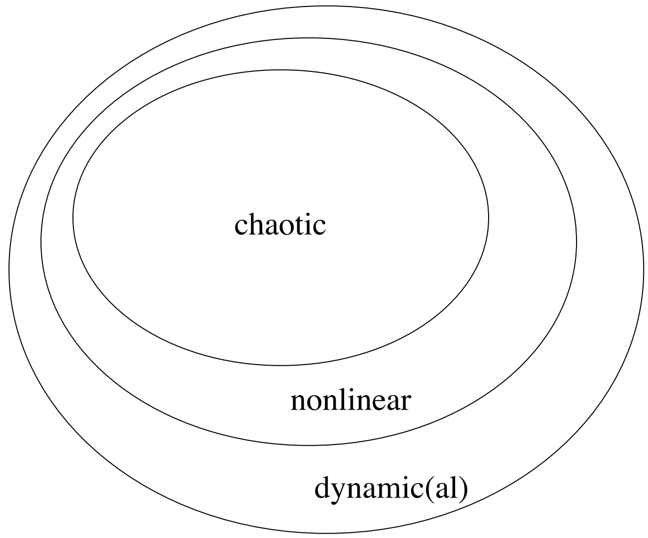


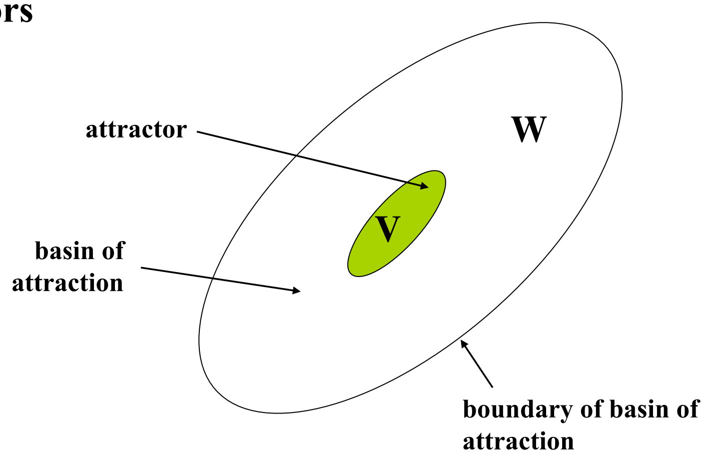
Yesterday was mostly about *maps.*

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation

Next: *flows.*

- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differential equations

Attractors



- Attractors exist only in dissipative systems!
- Dissipation ⟺ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

Conditions for chaos in continuous time systems:

Necessary:

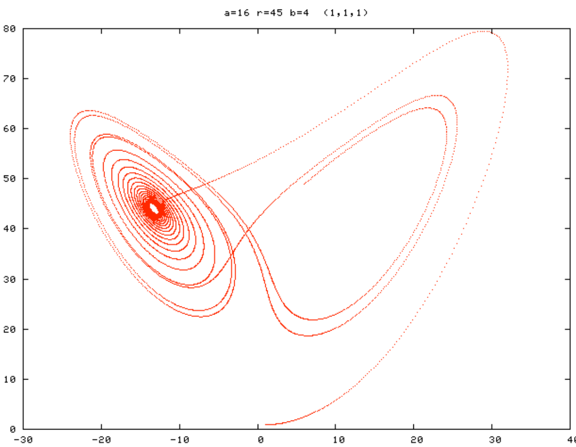
- Nonlinear
- At least three state-space dimensions

Necessary and sufficient:

- “Nonintegrable”
- i.e., cannot be solved in closed form

Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter



Deterministic Nonperiodic Flow[1]

Edward N. Lorenz

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

Abstract

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

J. Atm. Sci. **20:**130

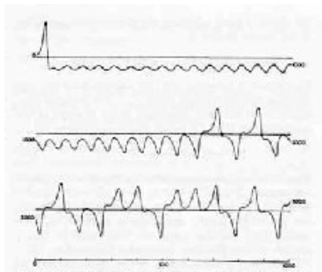


Fig. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

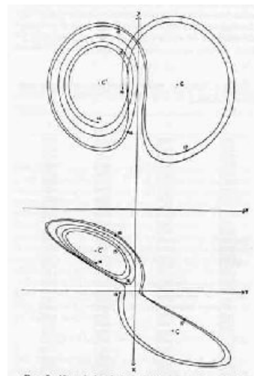


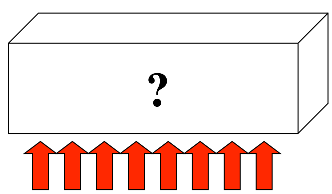
Fig. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

