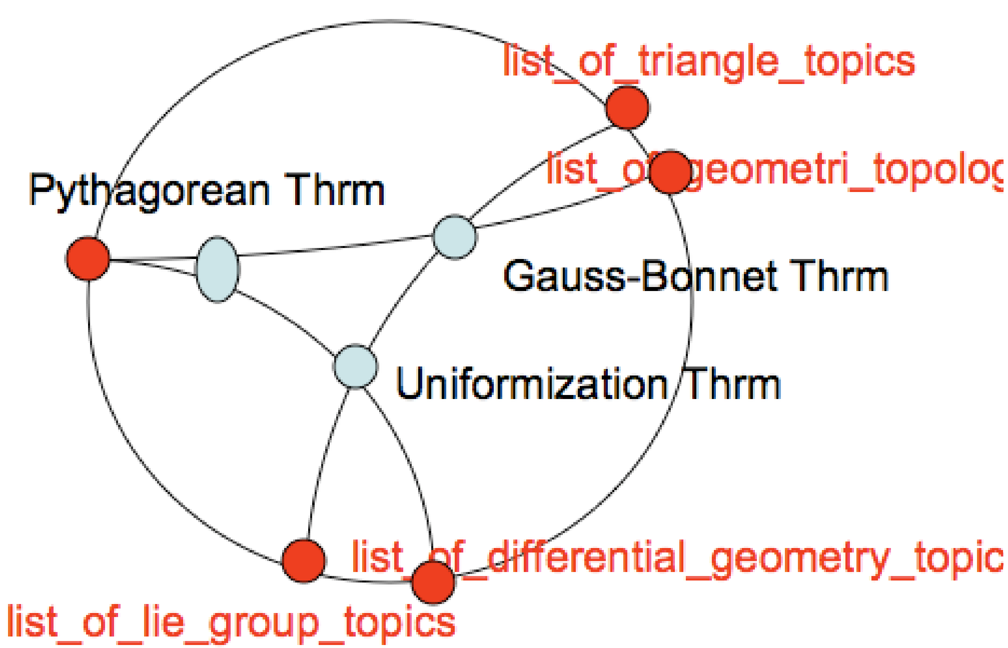
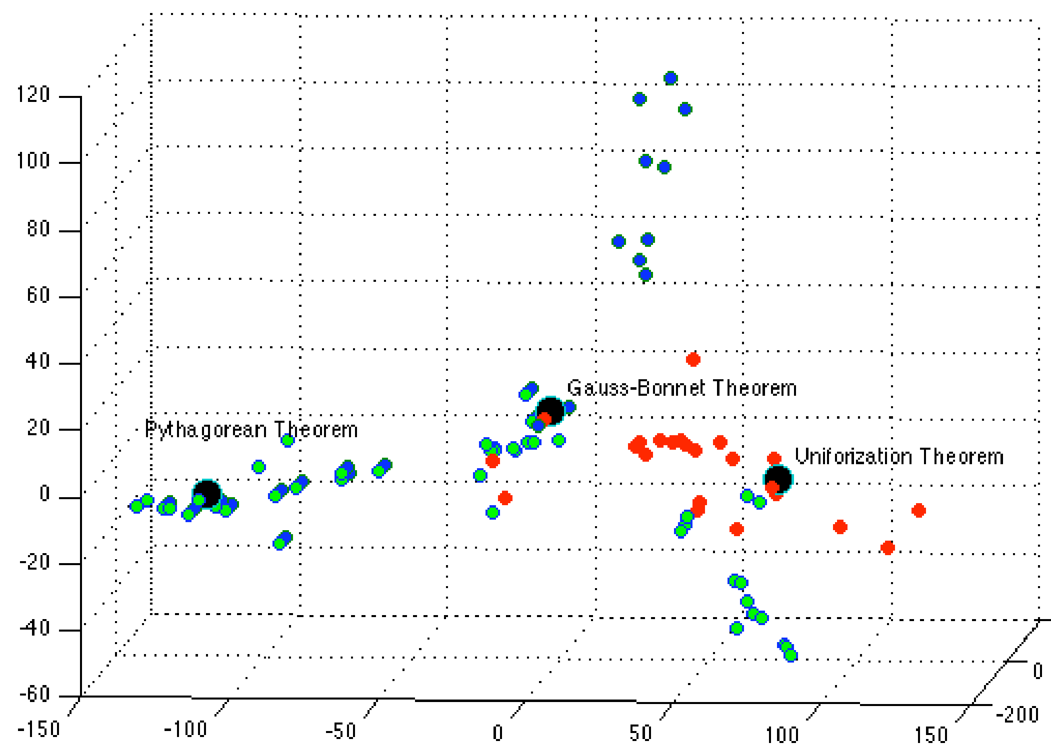
Complex Systems with Boundary and Non-Euclidean Geometry

