First Passage Time Statistics in the Logistic Model

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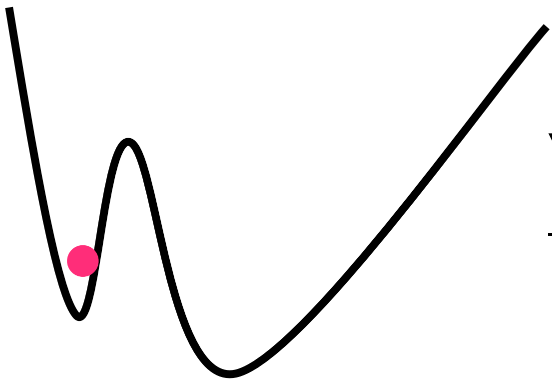
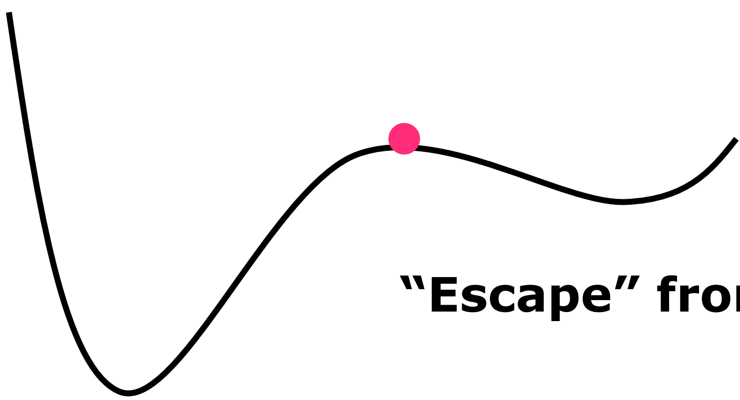
In general we can consider the **FIRST PASSAGE TIME** problem in different situations depending on the potential V(x) acting on the particle…



“Escape” from a sharp well
(Metastable problem)

$$\dot{x} = -\partial_x V(x) + \sqrt{\varepsilon}\,\xi(t)$$



“Escape” from an unstable point

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Passage times in standard Markov processes

1) Let the SDE be: $dx = -\partial_x V(x) dt + \sqrt{\varepsilon}\, \xi(t) dt$

2) If the noise is Gaussian and white: $\xi(t)\, dt = dw(t)$

The FPTD is reduced to study the Adjoint FP: $L^{+} = [-\partial_x V(x)]\partial_x + \varepsilon\, \partial^2{}_x$

3) In particular the "life-time" T from any unstable state Xo is the MFPT to reach a threshold value Ro starting from Xo.
This and similar problems can be studied from Dynkin's eq.:

$$L^{+} T(x_0) = -1; \; T(R_0) = 0$$

4) Any chance to study the problem if the noise is Non-Gaussian ?
5) Any chance to get analytical approximations for the FPTD from any unstable states ?

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The Stochastic Paths Perturbation Approach (SPPA)

From the SPPA we can get analytical approximations for the individual paths of any unstable stochastic process, x(t), by using the following approach:

1) Introduce into the SDE

$$\dot{x} = -\partial_x V(x) + \sqrt{\varepsilon}\,\xi(t)$$

the non-trivial transformation

$$\boxed{x(t) = z(t)\, y^{-\theta}(t)}$$

1) Solve iteratively the realization of each stochastic path $x(t)$

$$x(t) \approx O(\sqrt{\varepsilon})$$

2) From the random algebraic e.q.: $x(t_e) \approx O(1)$ **. It is possible,**

by invertion to get: $\boxed{t_e = f(\text{random numbers})}$

First Passage time statistics for the Logistic stochastic model

1) **Consider population Verhults' eq:** $\frac{d}{dt}n(t) = r\left(1 - \frac{n(t)}{K}\right)n(t); \quad \{n, r, K\} > 0$

2) **Then the SDE is:** $\boxed{\frac{d}{dt}n(t) = r\left(1 - \frac{n(t)}{K}\right)n(t) + \sqrt{\varepsilon}\,\xi(t); \quad n(t) \geq 0}$

3) **Introducing the transformation:** $\boxed{n(t) = z(t)\, y^{-1}(t)}$

4) **We get two coupled SDE:** $\boxed{\frac{dz}{dt} = rz + \sqrt{\varepsilon}\, y\xi\,; \qquad \frac{dy}{dt} = \frac{r}{K}z}$

5) **Iterate the solutions** $z(t), y(t)$ **With I.C.** **z(0)=0; y(0)=1**

6) **Find a random algebraic equation from the transition (escape problem)**

$$n(t) \approx O\left(\sqrt{\varepsilon}\right) \Rightarrow n(t_e) \approx O(K)$$

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The quasi deterministic paths for the Logistic model

1) Let $\xi(t)$ **be an arbitrary s.p. and the set:** $\dot{z} = rz + \sqrt{\varepsilon}\, y\xi\,;\quad \dot{y} = \dfrac{r}{K} z$

2) 1-order iteration:

$$\boxed{y(t) \approx 1;\quad z(t) \approx \sqrt{\varepsilon}\, e^{rt} \int_0^t e^{-rs} \xi(s) ds \equiv \sqrt{\varepsilon}\, e^{rt} h(t) \geq 0,\quad t \geq 0}$$

3) 2-order iteration:

$$y(t) - 1 \approx \frac{r}{K} \int_0^t z(s) ds = \frac{r\sqrt{\varepsilon}}{K} \int_0^t e^{rs} h(s) ds$$

$$\text{where:}\quad \dot{h} = e^{-rt} \xi(t);\quad h(t) \geq 0,\ h(0) = 0$$

4) Then the quasi-deterministic path is:

$$n(t) = \frac{z(t)}{Y(t)} \approx \frac{e^{rt} \sqrt{\varepsilon}\, h(t)}{1 + \dfrac{r\sqrt{\varepsilon}}{K} \displaystyle\int_0^t e^{rs} h(s) ds}$$

The escape problem for the logistic model (asymptotic approx.)

