A (Mostly) Informal Introduction to Computation Theory

- Computation theory is a different, more structural and less statistical approach to complexity, emergence, organization.
- Computation theory can be very elegant, rigorous, and mathematical.
- But I'll present little of the formalism. I think the math can obscure some of the basic ideas, which are really quite simple.

We'll begin with some examples in the form of a game:

- I'll give you the specification for a set
- I'll then show you an object, and you need to tell me if it's in the set or not

Example 1

The set $\mathcal{L}$ consists of all sequences of 0's and 1's of any length, except for those that have two 00's in a row.

Accept all sequences of 1's and 0's except for those which have two or more 0's in a row.

$$1110101101$$

$$1101101001$$

$$110110101011$$

Example 2:

The set $\mathcal{L}$ consists of all sequences of correctly balanced parentheses.

$$(\,(\,)\,(\,)\,)$$

$$(\,(\,)\,(\,)\,(\,)\,)\,(\,(\,)\,)$$

$$(\,(\,)\,(\,(\,)\,(\,)\,(\,)\,(\,)$$

This example is harder.

Example 3:

The set $\mathcal{L}$ consists of all sequences of 0's and 1's, except for those that contain a **prime** number of consecutive 0's!

$$1100011000001$$

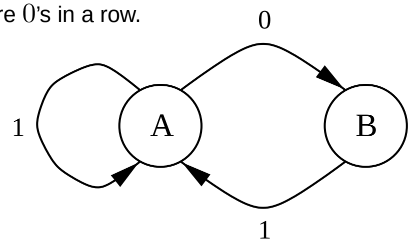
$$11000011$$

$$111000000000001$$

$$11\ \overbrace{00\cdots00}^{1031\text{ elements}}\ 11$$

What to learn from the examples

- There are qualitative differences between the procedures you just used to identify the strings on the previous slides.
- These distinctions lie at the heart of computation theory.
- We'll start by focusing on example 1.
- Your task was to accept all sequences of 1's and 0's except for those which have two or more 0's in a row.



- Sequence is OK if there exists a path through this machine
- Example: 1011001 is not in the set.

Finite State Machines

- The mathematical object on the previous page is known as a **Finite State Machine** or a **Finite Automaton**.
- Note that this two-state machine can correctly identify arbitrarily long sequences.
- The machine is a finite representation of the infinite set $\mathcal{L}$.

Some terminology and definitions

- A **Language** $\mathcal{L}$ is a set of words (symbol strings) formed from an **Alphabet** $\mathcal{A}$.
- We'll always assume a binary alphabet, $\mathcal{A} = \{0, 1\}$.

Big Idea: There is a correspondence between the rules needed to generate or describe a language, and the type of machine needed to recognize it.

Regular Expressions

- A **Regular Expression** is a way of writing down rules that generate a language.
- To generate a regexp, start with the symbols in $\mathcal{A}$.
- You can make new expressions via the following operations: grouping, concatenating, logical OR (denoted $+$), and closure $*$.
- Closure means 0 or more concatenations.
- Examples:
  1. $(0 + 1) = \{0, 1\}$
  2. $(0 + 1)^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$
  3. $(01)^* = \{\epsilon, 01, 0101, 010101, \ldots\}$
- (ϵ is the empty symbol.)

Regular Languages and FSM

- A language $\mathcal{L}$ is a **Regular Language** if and only if it can be generated by a regular expression.
- A puzzle: what is the regular expression that generates the language of example 1?

Two important results:

1. For any regular language, there is an FSM that recognizes it.
2. Any language generated by an FSM is regular.

Notes on terminology:

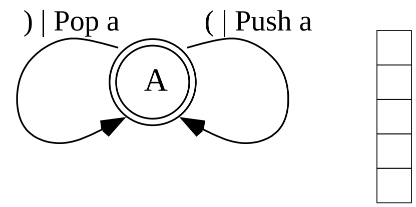
- A regular expression is a **rule**.
- A regular language is a **set**.
- A FSM is a **machine**.

Regular languages $\leftrightarrow$ FSM's is the first example of the correspondence between sets and the procedures or machines needed to recognize them.

Revisiting Parentheses

- This example is different than the last—you can't scan left to right unless you remember stuff.
- There is no FSM that can recognize this language. The problem is that as the string grows in length, the number of states necessary also grows.
- This task requires infinite memory. However, the memory only needs to be organized in a simple way.
- The parentheses language can be recognized by a device known as a **Pushdown Automata**.
- Put an object on the stack if you see a left paren $($ and take it off if you see a right paren $)$.
- If the stack is empty after scanning the sequence, then it is ok.