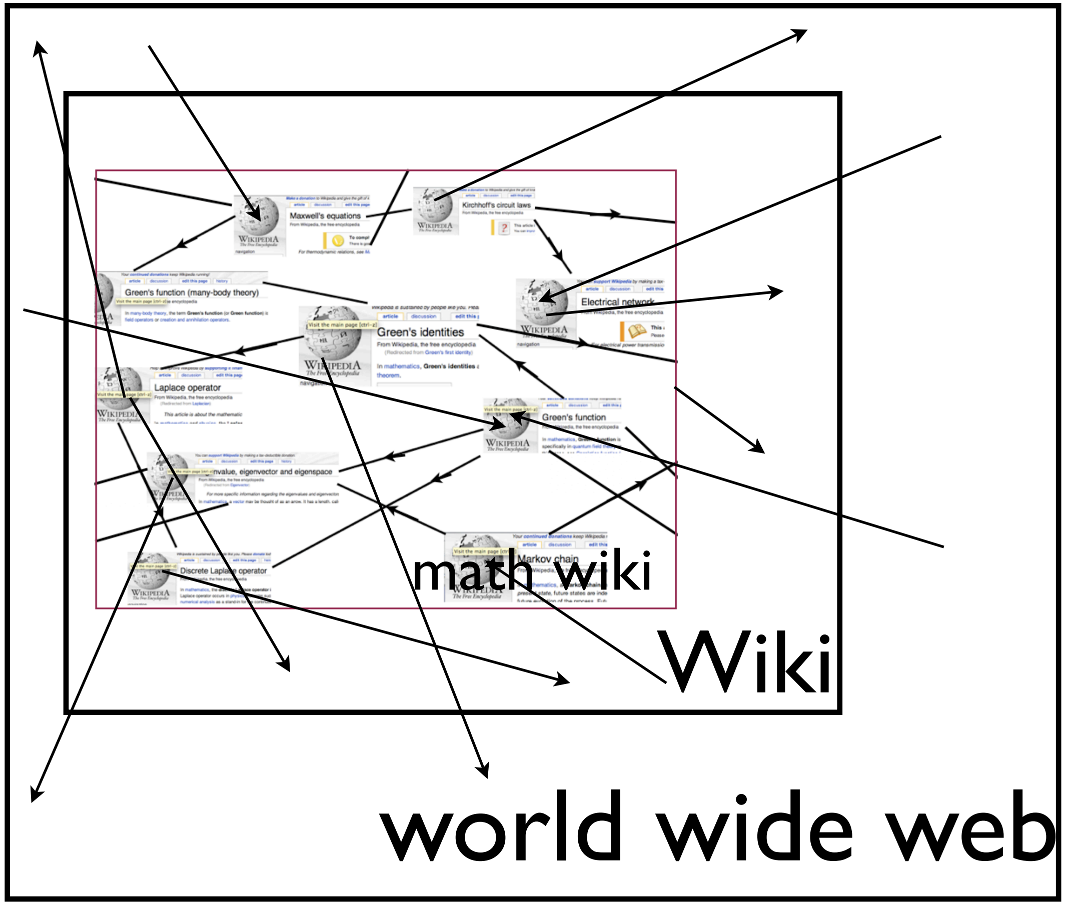
Lecture 2, CSSS10

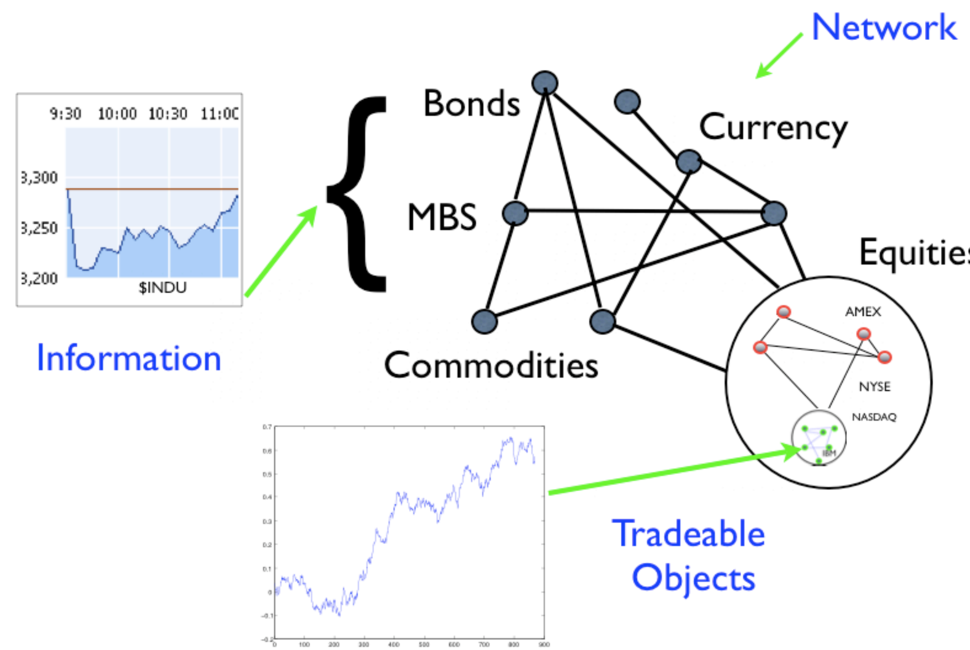




Greg Leibon
Memento, Inc
Dartmouth College

Systems with boundary





... usually the boundary at infinity

The Lists at ∞	
’list_of_abstract_algebra_topics’	’list_of_curve_topics’
’list_of_triangle_topics’	’list_of_mathematical_topics_in_quantum_theory’
’list_of_lie_group_topics’	’list_of_algebraic_coding_theory_topics’
’list_of_complex_analysis_topics’	’list_of_set_theory_topics’
’list_of_basic_probability_topics’	’list_of_fourier_analysis_topics’
’list_of_general_topology_topics’	’list_of_algorithm_general_topics’
’list_of_geometry_topics’	’list_of_partial_differential_equation_topics’
’list_of_numerical_computational_geometry_topics’	’list_of_topology_topics’
’list_of_geometric_topology_topics’	’list_of_group_theory_topics’
’list_of_computer_graphics_and_descriptive_geometry_topics’	’list_of_multivariable_calculus_topics’
’list_of_partition_topics’	’list_of_differential_geometry_topics’
’list_of_statistical_topics’	’list_of_variational_topics’
’list_of_stochastic_processes_topics’	’list_of_permutation_topics’
’list_of_linear_algebra_topics’	’list_of_algebraic_topology_topics’
’list_of_calculus_topics’	’list_of_homological_algebra_topics’
’list_of_exponential_topics’	’list_of_number_theory_topics’
’list_of_commutative_algebra_topics’	’list_of_recreational_number_theory_topics’
’list_of_computability_and_complexity_topics’	’list_of_basic_algebra_topics’
’list_of_boolean_algebra_topics’	’list_of_mathematical_logic_topics’
’list_of_representation_theory_topics’	’list_of_integration_and_measure_theory_topics’
’list_of_factorial_and_binomial_topics’	’list_of_string_theory_topics’
’list_of_numerical_analysis_topics’	’list_of_topics_related_to_%cf%80’
’list_of_real_analysis_topics’	’list_of_mathematical_topics_in_relativity’
’list_of_knot_theory_topics’	’list_of_trigonometry_topics’

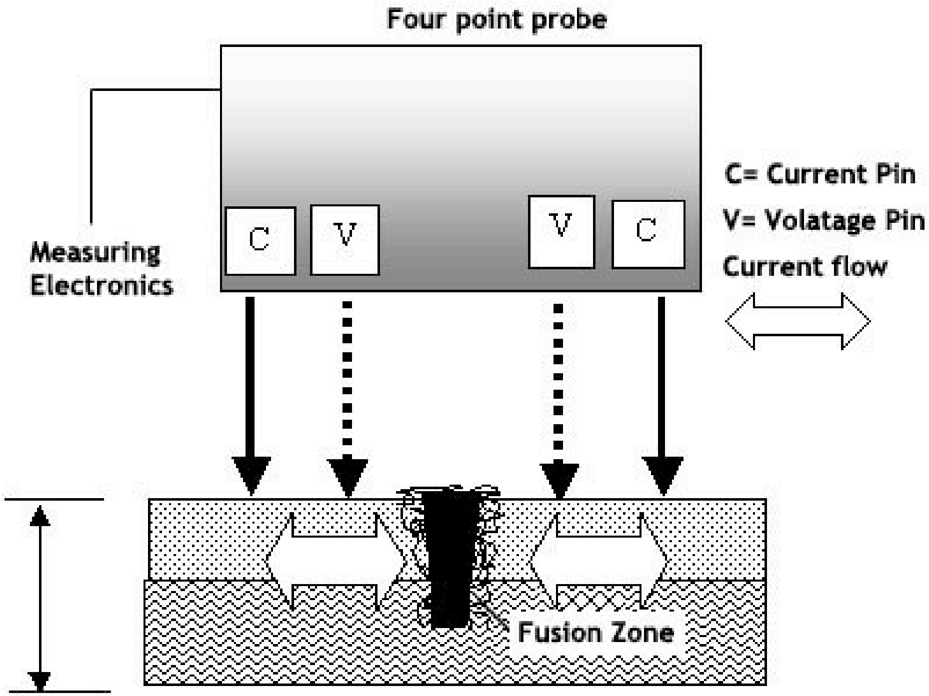
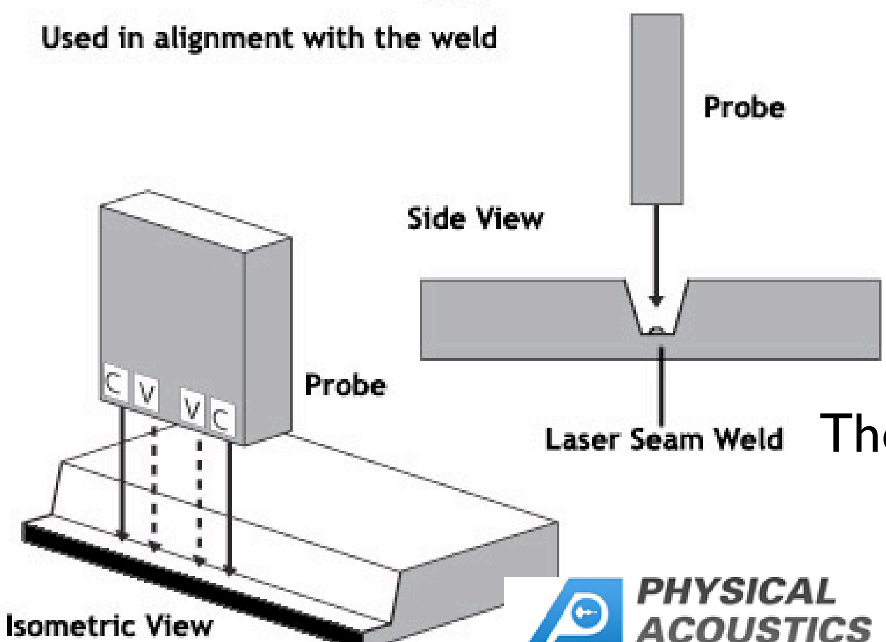


Figure 1. Four-point probe contact Resistivity Measurement Technique(Used across the weld)

OR

Used in alignment with the weld



PHYSICAL ACOUSTICS CORPORATION

Text

Let us go to the Riemann Sphere

$$S^2 = \mathbb{C} \cup \{\infty\}$$

$$V^d_{c,S^2}(z) = \frac{1}{2\pi} \log \left| \frac{z-c}{z-d} \right|$$

four point probe

$$[z, w; c, d]_{S^2} = V^d_{c,S^2}(z) - V^d_{c,S^2}(w)$$

$$V^d_{c,S^2}(z) - V^d_{c,S^2}(w) = \frac{1}{2\pi} \log |\{z, w, c, d\}|$$

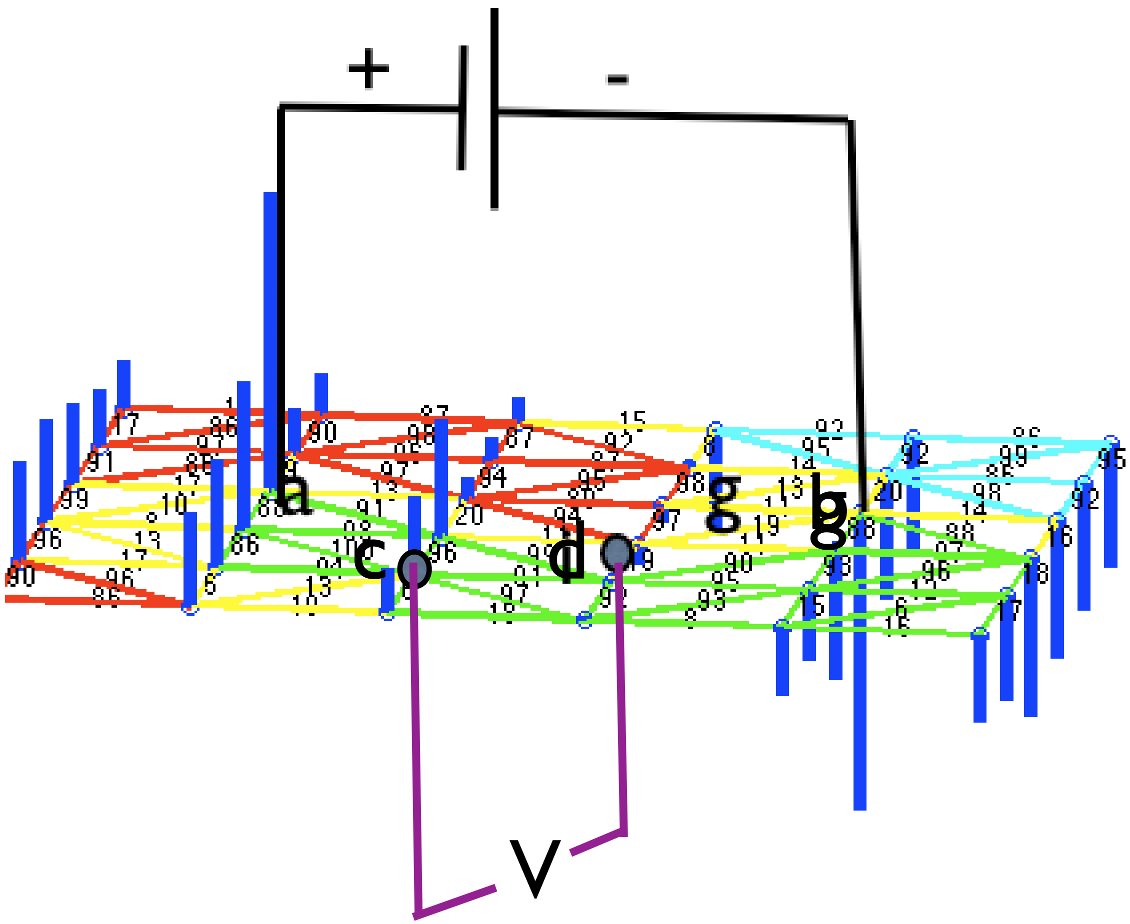
cross ratio

$$\{z, w, c, d\} = \frac{(z-c)(w-d)}{(z-d)(w-c)}$$

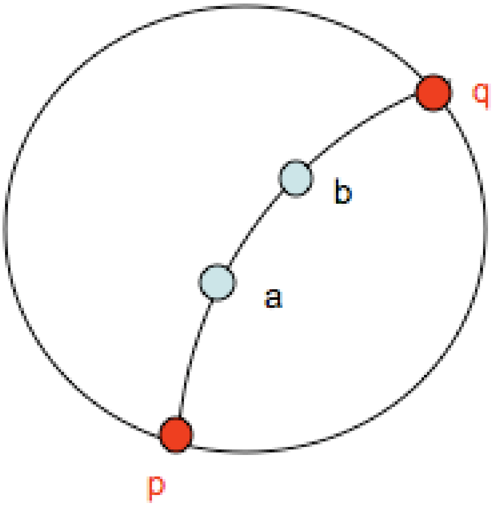
The unique invariant of Mobius transformation, i.e. conformal homeomorphisms of