Because $h(t)$ saturates we can approx.; $\quad n(t) \approx \dfrac{\sqrt{\varepsilon}\, e^{rt}\,\Omega}{\left(1 + \dfrac{r\sqrt{\varepsilon}}{K}\,\Omega \displaystyle\int_0^t e^{rs}\,ds\right)}$

$\boxed{h(\infty) \equiv \Omega \text{ is a random number!}}$

Then, when $n(t_e) \propto O(K)$ $\Rightarrow$ $t_e = \dfrac{1}{r}\log\left(\dfrac{K}{\sqrt{\varepsilon}\,\Omega}\right)$

Two important time scale:

***) First is the linear regime**

$n(t) \approx e^{rt} h(t) \approx O(\sqrt{\varepsilon}) \Rightarrow n(t_e) \propto O(K)$

then $P(\Omega) \Rightarrow \text{FPTD } P(t_e)$

***) Second is given by the non-linear deterministic evolution of a random I.C.: $\sqrt{\varepsilon}\,\Omega$ giving rise to the anomalous fluctuation**

$$\langle n(t)^2 \rangle \mapsto O(K)$$

***) We can also introduce an instanton-like approximation to study analitically this anomaluos fluctuations.**

On the noise perturbations and the pdf of Ω

Fluctuations of the Wiener class

1) **We have to solve the auxiliary SDE:**

$$\dot{h} = e^{-rt}\xi(t) \Rightarrow dh(t) = e^{-rt}dw(t);\ h(t) \geq 0$$

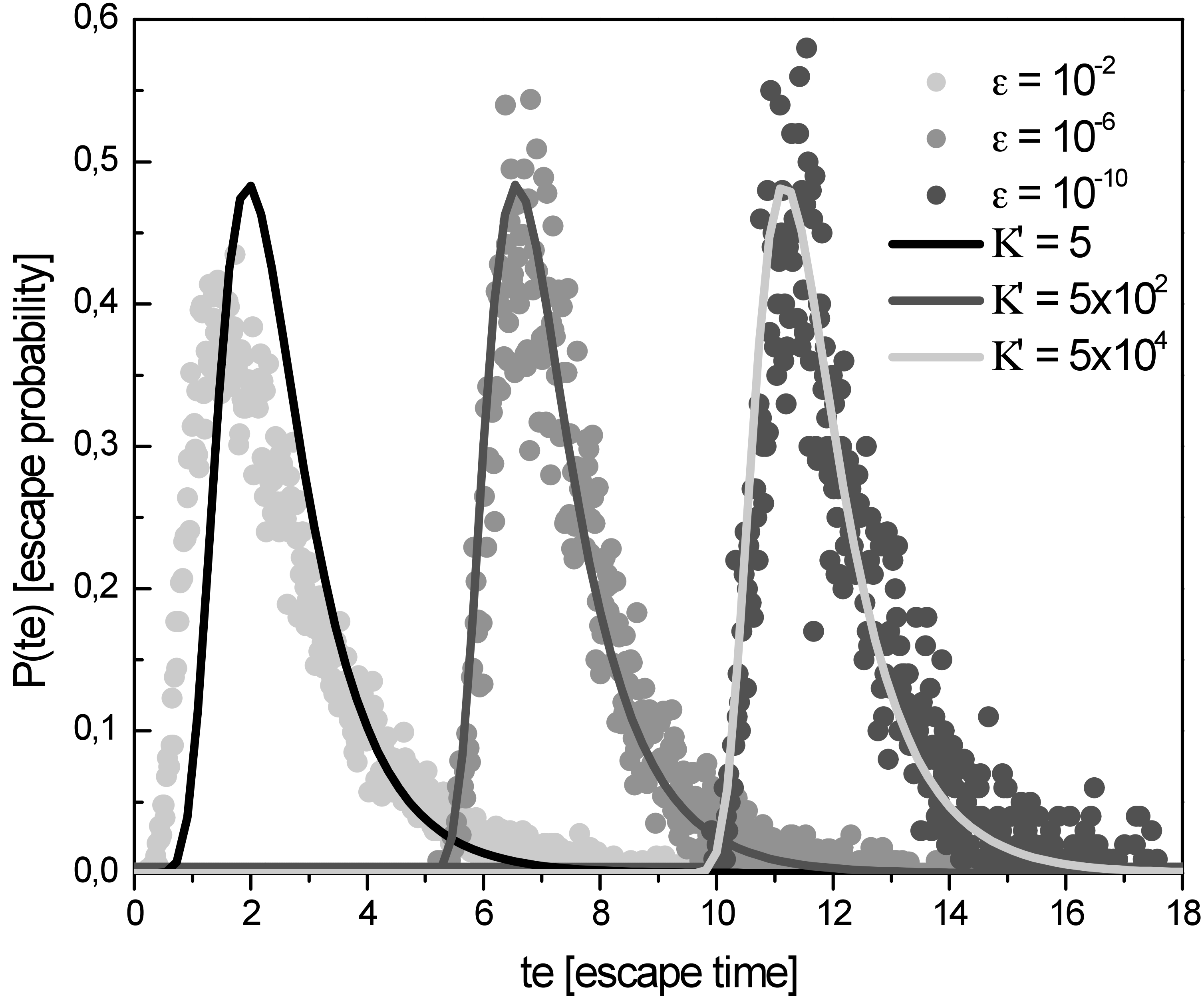
2) **Then, it is simple to calculate the stationary pdf of** $h(t)$ **so…**

