Games

Rajiv Sethi

SFI Complex Systems Summer School 2018

Overview

Foundations of Game Theory

- Players, Strategies, and Payoffs
- Nash Equilibrium

Examples and Experiments

Procedural Rationality

- Level-k Thinking
- Sampling Dynamics

Agent-based Computational Models

An Experiment

- Each person chooses a rational number in the interval $[0, 100]$
- We compute half the average
- Those closest to half the average, but not below, all get \$100
- Each of those below half the average pay \$100
- All others get \$0

Games

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Mixed strategies involve randomizations over the set of pure strategies

Nash Equilibium

A strategy profile $(s_1^*, ..., s_n^*) \in S_1 \times ... \times S_n$ is a Nash equilibrium if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for all i and all $s_i \in S_i$.

No player can improve her payoff with a unilateral change of strategy

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Equilibrium in the Half-the-Average game?

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	3, 3	0, 4
Defect	4, 0	1, 1

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Does a pure strategy equilibrium exist?

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Coordination and Hawk-Dove games have pure and mixed equilibria

Mixed Strategy Equilibria

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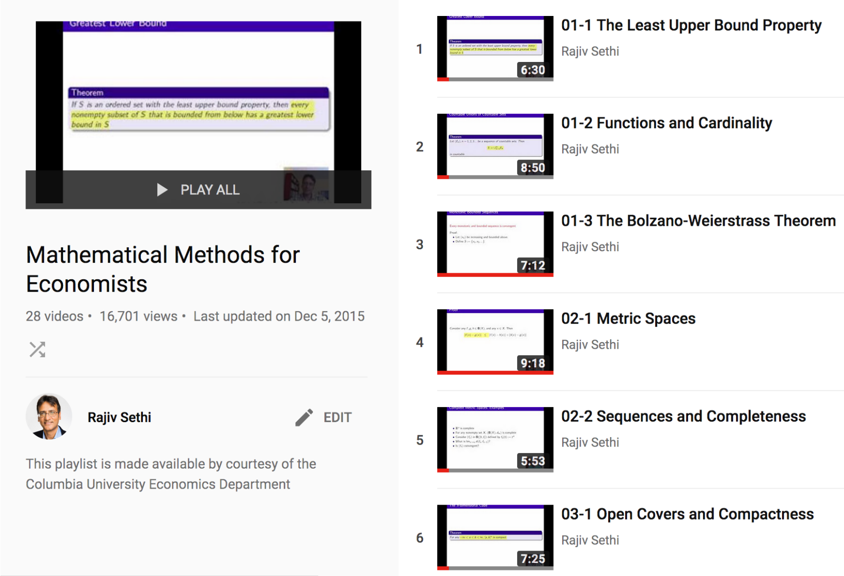
Symmetric mixed strategy equilibrium with distribution $(1/5, 3/5, 1/5)$

Equilibrium payoffs are $\frac{1}{4}$, collusive payoffs $\frac{5}{4}$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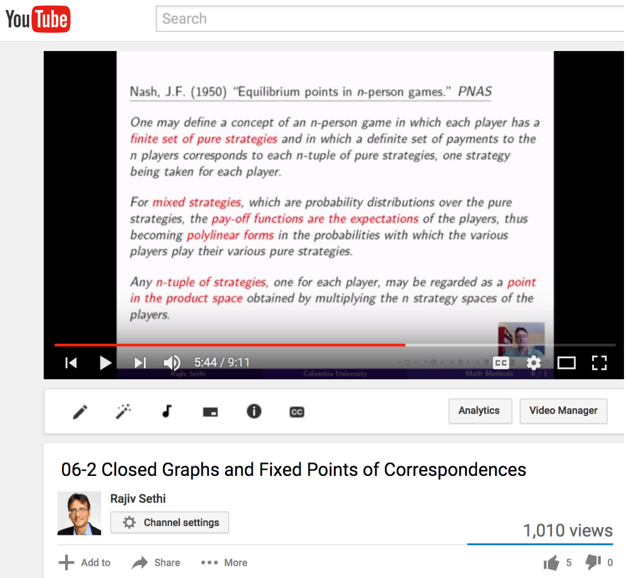
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Nash, J.F. (1950) "Equilibrium points in n-person games." PNAS
One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player.
For mixed strategies, which are probability distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.
Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players.
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Two Models of Procedural Rationality

Level-k Reasoning

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- Level-0 players choose uniformly at random
- Level-k players choose based on belief that others are level $k-1$
- Distribution of types implies distribution of strategies

