The price of anarchy in transportation networks

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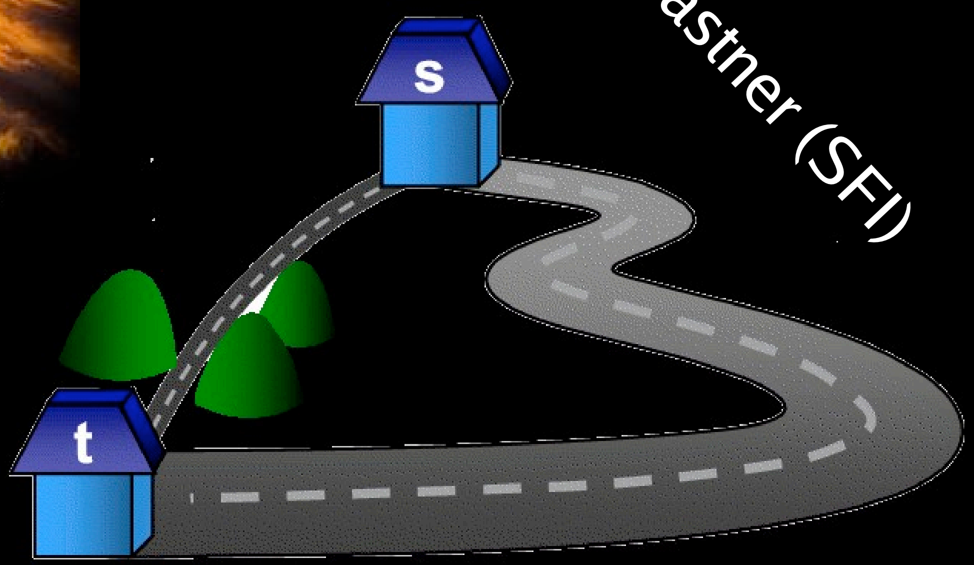


Network routing

In the late 90s, it was fashionable to discuss different approaches to finding routes in a network.

In general, the models dealt with “uncapacitated” networks: there were no limits for the permitted flow on each link.

In reality, links in a network cannot maintain an infinite amount of traffic without an increase in the cost for the users.

Examples:

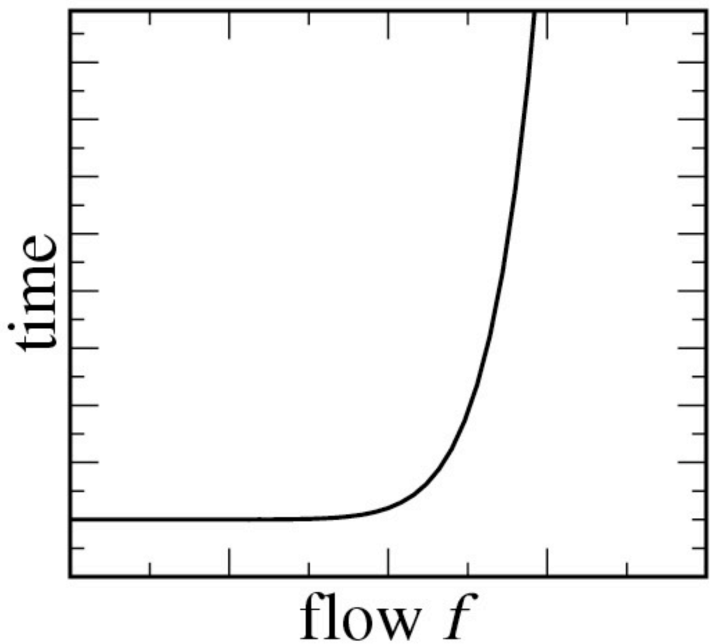
- travel time on a road network
- latency in Internet connections

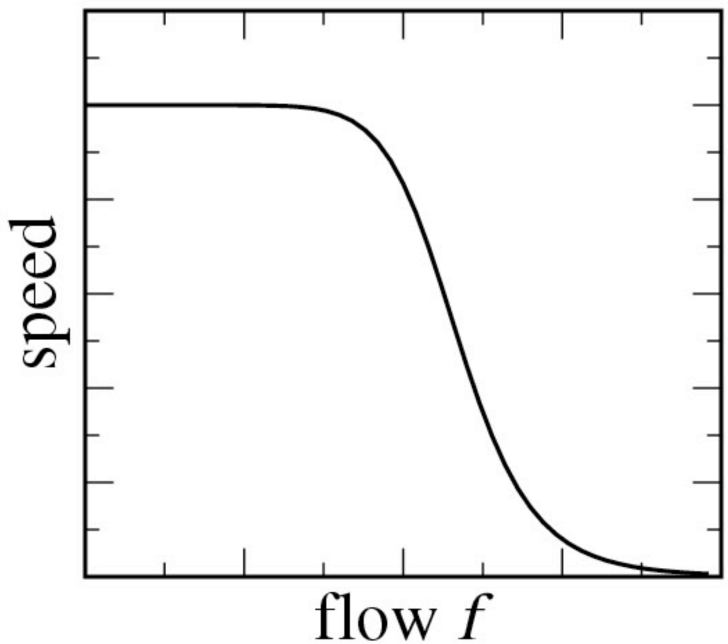
Congestion-dependent costs complicate matters . . .

Congestion-dependent costs

In our model, every link is associated with a travel time l_{ij}.

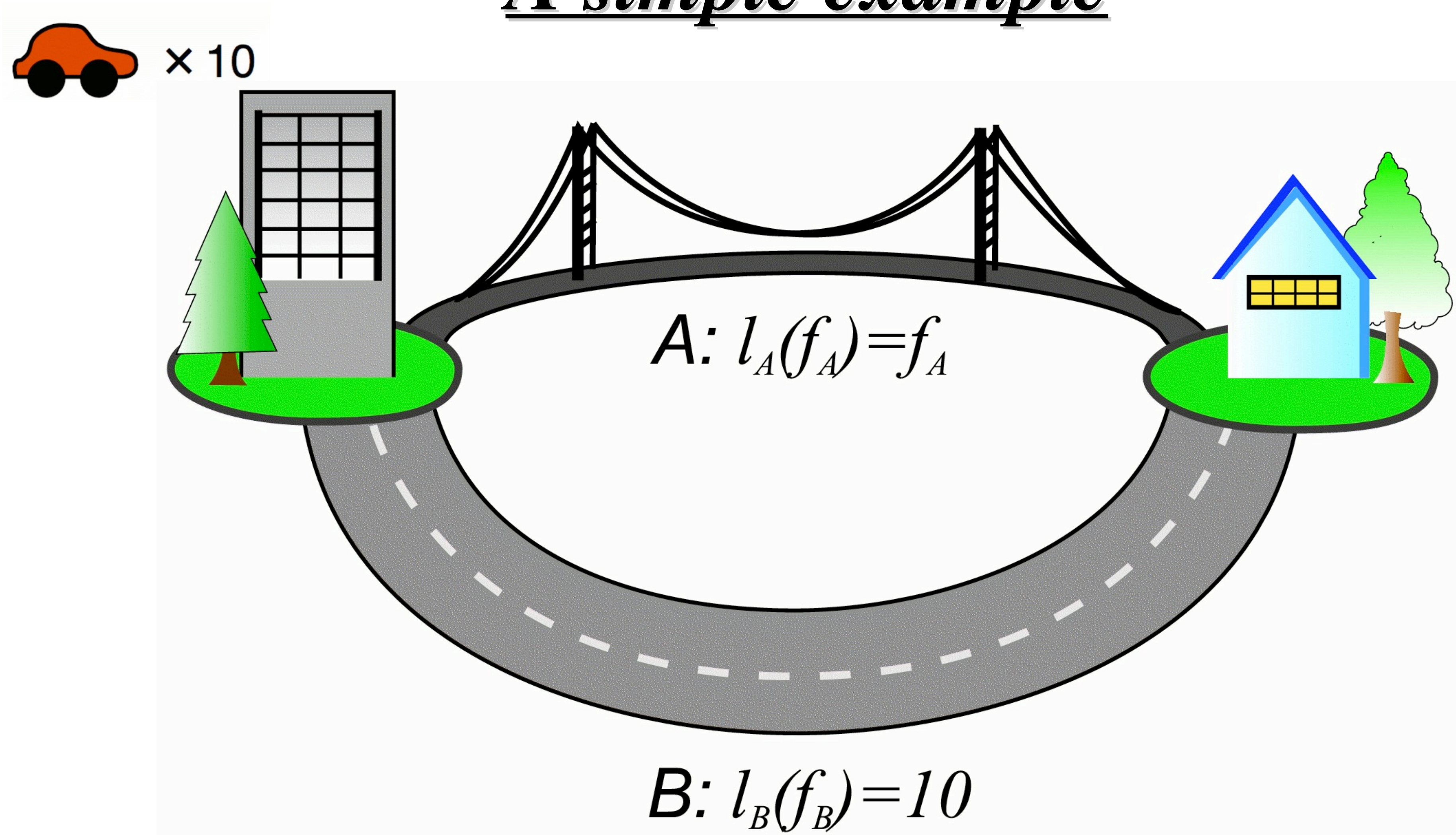
The travel time is a function of the flow f_{ij}, e.g., vehicles per hour.





