Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Summer School

June 2017

Liz Bradley

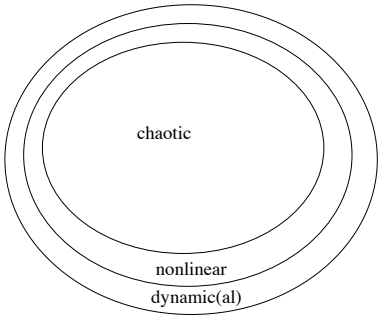
lizb@cs.colorado.edu

http://ayresriverblog.com

Chaos

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…



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Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

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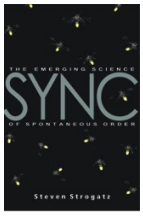
Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and "well-understood"

Where nonlinear dynamics turns up

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers

Where nonlinear dynamics turns up

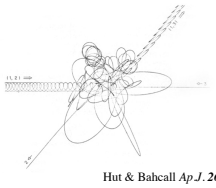
- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies



- and lots of other electronic, chemical, & biological systems

Where nonlinear dynamics turns up

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….



Hut & Bahcall *Ap J.* ***268****:319*

- Protein folding
- Population biology
- And many, many other fields (including yours)

- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: difference equation
- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differential equations

A useful graphical solution technique

- "cobweb" diagram
- *aka* return map
- *aka* correlation plot

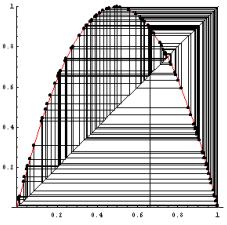
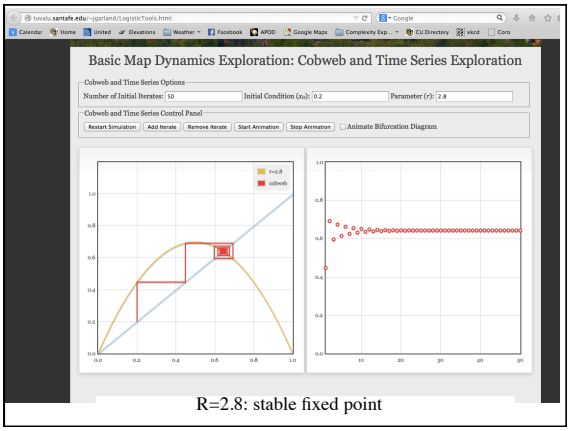