$$S^2 = \mathbb{C} \cup \{\infty\}$$

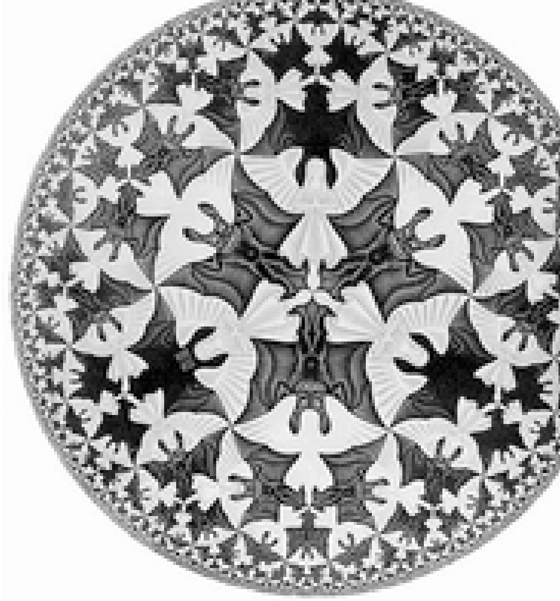
the cross potential



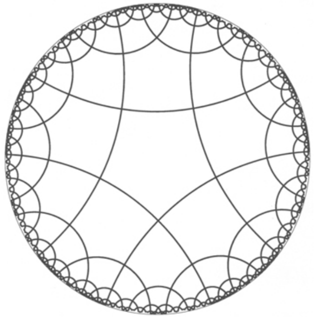
$$[a,b;c,d] = < V_c^d, V_a^b >_{Dir} = V_c^d(a) - V_c^d(b)$$



Cowgirl Hall of Fame Lemma

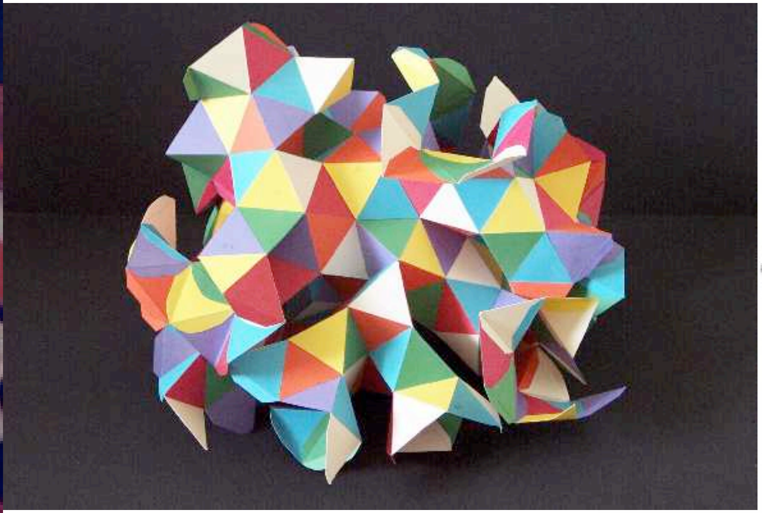


$$d_{hyp}(a,b) = max_{p,q \in \partial^\infty}[p,q;a,b]$$



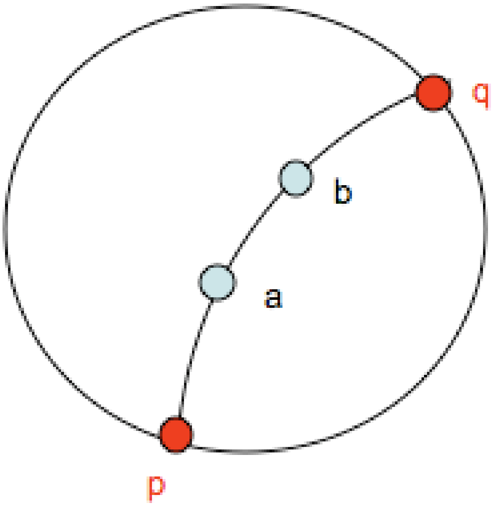
1. To draw a straight line from any point to any point.
2. To produce [extend] a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance [radius].
4. That all right angles are equal to one another.

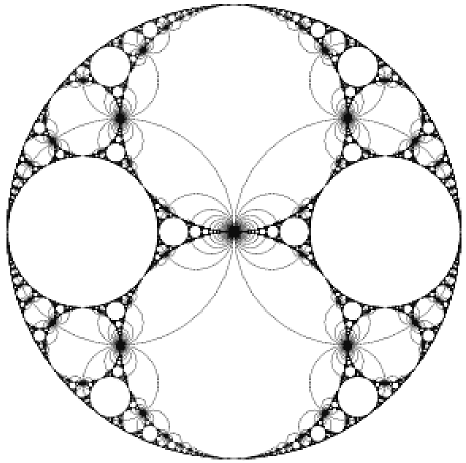
5. Through a point not on a given straight line, at most one line can be drawn that never meets the given line.



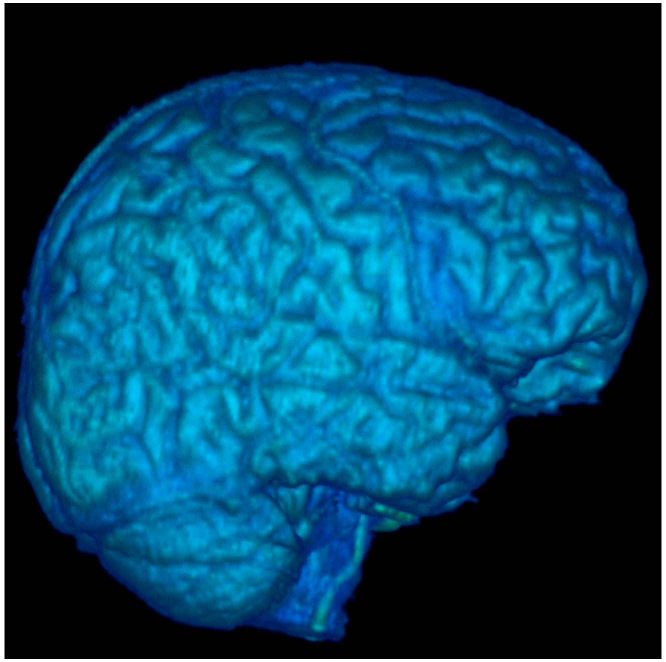
Daina Taimina

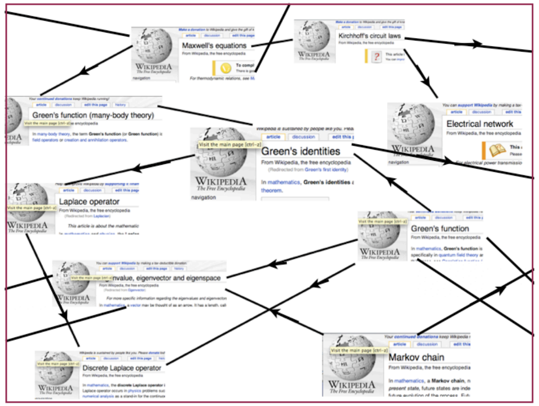
Cowgirl Hall of Fame Lemma



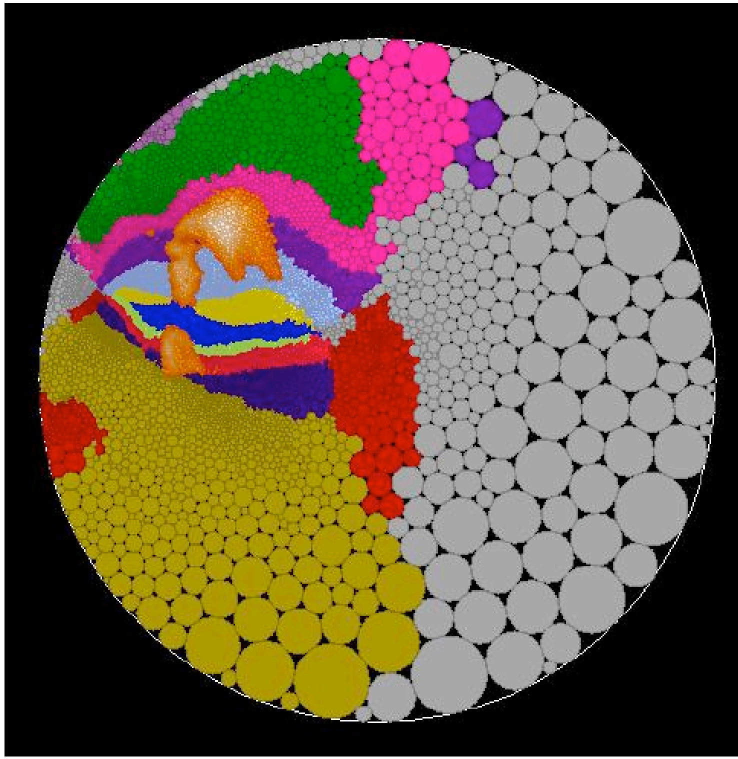


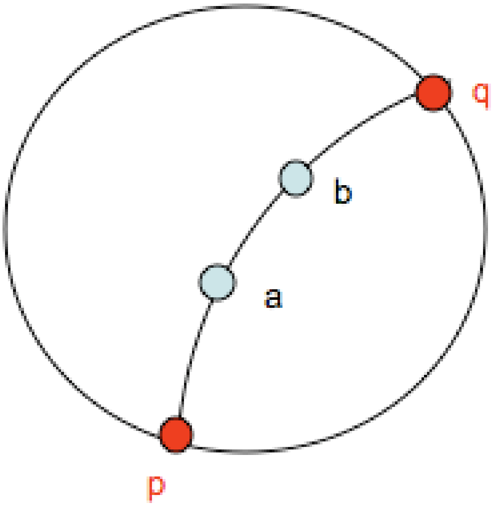
$$d_{hyp}(a,b) = max_{p,q \in \partial^\infty}[p,q;a,b]$$