- Equations:

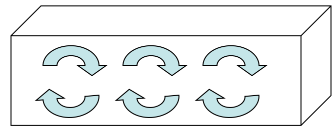
$x' = a(y-x)$

$y' = rx - y - xz$

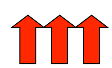
$z' = xy - bz$

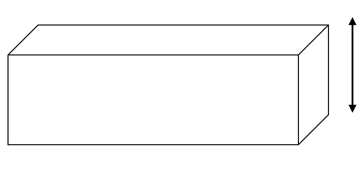


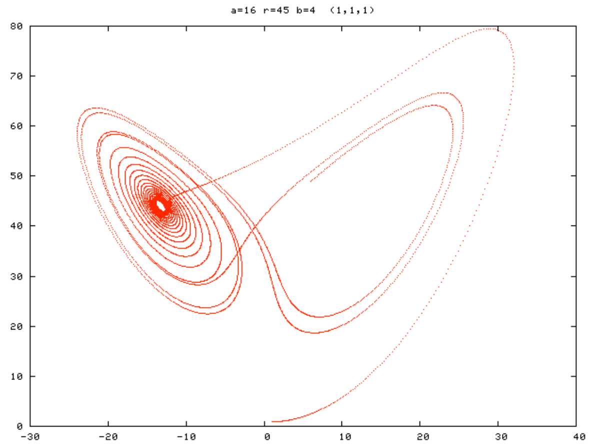
(first three terms of a Fourier expansion of the Navier-Stokes eqns)



- State variables:
  - x convective intensity
  - y temperature
  - z deviation from linearity in the vertical convection profile

- Parameters:
  - a Prandtl number - fluids property
  - r Rayleigh number - related to ΔT 
  - b aspect ratio of the fluid sheet