Pushdown Automata



- This is the PDA for the parentheses example
- If you see a "(", write (push) a symbol to the stack.
- If you see a ")", erase (pop) a symbol from the stack.
- The machine can only write to the top of the stack.
- This PDA can recognize balanced parentheses of any length.

Context-Free Languages

- The languages recognized by PDA are **context-free languages.**
- Regular languages are generated sequentiality—one symbol after the next.
- CFL's are generated by writing rules applied in parallel.
- For example, to generate the parentheses language, apply the following:

$$\begin{array}{rcl} W & \rightarrow & (V \\ V & \rightarrow & (VV \ \text{ or } \ ) \end{array}$$

- Start with W. The set of all possible applications of the above rules give you the set of all possible balanced parentheses.
- For example:

$$W, \ \ (V, \ \ ((VV, \ \ (()V \ \ (()(VV \ \ (()()V \ \ (()())$$

CFL Terminology

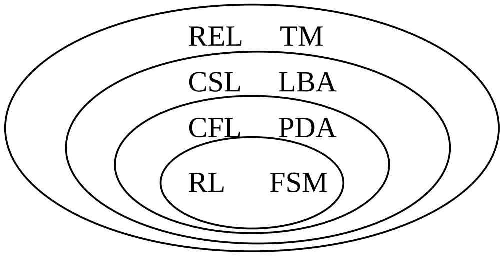
- $(,)$ are **terminals**, symbols in the alphabet $\mathcal{A}$.
- W, V are **variables**, symbols not in $\mathcal{A}$, to be eventually replaced by terminals.
- CFL's are context free in the sense that the production rule depends only on the variable, not on where the variable is in the string.

CFL Summary

- Every CFL can be recognized by a PDA, and every PDA produces a CFL.
- Also, FSM's are a proper subset of PDAs, and
- Regular Languages are a proper subset of CFL's
- We can thus divide languages into two classes, one of which is strictly more complex than the other.
- Are there even more complex languages? Yes ...

Chomsky Hierarchy

- The hierarchy continues:



- This hierarchy of languages/machines is known as the **Chomsky Hierarchy**.
- Each level in the hierarchy contains something new, and also contains all the languages at lower levels of the hierarchy.

Chomsky Hierarchy, terminology

- CSL = **Context Sensitive Language**. These are like CFL's, but allow transitions that depend on the position of the variable in the strings.
- LBA = **Linear Bounded Automata**. These are like PDA's, except:
  1. Controller can write anywhere on work tape.
  2. Work tape restricted to be a linear function of input.
- **Recursively Enumerable Languages** are those languages produced by an unrestricted grammar.
- An **Unrestricted Grammar** is like a CSL, but allows substitutions that shrink the length of the string.
- TM = **Turing Machines**. These are LBA's with linear tape restriction removed. These are the most powerful model of computation. (Example 3 requires a TM.) More on these later.

Chomsky Hierarchy, Conclusions

- Order languages (sets) by the type of machine needed to recognize elements of the language.
- There are qualitative difference between machines at different levels of the hierarchy.
- At lower levels of the hierarchy, there are algorithms for minimizing machines. (I.e., remove duplicate nodes.)
- The minimum machine can be viewed as a representation of the pattern contained in the language. The machine is a description of all the regularities.
- The size of the machine may be viewed as a measure of complexity.
- The machine itself reveals the "architecture" of the information processing.

Other computation theory notes

- It is possible to refine the Chomsky hierarchy with different sorts of machines. The result is a rich partial ordering of languages.
- To use computation theory as a basis for measuring complexity or structure, I think it's important to start at the bottom of the hierarchy and work your way up.

Computation Theory References

The basic material presented is quite standard and there are many references on it. Here are a few:

- Hopcroft and Ullman. Introduction to Automata Theory, Languages and Computation. Addison-Wesley. 1979. *A standard reference. Not my favorite, though. It's thorough and clear, but rather dense.*

- Brookshear. Theory of Computation: Formal Languages, Automata, and Complexity. Benjamin/Cummings. 1989. *I like this book. I find it much clearer than Hopcroft and Ullman.*

Computation theory applied to physical sequences

- Badii and Politi. Complexity: Hierarchical Structures and Scaling in Physics. Cambridge. 1997. *Excellent book, geared toward physics grad students. Closest thing to a textbook that covers topics similar to those I've covered throughout these lectures.*

- Bioinformatics textbooks?