$$\text{pdf } \mathrm{P}(\Omega) \mapsto \text{FPTD } P(t_e);$$

$$P(t'_e) = \frac{2K'}{erf(K')}\exp\left[-t'_e - K'^2\exp\left(-2t'_e\right)\right]$$

$$(t_e' = rt_e, K' = K\sqrt{r/\varepsilon})$$

3) **We also made comparisons with numerical simulations in the Verhults' model perturbed with Wiener fluctuations…**



P(te) [escape probability]
te [escape time]
ε = 10-2
ε = 10-6
ε = 10-10
K' = 5
K' = 5x102
K' = 5x104
0,0
0,1
0,2
0,3
0,4
0,5
0,6
0
2
4
6
8
10
12
14
16
18

On the Poisson noise perturbations and the pdf of Ω

1) As before we have to solve the auxiliar SDE:

$$\dot{h} = e^{-rt}\xi(t),\ \text{where}\ \xi(t)\ \text{is a Poisson noise, and}\ h(t) \geq 0$$

2) Then the functional of the process $h(t)$ will be

$$G_h\left[V(\bullet)\right] = e^{ik_0 h(0)} G_\xi\left[e^{-rt}\int_t^\infty V(s)ds\right]$$

3) Where $G_\xi\left[k(\bullet)\right]$ is the Poisson functional with parameters $\{\rho, A\}$

$$G_\xi\left[k(\bullet)\right] = \exp\left(-\rho\int_0^\infty [\exp\left(iA\,k(s)\right) - 1]ds\right)$$

4) Then the stationary pdf of $h(t)$ can be calculated by quadrature. Finally the FPTD follows from the SPPA and using the Jacobian transformation $\Omega \rightarrow t_e$ of by formula:

$$P(t_e) = \frac{1}{N} e^{-rt_e} P\left(\Omega = \frac{K\exp\left(-rt_e\right)}{\sqrt{\varepsilon}}\right);\ t_e \geq 0$$

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Concluding remarks

1) A general theory to work out the escape process from differents unstable points (normal-forms) has been presented using the Stochastic Path Perturbation Approach (SPPA).

2) This approach has been applied to study the first passage time distribution (FPTD) of a logistic model perturbed with arbitrary noise.

3) The stochastic Verhults' model has been worked out with two class of noise perturbations: Wiener and Poisson fluctuations.

4) Numerical simulations has been carried out in the case of having Gaussian perturbations, showing that the agreement is very good even for large noise intensity $\sqrt{\varepsilon}$.

5) Subcritical pitchfork bifurcations, as well as marginal unstable states have also been tackled with the SPPA to get the FPTD.

6) The escape problem in extended normal-form, and in periodically time-dependent dynamical systems have also been worked out.

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The passage time statistics is standard for Markov processes

Consider the general Markov problem: $\partial_t P(x,t) = L \bullet P(x,t)$

1) **The probability that at time t the path x(t) is still in the interval (a,b), given an initial condition xo at to=0, is the survival probability F(xo,t).**

2) **F(xo,t) obeys the adjoint equation:** $\boxed{\partial_t F(x_0,t) = L^{+} \bullet F(x_0,t)}$

3) **The FPTD (density) is given by:** $\boxed{f(x_0,t) = -\partial_t F(x_0,t)}$

4) **As a consequence the MFPT obeys the Dynkin equation** $\boxed{-1 = L^{+} \bullet T(x_0)}$

5) **These equations, completed with adequated BC, give the solution to the problem of calculating the PT statistics for Markov processes**

6) **However: In order to tackle analitically the FPT statistics, or to study the FPTD in non-Markov processes, non-stationary systems, many variables, etc.... , it is of interest to have approximations for the individual paths of the stochastic process.**