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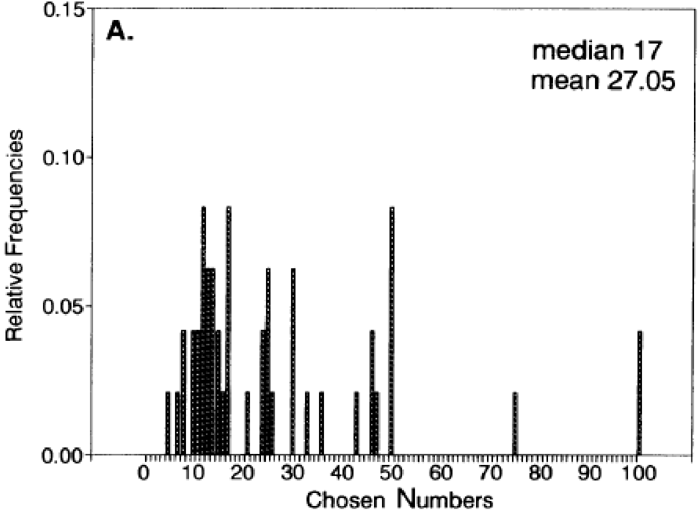
Fits behavior better than Nash in half-the-average and related games

Rosemarie Nagel (1995) Data



Level-k Equilibrium in the Public Goods Game

	H	M	L
H	6, 6	3, 7	0, 8
M	7, 3	4, 4	1, 5
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What about level-k for $k \geq 1$?

What does the model predict?

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Interpretation: steady state of a population with inflows and outflows

Sampling and Nash in All-Pay Auctions

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0	$\frac{5}{4}, \frac{5}{4}$	$0, \frac{3}{2}$	$0, \frac{1}{2}$
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$$w_0(p) = \left(\frac{1}{5} \times \frac{4}{5}\right) + \left(\frac{4}{5} \times \frac{1}{5} \times \frac{1}{5}\right) = \frac{24}{125} \neq \frac{1}{5}$$

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Which one should we expect to see?

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A stable sampling equilibrium is a stable rest point of the dynamics:

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Which of the equilibria in the public goods game is stable?

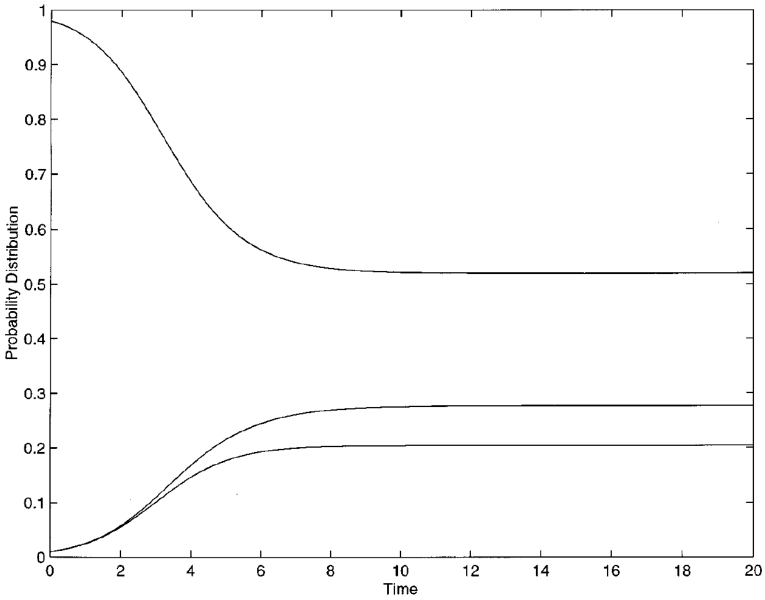

FIG. 1. Convergence to the interior $S(1)$ equilibrium.

Dynamics

Suppose initial state is $p = (\epsilon, \epsilon, 1 - 2\epsilon)$ with ϵ small

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So the strict Nash equilibrium is unstable

A Public Goods Game

3 players, 2 pure strategies $\{0, 1\}$ (contributions to a public good)

Define $S = a_1 + a_2 + a_3$ where a_i is player i contribution

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		$S - a_i$		
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a_i	0	1	$1 + \mu$	$1 + 2\mu$
	1	μ	2μ	3μ

where $3\mu > 1$

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If $\mu < 1$ NE has no contribution, if $\mu > 1$ then NE full contribution

All strict Nash equilibria are sampling equilibria

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	Nash	Stable Sampling
$\mu \in (1/3, 1/2)$	(1,0)	(1,0)
$\mu \in (1/2, 1)$	(1,0)	(0.72,0.28)
$\mu > 1$	(0,1)	(0.28,0.72)

So the strict Nash equilibrium is unstable even when it is efficient

Prediction: behavioral heterogeneity even in simple environments

Conclusions

Nash equilibrium works well in some cases, poorly in others

Same goes for alternative models (level k, sampling)

No solution concept has universal applicability

Approach to games must be context dependent

Further Reading

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