Emergent Behavior of Rock-Paper-Scissors Game

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Abstract

When rock-paper-scissors game is played by many people, there is an emergent behavior. Analytic model suggests that there are three phases, and computer simulation shows that phases are determined by how local interactions are. In this paper, analytic and computational model of RPS game, as well as applications to biological systems (E.coli and side-blotched lizards) are discussed.

INTRODUCTION

Rock-paper-scissors game, a game that is often played by children, has a simple rule; Rock wins against scissors, scissors win against paper, and paper wins against rock. If this game is done by many people over and over again, this simple interaction between people suggests a possibility of an emergent behavior. In particular, it has been of interest for many people when the game is played as follows; each person picks one strategy from rock, paper, and scissors randomely at the beginning → they randomely choose one person to play RPS game with → they keep their strategy if they win or tie, but change the strategy to the opponent's strategy if they lose → find the next opponent and repeat.

Of course it is unlikely that people play RPS game in this manner, but this situation is observed in some biological systems, such as in the system of E.coli[5], and in the system of side-blotched lizards[6]. In both cases, three different types of the same spieces have cyclic dominance, and a loser is displaced by a winner, through the fact that a winner can have offsprings but a loser cannot.

The above biological examples being part of the motivation, RPS game model was studied by several people. It turns out that a simple theory suggests that there are three different stable solutions: the solution where everybody has the same strategy, the solution where number of people with each strategy is 1/3 of total, and the solution where the number of people with each strategy oscillates in time[1]. For the first step and perhaps the best step to study this emergent behavior (best step according to V. Darley [4]), computer simulations were done, and succeeded in seeing an emergent behavior.

In this paper, I will discuss a simple analytical and computational model of RPS game that done in the manner described above, and biological systems that have interactions described in this manner.

SIMPLE ANALYTIC MODEL OF RPS GAME

THE GOVERNING EQUATIONS

Consider a system of many particles (or people) that can have one of three different strategies, S_1, S_2, or S_3 (which correspond to rock, paper and scissors), and those three strategies have a cyclic dominance $S_1 > S_2 > S_3 > S_1$.

Further, assume that when two particles encounter, the winner displaces the loser, hence the number of the winner strategy goes up by one and the number of the loser strategy goes down by one in this process.

Therefore, if we focus on one strategy, S_1 for instance, having many S_2 in the system increases the number of S_1, and having many S_3 reduces its number. In other words, S_1 gets successful with having S_2 around, and unsuccessful with having S_3 around.

In order to understand the collective behavior of the system, we will define 'fitness' f_i, which is a measure of how successful streategy i is in the system. If x_i is a density of the strategy i, then the rate of change in x_i, in terms of f_i, can be written as

$$\frac{\dot{x}_i}{x_i} = f_i(\mathbf{x}) - \bar{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\bar{f}(\mathbf{x})$ is the average fitness

$$\frac{\sum_i x_i f_i}{\sum_i x_i} \tag{2}$$

[1]. For convenience, normalization $\sum_i x_i = 1$ will be used from now on (so now x_i represents the ratio of the nuber of S_i to the total nuimber). The equation (1) shows that the number of S_i grows if its fitness f_i is greater than the average (i.e. if it is successful), and vice versa.

Fitness $f_i(\mathbf{x})$ certainly would depend on x_1, x_2, x_3, but it is not clear how it depends on them. As a simple model, we will assume that $f_i(\mathbf{x})$ is linear in $x_j, \forall j$. Then there exists a matrix $\mathbf{A}$ such that

$$f_i(\mathbf{x}) = A_{ij} x_j, \tag{3}$$

where Einstein convention is used[1]. Recall that fitness f_i is a measure of how successful S_i is, so A_{ij} should be positive if having S_j around S_i is comfortable for S_i (i.e. if i wins against j), and negative if it is the other way around, and how successful/unsuccessful it is should be represented by the magnitude of A_{ij}. In the case of the regular RPS game, it is comfortable for S_1 (rock) if there are S_2 (scissors) around, and uncomfotable with the same degree if there are S_3 (papers). Hence

$$A_{12} = k$$
$$A_{13} = -k$$

where k is some real number. Following the same analysis for all components, the whole matrix A can be written as

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}. \tag{4}$$

Note that $k = 1$ is used in normalization.

With this matrix $\mathbf{A}$, equation (1) becomes

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 - x_3) \\ \dot{x}_2 &= x_2(x_3 - x_1) \\ \dot{x}_3 &= x_1(x_1 - x_2). \end{aligned} \tag{5}$$

These three equations can now be solved to yield the behavior of RPS game.

