Sampling, Stability, and Public Goods

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Abstract

Most models of social preferences and bounded rationality that are effective in explaining efficiency-increasing departures from equilibrium behavior cannot easily account for similar deviations when they are efficiency-reducing. We show that the notion of sampling equilibrium, subject to a suitable stability refinement, can account for behavior in both efficiency-enhancing and efficiency-reducing conditions. In particular, in public goods games with dominant strategy equilibria, stable sampling equilibrium involves the play of dominated strategies with positive probability both when such behavior increases aggregate payoffs (relative to the standard prediction) and when it reduces aggregate payoffs. While the dominant strategy equilibrium prediction changes abruptly from zero contribution to full contribution as a parameter crosses a threshold, the stable sampling equilibrium remains fully mixed throughout. This is consistent with the available experimental evidence.

Keywords: bounded rationality; public goods; sampling equilibrium; sampling dynamics

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1 Introduction

Many important economic environments, including the private provision of public goods, common pool resource extraction, team production, and intra-household resource allocation often involve a trade-off between efficiency and equilibrium behavior. A significant experimental literature has explored the degree to which equilibrium predictions are validated in laboratory settings, and models of bounded rationality and other-regarding references have been developed to account for observed departures from these predictions.

The experimental literature has generally avoided calibrations of these models in which there is no trade-off between equilibrium and efficiency, presumably because such cases have been considered trivial or uninteresting. However, in the few instances in which this parameter space has been explored, the findings reveal very similar divergences from equilibrium play. That is, players appear to deviate from equilibrium behavior in much the same way regardless of whether such deviations raise or lower collective payoffs. And most models of social preferences and bounded rationality that are effective in explaining efficiency-increasing departures from equilibrium behavior cannot easily account for similar deviations when they are efficiency-reducing.

In this paper we show that a particular model of procedural rationality based on the sampling of actions can account for departures from standard equilibrium predictions in both the efficiency-enhancing and efficiency-reducing cases. The model involves the independent sampling of each available action by players, and the selection of the action that yields the best realized payoff. These realized payoffs depend on random draws from a distribution over the available actions. A sampling equilibrium is a distribution that is self-replicating in the following sense: the likelihood that a given action will be selected under the sampling procedure matches the probability assigned to it in the incumbent distribution. This may be interpreted as the steady state of a model with a large population of players with entry and exit in a manner described below.

We consider the predictions of this model for public goods games with dominant strategy equilibria that may or may not be efficient, and obtain the following results. When there are three or more players, the dominant strategy equilibrium is unstable under the sampling dynamics regardless of whether or not it is efficient. As a result, strictly dominated strategies are played with positive probability at any stable sampling equilibrium. While the dominant strategy equilibrium switches completely from zero contribution to full contribution as one moves from one regime to the other, the stable sampling equilibrium changes quantitatively but not qualitatively, and remains fully mixed. This is consistent with the available experimental evidence.

In the two player case (with each player having at least three strategies) the results are some-

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1The model of sampling equilibrium was developed by Osborne and Rubinstein [1998], and the stability refinement introduced by Sethi [2000].
what different. When the dominant strategy equilibrium is inefficient, it is also unstable under the sampling dynamics, so dominated strategies are played with positive probability and efficiency is increased. But unlike the case of many players, full contribution is stable under the sampling dynamics when it happens to also be efficient. In the two player case with an efficient dominant strategy equilibrium, therefore, the sampling approach generates the same prediction as the standard approach.

The intuition underlying these findings may be roughly summarized as follows. Since realized payoffs are determined by random draws from a population of actions, there is always a chance that a dominated strategy yields a higher payoff than a dominant strategy when they are independently sampled. A dominant strategy equilibrium will be unstable under the sampling dynamics if the deviation of a small share of the population away from this strategy results in a sufficiently great likelihood that the dominant strategy will not yield the highest realized payoff when sampled. Prior work has established that this instability can arise when the equilibrium is inefficient; our results show that this can happen also when it is efficient.

2 Related Literature

Among economic environments in which voluntary contributions serve collective ends are public goods (Bergstrom et al., 1986), common property (Ostrom et al., 1992), team production (Holmstrom, 1982), and allocation within families (Becker, 1981). The public goods case in particular has received a great deal of attention from experimentalists, typically using specifications where contribution to the public good is efficient but not individually rational (Andreoni, 1988; Isaac and Walker, 1988; Ledyard, 1995; Chaudhuri, 2011). A mismatch between standard predictions and experimental findings has given rise to several models of social preferences (Fehr and Schmidt, 1999; Bolton and Ockenfels, 2000), reciprocity (Rabin, 1993; Fischbacher et al., 2001; Dufwenberg and Kirchsteiger, 2004), and bounded rationality (Andreoni, 1995; Aumann, 1997; Houser and Kurzban, 2002).

Most relevant in motivating our work are experiments in which contributions to the public good are not only efficient but also individually rational, since these also show systematic deviations from dominant strategy choices (Saijo and Nakamura, 1995; Brandts and Schram, 2001; Kümmerli et al., 2010; Burton-Chellew and West, 2013). Neither standard social preferences nor most models of bounded rationality can explain this behavior. Efficiency-reducing departures from standard equilibrium predictions can be explained by envy or spite (Kockesen et al., 2000a,b), but such preferences cannot simultaneously account for efficiency-increasing departures in most environments.

