x,y,z}
The empty space looks the same from any point {x,y,z} and in any direction

⇒ Empty space is *invariant* under *translations* and *rotations.* This group is the Euclidean group

$$E_3 \simeq R^3 \rtimes O(3)$$

The infinite discrete symmetry of a lattice



Breaking the Euclidean group

G = Euclidean group of continuous rotations and translations
(E_3 = 6 parameter goup , E_2 is three parameter group)

$(R_1, a_1)\,(R_2, a_2) = (R_1R_2,\ a_1 + R_1a_2)$

H = Discrete symmetry group of crystal lattice
(Square lattice in d=2: $H = Z_4\ \underline{X}\ (Z \times Z)$)

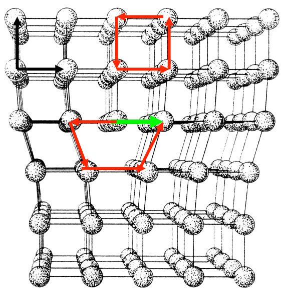
Excitations:

1. Goldstone modes → Phonons G/H (fundamental)
2. Solitons (defects) → $\pi_1(G/H) = \pi_0(H) = H$ (topological)

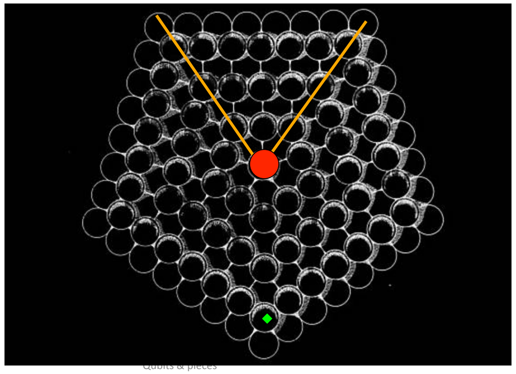
→ group H classifies line/point defects

Dislocatie: translational defect

- Defect (1, a)
- Burgersvector a



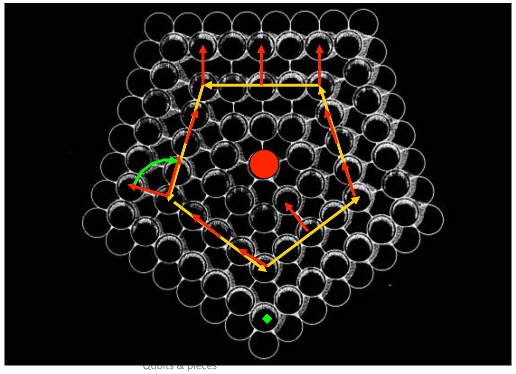
Disclination



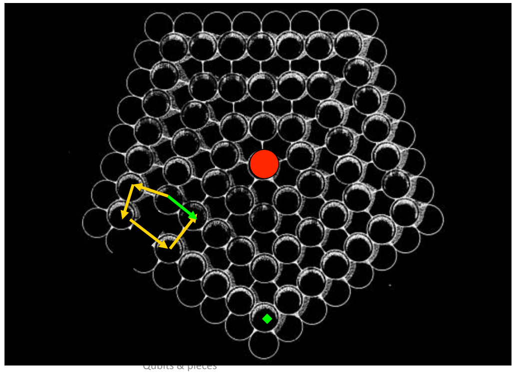
Disclination: Rotational defect

Defect (R, 0)

R=R(-π/2)



Two non-commuting defects



Braiding of noncommuting defects

T :

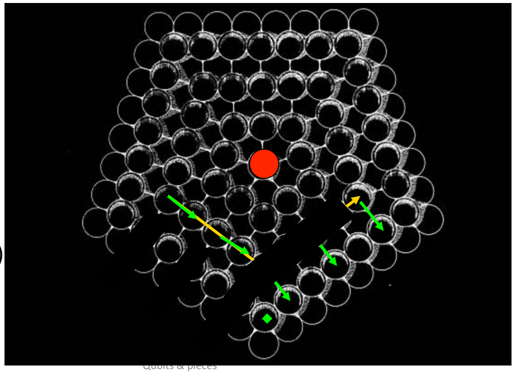
(1,a)(R,0) →

→

(1, a)(R,0)
(1,-a)(1, a)

= (R, a - Ra) (1, a)

= (R, a)



Braiding of noncommuting defects

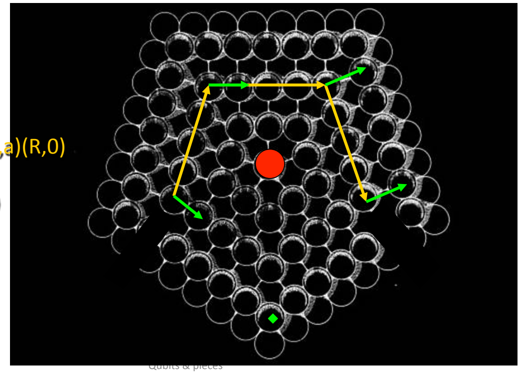
T^{-1} :

(1,a)(R,0) →

→(R,0)(R^{-1},0)(1,a)(R,0)

→ (R,0)(1,R^{-1} a)

= (R,a)



Summary

The notion of symmetry breaking has apotential for many applications.

the notion of symmetry
- of objects, spaces and equations
- discrete, continuous ; finite infinite

breaking symmetries
- order parameter
- phase transistions => critical point
- scale (conformal invariance at critical point
- classification of modes and defects
- duality