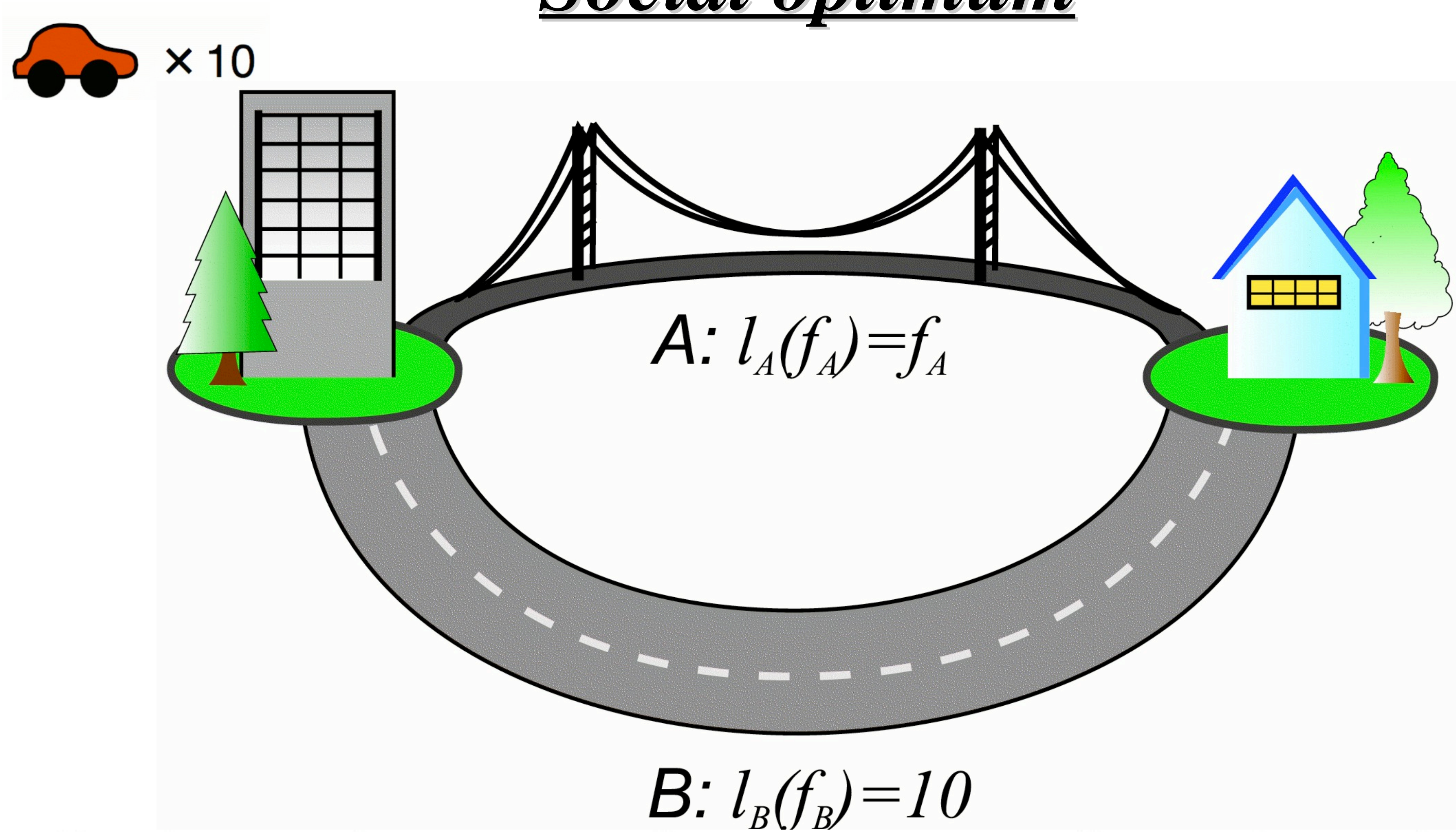
What are the consequences of such congestion-dependent costs?

A simple example



Suppose 10 cars travel per unit time from the left to the right.

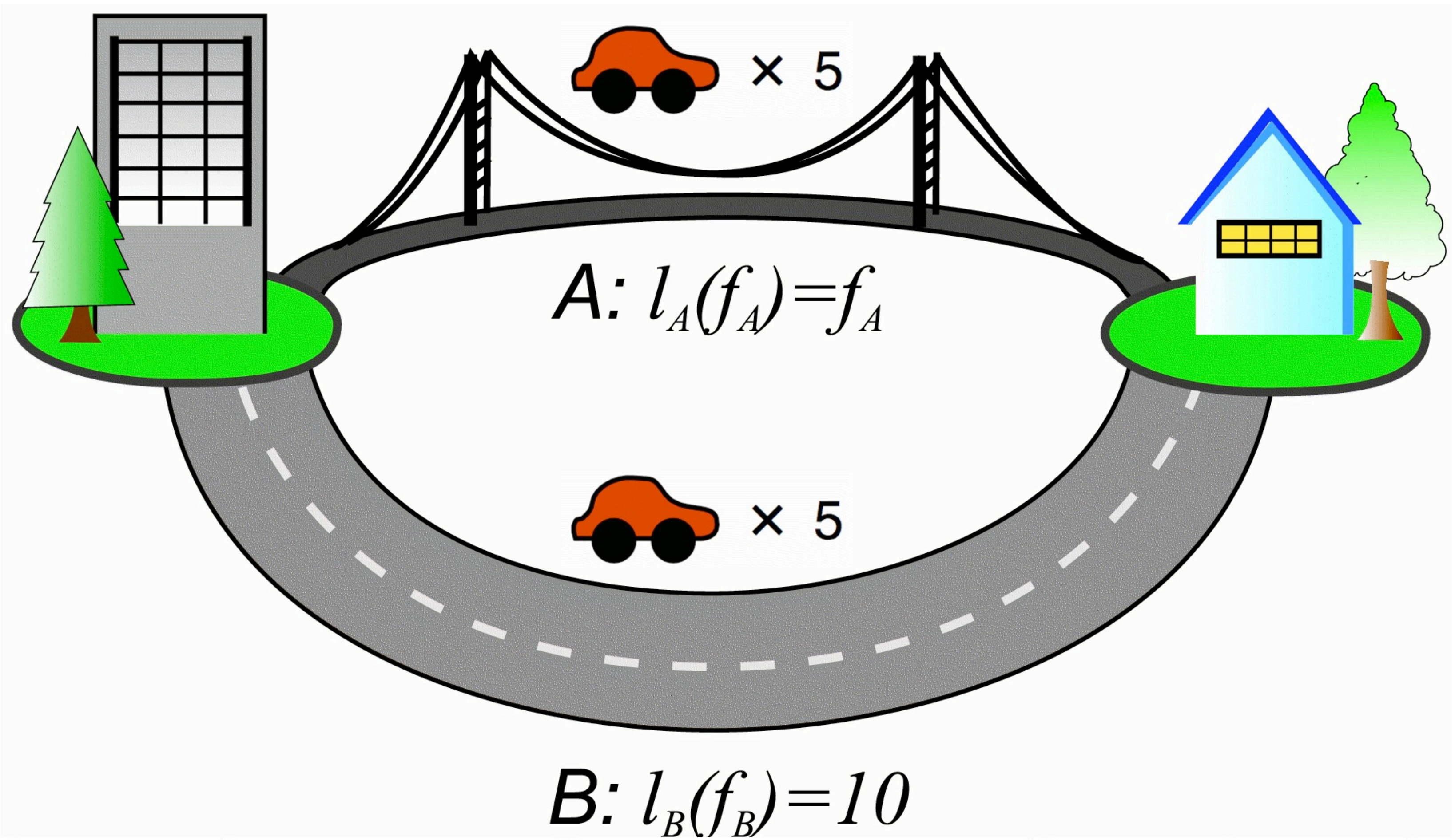
Social optimum



In the social optimum, drivers minimize the total travel time

$$C = l_A(f_A)f_A + l_B(f_B)f_B \qquad \longrightarrow \qquad f_A = f_B = 5.$$

Social optimum



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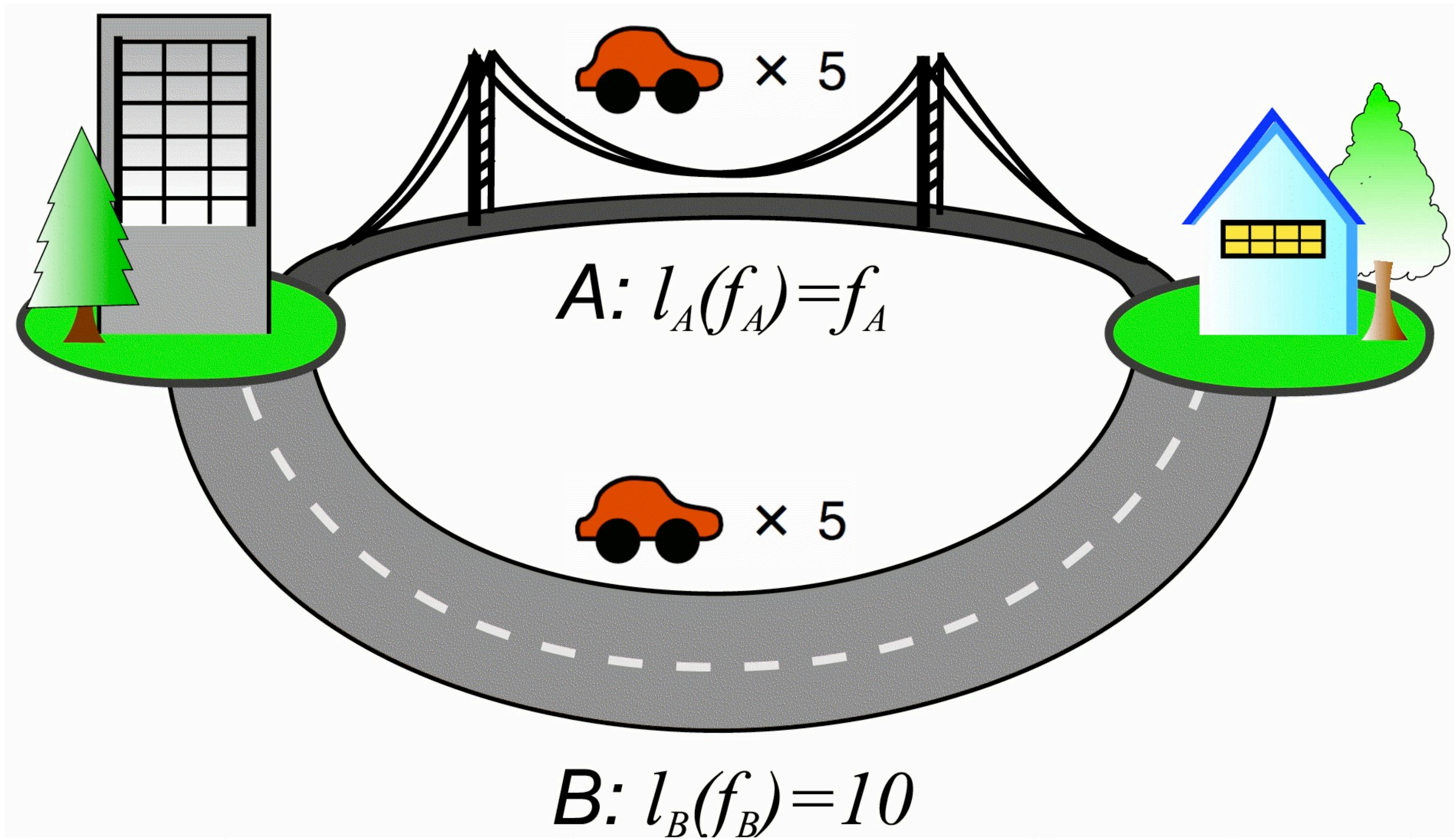