Bounded rationality models relying on sampling procedures assume that subjects do not con-
template the entire payoff function, relying instead on procedures based on some form of (actual or imagined) experimentation. One class of such models involves sampling a set of opponent actions, and then responding optimally to a distribution over these (Kosfeld et al., 2002; Droste et al., 2003; Selten and Chmura, 2008; Oyama et al., 2015). Since strictly dominated strategies can never be best responses to any opponent action, this class of procedures necessarily eliminates such strategies. Weakly dominated strategies, however, can survive.

An alternative approach involves sampling the payoff consequences of one’s own actions, which are independently evaluated against random draws of opponent actions. Osborne and Rubinstein (1998) introduced an equilibrium concept based on this idea: a sampling equilibrium is a distribution of actions that is self-replicating in the sense that the likelihood of an action being selected under the procedure matches the frequency with which it is being played in a large incumbent population from which one’s opponents are drawn. The set of sampling equilibria can be large, and includes all strict Nash equilibria, but this set can be refined by eliminating those that are unstable under a natural specification of out-of-equilibrium dynamics.

It is known that in public goods and common pool resource games, this stability refinement can result in the selection of sampling equilibria in which strictly dominated strategies are played with positive probability (Sethi, 2000; Cárdenas et al., 2015). Similarly, in the centipede game, stable sampling equilibrium involves the play of efficiency-increasing (weakly) dominated strategies (Sandholm et al., 2017). Our main contribution here is to show that such effects arise even when the dominant strategy equilibrium is itself efficient, in which case the refinement selects sampling equilibria in which efficiency-reducing dominated strategies are played with positive probability.

Prior attempts to account for the selection of efficiency-reducing dominated strategies have relied on confusion (Palfrey and Prisbrey, 1997; Brandts and Schram, 2001), maladaptive responses in laboratory settings (Hagen and Hammerstein, 2006; West et al., 2011), a distaste for the play of extreme strategies (Kümmerli et al., 2010), or social comparisons of the kind that give rise to envy or spite (Saijo and Nakamura, 1995; Brunton et al., 2001; Brandts et al., 2004). What we show here is that neither none of these factors are necessary to account for such behavior. In fact, the same mechanism giving rise to efficiency-enhancing outcomes in standard public goods games also results in efficiency-decreasing choices in public goods games with efficient dominant strategies.

3 Sampling Equilibrium and Dynamics

Consider a symmetric game with $n$ players and a set $A = \{a_1, \ldots, a_m\}$ of available actions for each player. Let $u(a_i, b)$ denote the payoff to a player choosing action $a_i \in A$ when the remaining $n - 1$ players choose the action profile $b \in A^{n-1}$. 
Let \( x = (x_1, \ldots, x_m) \) be a probability distribution on \( A \), and let \( v(a_i, x) \), denote random variables yielding \( u(a_i, b) \) with probability \( Pr(b) \) for each \( b \in A^{n-1} \). Here \( Pr(b) \) is the probability that the action profile chosen by the remaining \( n - 1 \) players is \( b \), assuming that each player chooses an action independently subject to the distribution \( x \) on \( A \).

Finally, let \( \omega(a_i, x) \) be the probability that, when each of the \( m \) random variables \( v(a_i, x) \) is drawn independently and exactly once, the action \( a_i \) yields the highest realized payoff. We assume that ties are broken uniformly at random, so \( \omega(a_i, x) \) is the probability that action \( a_i \) either yields the strictly best outcome, or is weakly best and selected under the tie-breaking rule.

A sampling equilibrium is defined as a probability distribution \( x^* \) on the set of actions \( A \) with the property that
\[
\omega(a_i, x^*) = x^*_i \quad \text{for every action } a_i \in A.
\]
This concept was introduced by Osborne and Rubinstein (1998), and may be understood as follows. Suppose that a large population of players is such that the proportion playing action \( a_i \) is \( x_i \), and a new entrant picks an action by sampling each available choice exactly once, then settling on the one that resulted in the highest realized payoff (either strictly or after the breaking of ties). Whenever an action is sampled, all opponent actions are drawn independently from the population subject to the distribution \( x \). If \( x \) is a sampling equilibrium, then the likelihood that \( a_i \) will be selected under the sampling procedure is precisely \( x_i \) for each \( i \).

This naturally leads to an interpretation of a sampling equilibrium as a rest point of a dynamic process in which actions increase in population frequency if they result in the highest realized payoff with greater likelihood than they are currently being played. These sampling dynamics were introduced and explored in Sethi (2000), and further studied by Sandholm et al. (2017), who refer to them as best experienced payoff dynamics.

At any population state \( x \), the sampling dynamics take the following form:
\[
\dot{x}_i(t) = F(x(t)),
\]
where \( F \) satisfies the monotonicity condition
\[
\omega(a_i, x(t)) > x_i(t) \iff \dot{x}_i(t) > 0.
\]
Here, \( \dot{x}_i(t) \) represents the rate of change of the distribution of actions in the population at time \( t \). An example of sampling dynamics is given by:
\[
\dot{x}_i(t) = \omega(a_i, x(t)) - x_i(t).
\]
It is easily verified that any strict Nash equilibrium must also be a sampling equilibrium, although there may exist sampling equilibria that are not Nash equilibria, and indeed involve the playing with positive probability of strictly dominated actions.
The set of sampling equilibria can be refined by selecting those that are stable with respect to the sampling dynamics. As shown in Sethi (2000) and Cárdenas et al. (2015), this refinement can sometimes eliminate dominant strategy equilibria, and give rise to the play of dominated strategies with positive probability in the unique stable sampling equilibrium. This feature of sampling dynamics plays a key role in our analysis here. In fact, we show that this effect arises even when a dominant strategy equilibrium is efficient.