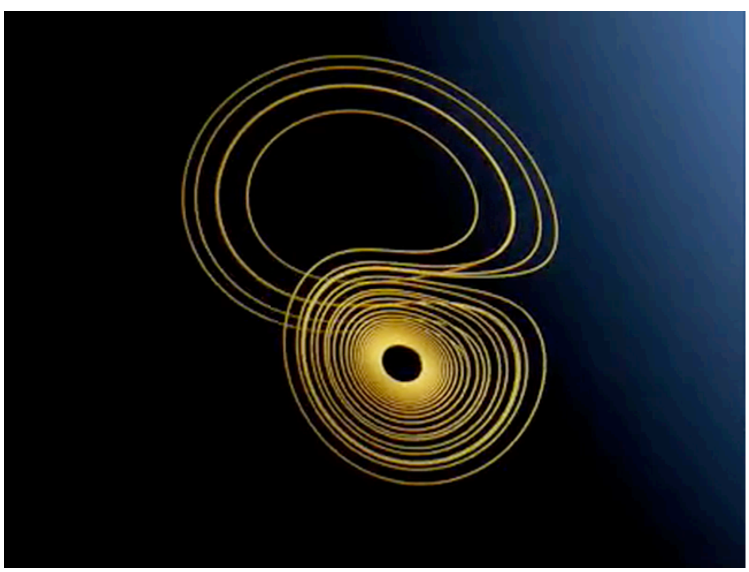


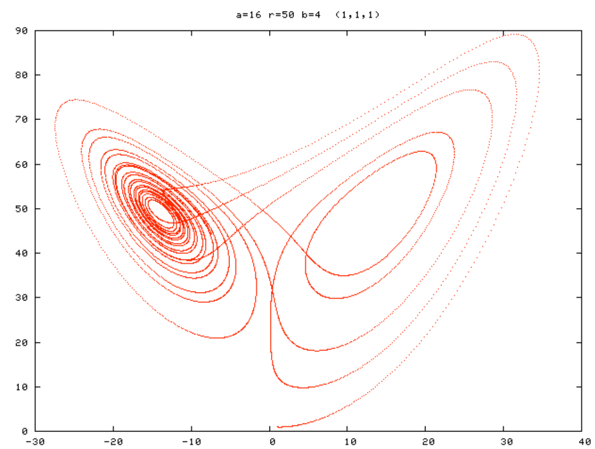


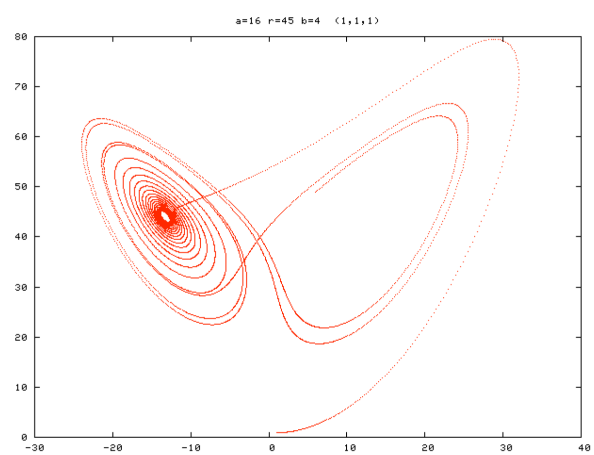
$x' = a(y-x)$

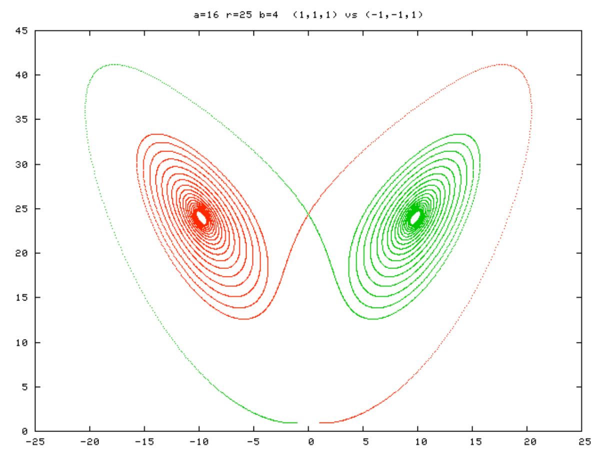
$y' = rx - y - xz$

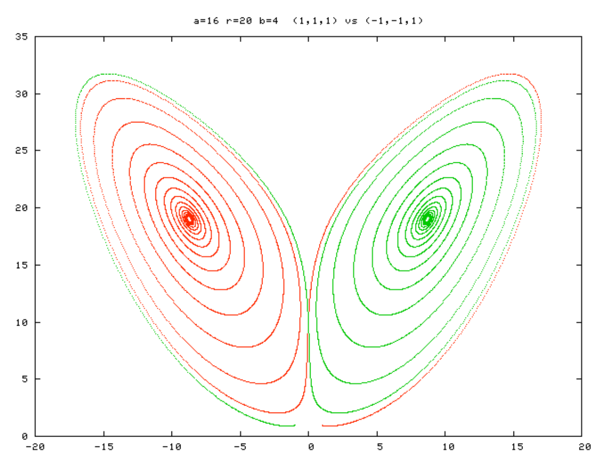
$z' = xy - bz$

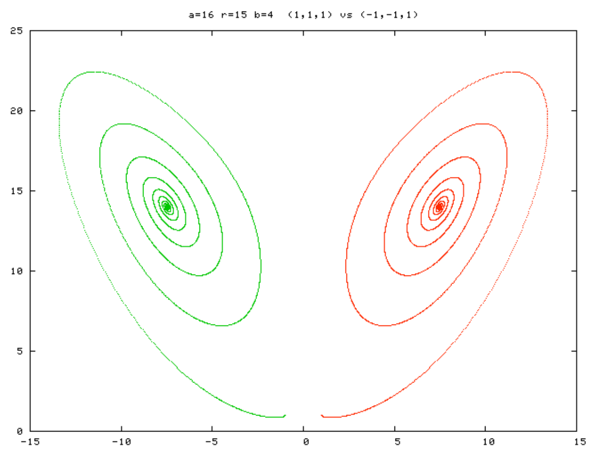



a=16 r=50 b=4 (1,1,1)
90
80
70
60
50
40
30
20
10
0
-30
-20
-10
0
10
20
30
40


a=16 r=45 b=4 (1,1,1)
80
70
60
50
40
30
20
10
0
-30
-20
-10
0
10
20
30
40


a=16 r=25 b=4 (1,1,1) vs (-1,-1,1)


a=16 r=20 b=4 (1,1,1) vs (-1,-1,1)



Attractors

<u>Four types:</u>

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

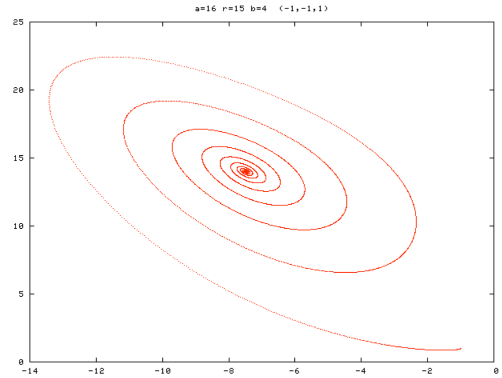
A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc.