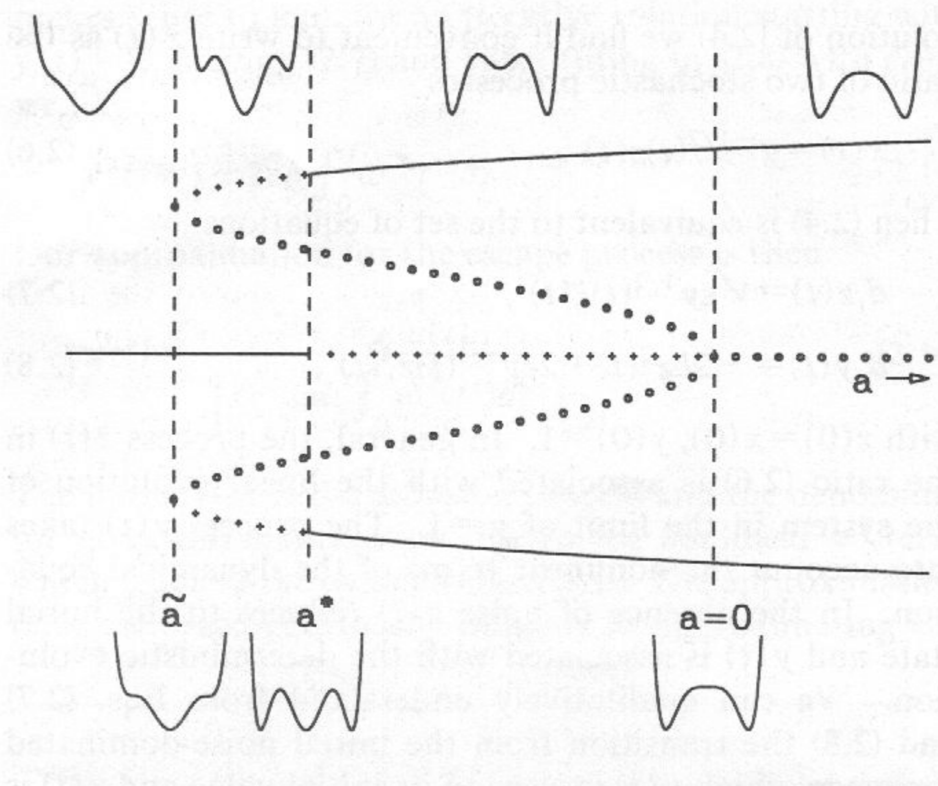


FIG. 1. Bifurcation diagram corresponding to the sixth-order potential $V(x)$ defined in (2.3). The solid lines denote stable states. Dots and crosses stand, respectively, for unstable and metastable states, where $\tilde{a} = -b^2/(4c)$ and $a^* = -(\frac{3}{4})b^2/(4c)$.

$$V(x) = \frac{-a}{2}x^2 - \frac{b}{4}x^4 + \frac{c}{6}x^6$$

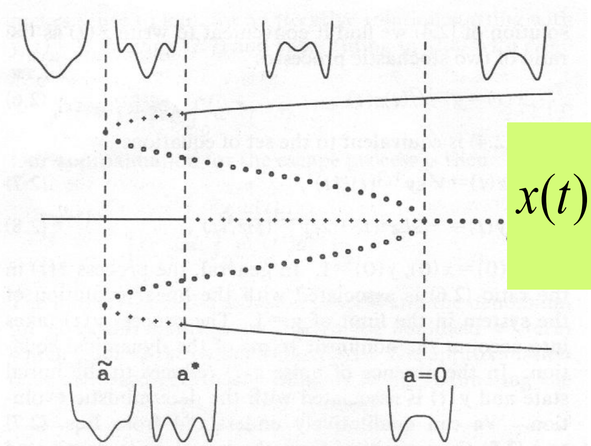
How to get individual paths for the stochastic process x(t)

$$dx = \left(ax + bx^3 - cx^5\right)dt + \sqrt{\varepsilon}\,\xi(t)dt; \quad \{\mathrm{a}, b, c\} > 0$$

Escape processes from a subcritical bifurcation

Here also the delicates mechanism of escape is non-Gaussian !

$$dx = (bx^3 - cx^5)dt + \sqrt{\varepsilon}\, dw(t); \;\; \{b,c\} > 0$$



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$$x(t) = \frac{z(t)}{\sqrt{y(t)}}$$

$$\mathrm{x}(t) \approx \left[\frac{\sqrt{\varepsilon}\, w(t)}{[1 - 2\varepsilon b \int_0^t w(s)^2\, ds]^{1/2}} \right]$$

Using the Wiener scaling we can study the escape process $\mathrm{x}(t_e) = \infty$ **solving the algebraic random eq.:**

$$0 = 1 - 2b\varepsilon\, t_e^{\,2}\,\Omega(1); \;\; \Omega(1) \equiv \int_0^1 w(s)^2\, ds; \;\; G_\Omega(\lambda) \equiv \left\langle e^{-\lambda\Omega} \right\rangle = 1/\sqrt{\cosh 2\sqrt{\lambda}}$$

$$\text{pdf } \mathrm{P}(\Omega) \longrightarrow \text{FPTD } f(t_e)$$

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Escape processes from a subcritical bifurcation