4 Public Goods

Suppose that each of \( n \) players has a finite set of actions \( A = \{a_1, a_2, \ldots, a_m\} = \{0, 1, \ldots, e\} \), interpreted as contributions to the provision of a public good. The payoff to a player who chooses action \( a_i \in A \) when the remaining players choose the action profile \( b \in A^{n-1} \) is

\[
u(a_i, b) = (e - a_i) + f(S),
\]

where

\[
S = a_i + \sum_{j=1}^{n-1} b_j
\]

is the aggregate contribution, and \( f \) is (weakly) concave and satisfies \( f(0) = 0 \).

A special case with linear payoffs arises when each unit contributed yields some fixed amount \( \mu \) to each player, in which case

\[
u(a_i, b) = (e - a_i) + \mu S.
\]

As long as \( \mu n > 1 \) and \( \mu < 1 \), this represents a social dilemma. The condition \( \mu n > 1 \) ensures that is socially optimal for all to contribute fully to the public good, while \( \mu < 1 \) implies that is individually rational to contribute nothing.

We are also interested in the case \( \mu > 1 \), for which there is no tension between standard equilibrium concepts and efficiency. The Nash equilibrium then is to fully contribute to the public good. Yet, even in this case, there is experimental evidence showing that full cooperation is not reached in the lab (Saijo and Nakamura [1995] Kümmerli et al. [2010] Burton-Chellew and West 2013). As we show below, stable sampling equilibria involve less than complete contribution levels under very general conditions, and can therefore account for this behavior.
5 Examples

We begin with two examples to illustrate the concepts and intuitions used in the more general results below.

5.1 Three Players and Two Actions

First consider the case of a linear public goods game with three players and two actions per player. The set of actions is \( A = \{a_1, a_2\} = \{0, 1\} \), and the payoffs to a player choosing action \( a_i \) given the aggregate contribution of the other two players is shown in the following table:

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1 + ( \mu )</td>
<td>1 + 2( \mu )</td>
</tr>
<tr>
<td>1</td>
<td>( \mu )</td>
<td>2( \mu )</td>
<td>3( \mu )</td>
</tr>
</tbody>
</table>

This is a standard public goods game when \( \mu \in (1/3, 1) \), with a dominant strategy equilibrium involving no contribution, and an efficient action profile with full contribution by all players. For \( \mu > 1 \) the game has an efficient dominant strategy equilibrium with full contribution.

The solution concept of stable sampling equilibrium yields quite different predictions. For \( \mu \in (1/3, 1/2) \), there is a unique stable sampling equilibrium that corresponds to the Nash equilibrium. To see this, consider the population state \( x = (1 - \varepsilon, \varepsilon) \), and note that the action \( a_2 \) yields the largest payoff only if there are no contributions when \( a_1 \) is sampled, while the other two players both contribute when \( a_2 \) is sampled. Hence we have

\[
\omega(a_2, x) = \varepsilon^2 (1 - \varepsilon)^2,
\]

which is lower than the probability \( \varepsilon \) with which \( a_2 \) is being chosen at state \( x \). As a result, \( \dot{x}_2 < 0 \) for any \( \varepsilon > 0 \), and all trajectories converge to the state with no contributions under the sampling dynamics.

The Nash equilibrium continues to be a sampling equilibrium for \( \mu \in (1/2, 1) \), but becomes unstable. To see why, again consider the population state \( x = (1 - \varepsilon, \varepsilon) \), which is close to the zero contribution state when \( \varepsilon \) is small. Note that the action \( a_2 \) is selected under sampling if the following event occurs: there are no contributions from others when \( a_1 \) is sampled, while \( a_2 \) is sampled. Hence there is at least one unit contributed when \( a_2 \) is sampled. Hence

\[
\omega(a_2, x) \geq (1 - \varepsilon)^2 (1 - (1 - \varepsilon)^2) = 2\varepsilon + o(\varepsilon^2),
\]

where \( o(\varepsilon^2) \) denotes terms that are second order or higher in \( \varepsilon \). For \( \varepsilon \) sufficiently small, we therefore have \( \omega(a_2, x) > \varepsilon = x_2 \) and so \( \dot{x}_2 > 0 > \dot{x}_1 \). That is, the proportion of the population that
contributes grows under sampling dynamics when this proportion is sufficiently small, rendering the equilibrium with zero contributions unstable.

If the dominant strategy equilibrium is unstable, then positive contributions must arise in any stable sampling equilibrium. In fact, it can be shown by straightforward computation that there is a unique stable sampling equilibrium at $(0.72, 0.28)$. That is, about 28% of the population contributes at the stable sampling equilibrium, even though contribution is a dominated strategy.