(Monica K. Hurdal)





Cowgirl Hall of Fame Lemma

$$d_{hyp}(a,b) = max_{p,q \in \partial^\infty}[p,q;a,b]$$

proof: In the Poincare disk model, the geodesic through a and b is a circular arc that intersects the ∂D^2 (the unit circle) at right angles. Furthermore, from exercise 1, $d_{hyp,D^2}(z,w) = \log|\{p,q,z,w\}|$ where p and q are the points where this geodesic intersects ∂D^2. So all we need to show is that among all the p and q on disk's boundary that these p and q maximize the $\log|\{p,q,z,w\}|$. To do this we note that the cross ratio is invariant under orientation preserving conformal homeomorphisms and that the orientation preserving conformal self maps of the disk are a subset of these homeomorphisms that form the hyperbolic isometries. Furthermore, the hyperbolic isometries are transitive on the unit tangent bundle; so we can send our points to $a = 0$, $b = x$ with $0 < x < 1$ and hence force the geodesic to be $[-1,1]$. Other choices of p and q could be written as $p = e^{i\theta}$ and $q = e^{i\phi}$, and so we need to show that

$$\log|\{p,q,0,x\}| = \log\left|\frac{(e^{i\theta}-0)(e^{i\phi}-x)}{(e^{i\theta}-x)(e^{i\phi}-0)}\right| = \log|e^{i\phi}-x| - \log|(e^{i\theta}-x)|$$

is maximized at $p = -1$ and $q = 1$. But since $|z-x|$ is the Euclidean distance from x to z and log monotonically increases, we see that is indeed true. **Q.E.D**

On any network.....

$$d_{hyp}(a,b) = max_{p,q \in \partial^{\infty}}[p,q;a,b]$$

For God's sake, please give it up. Fear it no less than the sensual passion, because it, too, may take up all your time and deprive you of your health, peace of mind and happiness in life.

-Farkas Bolya

[A letter to his son János urging him to give up work on non-Euclidean geometry.

$$d_{hyp}(a,b) = max_{p,q\in\partial^\infty}[p,q;a,b]$$

Hyperbolic Metric Lemma: d_{hyp} is a psuedo-metric.

proof:

1. d_{hyp} is non-negative since $[p,q;a,b] = -[q,p;a,b]$, so the max is non-negative.
2. d_{hyp} is symmetric since $[p;q;b,a] = [q,p;a,b]$, so $max_{p,q\in\partial^\infty}[p,q;a,b]$ and $max_{p,q\in\partial^\infty}[p,q;b,a]$ will agree.
3. To see d_{hyp} satisfies the triangle inequality first notice that $[p,q;a,c] = [p,q;a,b] + [p,q;b,c]$ since

   $$[a,b;c,d] = V_p^q(a) - V_p^q(c) = V_p^q(c) - V_p^q(a) + (V_p^q(b) - V_p^q(b)) = [p,q;a,b] + [p,q;b,c].$$

   So using the p and q that maximize $[p,q;a,c]$, we have

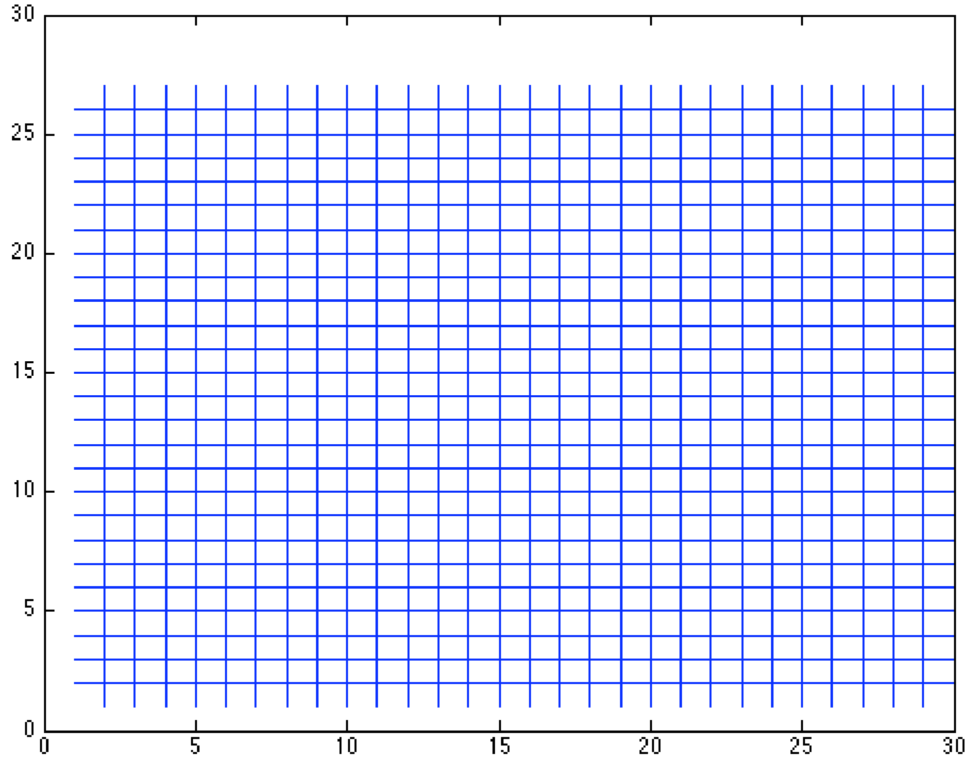
   $$d_{hyp}(a,c) = [p,q;a,c] = [p,q;a,b] + [p,q;b,c]$$

   $$<= d_{hyp}(a,b) + d_{hyp}(b,c).$$

Q.E.D

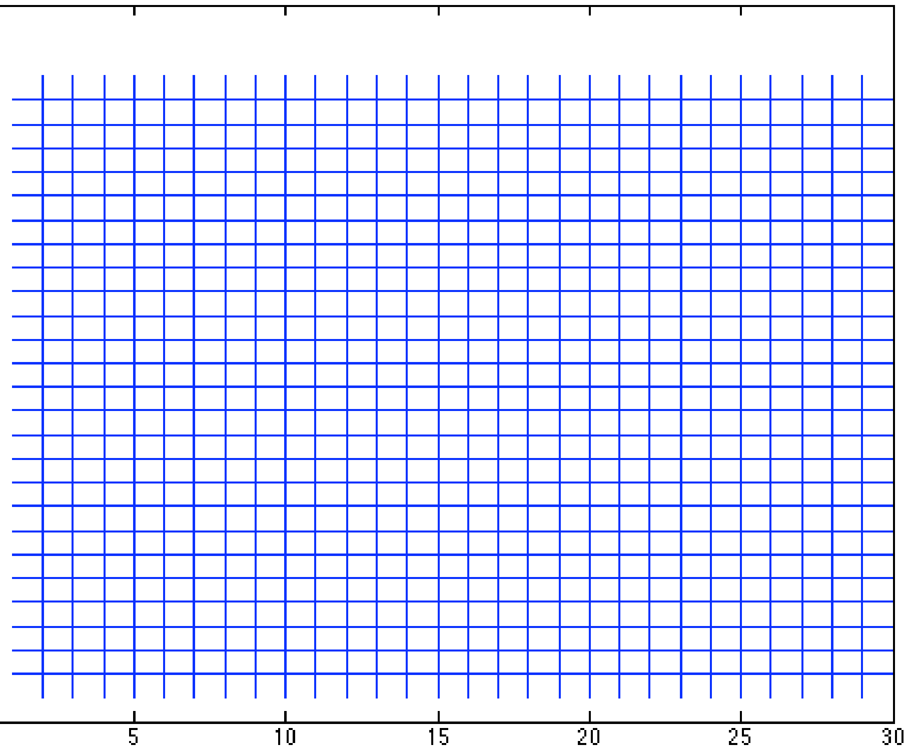
Kakawa’s Salted Carmel time!