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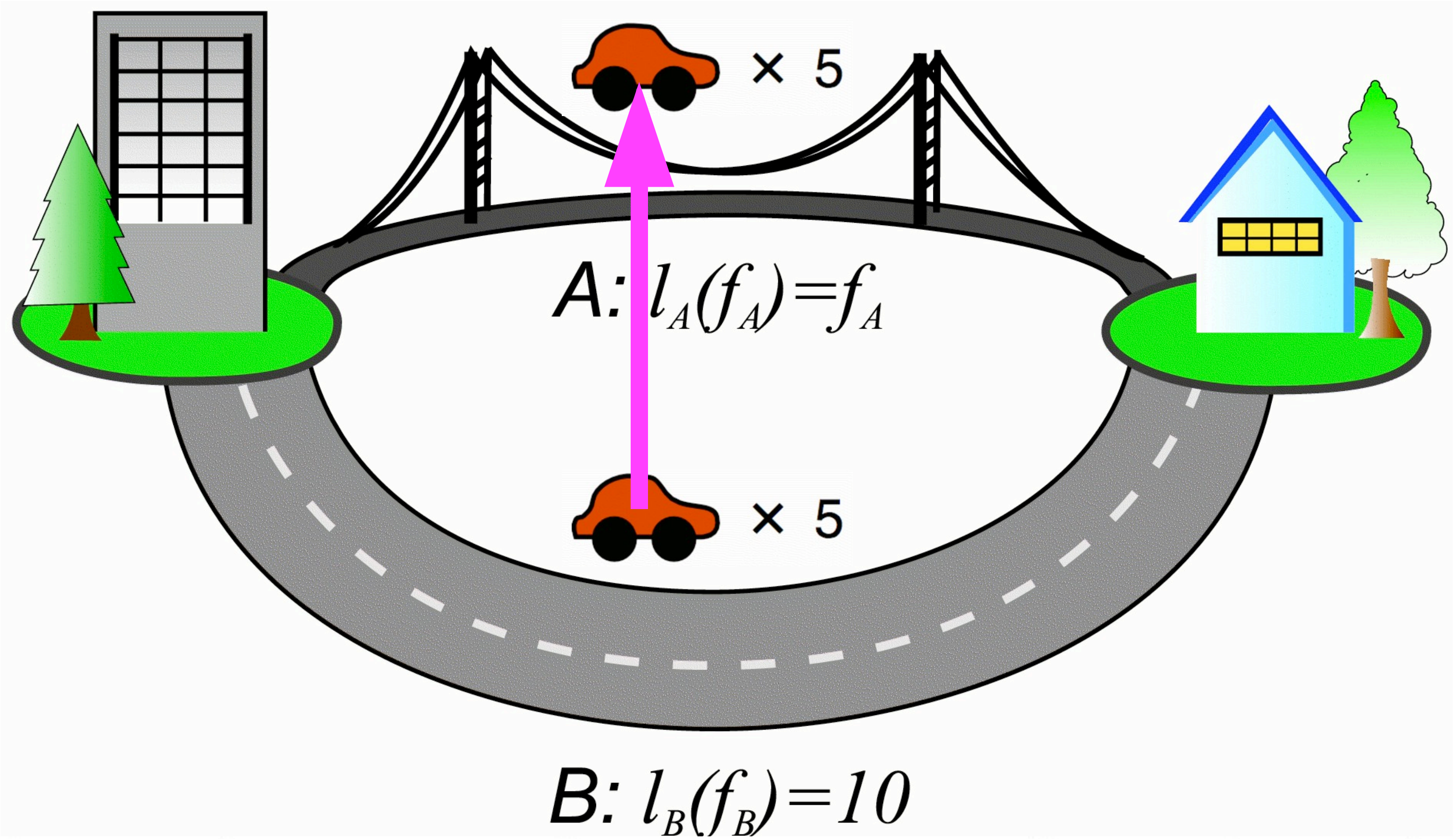
However, drivers on B pay more than they would on A, so there is an incentive to change paths.

Social optimum



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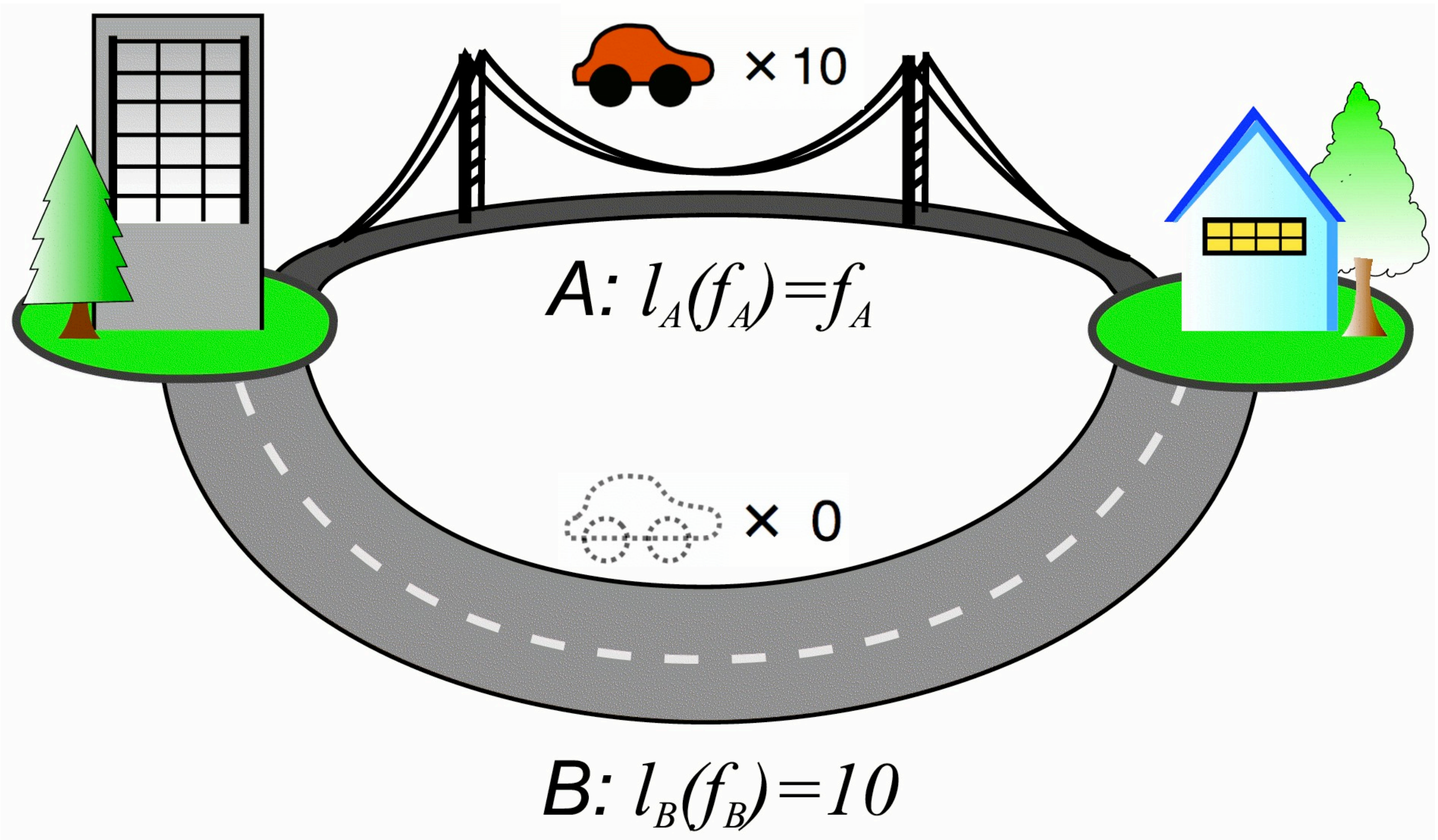
Nash equilibrium



In the Nash equilibrium with $f_A = 10$ and $f_B = 0$, no driver can reduce his cost unilaterally.

However, the cost paid by all users together has increased from $C = 75$ to $C = 100$.

What are realistic routing strategies?

The previous example is also known as *Pigou-Knight-Downs-Thomson paradox*.

It is not strictly paradoxical. Often, social optimum $\neq$ Nash equilibrium.

But what is more likely to be seen in reality?

Traditional assumption – *Wardrop's principle* (1952):
"The journey times in all routes actually used are equal or less than those which would be experienced by a single vehicle on any unused route." $\longrightarrow$ Nash equilibrium!