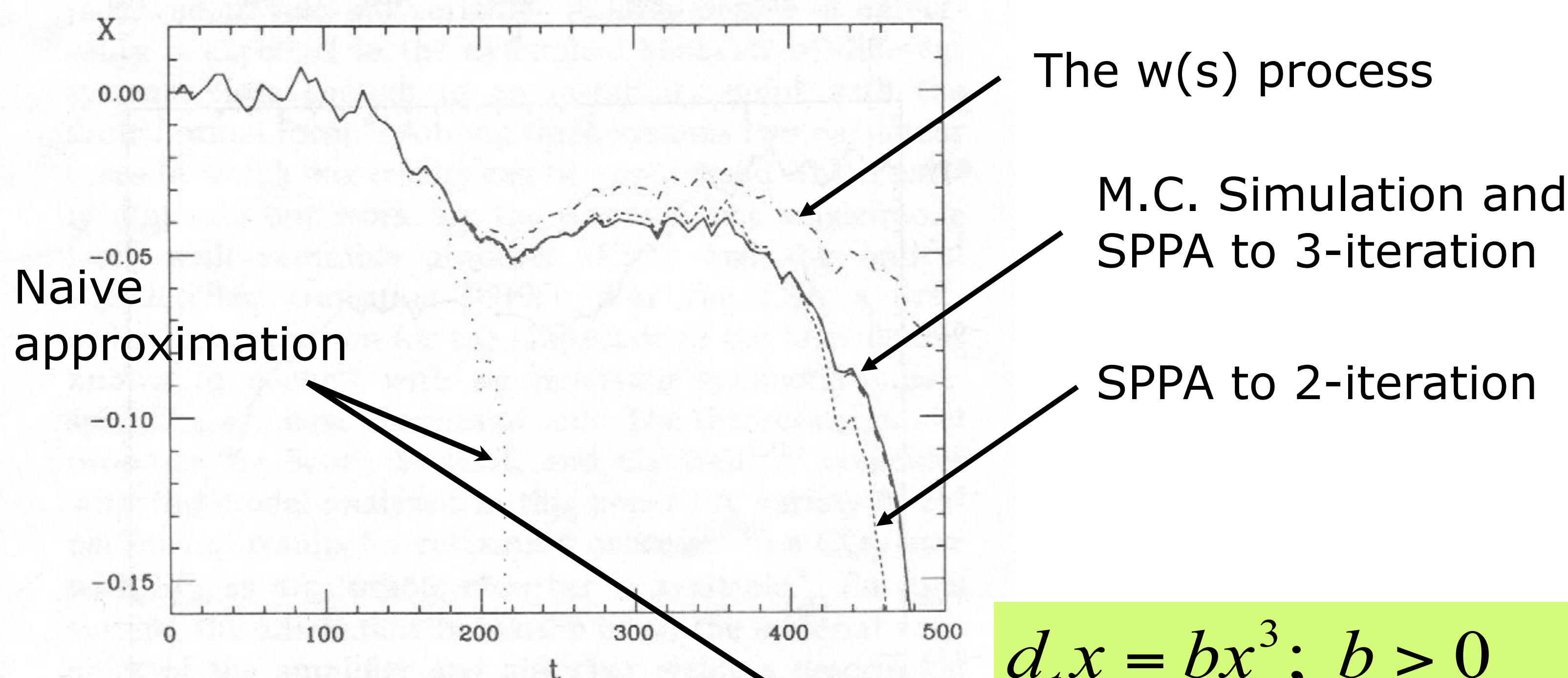


FIG. 3. Different stochastic paths of a given realization of the noise. The solid line corresponds to a simulation of the exact process (2.4). The dot-dashed line corresponds to a Wiener process. The dots correspond to the approximation (2.5), the small dashed line corresponds to the approximation (2.14), and the long dashed line to the approximation (2.15). The value of the parameters are given in Ref. 19.

$$d_t x = b x^3;\ b > 0$$

with $\sqrt{\varepsilon}\, w(t)$ as random IC

$$x(t) = \frac{\sqrt{\varepsilon}\, w(t)}{\sqrt{1 - 2\varepsilon b t\, w(t)^2}}$$

A non-symmetric normal from case

1) **Consider the SDE:** $dx = \left(ax^2 + b\right)dt + \sqrt{\varepsilon}\, dw(t);\ \ a > 0, b > 0$

(The particular marginal case b=0, will be solved apart.)

1) **Propose the non-linear transformation:** $x(t) = H(t)Y(t)^{-1}$

2) **Then we obtain the equivalent set of SDEs.:**

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{dt}}H(t) = bY(t) + \sqrt{\varepsilon}\,Y(t)\xi(t); \qquad \frac{\mathrm{d}}{\mathrm{dt}}Y(t) = -aH(t)$$

3) **Doing the iterations (noise intensity) as before we get:**

$$\mathrm{x}(t) \approx \left[{[bt + \sqrt{\varepsilon}\,w(t)]} \Big/ {[1 - (abt^2)/2 - a\sqrt{\varepsilon}\,\Omega(t)]} \right]$$

$$\Omega(t) = \int_0^{\mathrm{t}} w(s)\, ds$$

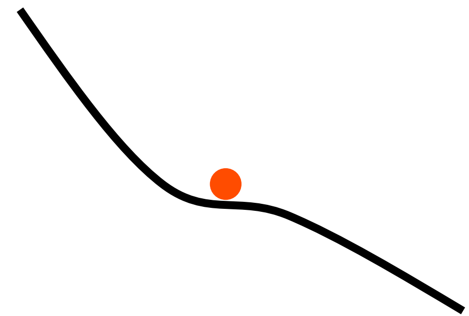
4) **Using the Wiener scaling we can solve for t, for example:**

$$\Omega(t) = \int_0^{\mathrm{t}} w(s)\, ds = t^{3/2} \int_0^{1} w(s)\, ds$$

A non-symmetric normal from case

1) **From the previous iterations and scaling out the Wiener integral we get (where $\Omega(1)$ is a Gaussian r.v.)**

$$\mathrm{x}(t) \approx \left[\frac{bt + \sqrt{\varepsilon}\, w(t)}{1 - (abt^2)/2 - a\sqrt{\varepsilon}\, t^{3/2}\,\Omega(1)} \right]$$