Put the boundary at infinity

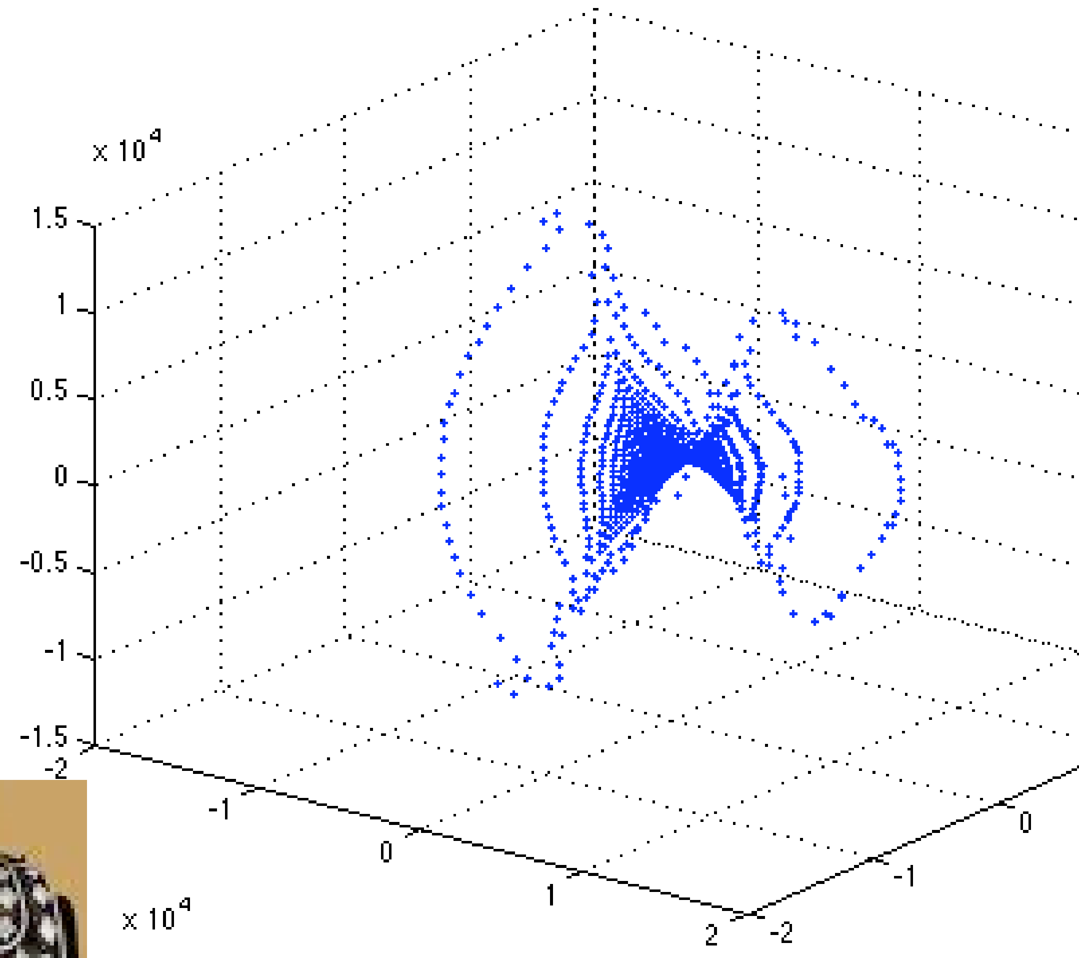


What does it look like?

Boundary at infinity

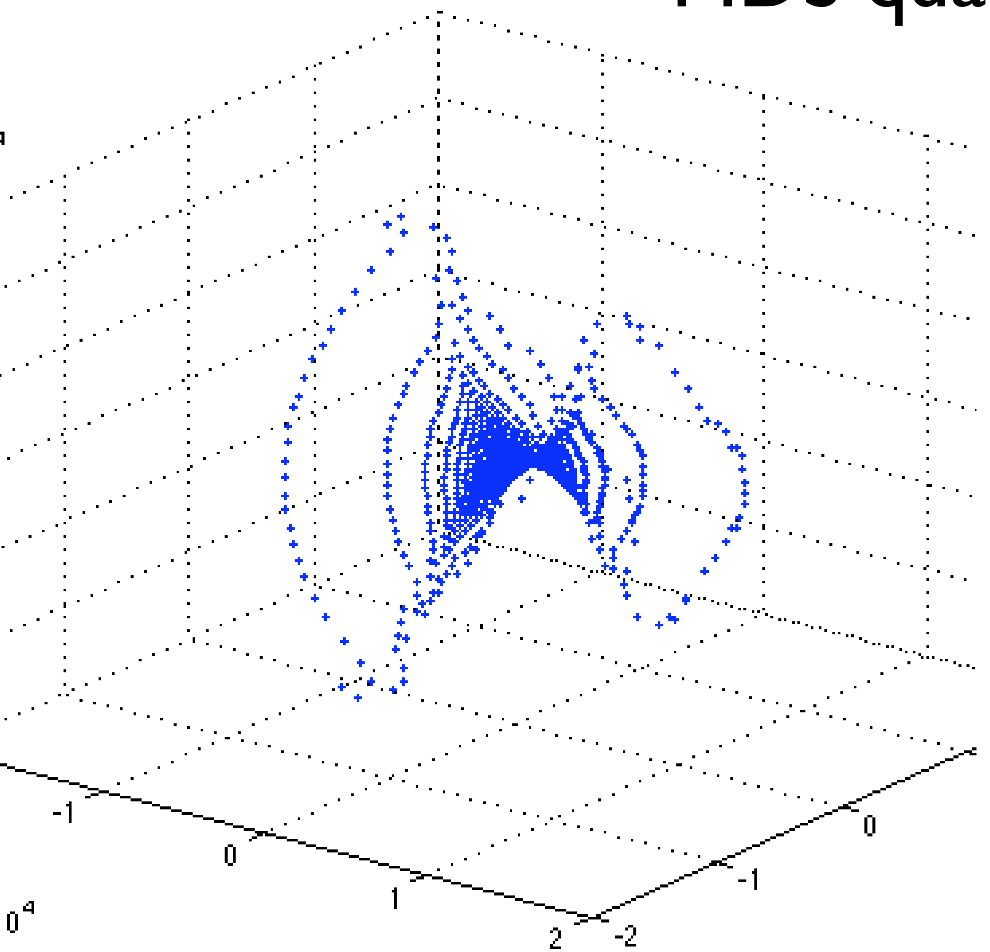


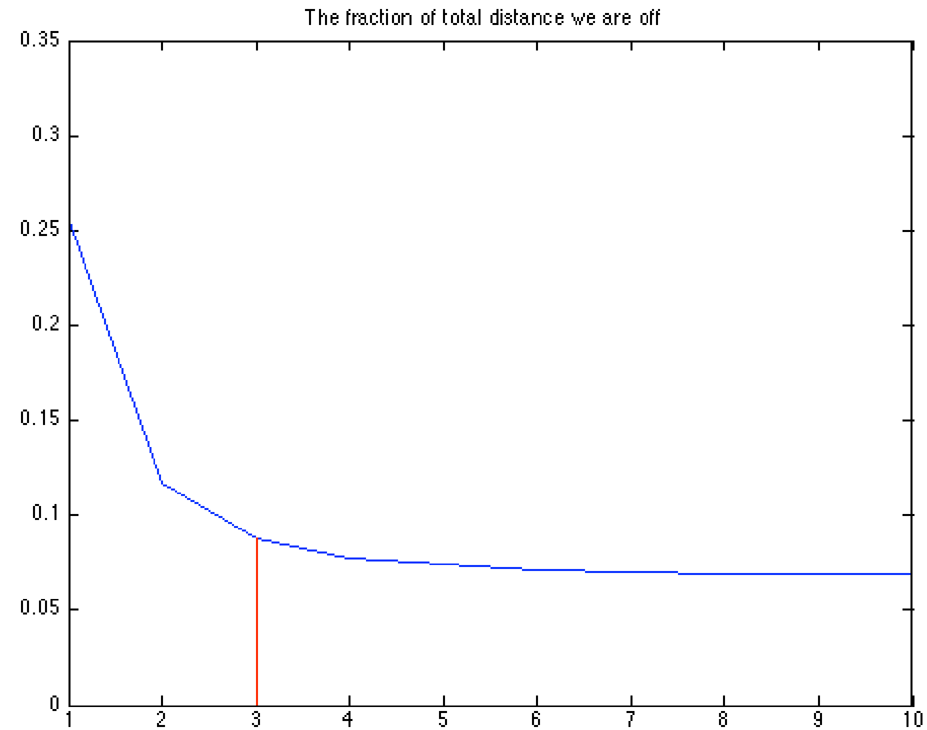
Indeed it is hyperbolic



(open UniformHyperbolicChain.fig)

MDS quality as a function of dimension





Hilbert's theorem (1901) states that there exists no complete regular surface S of constant negative Gaussian curvature K immersed in R^3. This theorem answers the question for the negative case of which surfaces in R^3 can be obtained by isometrically immersing complete manifolds with constant curvature.