This assumption is essentially supported by recent psychology experiments (Selten et al., Helbing et al.).

How can the travel times be calculated?

The social optimum is the solution of:

Minimize the total travel time $C = \sum_{\text{link } i \to j} l_{ij}(f_{ij}) f_{ij}$ subject to

$$\sum_{j:\ \exists \text{ link } i \to j} f_{ij} - \sum_{j:\ \exists \text{ link } j \to i} f_{ji} = \begin{cases} F & \text{if } i = s \text{ (origin)} \\ -F & \text{if } i = t \text{ (destination)} \\ 0, & \end{cases} \quad (1)$$

$$f_{ij} \geq 0 \text{ for all links } i \to j. \quad (2)$$

If $l_{ij}(f_{ij}) f_{ij}$ convex $\to$ capacity-scaling minimum cost flow algorithm.

The flows in the Nash equilibrium minimize $\tilde{C} = \sum_{\text{link } i \to j} \int_0^{f_{ij}} l_{ij}(f') df'$.

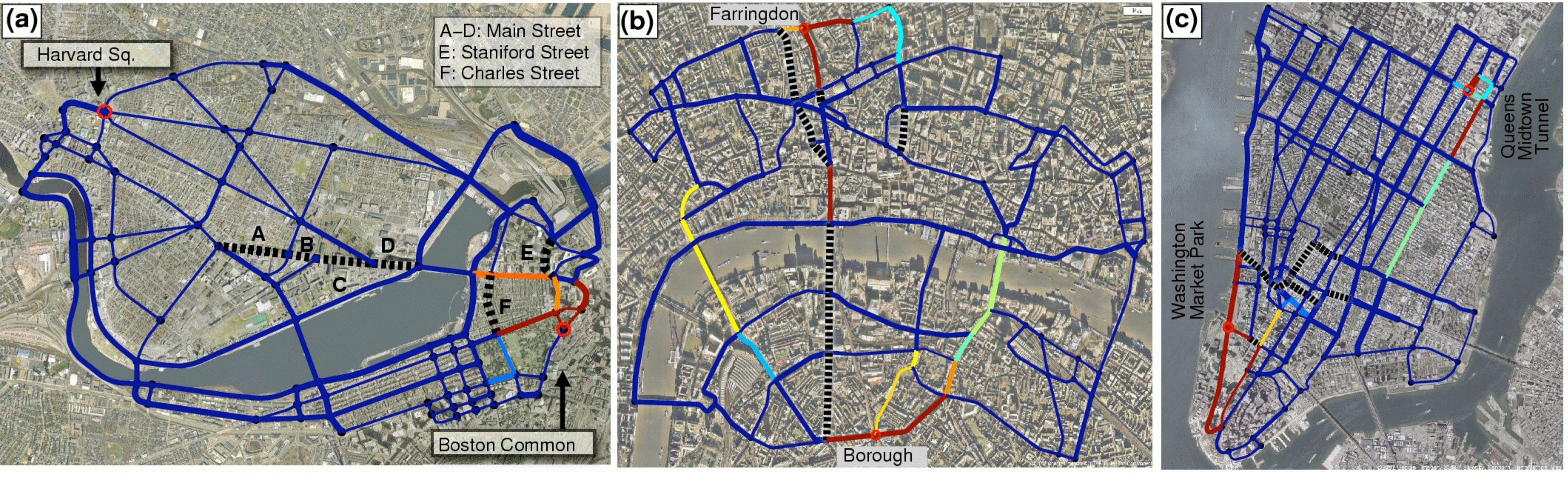
There is exactly one Nash equilibrium.

The Price of Anarchy

The *Price of Anarchy* is the ratio of the Nash equilibrium cost to the social minimum (Papadimitriou 2001),

$$\mathrm{PoA} = \frac{\sum l_{ij}(f_{ij}^{NE}) \cdot f_{ij}^{NE}}{\sum l_{ij}(f_{ij}^{SO}) \cdot f_{ij}^{SO}}.$$

We wish to calculate the PoA for several real road networks.



The Price of Anarchy

Travel times are assumed to follow the *Bureau of Public Roads (BPR) function*,

$$l_{ij} = \frac{d_{ij}}{v_{ij}} \left[1 + \alpha \left(\frac{f_{ij}}{p_{ij}} \right)^{\beta} \right].$$

d_{ij}: distance, v_{ij}: speed, p_{ij}: capacity.
The parameters α and β have been fitted to empirical data by Singh (1999) as $\alpha = 0.2$, $\beta = 10$. With these values, we find:

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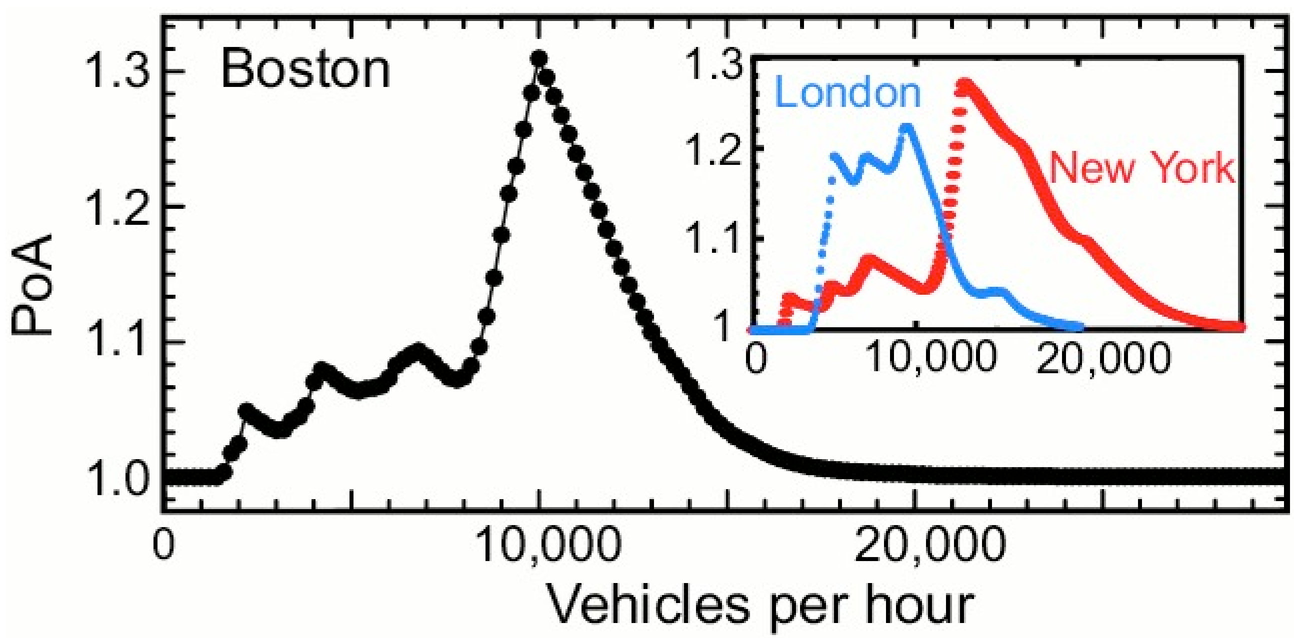
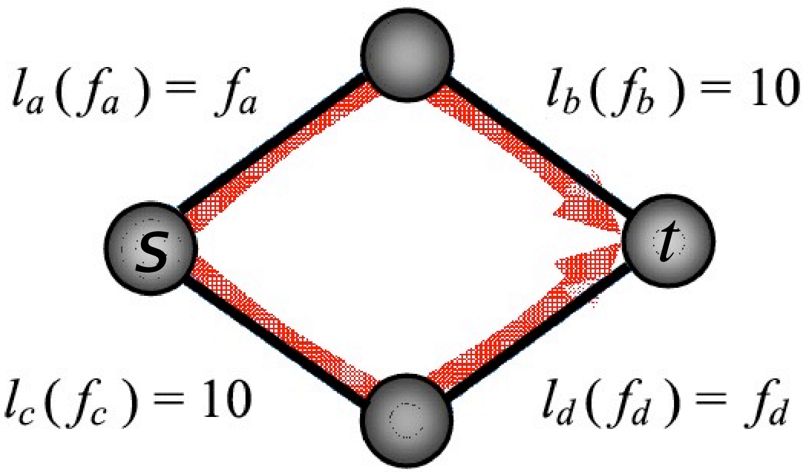


More roads, less congestion?



Suppose there is a total flow of 10 from s to t.