Image from Doug Ravenel's website at URochester

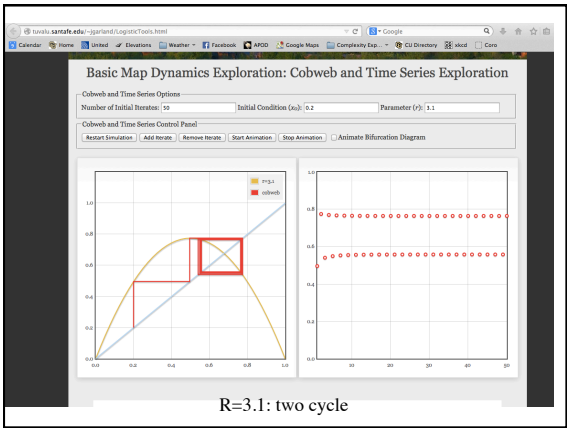


R=2.8: stable fixed point

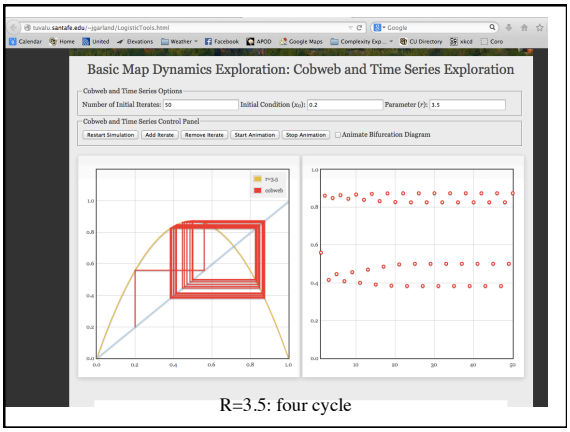
Bifurcations

Qualitative changes in the dynamics caused by changes in *parameters:*

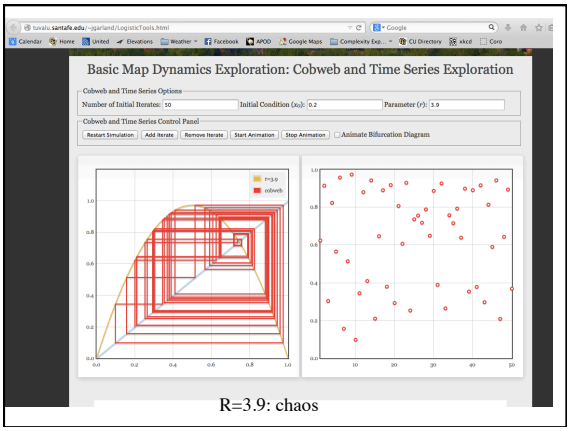
- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- Logistic map: R parameter…



R=3.1: two cycle



R=3.5: four cycle



R=3.9: chaos

Chaos

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- sensitive dependence on initial conditions — **SDOIC**
- characteristic structure…

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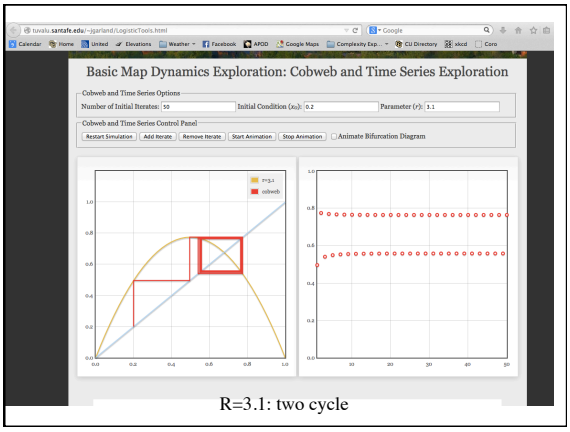
Showing all of that on one plot: the "bifurcation diagram"…



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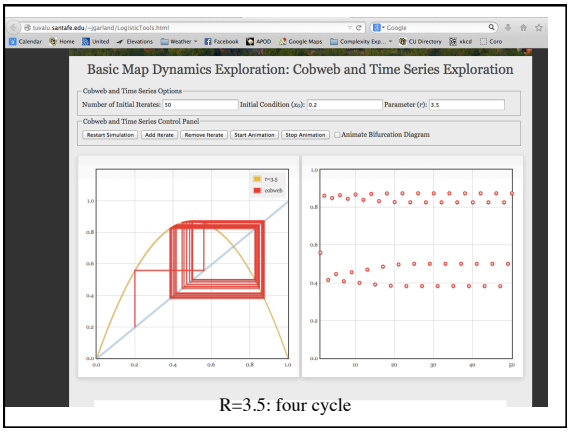

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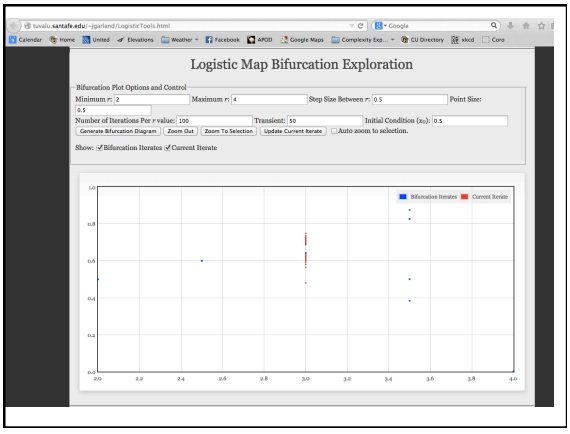