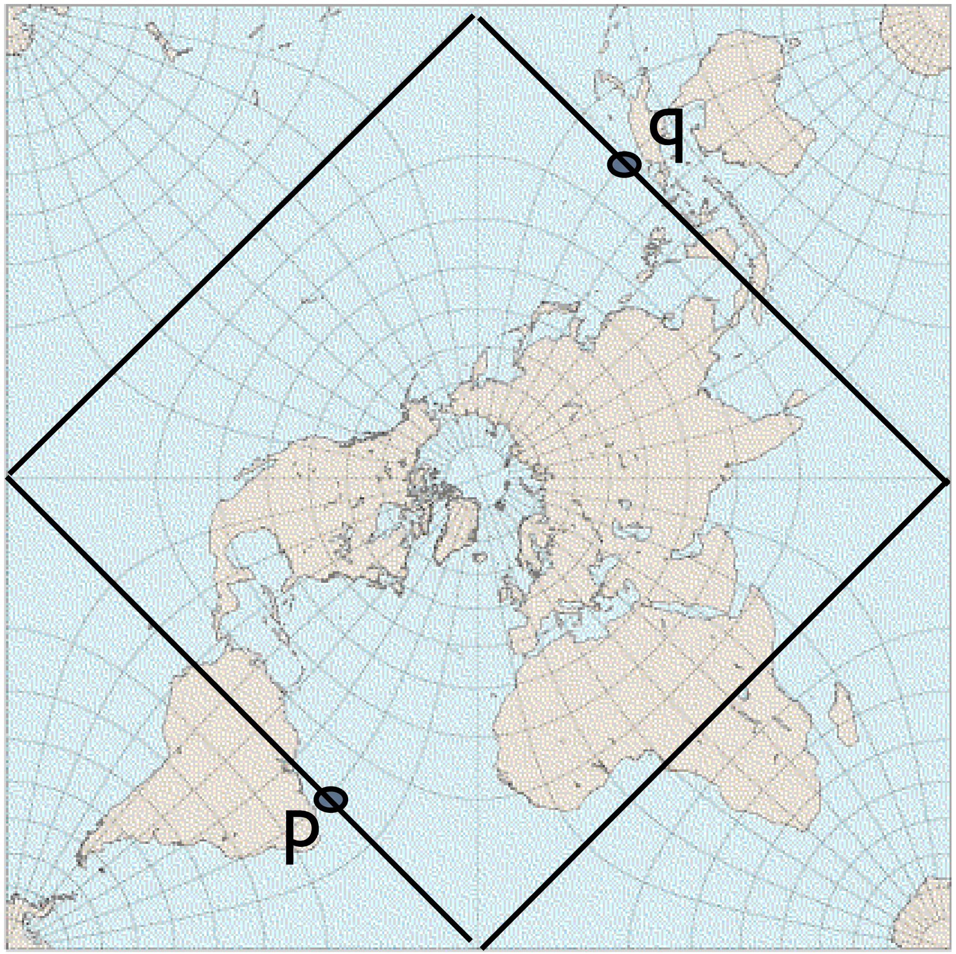
Compare Nash's Embedding theorem

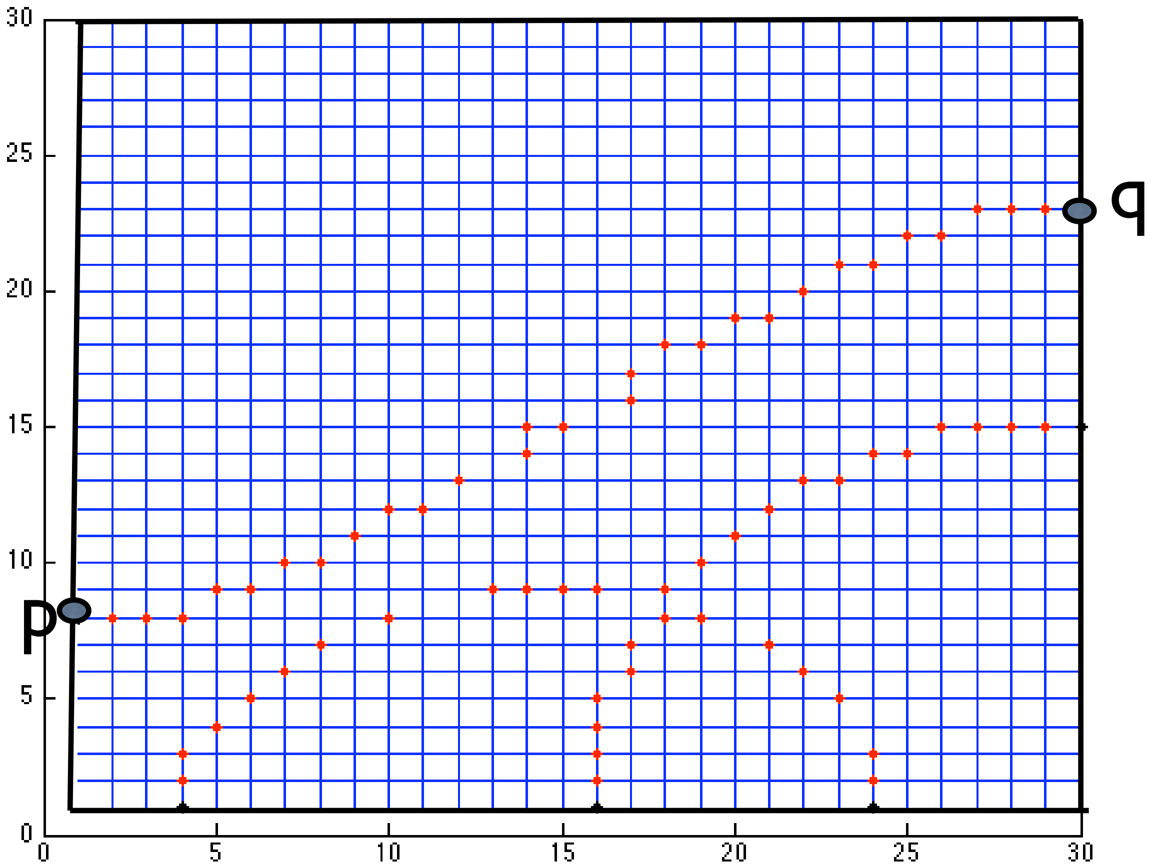
$$(p(a,b),q(a,b)) = argmax(max_{p,q\in A}[a;b;p;q]).$$

oriented geodesic current

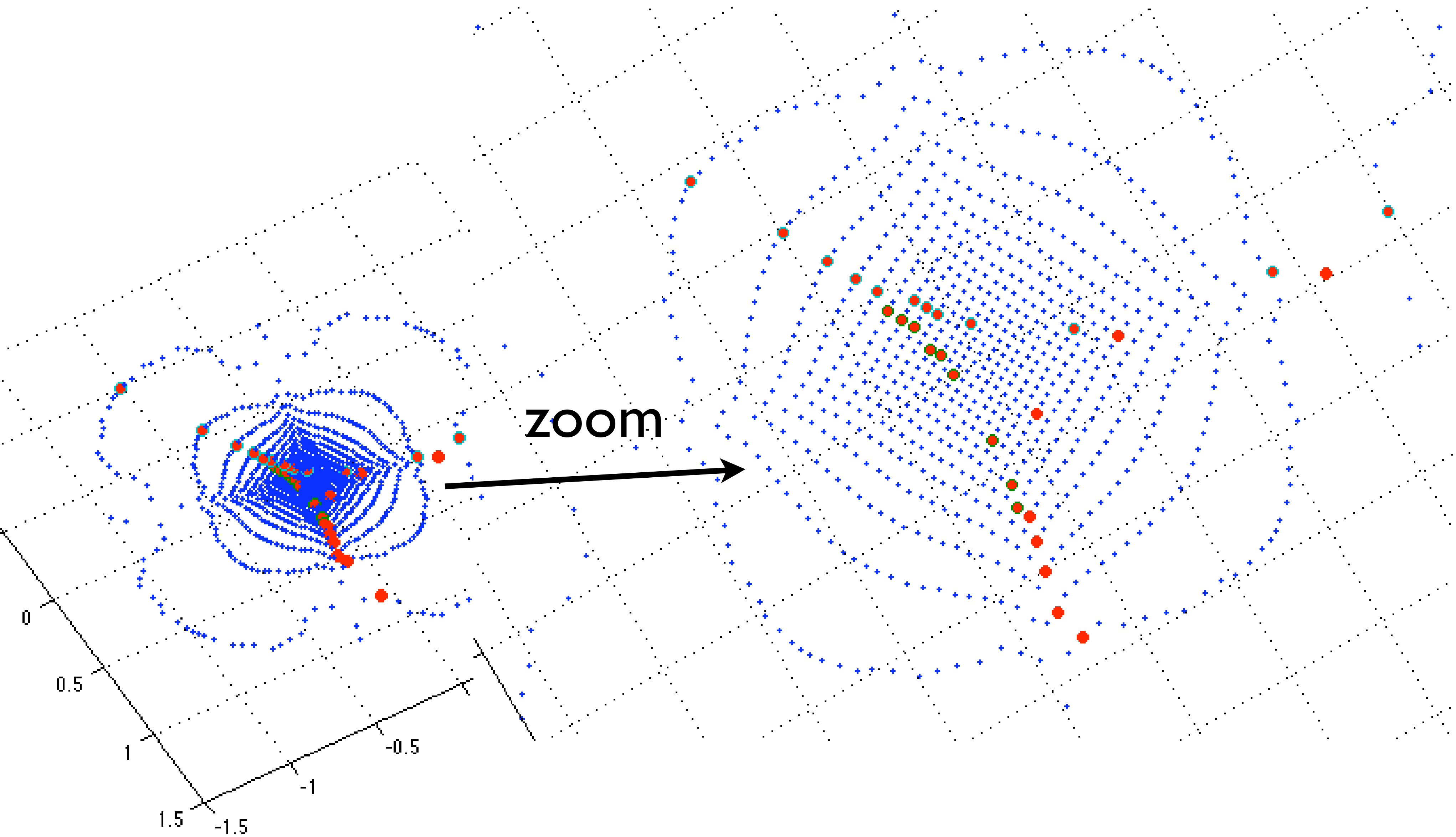
$$g_{p,q} = \{(a,b) \mid (p(a,b),q(a,b)) = (p,q)\}$$

quincuncial
Charlse Peirce (1879)