Now consider the public goods game with $\mu > 1$. The Nash equilibrium in which all players contribute is efficient, and is also a sampling equilibrium, but one that is unstable under the dynamics. To see why, consider the population state $x = (\epsilon, 1-\epsilon)$, which is close to the full contribution state when $\epsilon$ is small. Now consider the following event: the two other players both contribute when $a_1$ is sampled, and at most one other player contributes when $a_2$ is sampled. In this case $a_1$ is selected under the sampling procedure, so we have:

$$\omega(a_1, x) \geq (1-\epsilon)^2(1-(1-\epsilon)^2) = 2\epsilon + o(\epsilon^2).$$

For $\epsilon$ sufficiently small, we therefore have $\omega(a_1, x) > \epsilon = x_1$ and so $\dot{x}_1 > 0$. That is, the equilibrium with full contribution is unstable, despite the fact that it involves the play of an efficient dominant strategy.

In this case the only stable equilibrium is at $x = (0.28, 0.72)$, which involves higher contribution levels than in the standard public goods game. While the Nash equilibrium switches completely from zero to full contribution, the stable sampling equilibrium changes quantitatively but not qualitatively.

These properties are summarized in the following table:

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Nash</th>
<th>Stable Sampling</th>
<th>Efficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1/3, 1/2)$</td>
<td>$(1,0)$</td>
<td>$(1,0)$</td>
<td>$(0,1)$</td>
</tr>
<tr>
<td>$(1/2, 1)$</td>
<td>$(1,0)$</td>
<td>$(0.72, 0.28)$</td>
<td>$(0,1)$</td>
</tr>
<tr>
<td>$&gt; 1$</td>
<td>$(0,1)$</td>
<td>$(0.28, 0.72)$</td>
<td>$(0,1)$</td>
</tr>
</tbody>
</table>

With two actions (regardless of the number of players) the sampling dynamics are one dimensional and easily characterized. When there are three actions, the dynamics are planar flows and the analysis is a little more involved.
5.2 Two Players and Three Actions

Consider the case of two players and three actions, with $A = \{a_1, a_2, a_3\} = \{0, 1, 2\}$. The payoffs to a player choosing a contribution level $a_i$, given the contribution $S - a_i$ of the other player, is shown in the following table:

<table>
<thead>
<tr>
<th>$S - a_i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>$2 + \mu$</td>
<td>$2 + 2\mu$</td>
</tr>
<tr>
<td>1</td>
<td>$1 + \mu$</td>
<td>$1 + 2\mu$</td>
<td>$1 + 3\mu$</td>
</tr>
<tr>
<td>2</td>
<td>$2\mu$</td>
<td>$3\mu$</td>
<td>$4\mu$</td>
</tr>
</tbody>
</table>

Here we have a standard public goods game if $\mu \in (1/2, 1)$ and an efficient dominant strategy if $\mu > 1$.

Since the set of sampling equilibria and associated dynamics depend only on ordinal properties of the payoff function, and we can obtain a complete characterization by separately considering just a small set of distinct cases.

Let $x = (1 - x_2 - x_3, x_2, x_3)$ denote the population state, and consider the case $\mu \in (1/2, 2/3)$. The action $a_2$ will be best under sampling if the opponent contribution is greater when $a_2$ is sampled than when $a_1$ is sampled, and at least as great when $a_2$ is sampled than when $a_3$ is sampled. This can happen in two different ways: (i) the opponent contributes zero when $a_1$ is sampled, 1 when $a_2$ is sampled, and at most 1 when $a_3$ is sampled, and (ii) the opponent contributes at most 1 when $a_1$ is sampled, and 2 when $a_2$ is sampled. Hence

$$\omega(a_2, x) = (1 - x_2 - x_3)x_2(1 - x_3) + (1 - x_3)x_3,$$

(1)

Meanwhile, action $a_3$ is best under sampling if the opponent contributes zero when $a_1$ is sampled, contributes at most 1 when $a_2$ is sampled, and 2 when $a_3$ is sampled. Hence

$$\omega(a_3, x) = (1 - x_2 - x_3)(1 - x_3)x_3.$$ 

We can compute all sampling equilibria by analytically solving the two simultaneous polynomial equations $\omega(a_2, x) = x_2$ and $\omega(a_3, x) = x_3$. Only two such conditions are needed since the third is given by $x_1 = 1 - x_2 - x_3$.

There is clearly a rest point at $x_2 = x_3 = 0$, which corresponds to the dominant strategy equilibrium, with zero contributions. It can be verified that there exist no other real valued solutions to the polynomial system in the unit simplex, so this sampling equilibrium is unique. Stability then follows from the fact that the equilibrium is at a corner of the unit simplex, and therefore cannot be enclosed by a limit cycle. Since all trajectories in a planar flow must converge to a point or a
limit cycle, and all cycles must enclose a rest point, we deduce that all trajectories converge to the unique sampling equilibrium, which is therefore globally stable.\footnote{These claims follow from the Poincaré-Bendixson Theorem for planar flows, and the Poincaré-Hopf Index Theorem; see Guckenheimer and Holmes (1983) for details.}

Next consider the case $\mu \in (2/3, 1)$. The action $a_2$ will be best under sampling under exactly the same conditions as in the previous case, so $\omega(a_2, x)$ remains as shown in \footnote{In fact, when $\mu > 2$ the probability that the efficient dominant strategy is selected under sampling is even greater, so the dominant strategy equilibrium remains stable.}. However, action $a_3$ has a greater probability of being selected; it will be best under sampling if the opponent contribution is greater when $a_3$ is sampled than when $a_1$ and $a_2$ are sampled. This can happen in two different ways: (i) the opponent contributes one when $a_3$ is sampled, and zero when $a_1$ and $a_2$ are sampled, and (ii) the opponent contributes two when $a_3$ is sampled, and at most 1 when $a_1$ and $a_2$ are sampled. Hence

$$\omega(a_3, x) = (1 - x_2 - x_3)^2 x_2 + (1 - x_3)^2 x_3.$$ 

We can compute all sampling equilibria by analytically solving two simultaneous polynomial equations as before. In this case we have three real roots, two of which are in the unit simplex: $x = (1, 0, 0)$ and $x = (0.52, 0.28, 0.20)$. The Jacobian evaluated at the first of these is

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $-1$ and 1. This equilibrium is therefore \textit{unstable} under the sampling dynamics.