Nash flow:

$$f_a = f_b = f_c = f_d = 5.$$

cost: $C_{Nash} = 150.$

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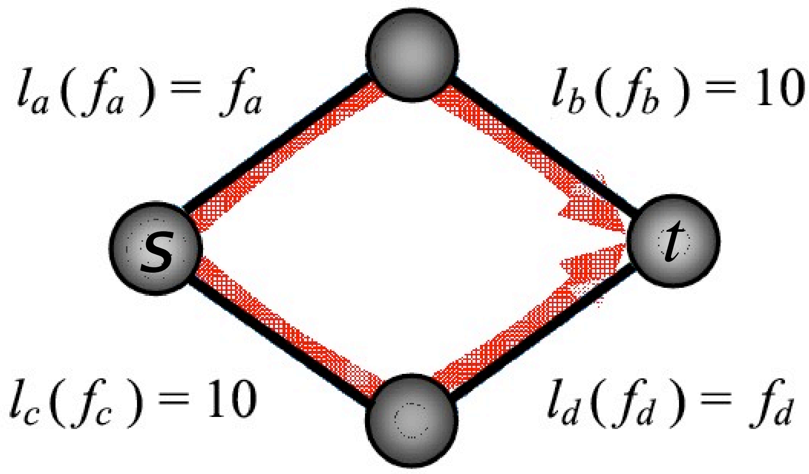

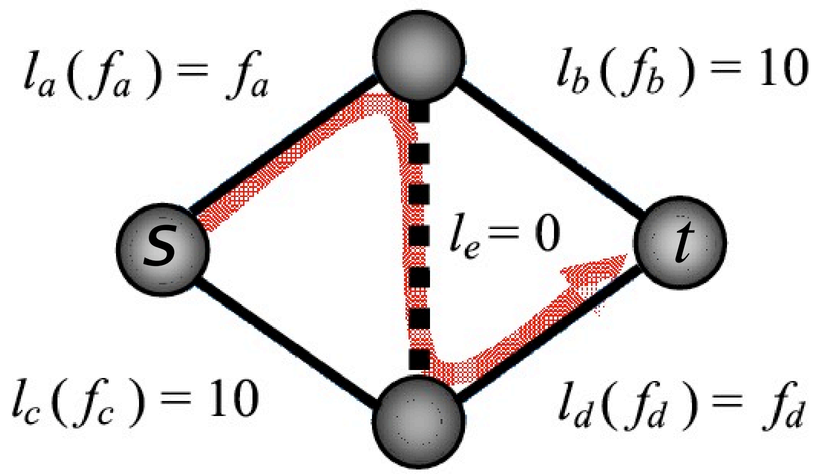

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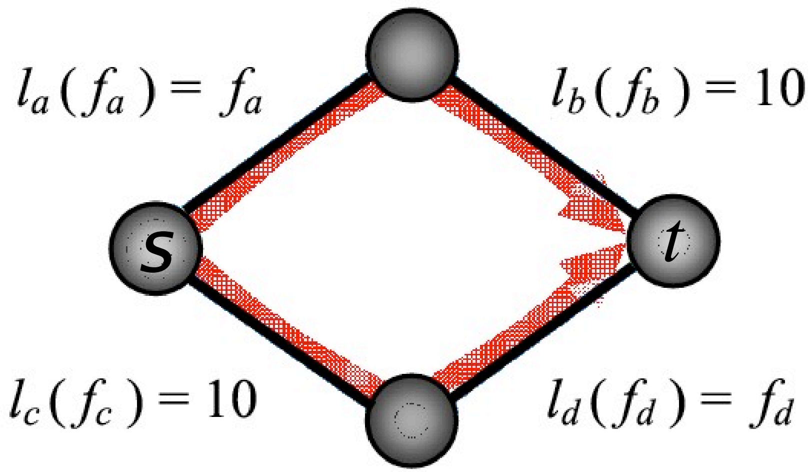

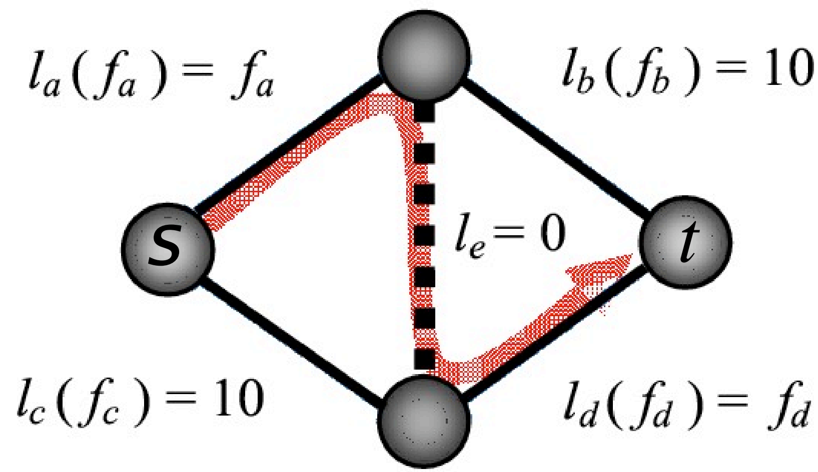

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cost: $C_{Nash} = 200.$

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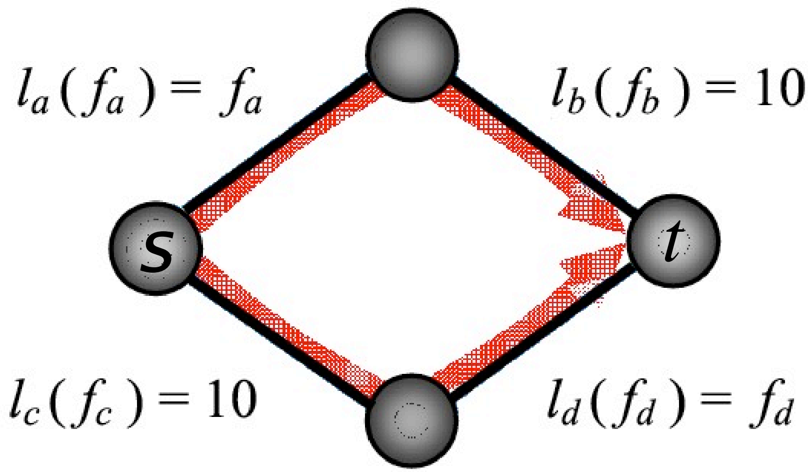

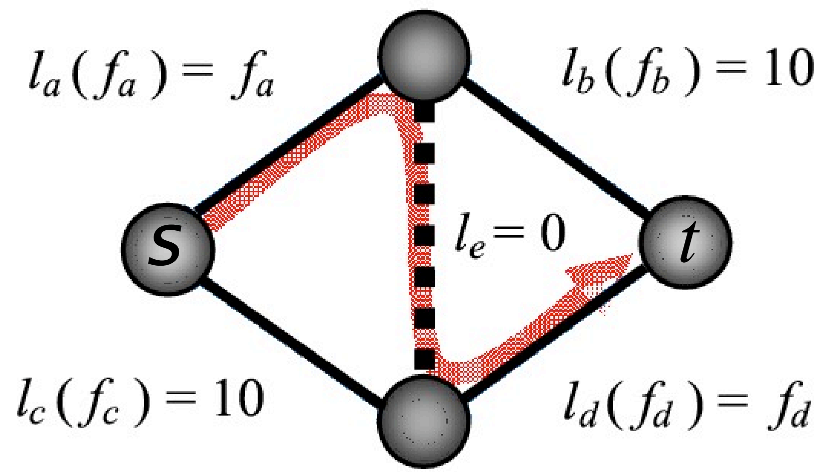

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cost: $C_{Nash} = 150.$

$$f_a = f_e = f_d = 10,$$

$$f_b = f_c = 0.$$

cost: $C_{Nash} = 200.$

Braess's paradox (1968):

In Nash flows, network improvements can degrade network performance.

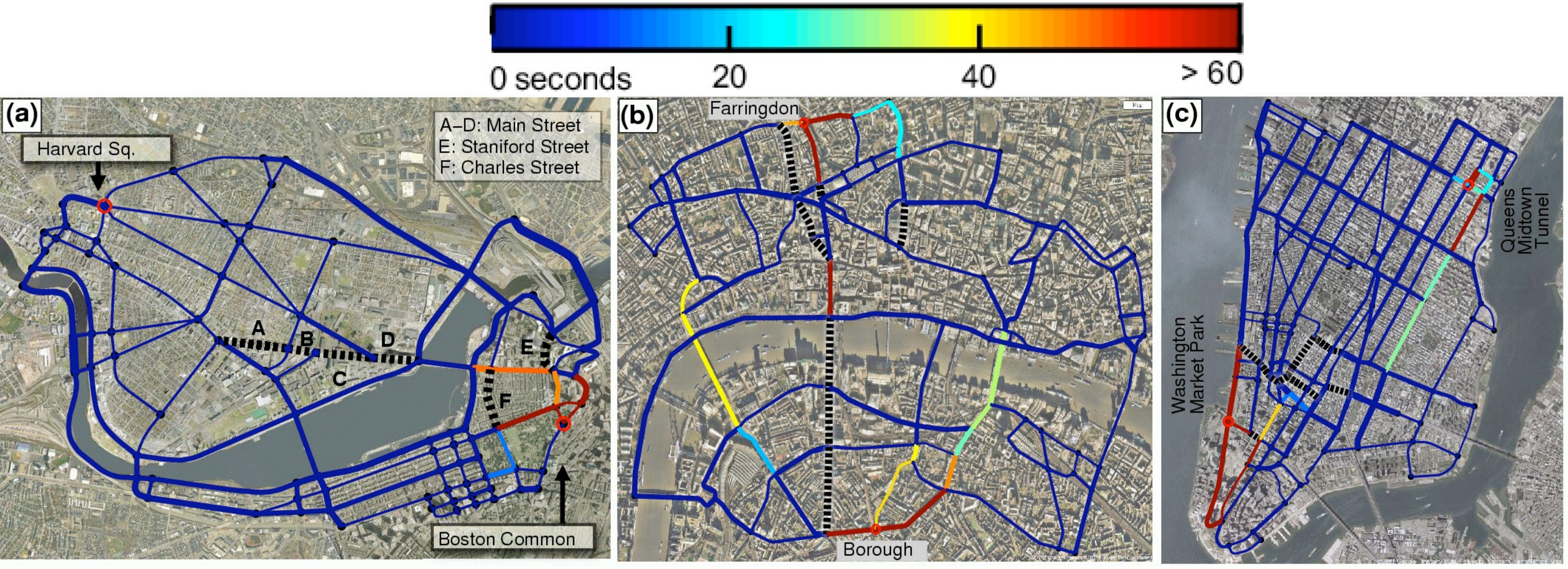
Braess's paradox

Is this "tragedy of the open road" (Samuelson 1992) a purely academic construction or a realistic problem?