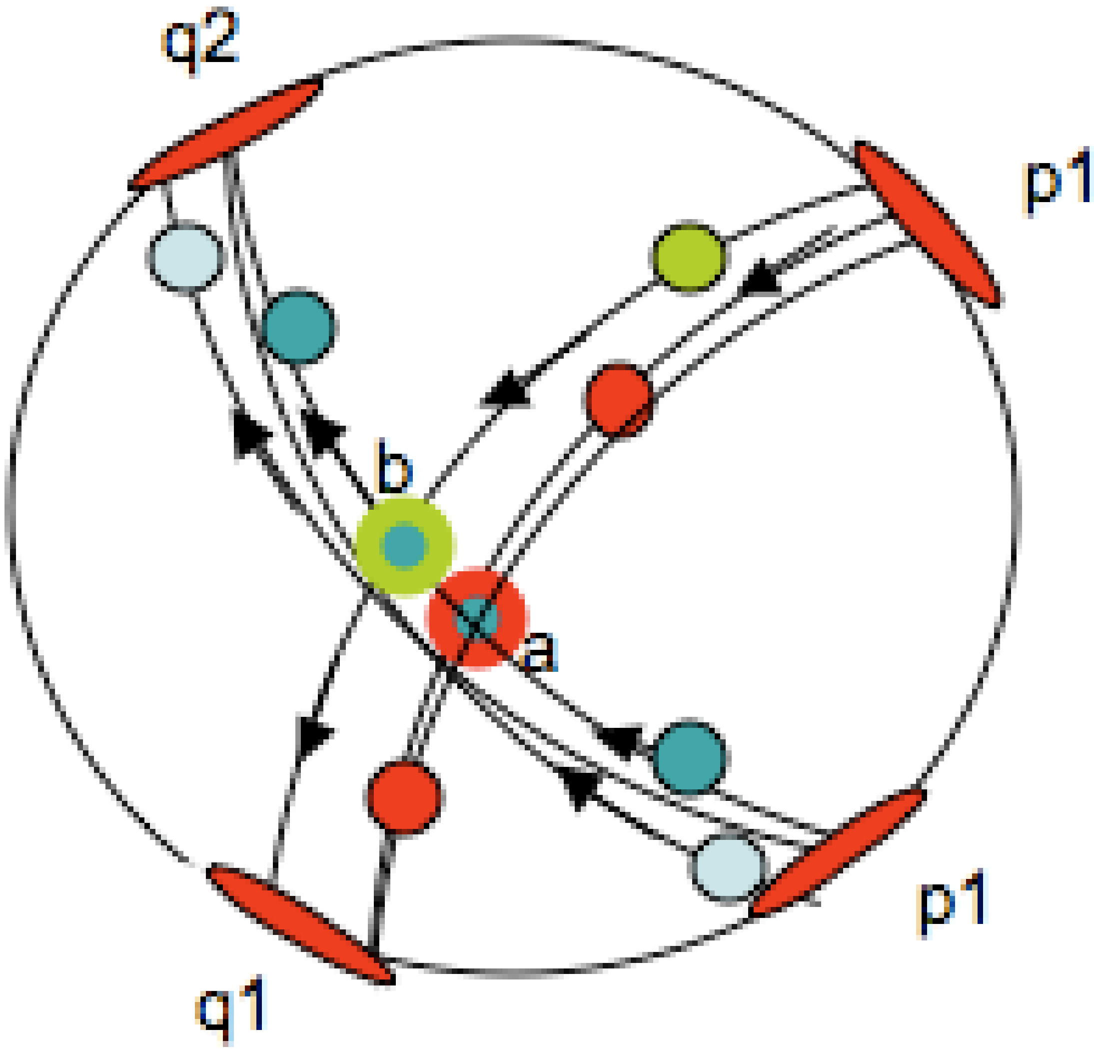


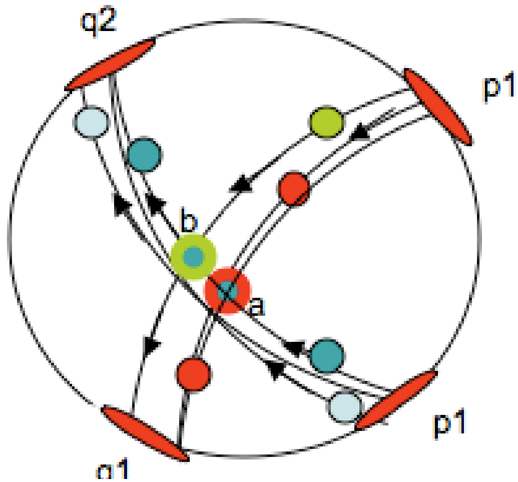
Geodesic Current Lemma If $(a,b) \in g_{p,q}$ and $(b,c) \in g_{p,q}$, then $(a,c) \in g_{p,q}$.



open ThreeCentralGeodescis.fig

Geodesic Currents





Geodesic Current Consistency Lemma If $(a,b) \in g_{p,q}$ and $(b,c) \in g_{p,q}$, then $(a,c) \in g_{p,q}$.

proof: If (a,c) is not in $g_{p,q}$, then (p,q) is not in $argmax(max_{p_1,q_1 \in A}[p_1,q_1;a,c])$ and there exist (p_0,q_0) such that

$$[p,q;a,c] < [p_0,q_0;a,c].$$

As in the proof of the hyperbolic metric lemma, we have

$$[p_0,q_0;a,c] = [p_0,q_0;a,b] + [p_0,q_0;b,c]$$

which by the definition of the distance must satisfy

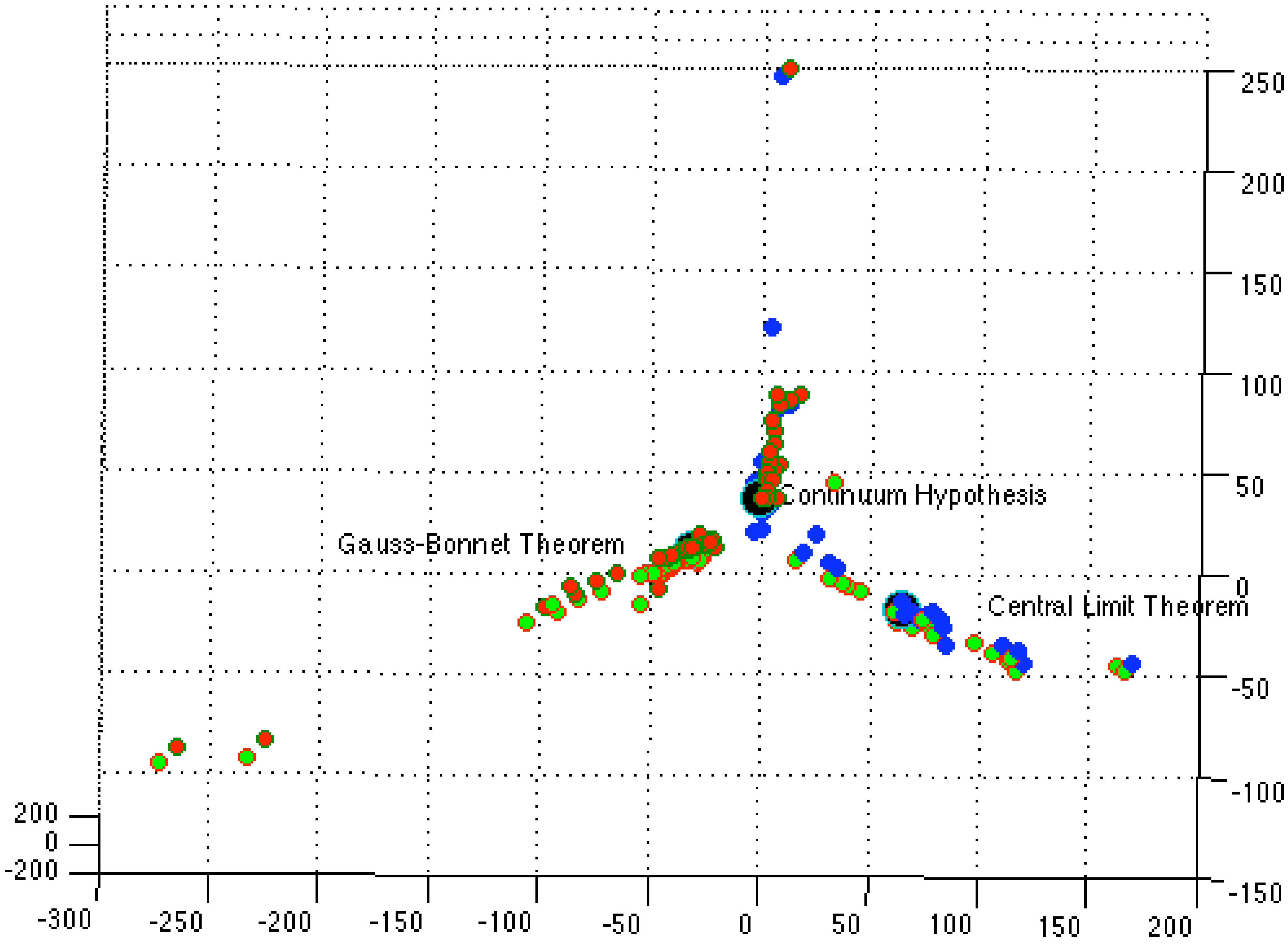
$$[p_0,q_0;a,b] + [p_0,q_0;b,c] <= d(a,b) + d(b,c) = [p,q;a,b] + [p,q;b;c],$$

which, in turn, satisfies

$$[p,q;a,b] + [p,q;b;c] = [p,q;a,c].$$

So in the end assuming (a,c) is not in $g_{p,q}$ results in $[p,q;a,c] < [p,q;a,c]$, a contradiction. **Q.E.D**

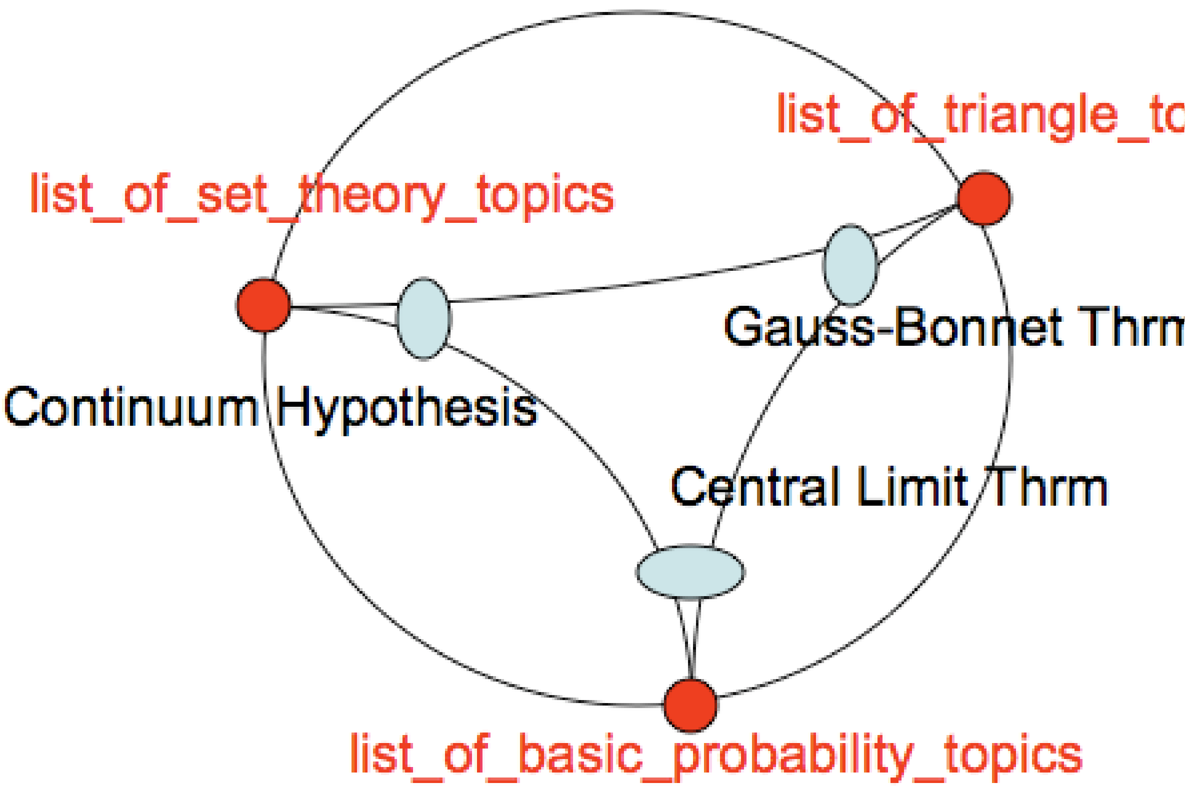
Geodesic from 'central_limit_theorem' to 'continuum_hypothesis'	
Start at top	Continue to end
'central_limit_theorem'	'effective_descriptive_set_theory'
'stretched_exponential_function'	'small_veblen_ordinal'
'monty_hall_problem'	'feferman%e2%80%93sch%c3%bctte_ordinal'
'spoof_(game)'	'ackermann_ordinal'
'probability_surveys'	'diamond_principle'
'studia_mathematica'	'grzegorczyk_hierarchy'
'montgomery%27s_pair_correlation_conjecture'	'codomain'
'entropy_power_inequality'	'conference_board_of_the_mathematical_sciences'
'president_of_the_institute_of_mathematical_statistics'	'finite'
'factorization_lemma'	'jensen%27s_covering_theorem'
'fuzzy_measure_theory'	'dynkin_system'
'pi_system'	'balls_and_vase_problem'
'shattering'	'large_veblen_ordinal'
'theory_of_conjoint_measurement'	'computable_real_function'
'linear_partial_information'	'church%e2%80%93kleene_ordinal'
'latin_square_property'	'ramified_forcing'
'm._riesz_extension_theorem'	'dynkin%27s_lemma'
'covering_theorem'	'hilbert%27s_paradox_of_the_grand_hotel'
'transitivity_(mathematics)'	'ordinal_notation'
'operation_(mathematics)'	'ontological_maximalism'
	'continuum_hypothesis'