When $\mu > 1$ there is an efficient dominant strategy equilibrium and, in contrast with the three-player two-action case, this equilibrium is \textit{stable}. We shall show this for the case of $\mu \in (1, 2)$; the last remaining case of $\mu > 2$ follows the same logic\footnote{In fact, when $\mu > 2$ the probability that the efficient dominant strategy is selected under sampling is even greater, so the dominant strategy equilibrium remains stable.}

Given $\mu \in (1, 2)$ the action $a_1$ will be best under sampling if the opponent contribution is greater when $a_1$ is sampled than when $a_2$ and $a_3$ are sampled. This can happen in two different ways: (i) the opponent contributes 1 when $a_1$ is sampled, and zero when $a_2$ and $a_3$ are sampled, and (ii) the opponent contributes 2 when $a_1$ is sampled, and at most 1 when $a_2$ and $a_3$ are sampled. Hence

$$\omega(a_1, x) = x_1^2 x_2 + (1 - x_1 - x_2)(x_1 + x_2)^2.$$ 

Similarly, action $a_2$ is best under sampling if the opponent contribution is greater when $a_2$ is sampled when $a_3$ is sampled, and is not below the contribution when $a_1$ is sampled. This can happen in two ways: (i) the opponent contributes 1 when $a_2$ is sampled, zero when $a_3$ is sampled, and at most 1 when $a_1$ is sampled, and (ii) the opponent contributes 2 when $a_2$ is sampled, and at most 1 when $a_3$ is sampled. We thus obtain

$$\omega(a_2, x) = x_1 x_2 (x_1 + x_2) + (1 - x_1 - x_2)(x_1 + x_2).$$
We can compute all sampling equilibria by analytically solving the two simultaneous polynomial equations \( \omega(a_1, x) = x_1 \) and \( \omega(a_2, x) = x_2 \). There is clearly a rest point at \( x_1 = x_2 = 0 \), which corresponds to the dominant strategy equilibrium, with full contribution. It can be verified that there exist no other real valued solutions to the polynomial system, so this sampling equilibrium is unique. As in the case of \( \mu \in (1/2, 2/3) \), stability follows from the fact that this equilibrium is at a corner of the simplex, and therefore cannot be enclosed by a limit cycle.

These properties are summarized in the following table:

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>Nash</th>
<th>Stable Sampling</th>
<th>Efficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu \in (1/2, 2/3) )</td>
<td>(1,0,0)</td>
<td>(1,0,0)</td>
<td>(0,0,1)</td>
</tr>
<tr>
<td>( \mu \in (2/3, 1) )</td>
<td>(1,0,0)</td>
<td>(0.52,0.28,0.20)</td>
<td>(0,0,1)</td>
</tr>
<tr>
<td>( \mu &gt; 1 )</td>
<td>(0,0,1)</td>
<td>(0,0,1)</td>
<td>(0,0,1)</td>
</tr>
</tbody>
</table>

These examples show that dominant strategy equilibria are sometimes stable and sometimes not, though there is no easy or intuitive way of determining which of these applies in any given case. The section to follow establishes more general conditions for stability.

6 Main Results

We begin with the case of three or more players. The first result identifies sufficient conditions for instability of dominant strategy equilibria in standard public goods games (see the appendix for all proofs).

**Theorem 1.** Suppose \( n \geq 3 \). If \( f(1) < 1 < f(2) \), there is an inefficient dominant strategy equilibrium with zero contribution that is unstable under the sampling dynamics.

The condition \( f(1) < 1 \) implies that no player profits from contributing anything if no opponent contributes. With concavity of \( f \), this ensures that there is a dominant strategy equilibrium with zero contribution. Meanwhile \( f(2) > 1 \) implies that this equilibrium is inefficient. It also implies that if at least one opponent chooses to contribute at least one unit to the public good when action \( a_2 \) is sampled, while all opponents contribute nothing when \( a_1 \) is sampled, then \( a_1 \) will not be best under sampling. With three or more players, this can happen in at least two different ways, and this is enough to ensure that the probability that \( a_1 \) is best under sampling is smaller than the probability with which it is currently being played at population states in which \( a_1 \) is played with probability close to 1. This, in turn, ensures local instability of the equilibrium.\(^4\)

\(^4\)Specifically, the condition \( f(2) > 1 \) ensures that the action \( a_1 \) is inferior in the sense of Sethi (2000), and with \( n \geq 3 \) this is enough to ensure instability.
It might be thought that the instability of the dominant strategy equilibrium arises from its inefficiency, which allows for higher payoffs when other actions are sampled. But, in fact, instability can arise even if the dominant strategy equilibrium is efficient:

**Theorem 2.** Suppose \( n \geq 3 \). If \( f(ne) > f(ne - 1) + 1 \), there is an efficient dominant strategy equilibrium with maximal contribution that is unstable under the sampling dynamics.