2) **The escape times defined by** $\mathrm{x}(t_e) = \infty$

can be obtained from the random algebraic equation:

$$0 = 1 - (abt_e{}^2)/2 - a\sqrt{\varepsilon}\, t_e{}^{3/2}\,\Omega(1)$$

3) **Then we get the FPTD:**

$$f\,(t_e) = P(\Omega)\left|\frac{d\Omega}{dt_e}\right|, \text{ and we can calculate: } \langle t_e \rangle; \left\langle \left(t_e - \langle t_e \rangle\right)^2 \right\rangle, \text{ etc.}$$

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A non-symmetric normal from case

$$K = b^3/\left(a\varepsilon^2\right) \text{ (natural parameter)}; \;\; \tau = \sqrt{2/ab} \text{ (deterministic escape)}$$

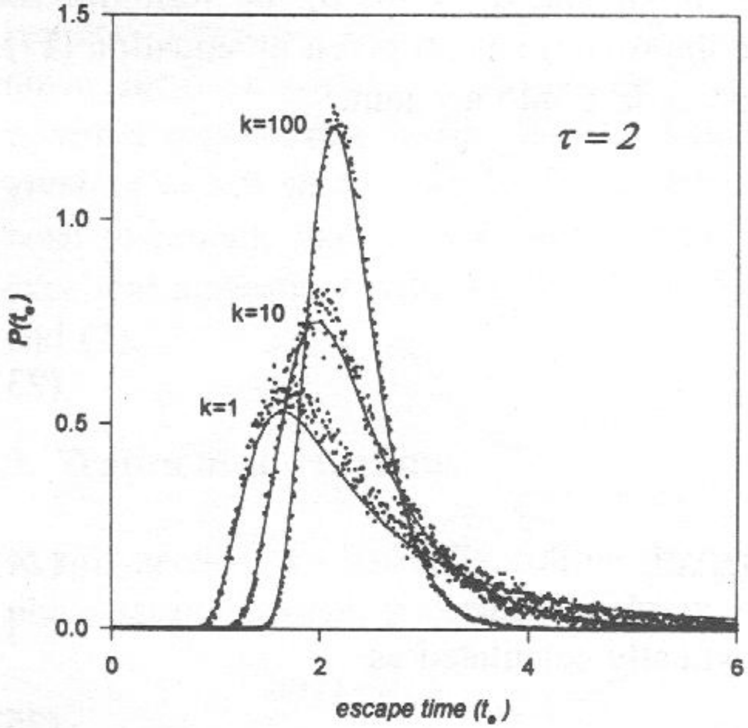

Figure 1. Plot of FPTD $P_K(\tau = 2, t_c)$ as a function of t_c for several values of K (=1, 10, 100). The dots show the Monte Carlo simulations of the SDE (1).

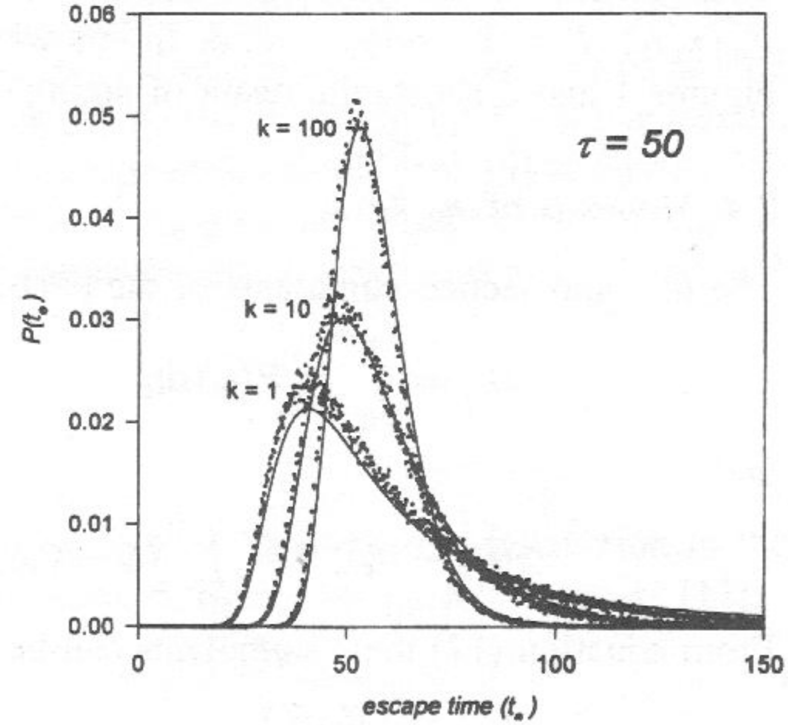

Figure 2. Plot of FPTD $P_K(\tau = 50, t_c)$ as a function of t_c for several values of K (=1, 10, 100). The dots show the Monte Carlo simulations of the SDE (1).