SOLUTIONS

Looking at the equation (5), it is easy to see two stable solutions, where $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$:

$$x_i = 0, x_j = 0, x_k \neq 0 \; (i \neq j \neq k) \tag{6}$$

and

$$x_1 = x_2 = x_3. \tag{7}$$

The more general solution of x_1, x_2, x_3 in the equations (5) can be seen most conveniently by considering the sum $x_1 + x_2 + x_3$ and the product $x_1x_2x_3$. Using the equations (5), it is easy to show that

$$\frac{d}{dt}(x_1 + x_2 + x_3) = 0 \tag{8}$$

and

$$\frac{d(x_1x_2x_3)}{dt} = 0. \tag{9}$$

Hence the densities satisfy equations

$$x_1 + x_2 + x_3 = N \tag{10}$$

and

$$x_1x_2x_3 = A, \tag{11}$$

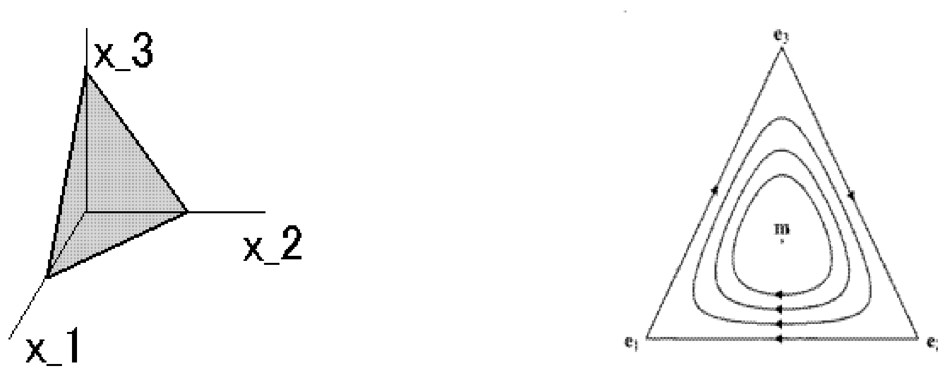

Figure 1: The solution for the equation (5) must lie on the grey plane indicated in the left figure, due to the fact that $x_1 + x_2 + x_3 = constant$. The right figure shows trajectories of the solutions for different N and A on a triangle in the left figure. e_i's correspond to x_i's here. (The picture taken from [1].)

where N and A are some positive constants. If we plot x_1, x_2, x_3 with themselves being the axis, all trajectories have to be on the plane given by the equation (10). Moreover, combining the equation (11) with the equation (10) suggests that there are only two possible values for x_i for each x_j (i.e. eliminating one variable using the two equations gives a quadratic equation for the remaining variables, which can be solved for one of them. It can be shown that two positive solutions exist.) Therefore, for given N and A, possible values of x_1, x_2, x_3 must lie on a closed path shown in Figure 1.

Note that solution (6), (7) and the solution seen in Figure 1 all have completely different features. The solution (6) is the case where everybody has the same strategy and the system stays that way forever (called an absorbing phase[3]). The solution (7) is the case where the densities of strategies is the same for all three kinds (called a self-organizing phase[3]). Lastly, in the final solution (except for the special limits of this solution, which correspond to (6) and (7)), densities of x_1, x_2, x_3 oscillate, preserving the sum and product of densities of all three of them (called an oscillating phase[3]). These are phases of RPS games, and how these phases emerge have been studied with computer simulationtions, as seen in the next section.

COMPUTER SIMULATION OF RPS GAME

A Szalnoki and G Szabo studied phase transitions of RPS games using a computer simulation. The simulation was done on a network that consists of sites where each can occupy one strategy (rock, paper, or scissors), and each site being connected to z other sites. In particular, a phase diagram was made for z=3 honeycomb lattice case, so here we will focus on z=3 case. Starting from a random initial condition, a link between two sites are chosen randomely at each time step and compare the strategies. The winning strategy occupies the other site and the same procedure is repeated until the system goes to a steadt state[2]. The authors of [2] and [3] seem to have suspected that how local the interactions are controls the system. So they decided to make the following two modifications to the regular honeycomb lattice.

In the simulation, Q portion of the links that are connecting nearest neighbors were replaced by links that connect sites that are not the nearest neibours to each other. This means that the lattice is the natural honeycomb lattice if Q=0, and is completely random if Q=1. This type of randomness is called quenched randomness[2].

In addition to having quenched randomness, annealed (temporal) randomness P was introduced. This is the probability that standard links (ones connecting nearest neighbors) are replaced by random ones at each time step in the simulation[2].