The condition \( f(ne) > f(ne - 1) + 1 \) ensures that a player prefers to contribute the full endowment when all others are doing so, relative to contributing one unit less. Concavity of \( f \) then ensures that full contribution is a dominant strategy for all players. But the equilibrium is locally unstable.

For some intuition, consider a population state in which a share \( 1 - \eta \) chooses full contribution, where \( \eta > 0 \). Clearly \( a_m \) will be best under sampling if all other players also choose full contribution when \( a_m \) is sampled. The probability of this is \( (1 - \eta)^{n-1} \), which is less than \( 1 - \eta \) when \( n \geq 3 \). Of course there are many other events under which \( a_m \) will be best under sampling, but as we show in the proof of the result, the combined probability of these does not allow \( a_m \) to be best under sampling with (even weakly) greater probability than \( 1 - \eta \) when \( \eta \) is sufficiently small. And this ensures local instability of the equilibrium. This intuition suggests that the two-person case is likely to have different stability properties, and indeed we have seen in our examples—and show more generally below—that it does.

Taken together, these results establish that when there are three or more players, sampling dynamics are efficiency enhancing in a standard public goods game, but efficiency reducing when the game has an efficient dominant strategy equilibrium.

Next consider games with two players. The following result identifies conditions under which an inefficient dominant strategy equilibrium is unstable.

**Theorem 3.** Suppose \( n = 2 \) and \( e \geq 2 \). If \( f(1) < 1 \) and \( f(3) > 2 \), there is an inefficient dominant strategy equilibrium with zero contribution that is unstable under the sampling dynamics.

As before, the condition \( f(1) < 1 \) together with concavity of \( f \) ensures that there is a dominant strategy equilibrium with zero contribution, and \( f(3) > 2 \) implies that it is inefficient. Furthermore, \( f(3) > 2 \) implies \( f(2) > 1 \) since \( f \) is concave. That is, there are two different actions, \( a_2 \) and \( a_3 \), that can each be best under sampling if the opponent contributes at least one unit when they are sampled, and none when all others are sampled (note that these are mutually exclusive events). As we show in the proof of the result, this allows the probability of \( a_1 \) being best to fall below the probability with which it is currently being played when the latter probability is close to 1, which ensures instability of the equilibrium. Note that this reasoning requires there to be at
least three actions, so we must have \( e \geq 2 \).

As we have seen in our examples, the greatest difference between the two-player case and that of three or more players arises when there is an efficient dominant strategy equilibrium. The following result generalizes this insight.

**Theorem 4.** Suppose \( n = 2 \). If \( 1 + f(2e - 1) < f(2e) \), then there is an efficient dominant strategy equilibrium with full contribution that is globally stable under the sampling dynamics.

The intuition for this result is quite straightforward. When there is an efficient dominant strategy equilibrium, action \( a_m \) will be best under sampling if the opponent chooses also chooses \( a_m \) when full contribution is sampled. This happens with precisely the probability with which \( a_m \) is being played at the current population state. But there are other ways in which \( a_m \) could be best under sampling—for instance if the opponent contribution is not smaller when \( a_m \) is sampled than it is when all other actions are sampled. This means that the probability that \( a_m \) is selected under sampling strictly exceeds the probability with which it is being played at all population states other than the equilibrium itself. The equilibrium is therefore globally stable.

The simplicity and transparency of this intuition makes it all the more surprising that it does not apply when there are more than two players.

### 7 Discussion

Decades of experimental research on social dilemmas, including public goods and common-pool resource games, has consistently found that subjects choose dominated strategies with high frequency. A few studies have documented this effect even when there is no trade-off between equilibrium and efficiency. Models of altruism, inequality-aversion, and reciprocity cannot account for such behavior in games with efficient dominant strategies, while models of envy and spite are not consistent with experimental findings in standard public goods games. Our main contribution here has been to show that a model of sampling equilibrium, suitably refined, can simultaneously account for experimental behavior in public goods games with inefficient and efficient dominant strategies.

When payoff functions are simple and transparent, and players have ample time to make choices, they may well be able to carry out the deductive exercises necessary to eliminate dominant strategies. In such settings we expect sampling equilibrium to have limited value. Conversely, the approach of stable sampling equilibrium is likely to be most useful in environments in

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\(^5\)The condition \( f(3) > 2 \), which implies \( f(2) > 1 \) when \( f \) is concave, ensures that the action \( a_1 \) is *twice inferior* in the sense of Sethi (2000), and this is enough to ensure instability.
which players have only partial information about the mapping from action profiles to payoffs, or face complex payoff functions that are difficult to fully grasp and analyze. An example of such a setting is the non-linear common pool resource game studied by Apesteguia (2006), in which subjects were assigned to one of two experimental conditions: full and null information on the payoff structure. Apesteguia found that aggregate behavior was not significantly different between the two treatments, which illustrates the difficulty of reasoning based on the complete mapping from action profiles to payoffs in games with many players and complex, nonlinear structures.