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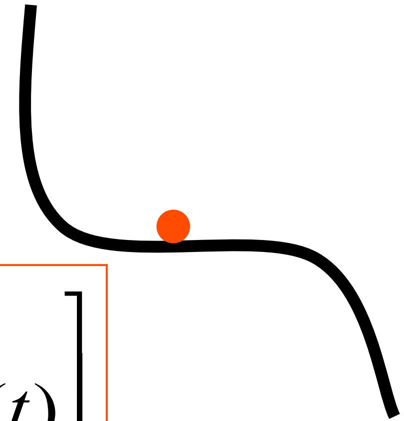
A non-symmetric normal from case

Marginal case (b=0)



1) **Let the SDE** $dx = ax^2 dt + \sqrt{\varepsilon}\, dw(t);\ a > 0$

2) **Doing a SECOND order iterations we get:**

$$\mathrm{x}(t) \approx \left[\sqrt{\varepsilon}\, w(t) \Big/ 1 - a\sqrt{\varepsilon}\,\Omega(t) + a^2 \varepsilon^2 \Theta(t) \right]$$

3) **Where:**

$$\Omega(t) = \int_0^t w(s)\, ds; \quad \eta(\mathrm{t}) = \int_0^t \Omega(s)\, dw(s); \quad \Theta(\mathrm{t}) = \int_0^t \eta(s)\, ds$$

4) **Using the Wiener scaling we can study the escape** $x(t_e) = \infty$

solving the random algebraic eq. $0 = 1 - a\sqrt{\varepsilon}\, t_e^{\,3/2}\,\Omega(1) + a^2 \varepsilon^2 t_e^{\,3}\,\Theta(1)$

Then we get:

$$\mathrm{pdf\ P}(\Omega, \Theta) \longrightarrow \mathrm{FPTD}\ f(t_e)$$

A non-symmetric normal from case

Marginal case (b=0)

$$A = a\sqrt{\varepsilon} \quad \text{(natural parameter); (there is not deterministic escape)}$$

$O\left(a\sqrt{\varepsilon}\right)$

$O\left(a^2\varepsilon\right)$

Stochastic escape processes 2291

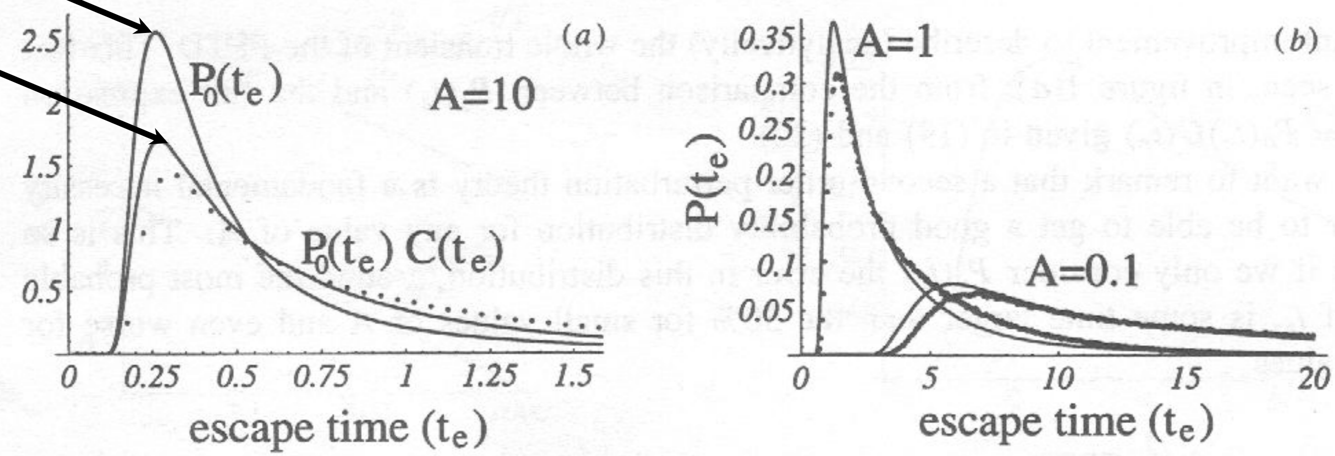


Figure 1. (*a*) Plot of FPTD $P_A(t_e) \equiv P_o(t_e)C(t_e)$, coming from the present second-order perturbation theory (19), as a function of t_e for $A = 10$. The dotted curve represents the Monte Carlo simulations of the SDE (1) with $b = 0$, having reached $X_{at} = 1$ for the first time. The corresponding $\mathcal{O}(\sqrt{\epsilon})$ probability distribution $P_o(t_e)$ is also shown. Details of the simulation are given in [5]. (*b*) Plot of FPTD $P_A(t_e)$ as a function of t_e for two values of $A(= 0.1, 1)$, the dotted curve represents the Monte Carlo simulations of the SDE (1) with $b = 0$.