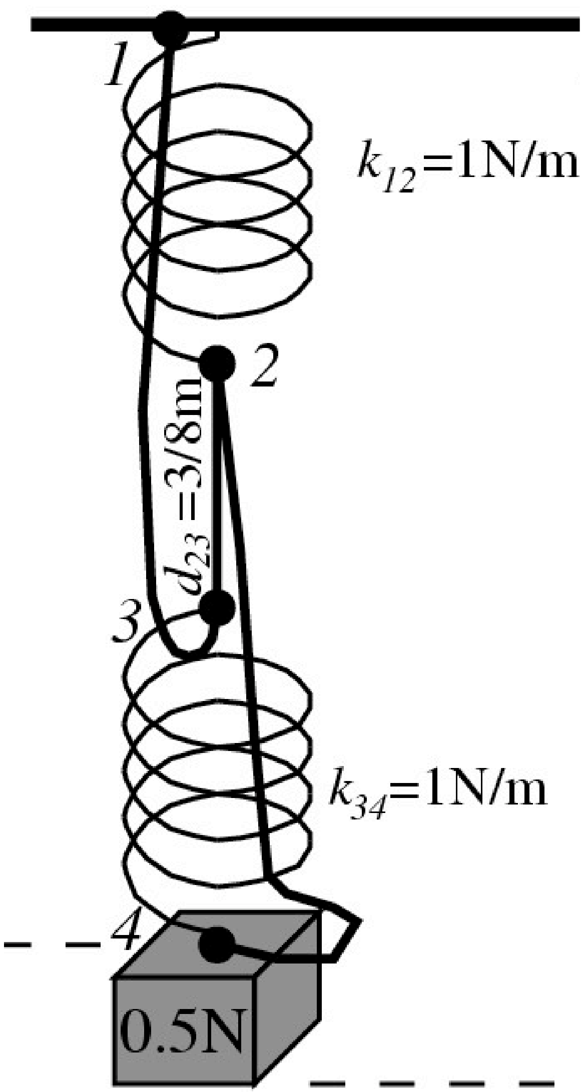
- Stuttgart: "At the end of 1968, streets around the Schloßplatz were opened to traffic and not used in the anticipated manner. A traffic chaos during peak hours ensued. It was only solved by closing the lower Königstraße." (Knödel 1969)
- Winnipeg: "This phenomenon may occur in real life." (Fisk and Pallottino 1980)
- New York: "What if they closed 42nd Street and nobody noticed?" (New York Times 1990)
- Laboratory experiments "strongly reject the hypothesis that the paradox is of marginal value." (Rapoport et al. 2005)

Nash flow after link removal

We investigate how the additional travel time increases after one edge is removed (blue to red). If the travel time decreases (Braess's paradox) the edge is marked as a black dotted line.

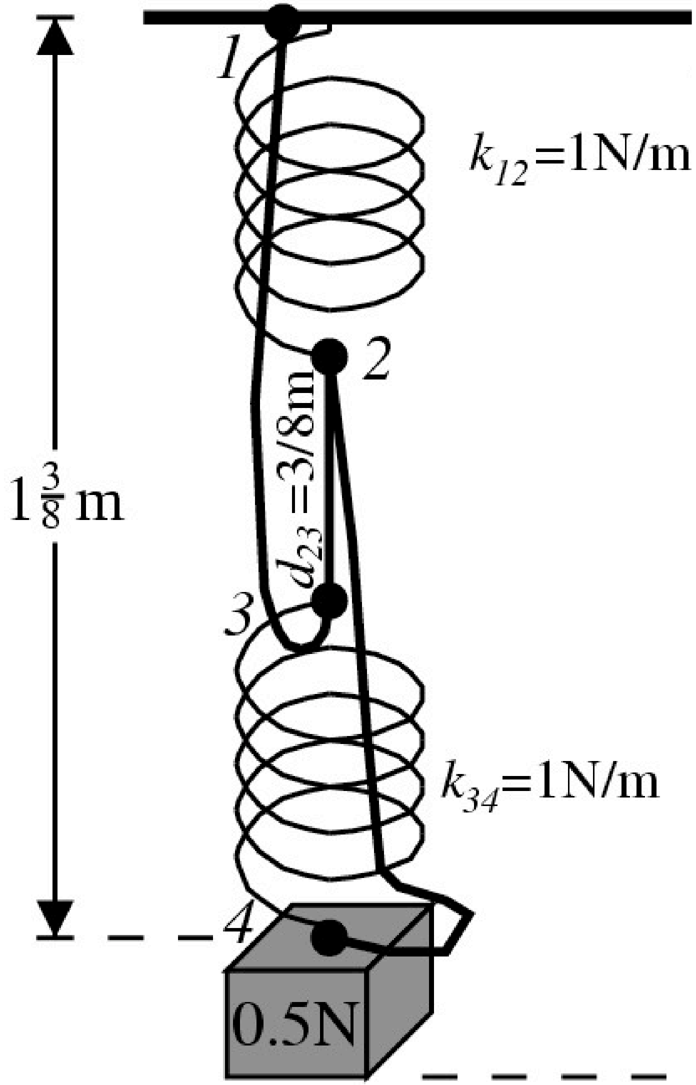


What is the connection to physics?



Suppose the two slack strings have a length of 1m each.

What is the connection to physics?



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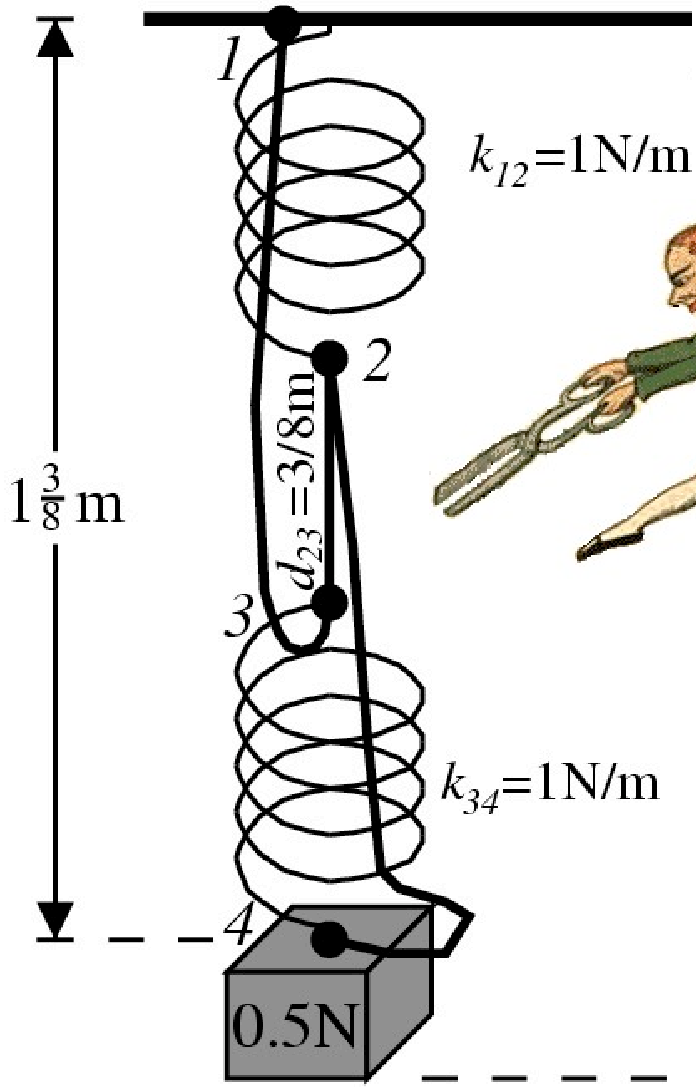
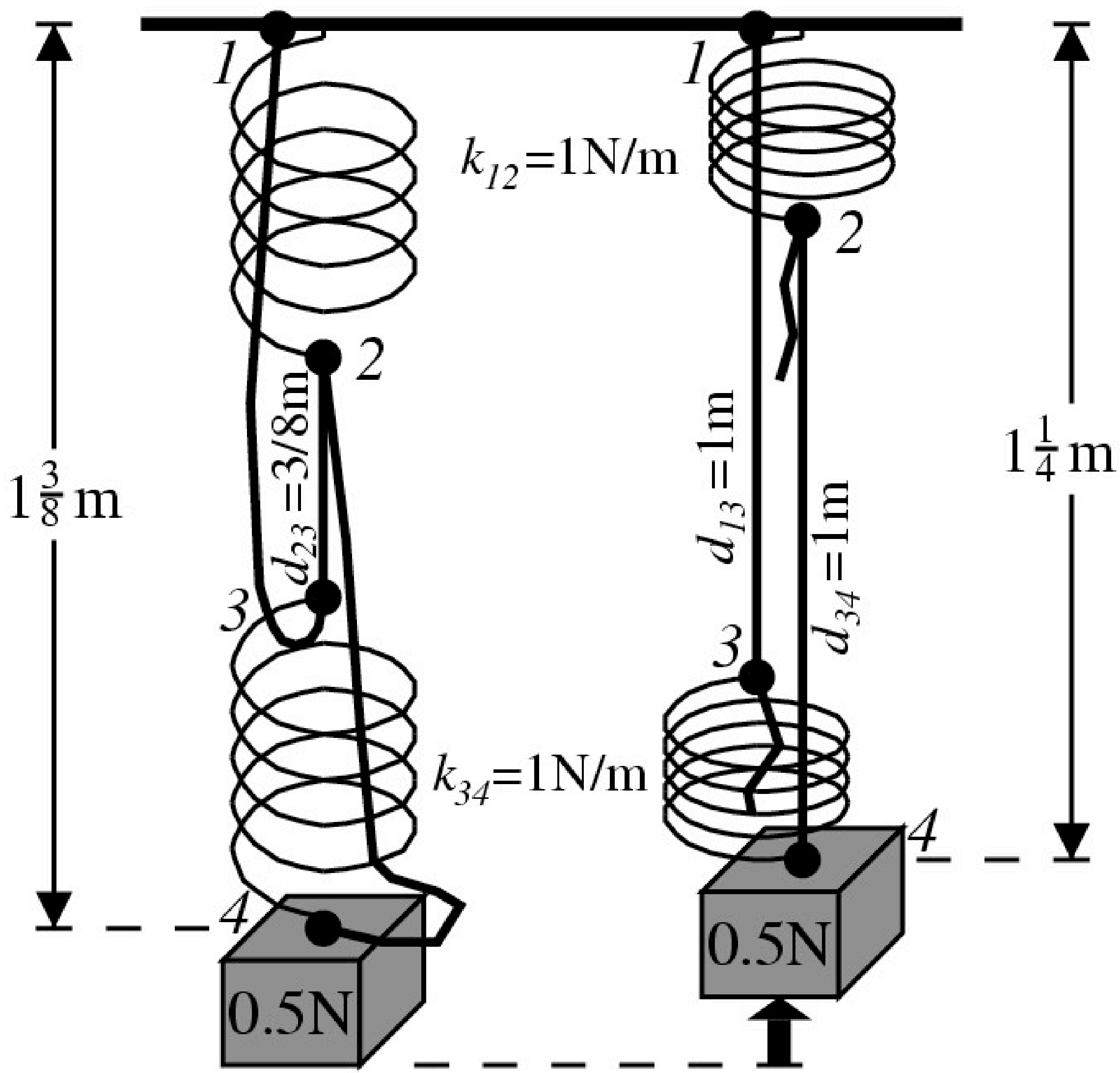


Suppose the two slack strings have a length of 1m each.