When standard links were replaced by random ones in both procedures described above, the number of links connected to each site was kept fixed to 3[2].

Suspecting that P and Q are the control parameters of RPS games, the authors of [3] performed simulations with various Q and P values. As a result, all phases in the last section (absorbing, self-organizing, and oscillating phases) were observed. The system is in absorbing phase for small Q and P, absorbing state for large Q and P, and oscillating phase in between. The phase diagram is shown in Figure 2.

This simulation shows that there is an emergent behavior in rock-paper-scissors game, and control parameters are (at least) quenched randomeness Q and annealed randomeness P–the phase of the system is determined by how randomely sites can interact with other sites.

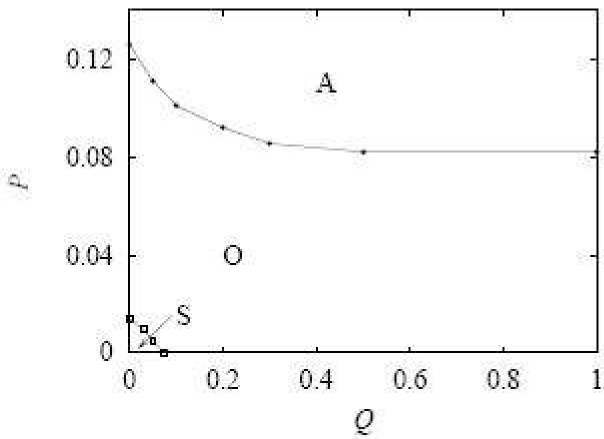

Figure 2: Phase diagram of RPS game by A. Szolnoki and G. Szabo. Q is the portion of the lattice that is randomely connected, and P is the probability that a regular link is replaced by a random one in each timestep. Small Q and P corresponds to a self-orgnizing phase(number of strategies is the same for all three), large Q and P corresponds to an absorbing phase(all sites have the same strategy), and in between, there is an oscillating phase(number of each strategy oscillates).

APPLICATION TO BIOLOGICAL SYSTEMS

ESCHERICHIA COLI

Rock-paper-scissors game relationship is occationally seen in biological systems. One example is a collection of Escherichia coli, normally called E.coli[5]. There are three types of E.coli:

- type C–has the ability to create toxin called colicin, but it is rresistant to the colicin
- type R–resistant to colicin, but it does not have the ability to make colicin
- type S–gets killed when exposed to colicin

Type R bacteria grow more rapidly than type C bacteria, because not having the ability to create colicin makes it easier to grow. So if those two types of bacteria are put in a same container, then type R 'wins' and displace type C. Type S bacteria grow even more rapidly than type R bacteria because type S bacteria absorb nutrients more efficiently. Hence S wins against R. Lastly, C wins against S beacuse the colicin of type C bacteria kills type S bacteria. Hence, these three types of bacteria have the feature of rock-paper-scissors ($S > R > C > S$).

The authors of [5] put those three types of bacteria together in a container (which makes the situation be ideally described by the computer simulation in the last section). The number of each type of bacteria was measured every one day, for seven days. The experiment was done with three different conditions: Static plate condition (bacteria can interact with ones close by only), Flask condition (the container was shaken frequently so that bacteria interactions are not local), and Mixed condition (somewhere in between the last two extrema). The result is in Figure 3.

Two of the three states that were suggested in the computer simulation were observed. The number of bacteria is roughly the same and stays constant if bacteria can interact only locally (Static plate condition gives self-organized state), one type dominate if the interaction is allowed at random locations (Flask condition gives absorbing state). The result for the mixed plate was also an absorbing state, which suggests that the randomness was above critical point, if the simulation in the last section corresponds to this particular case close enough.

