Yesterday was mostly about maps.

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation
Next: flows.

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations

Attractors

- Attractors exist only in dissipative systems!
- Dissipation → contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic attractors
Conditions for chaos in continuous time systems:

**Necessary:**
- Nonlinear
- At least three state-space dimensions

**Necessary and sufficient:**
- “Nonintegrable”
- i.e., cannot be solved in closed form

Concepts: review
- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter
Deterministic Nonperiodic Flow

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Manuscript received 18 November 1962, in revised form 7 January 1963

ABSTRACT

Unlike systems of deterministic ordinary differential equations, which are designed to represent bounded solutions, solutions of these equations can be identified with trajectories to extreme limit points. For these systems with bounded solutions, it is found that nonperiodic solutions are generally unstable with respect to small perturbations, so that slightly differing initial states can result in extreme differences. Systems with bounded solutions are shown to possess bounded, numerical solutions. A simple system representing cellular oscillation is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very long-range weather prediction is examined in the light of these results.

J. Atmos. Sci. 26:130
• Equations:

\[ x' = a(y-x) \]
\[ y' = rx -y -xz \]
\[ z' = xy - bz \]

(first three terms of a Fourier expansion of the Navier-Stokes eqns)
• State variables:
  ▪ $x$ convective intensity
  ▪ $y$ temperature
  ▪ $z$ deviation from linearity in the vertical convection profile

• Parameters:
  ▪ $a$ Prandtl number - fluids property
  ▪ $r$ Rayleigh number - related to $\Delta T$
  ▪ $b$ aspect ratio of the fluid sheet
\[ x' = a(y-x) \]
\[ y' = rx - y - xz \]
\[ z' = xy - bz \]

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Attractors

Four types:
- fixed points
- limit cycles (aka periodic orbits)
- quasiperiodic orbits
- chaotic attractors

A nonlinear system can have any number of attractors, of all types, sprinkled around its state space
Their basins of attraction partition the state space
And there’s no way, a priori, to know where they are, how many there are, what types, etc.
Attractors:

• Fixed point

Attractors:

• Limit cycle

e.g., wall voltage!
Attractors:

• Quasi-periodic orbit…

“Strange” or chaotic attractors:

• often fractal
• covered densely by trajectories
• exponential divergence of neighboring trajectories…
Lyapunov exponents:

- positive $\lambda$ is a signature of chaos
- negative $\lambda$ compress state space; positive $\lambda$ stretch it
- nonlinear analogs of eigenvalues: one $\lambda$ for each dimension
- $\Sigma \lambda < 0$ for dissipative systems
- long-term average in definition; biggest one dominates as $t \to \infty$
- $\lambda$ are same for all ICs in one basin

“Strange” or chaotic attractors:

- exponential divergence of neighboring trajectories
- often fractal
- covered densely by trajectories
- contain an infinite number of “unstable periodic orbits”…
Unstable periodic orbits (UPOs):
Attractor “bones”…

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Poincare recurrence

Crutchfield et al.
Chaos 255:46