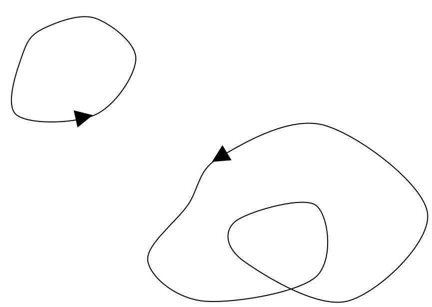
Attractors:

- Fixed point



Attractors:

- Limit cycle

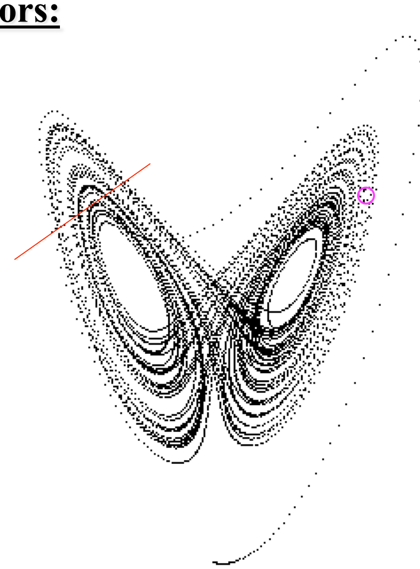


e.g., wall voltage!

Attractors:

• Quasi-periodic orbit…

“Strange” or chaotic attractors:



• *often* fractal

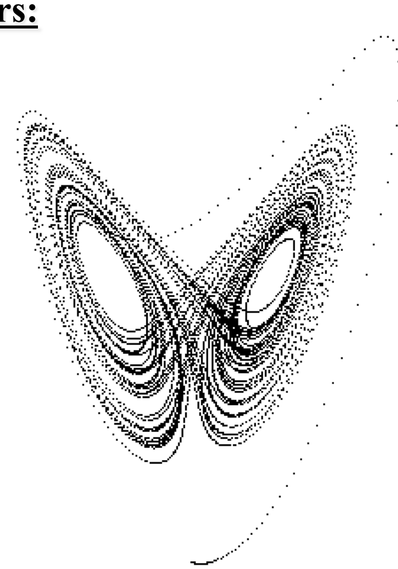
• covered densely by trajectories

• exponential divergence of neighboring trajectories…

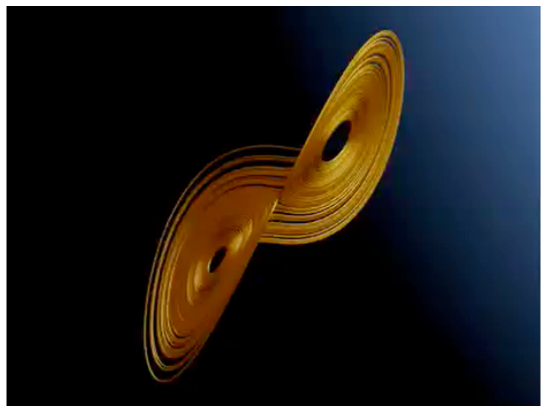
Lyapunov exponents:

- *positive λ is a signature of chaos*
- negative λ compress state space; positive λ stretch it
- nonlinear analogs of eigenvalues: one λ for each dimension
- $\Sigma\lambda < 0$ for dissipative systems
- long-term average in definition; biggest one dominates as t → infinity
- λ are same for all ICs in one basin

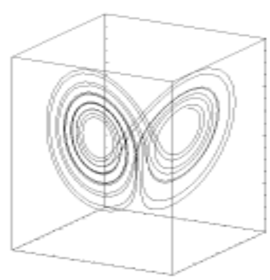
“Strange” or chaotic attractors:

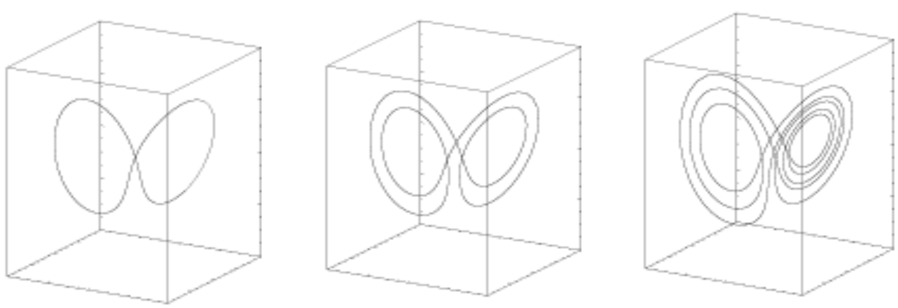


- exponential divergence of neighboring trajectories
- *often* fractal
- covered densely by trajectories
- contain an infinite number of “unstable periodic orbits”…