Let us now cut the taut string.

What will happen to the weight?

What is the connection to physics?



The weight moves up. Why?

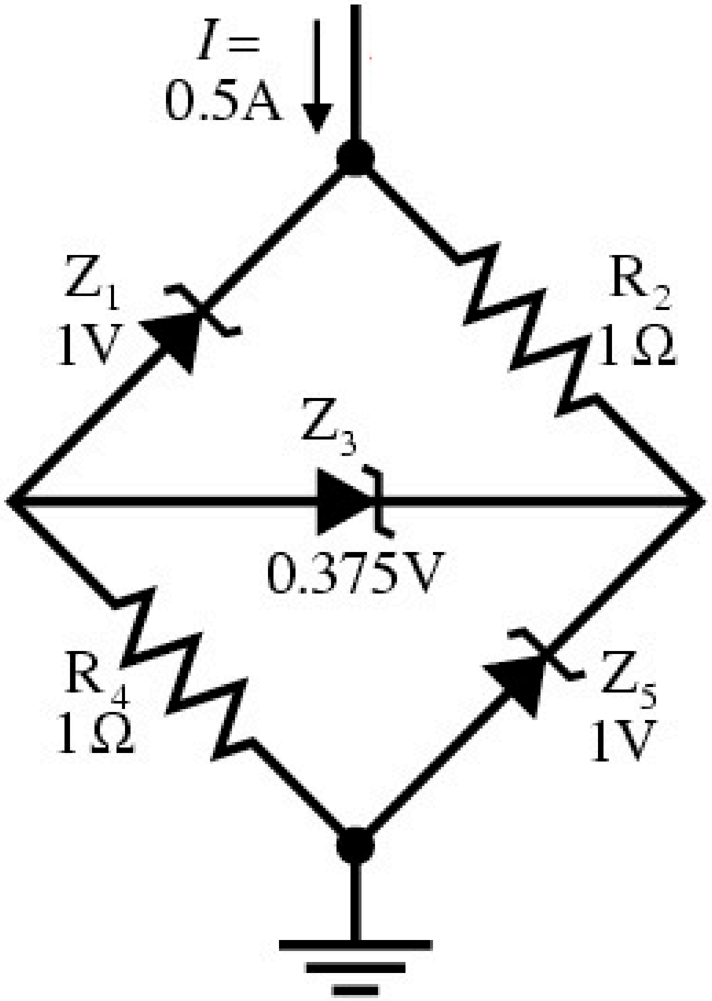
The mathematics is exactly the same as in Braess's paradox:

tension $\leftrightarrow$ flow,
extension $\leftrightarrow$ travel time.

The extension is the same regardless of the path. $\leftrightarrow$ Wardrop's principle.

Example from Cohen and Horowitz, Nature 1991.

A paradoxical electrical circuit



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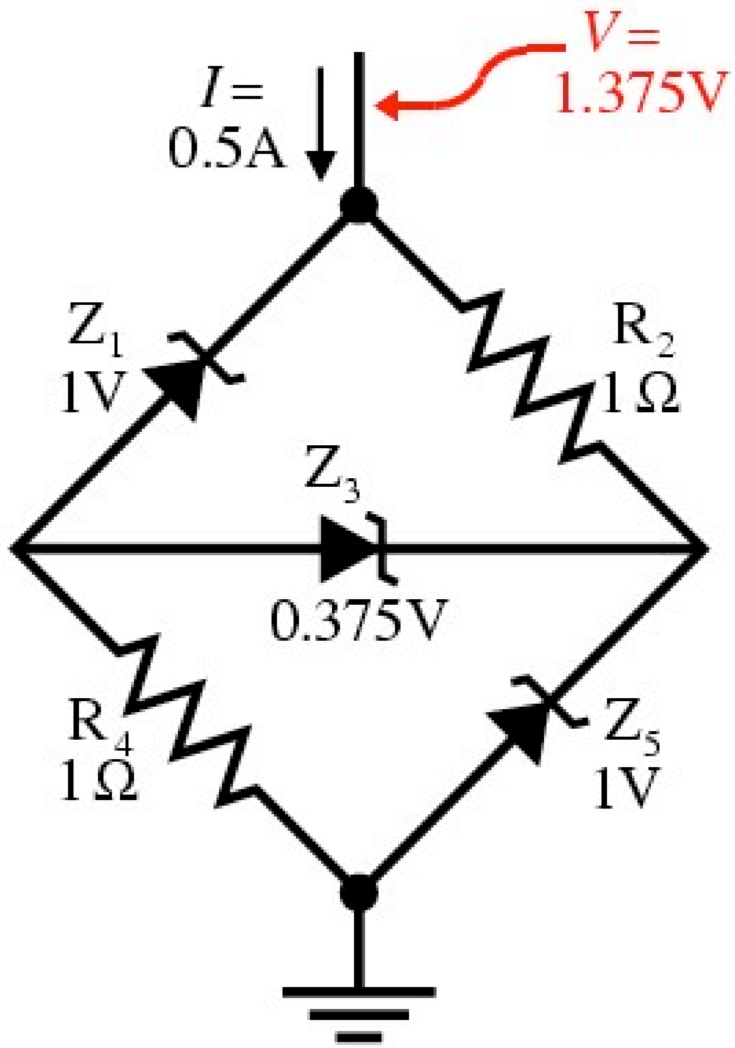

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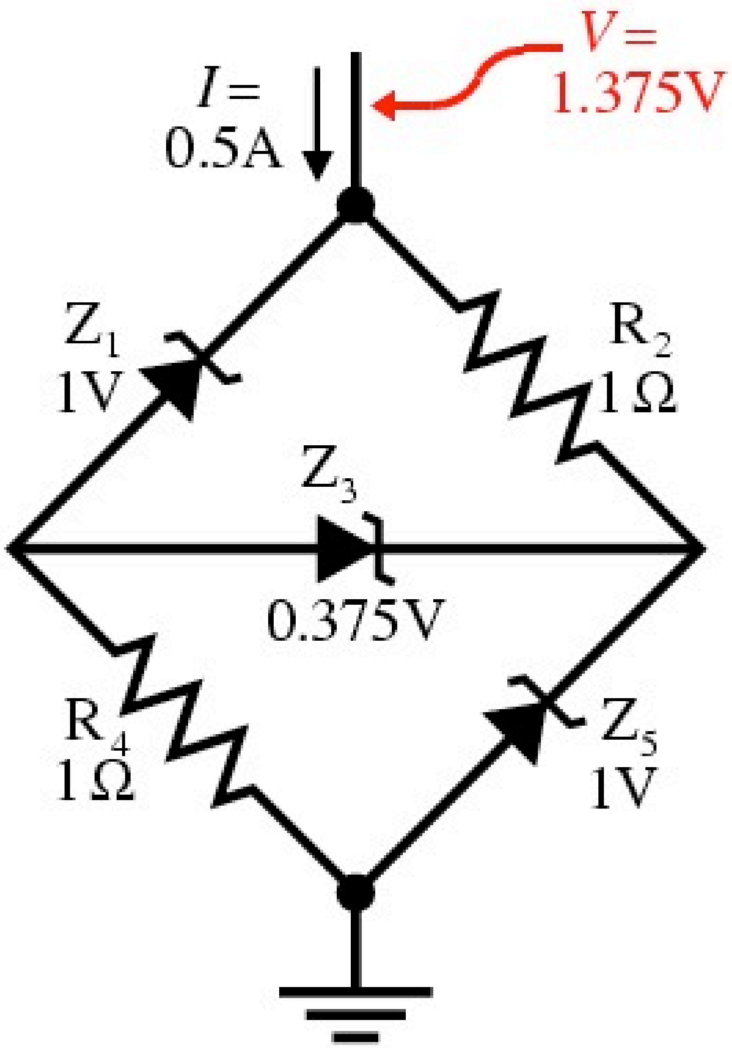