We have considered the case in which each action is sampled just once before a choice is made based on the best realized payoff. Analogously, one could allow for \( k \) samples for each action, resulting in what Osborne and Rubinstein (1998) call \( S(k) \) equilibrium. It is easily seen that as \( k \) approaches infinity, the average realized payoff approaches the expected payoff from an action, so \( S(k) \) equilibrium approaches Nash equilibrium. We therefore expect that when players have the opportunity and incentive to sample each action many times, dominated strategies will be played with much lower frequency.

Since the case of efficient dominant strategies is not usually tested in experimental public goods games, models of social preferences and bounded rationality are seldom calibrated to these environments. Our findings suggest that there is much to be gained by doing so. More generally, as proposed by Burton-Chellew and West (2013) and Rand and Kraft-Todd (2014), there is much to be learned from further experimental work with efficient public goods games in deeping our understanding of imperfect cooperative behavior.
Appendix

Proof of Theorem 1. Suppose \( f(1) < 1 < f(2) \). Then the dominant strategy is to contribute zero because the (weak) concavity of \( f \) implies that at any action profile \((a_i, b)\), with \( a_i \in \{1, \ldots, e\} \), we have

\[
f\left(a_i + \sum_{j=1}^{n-1} b_j\right) < 1 + f\left(a_i - 1 + \sum_{j=1}^{n-1} b_j\right).
\]

That is, regardless of the total contribution of others, a player can increase her payoffs by reducing her contribution by one unit, and the dominant strategy equilibrium is accordingly \( x^* = (1, 0, \ldots, 0) \). The equilibrium is inefficient because if any two players were to contribute an additional unit, the resulting net benefits would be positive for both of them.

To show that the equilibrium is unstable, consider a population state \( x = (1 - \eta, \eta_2, \ldots, \eta_m) \), where \( \eta_i \) is the population share of individuals taking action \( a_i \), and

\[
\eta = \sum_{i=2}^{m} \eta_i.
\]

Let \( \omega(a_1, x) \) denote the probability that zero contribution yields the highest payoff under sampling at this state, and define the event \( E_0 \) as follows: when \( a_1 \) is sampled, all other players also choose to contribute nothing, while when \( a_2 \) is sampled, exactly one other player makes a positive contribution. This contribution could be anything between 1 and \( e \), resulting in a payoff of at least \((e - 1) + f(2) > e\). That is, under the event \( E_0 \), a higher payoff is obtained when \( a_2 \) is sampled than when \( a_1 \) is sampled, so zero contribution does not yield the highest payoff under this event. The probability of the event \( E_0 \) is

\[
\Pr(E_0) = (1 - \eta)^{n-1} \times (n - 1)\eta(1 - \eta)^{n-2}.
\]

The first factor is the probability of zero contributions by all opponents when \( a_1 \) is sampled, and the second is the probability that exactly one of the \( n - 1 \) other subjects selects an action other than \( a_1 \). This probability can be rewritten as

\[
\Pr(E_0) = (n - 1)\eta(1 - \eta)^{2n-3}.
\]

Thus, since \( n \geq 3 \), the probability that \( a_1 \) is not selected under sampling is

\[
1 - \omega(a_1, x) \geq (n - 1)\eta(1 - \eta)^{2n-3} \geq 2\eta + o(\eta^2),
\]

where \( o(\eta^2) \) denoted terms that are second order or higher in \( \eta \). For \( \eta \) sufficiently small, we therefore have \( 1 - \omega(a_1, x) > \eta \), and hence

\[
\omega(a_1, x) < 1 - \eta = x_1
\]
For \( \eta \) sufficiently small, therefore, \( \dot{x}_1 < 0 \). That is, there exists a neighborhood \( N_0 \) of \( x^* \) such that \( \dot{x}_1 < 0 \) at all points \( x \in N_0 \setminus x^* \). The dominant strategy equilibrium is therefore unstable. \( \square \)

**Proof of Theorem 2.** Suppose \( f(ne) > f(ne - 1) + 1 \). In this case the dominant strategy involves contribution of the entire endowment, because the (weak) concavity of \( f \) implies that at any action profile \((a_i, b)\), with \( a_i \in \{1, \ldots, e\} \), we have

\[
1 + f \left( a_i - 1 + \sum_{j=1}^{n-1} b_j \right) < f \left( a_i + \sum_{j=1}^{n-1} b_j \right).
\]

That is, the efficient dominant strategy equilibrium is \( x^* = (0, \ldots, 0, 1) \).

Consider a population state \( x = (\varepsilon_1, \ldots, \varepsilon_{m-1}, 1 - \varepsilon) \), where \( \varepsilon_i \) is the population share of individuals taking action \( a_i \), and

\[
\varepsilon = \sum_{i=1}^{m-1} \varepsilon_i.
\]

Let \( \omega(a_m, x) \) denote the probability that full contribution yields the highest payoff under sampling at this state, and define the event \( E_\varepsilon \) as follows: when \( a_m \) is sampled, exactly one other player chooses less than full contribution, while when \( a_{m-1} \) is sampled, all other players choose full contribution. In this case the payoff obtained when \( a_m \) is sampled is at most \( f(ne - 1) \), while the payoff from sampling \( a_{m-1} \) is \( 1 + f(ne - 1) \), so action \( a_m \) is not the best under the sampling procedure under this event. The probability of \( E_\varepsilon \) is

\[
\Pr(E_\varepsilon) = (1 - \varepsilon)^{n-1} \times (n - 1) \varepsilon (1 - \varepsilon)^{n-2}.
\]