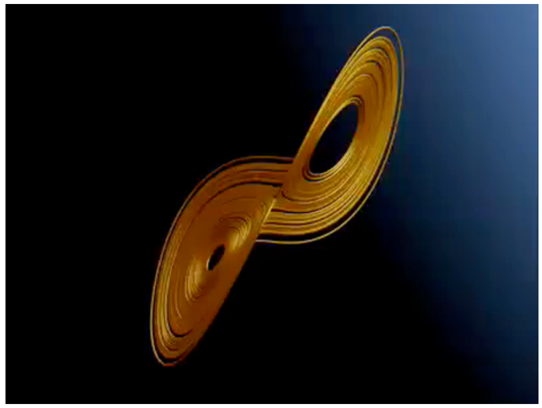


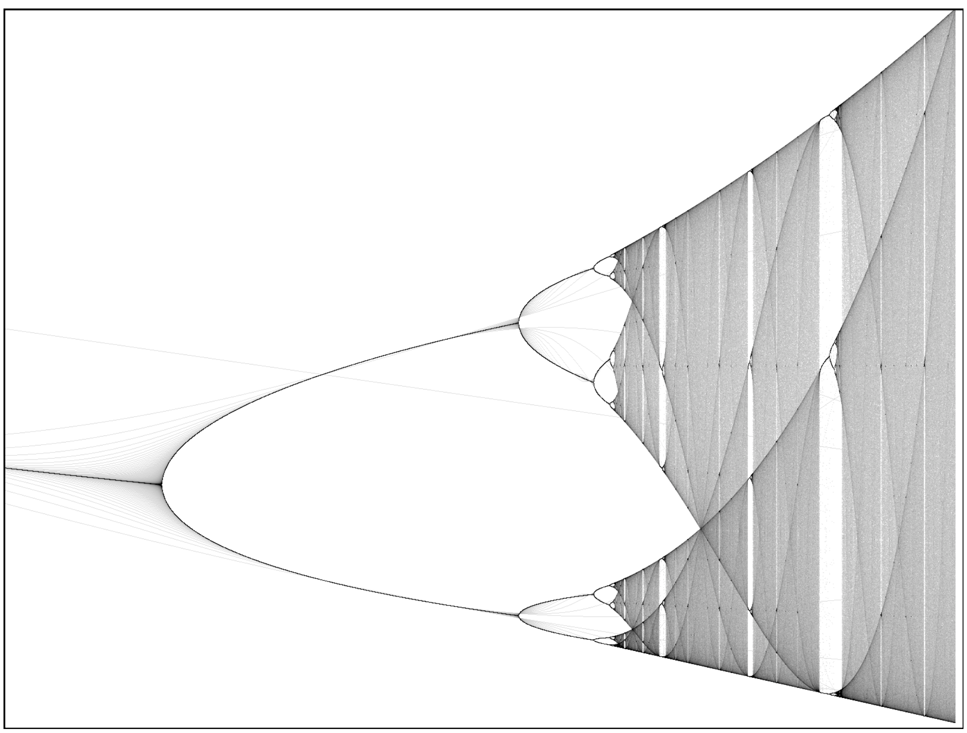
Unstable periodic orbits (UPOs):



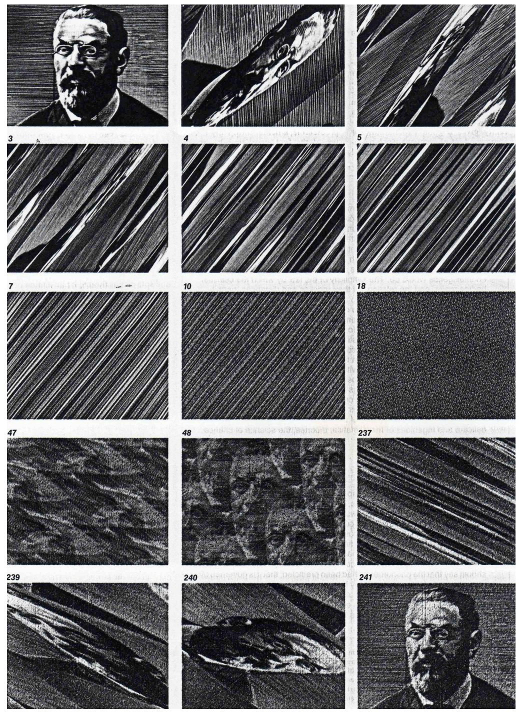


Attractor "bones"…





Poincare recurrence



Crutchfield *et al*.
Chaos **255:**46