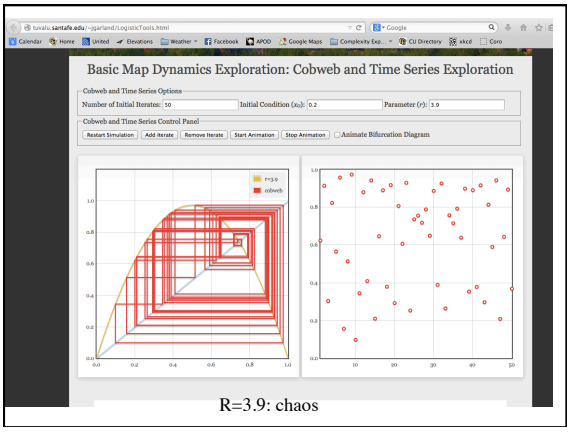

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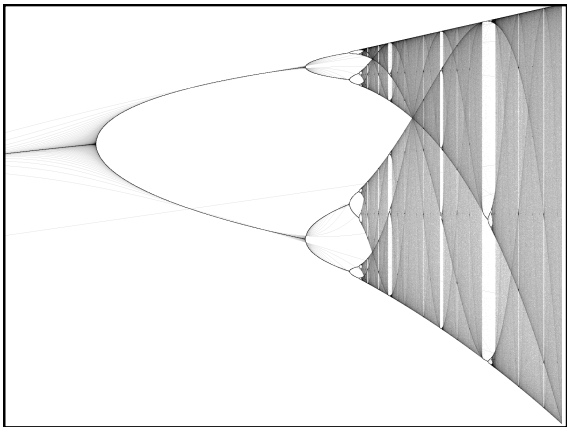
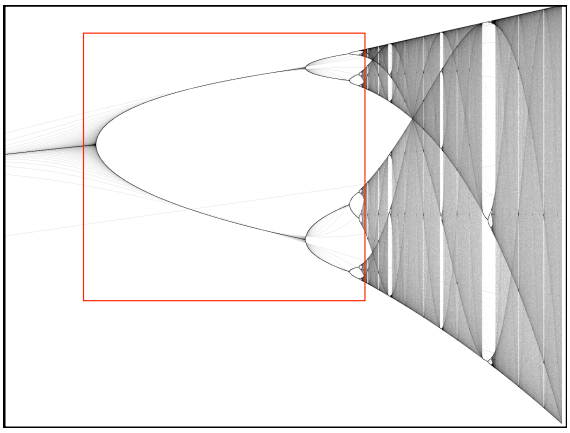
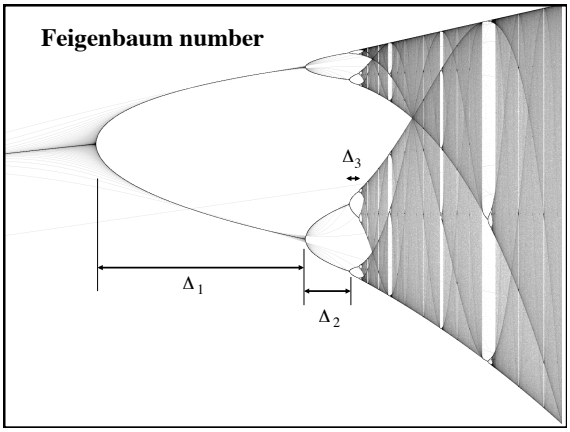


- chaos
- veils/bands: places where chaotic attractor is dense



- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- *period-doubling cascade @ low R*

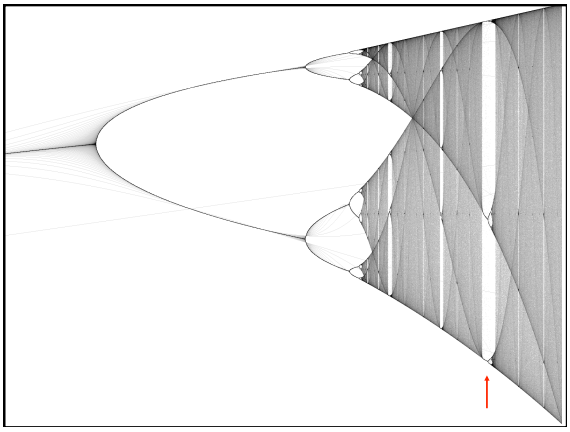


Universality!