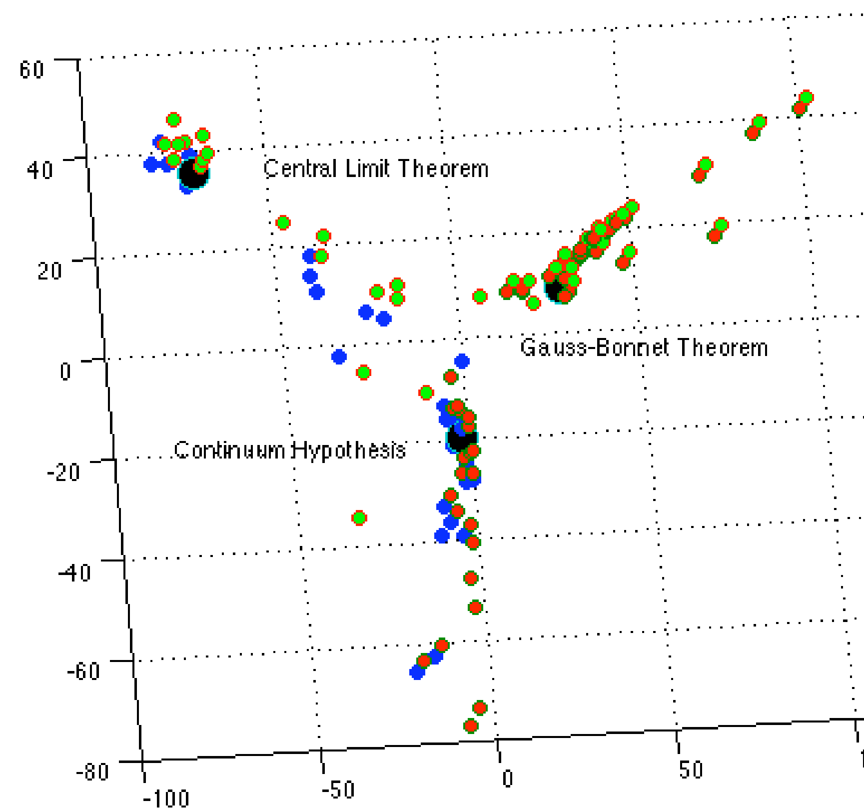


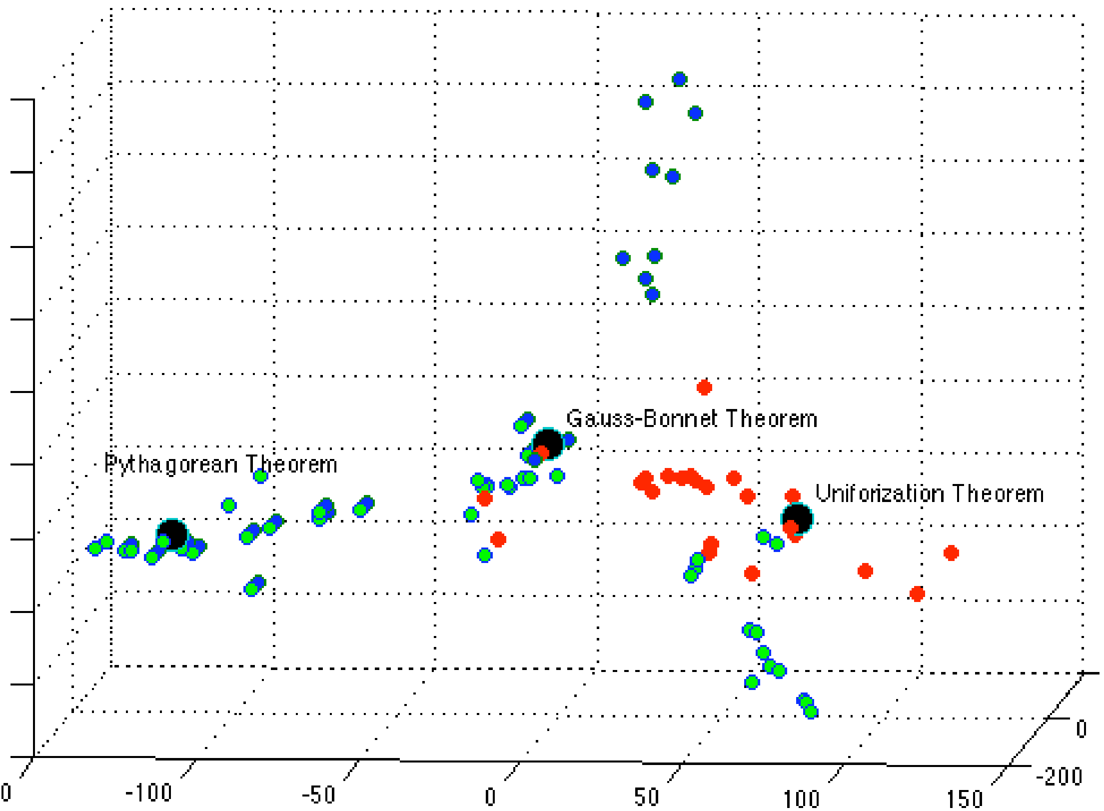
Continuum Hypothesis
Gauss-Bonnet Theorem
Central Limit Theorem
250
200
150
100
50
0
-50
-100
-150
200
0
-200
-300
-250
-200
-150
-100
-50
0
50
100
150
200

“There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.”

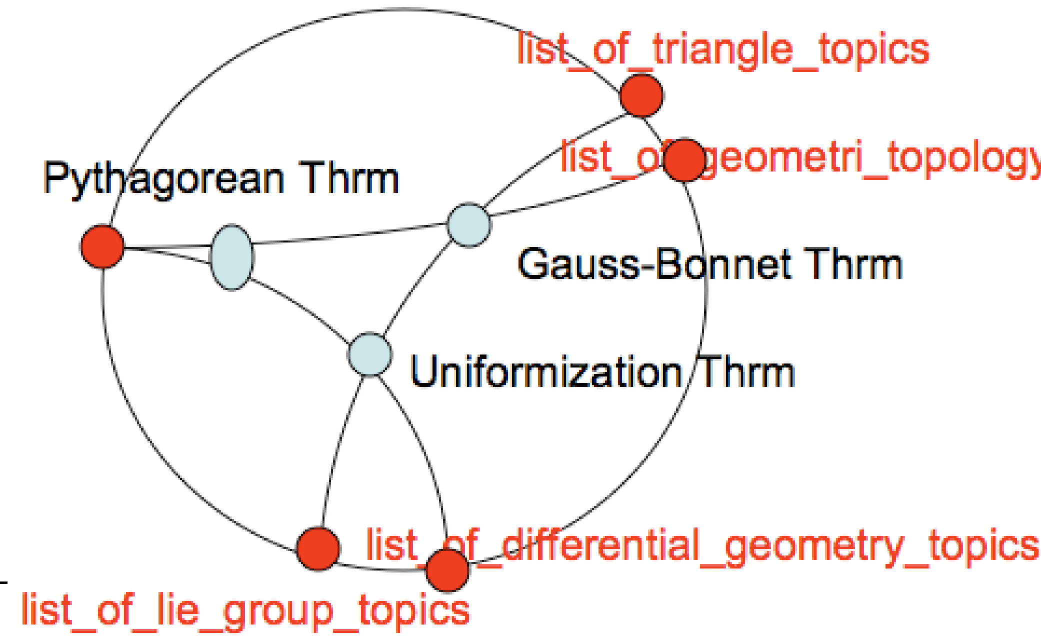
- Nikolai Lobachevsky







Pythagorean Theorem
Gauss-Bonnet Theorem
Uniforization Theorem
-100
-50
0
50
100
150
0
-200



list_of_triangle_topics
Pythagorean Thrm
list_of_geometri_topology
Gauss-Bonnet Thrm
Uniformization Thrm
list_of_differential_geometry_topics
list_of_lie_group_topics

“I have not had a moment's peace or happiness in respect to electromagnetic theory since November 28, 1846. All this time I have been liable to fits of ether dipsomania, kept away at intervals only by rigorous abstention from thought on the subject.”

-Lord Kelvin (to FitzGerald 1896)

* **Dipsomania** is a term USUALLY related to an incontrollable craving for alcohol.... the obsession is so compulsive that the dipsomanic will ingest whatever intoxifying liquid is at hand, whether it is fit for consumption or not. Dipsomania differs from alcoholism in that it is an uncontrollable periodic lust for alcohol, with, in the interim, no desire for alcoholic beverages.

Ether Dipsomania is a term related to an incontrollable craving for a consistent and appealing theory of something in the form of an analogon to the theory of electromagnetism...the obsession is so compulsive that the ether dipsomaniac will....

Kakawa's Salted Carmel time!

Do you have a boundary?