Escape processes for extended systems…

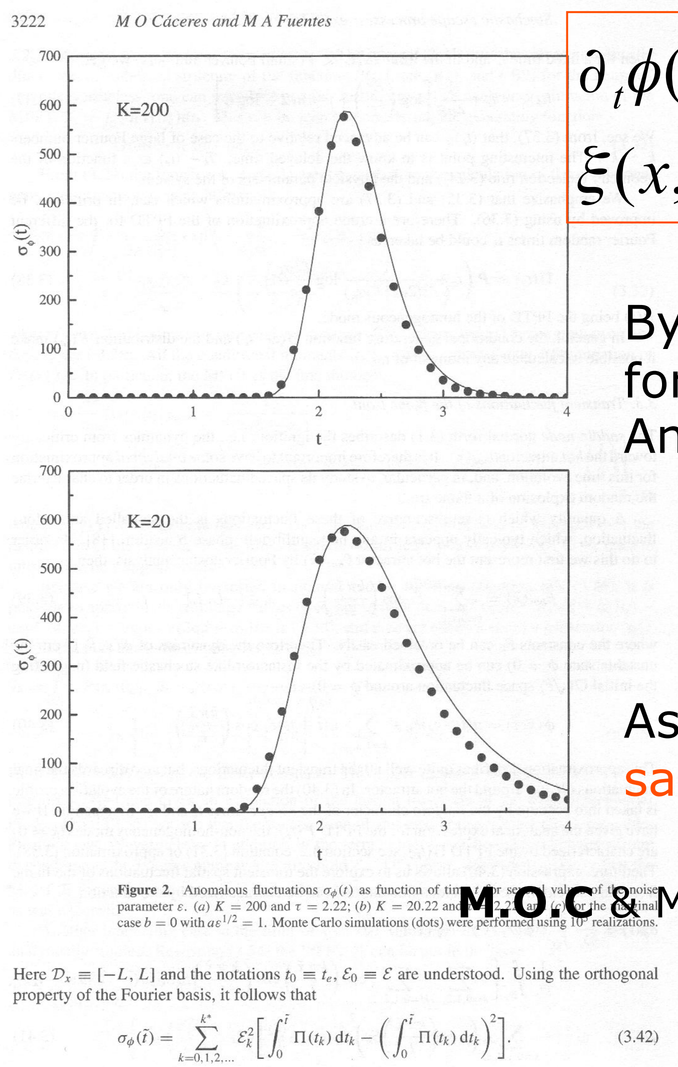

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Figure 2. Anomalous fluctuations $\sigma_\phi(t)$ as function of time for several values of the noise parameter ε. (a) $K = 200$ and $\tau = 2.22$; (b) $K = 20.22$ and $\tau = 2.22$, for the nominal case $b = 0$ with $a\varepsilon^{1/2} = 1$. Monte Carlo simulations (dots) were performed using 10^3 realizations.

Here $\mathcal{D}_x \equiv [-L, L]$ and the notations $t_0 \equiv t_e$, $\mathcal{E}_0 \equiv \mathcal{E}$ are understood. Using the orthogonal property of the Fourier basis, it follows that

$$\sigma_\phi(\bar{t}) = \sum_{k=0,1,2,\ldots}^{k^*} \mathcal{E}_k^2 \left[\int_0^{\bar{t}} \Pi(t_k)\, dt_k - \left(\int_0^{\bar{t}} \Pi(t_k)\, dt_k \right)^2 \right]. \quad (3.42)$$

$$\partial_t \phi(x,t) = D\partial^2{}_x\, \phi(x,t) + a\phi(x,t)^2 + b + \sqrt{\varepsilon}\,\xi(x,t)$$

$\xi(x,t) =$ is zero mean Gaussian white field

By introducing an "instanton-like" approximation for the field, it is possible to study the Anomalous fluctuations of the stochastic field

$$\sigma_\phi(\mathrm{t}) = \frac{1}{D_x} \int_{D_x} \left\langle \phi(x,t)^2 \right\rangle - \left\langle \phi(x,t) \right\rangle^2 dx$$

Associated to the "escape" (life-time) from a saddle-node extended normal form

M.O.C & M.A. Fuentes; J. Phys. A Math. & Gen. 32, 3209, (99).

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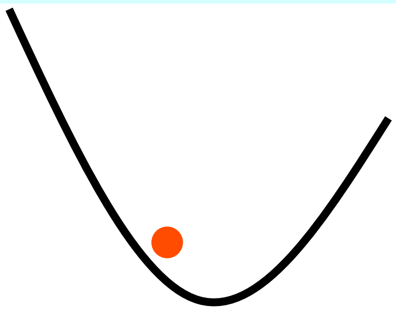
Escape processes for non-stationary systems…

Stochastic periodically driven instability

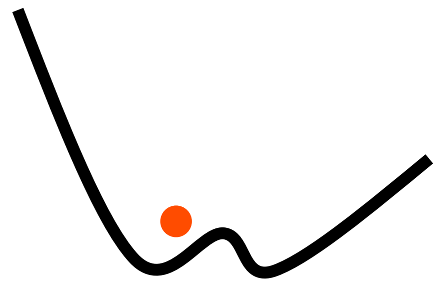
Consider the stochastic amplitude equation (Swift-Hohenberg-Ahler, PRL88, PRA91):

$$\frac{dq}{dt} = [a + b\cos(t)]q - q^3 + \sqrt{d}\,\xi(t),$$

Picture for the time-dependent associated potential in the F-P dynamics…



$$[a + b\cos(t_1)] < 0,$$



$$[a + b\cos(t_2)] > 0,$$

Physical Motivation: to get Correlation times, Switching time…

[a+b cos(t)] is the time-dependent Rayleigh number,in the evolution eq. for the **slowest one-mode q(t) (order parameter)** in a finite cilindrical cell; G. Ahlers et al. experiment. Fluid Mech. (81); A. Becker, **M.O.C.** & L. Kramer PRA, (1991); **M.O.C** & A. Lobos, JPA (2006).