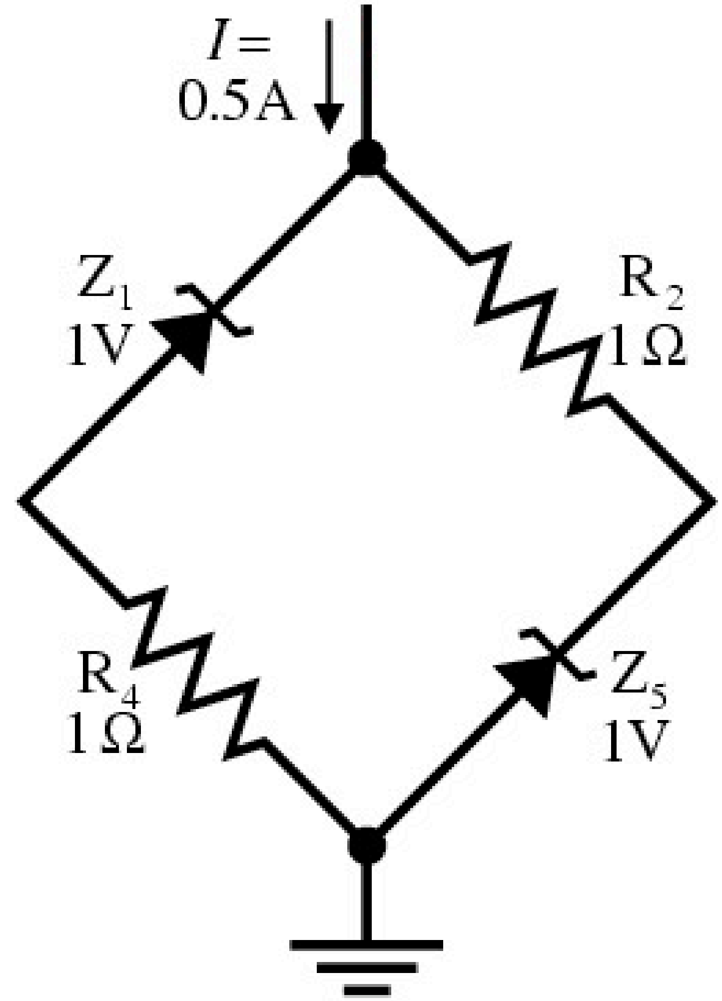

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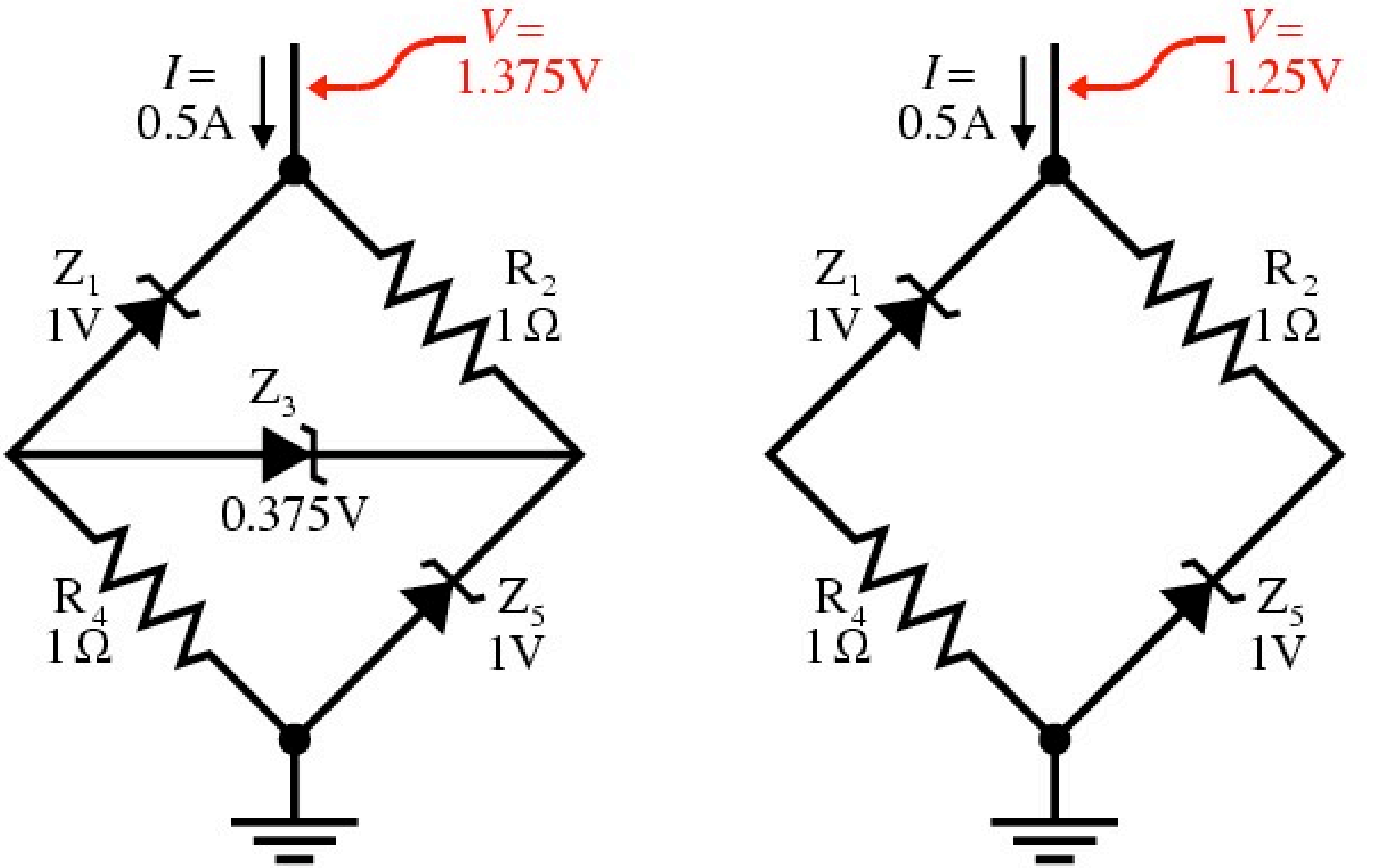


Removing a path increases the circuit's conductance (I/V).

electric current ↔ traffic flow, voltage drop ↔ travel time
Kirchhoff's rules ↔ Wardrop's principle

Braess's paradox in the recent mathematics literature

Roughgarden (2001):

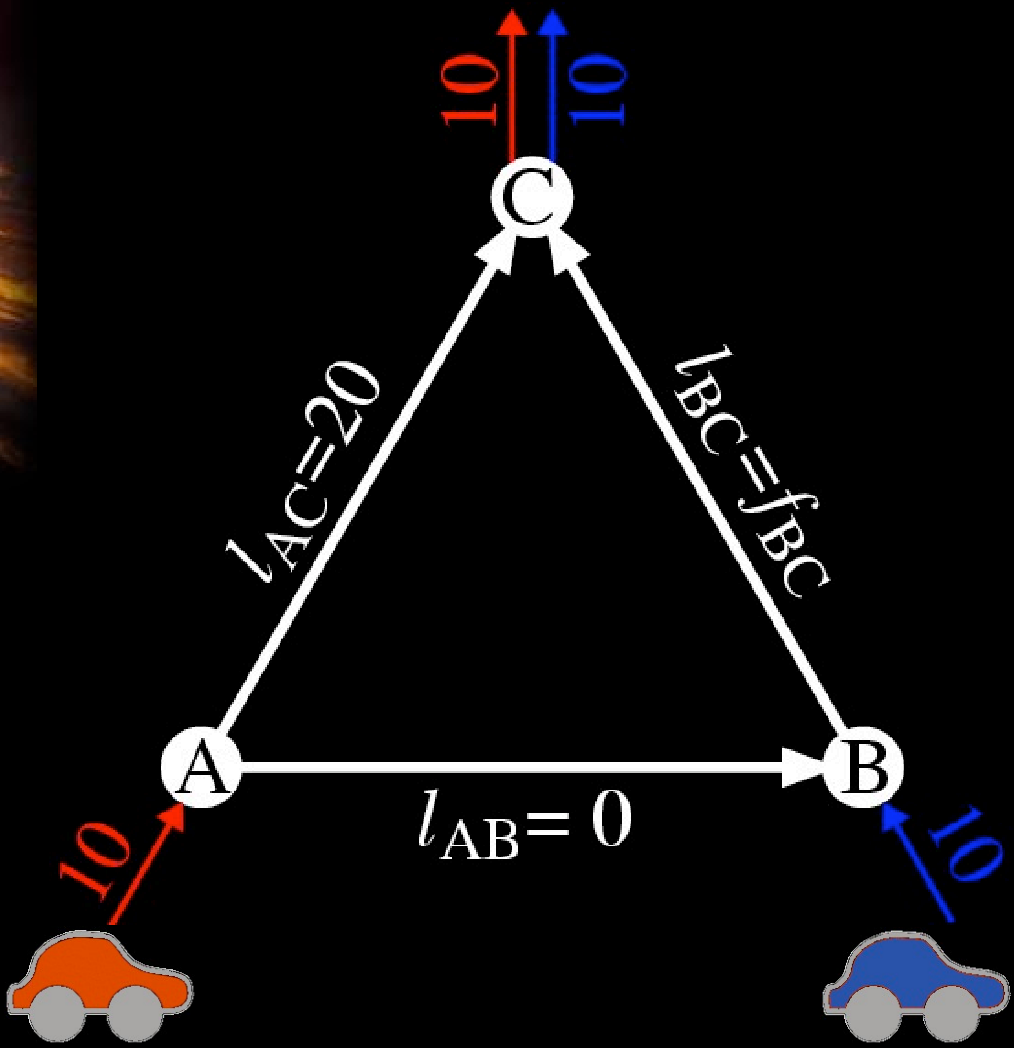
- Finding the optimal set of link removals for arbitrary networks with arbitrary cost functions is NP-hard.
- The only algorithm with a worst-case performance guarantee is to open all links (approximation ratio = number of nodes/2).

Milchtaich (2005):

- In an undirected two-terminal network, Braess's paradox cannot occur if the network is series-parallel. (A network is series-parallel if there are no two paths from origin to destination passing through any link in opposite directions.)
- For every non-series-parallel network, there exist cost functions creating Braess's paradox.

Open questions

Are there heuristics which perform well on average?



Multiple origins and destinations?