The first factor is the probability that all others choose full contribution when \( a_{m-1} \) is sampled, and the second is the probability that exactly one of the other \( n - 1 \) players chooses an action other than full contribution when \( a_m \) is sampled. This can be written as

\[
\Pr(E_\varepsilon) = (n - 1)(1 - \varepsilon)^{2n-3} \varepsilon.
\]

Thus, since \( n \geq 3 \), the probability that \( a_m \) is not best under sampling is

\[
1 - \omega(a_m, x) \geq (n - 1) \varepsilon (1 - \varepsilon)^{2n-3} \geq 2 \varepsilon + o(\varepsilon^2).
\]
For $\varepsilon$ sufficiently small, we therefore have $1 - \omega(a_m, x) > \varepsilon$, and hence

$$\omega(a_m, x) < 1 - \varepsilon = x_m$$

For $\varepsilon$ sufficiently small, therefore, $\dot{x}_m < 0$. That is, there exists a neighborhood $N_\varepsilon$ of $x^*$ such that $\dot{x}_m < 0$ at all points $x \in N_\varepsilon \setminus x^*$. The dominant strategy equilibrium is therefore unstable.  

Proof of Theorem 3. Suppose $f(1) < 1$ and $f(3) > 2$. The first of these conditions, together with concavity of $f$, ensures that there is a dominant strategy equilibrium at $x^* = (1, 0, ..., 0)$. The second condition ensures that this equilibrium is inefficient. Furthermore, these two conditions, together with the concavity of $f$, imply that $f(2) > 1$ since

$$f(2) \geq \frac{1}{2} (f(3) + f(1)) > 1 + \frac{1}{2} f(1) > 1.$$ 

Now consider a population state $x = (1 - \eta, \eta_2, ..., \eta_m)$, where $\eta_i$ is the population share of individuals taking action $a_i$, and

$$\eta = \sum_{i=2}^{m} \eta_i.$$ 

As before, let $\omega(a_1, x)$ denote the probability that action $a_1$ (involving zero contribution) is best under sampling at this state, and define two events $M_0$ and $M_1$ as follows. $M_0$ is the event that when action $a_2$ is sampled, the other player chooses some positive contribution; when every other action, including $a_1$, is sampled, the other player contributes zero. The payoff from sampling $a_2$ is at least $(e - 1) + f(2)$, which exceeds the payoff $e$ from sampling $a_1$, since $f(2) > 1$. The probability of $M_0$ is $(1 - \eta)^{m-1} \eta$.

$M_1$ is the following event: when action $a_3$ is sampled, the other player chooses some positive contribution, while when every other action, including $a_1$, is sampled, the other player contributes nothing. The payoff from sampling $a_3$ is at least $(e - 2) + f(3)$, which again exceeds $e$, the payoff from sampling $a_1$, since $f(3) > 2$. The probability of $M_1$ is also $(1 - \eta)^{m-1} \eta$.

Since $M_0$ and $M_1$ are mutually exclusive, and each of them results in an action other than $a_1$ being best under sampling, we have

$$1 - \omega(a_1, x) \geq \Pr(\mathcal{M}_0 \cup \mathcal{M}_1) = 2\eta(1 - \eta)^{m-1} = 2\eta + o(\eta^2).$$

For $\eta$ sufficiently small, we therefore have $1 - \omega(a_1, x) > \eta$, and hence

$$\omega(a_1, x) < 1 - \eta = x_1.$$ 

For $\eta$ sufficiently small, therefore, $\dot{x}_1 < 0$. That is, there exists a neighborhood $N_0$ of $x^*$ such that $\dot{x}_1 < 0$ at all points $x \in N_0 \setminus x^*$. The equilibrium is therefore unstable.  

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Proof of Theorem 4. Suppose that $1 + f(2e - 1) < f(2e)$, in which case the concavity of $f$ implies that for any action $a_i \in \{a_2, \ldots, a_m\}$ and any opponent action $b$,

$$1 + f(a_i - 1 + b) < f(a_i + b).$$

That is, $a_m$ is a dominant strategy, and there is an efficient dominant strategy equilibrium with full contribution at $x^* = (0, \ldots, 0, 1)$.

Consider a population state $x = (\epsilon_1, \ldots, \epsilon_{m-1}, 1 - \epsilon)$, where $\epsilon_i$ is the population share of individuals choosing action $a_i$, and

$$\epsilon = \sum_{i=1}^{m-1} \epsilon_i.$$

As before, let $\omega(a_m, x)$ denote the probability that action $a_m$ (full contribution) is best under sampling. When $a_m$ is sampled and the other player chooses a different action $a_k$, the resulting payoff is $f(e + a_k)$. The action $a_m$ will be best under sampling if, when every other action $a_i$ is sampled, the opponent chooses a contribution level at most $a_k$. In this case the probability that $a_m$ yields the largest payoff satisfies

$$\omega(a_m, x) \geq (1 - \epsilon) + \epsilon_{m-1} \left( \sum_{i=1}^{m-1} \epsilon_i \right)^{m-1} + \ldots + \epsilon_2 \left( \sum_{i=1}^{m-2} \epsilon_i \right)^{m-1} + \epsilon_1^m$$

for all $\epsilon > 0$. Hence

$$\omega(a_m, x) > 1 - \epsilon = x_m,$$

and $x_m > 0$ at all population states other than the dominant strategy equilibrium. The equilibrium is therefore globally stable. □
References