Feigenbaum number and many other interesting chaotic/dynamical properties hold *for any 1D map with a quadratic maximum*.

Proof: renormalizations. See Strogatz §10.7

Don't take this too far, though…

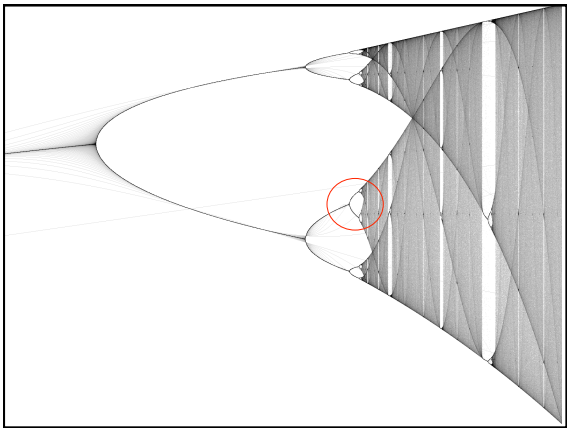


- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- *windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)*

There's something very special about 3…

- Sarkovskii (1964)

  3, 5, 7, …3x2, 5x2, …$3x2^2$, $5x2^2$, … 2^2, 2, 1
- Yorke (1975)
- Metropolis *et al.* (1973)



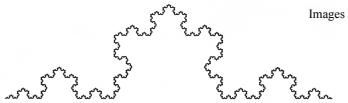
- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
- *small copies of object embedded in it (fractal)*

(lots of other interesting stuff, too — e.g., Misiurewicz points)

Fractals

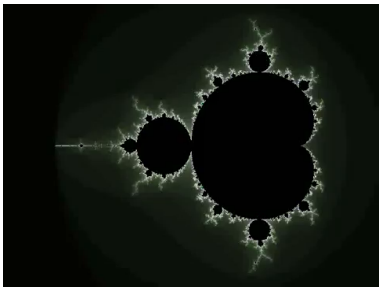
- non-integer Hausdorff dimension
- self-similar

Images from Gleick



Canonical example: the Cantor set!

Another canonical example: the Mandelbrot set



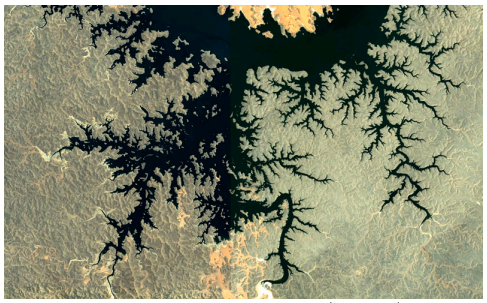
www.youtube.com/watch?v=G_GBwuYuOOs

Fractals basins and basin boundaries

Newton's method on $x^4 - 1 = 0$

From Strogatz

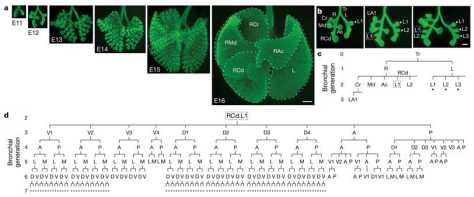
Fractals in the wild



paulbourke.net/fractals/googleearth/

See also: coastlines, trees, lungs, clouds, snowflakes …

~23 generations of branches in the mammalian lung

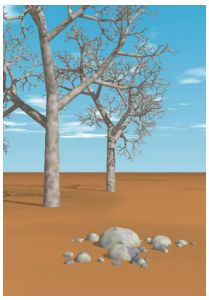


2^{23} ~ 8 million branching events!
how does the epithelium fold 8 million times?

Metzger et al, Nature, 2008

Slide courtesy of Celeste Nelson

Fractals in computer graphics



Matthew Ward, WPI
davis.wpi.edu/~matt/courses/fractals/trees.html

Fractals and chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.

But **not** "*all.*"