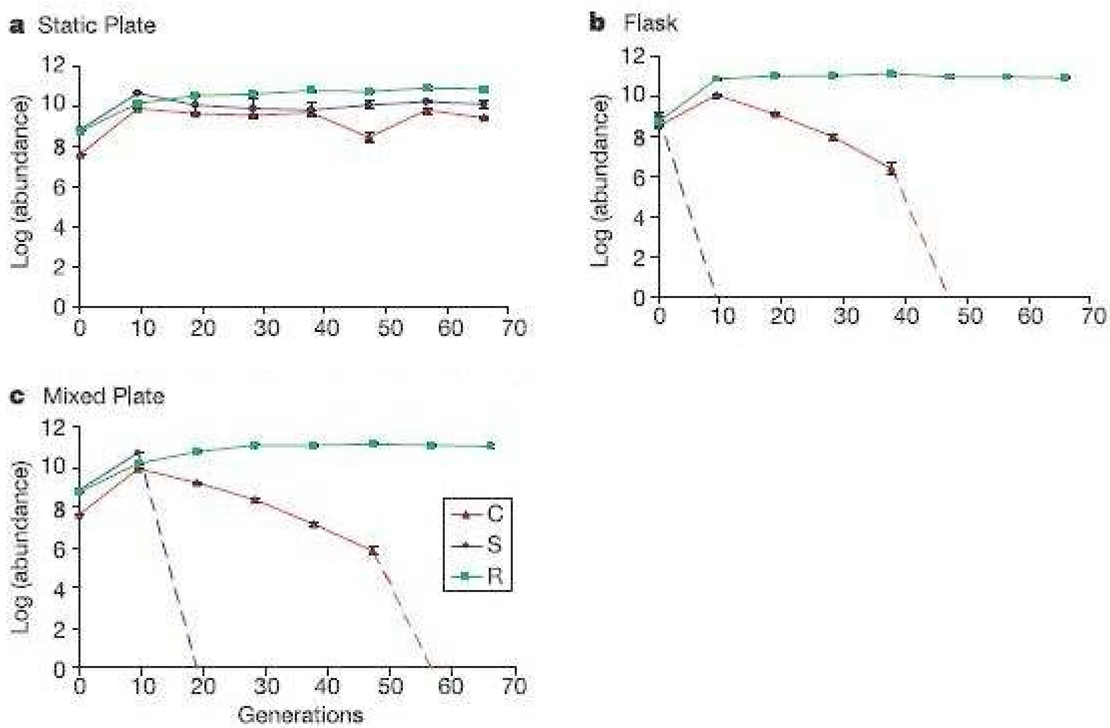

Figure 3: Numbers of all three types of bacteria as a function of time, when they are put together. The numbers on the horizontal axis are number of bacteria generations, which corresponds to 10 time days. Graph 'a' is for the case where the interaction between bacteria is local, graph 'b' is for the case where the interaction is global, and graph 'c' is for between the last two cases. Figures are taken from [5].

SIDE-BLOTCHED LIZARD

Another example of rock-paper-scissors game in biological systems is a collection of side-blotched lizards. This type of lizards have three different types of males that can be distinguished with the color of their throat[6]:

- orange throat–They have largest teritorries and posses a large number of females, and they are physically the strongest among three typese.
- blue throat–They do not guard as many females, but because of the small number of females, they guard females more carefully. Hence they win against yellow throat males.
- yellow throat–They are the weakest of all types. However, they look similar to females, so they try to sneak into territories of other males.

Orange throat males win against blue throat males, simply due to their strength. Blue throat males lose against orange throat males, but can win aganst yellow throat males because of strength, and because they guard females closely, which makes it difficult for yellow throat males to sneak in. Yellow throat males cannot sneak into blue throat males' territories due to a heavy guard, but can sneal into orange throat males' territories and steal females. This shows that males of side-blotch lizard are in the rock-paper-scissors situation ($Orange > Blue > Yellow > Orange$). The winner gets to mate with a female, which results in taking the place of the loser through making offsprings. The number of each type of males were measured through the year 1990 to 1999 by Sinervo et. al, and the result is shown in Figure 4.

As in the figure , the state seems to be somewhat close to an oscillating state, but not close eonugh to say so confidently. In fact, this system is not a simple RPS game because there are two types of females, and females control the system as well by choosing their mate. Nevertheless, the feature of RPS game is reflected in the plot, though not close to 100%, and it seems that the system is the closest to the oscillating phase.

Conclusion

Simple rock-paper-scissors model have three different phases: Absorbing phase (everybody has the same strategy), self-organizing phase (number is the same for all strategies), and oscillating phase (number oscillates for all strategies). The control parameters are quenched randomeness and annealed

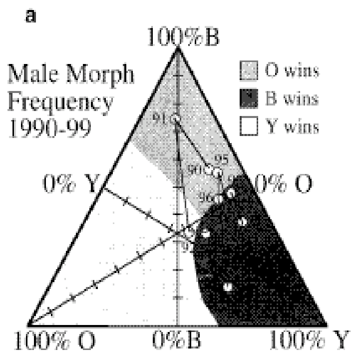

Figure 4: The percentage of each type of lizards (called frequency here). White dots indicate the number of all three types, and colors indicate which type has the most advantage if the dot is there.

randomeness. This emergent behavior was seen in the simulation by A Szolnoki and G Szabo. In biological systems, the behavior was seen in collection of E.coli and side-blotched lizards.

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