Phase Transitions in Physics and Computer Science

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Magnetism

- When cold enough, Iron will stay magnetized, and even magnetize spontaneously
- But above a critical temperature, it suddenly ceases to be magnetic
- Interactions between atoms remain the same, but global behavior changes!
- Like water freezing, outbreaks becoming epidemics, opinions changing...

The Ising model

- Lattice (e.g. square) with n sites
- Each has a "spin" $s_i = \pm 1$, "up" or "down"
- Energy is a sum over neighboring pairs:

$$E = -\sum_{ij} s_i s_j$$

- Lowest energy: all up or all down
- Highest energy: checkerboard

Boltzmann Distribution

- At thermodynamic equilibrium, temperature T
- Higher-energy states are less likely:

$$P(s) \sim e^{-E(s)/T}$$

- When $T \to 0$, only lowest energies appear
- When $T \to \infty$, all states are equally likely

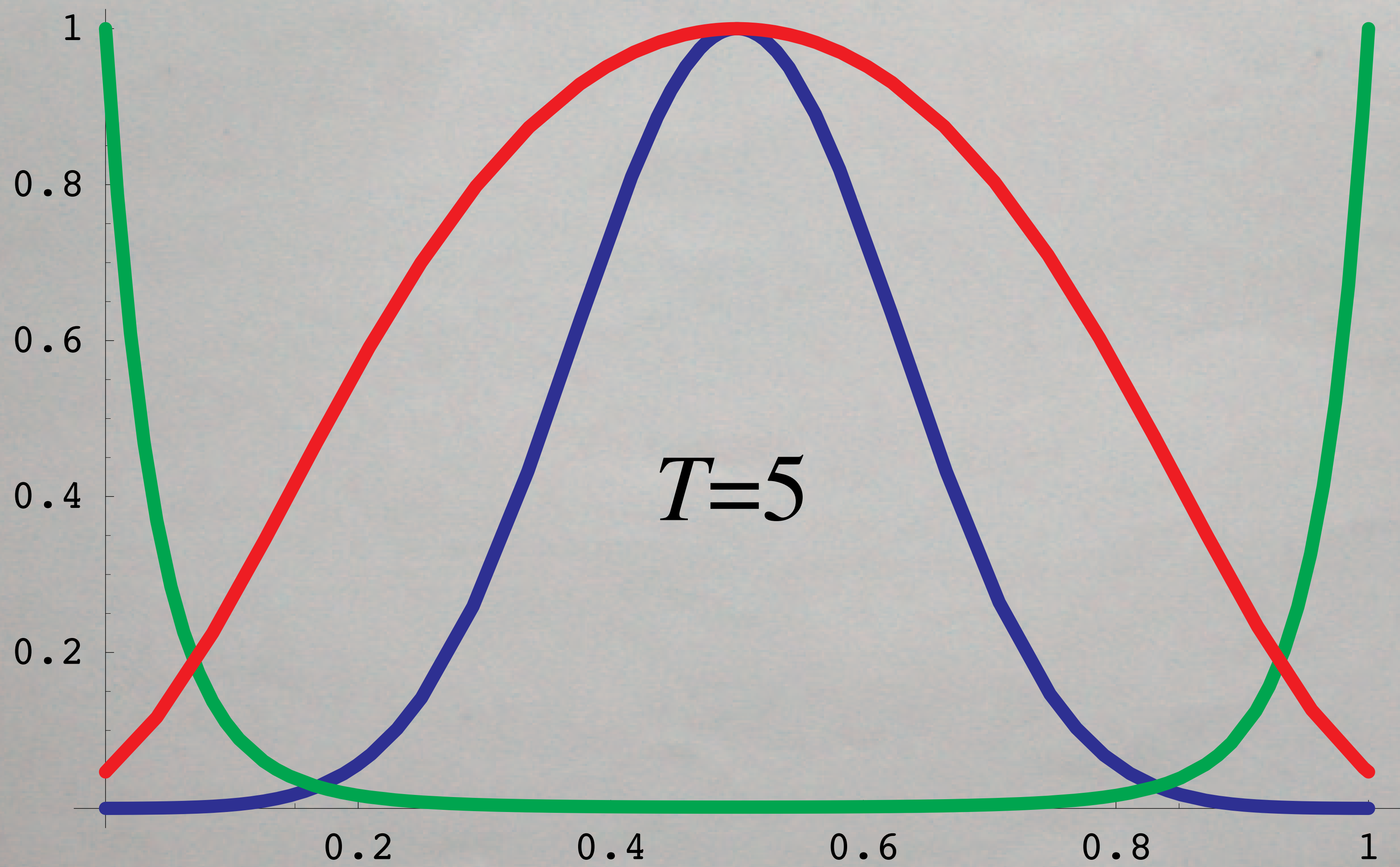
What Happens

- Below critical temperature, the system "magnetizes": mostly up or mostly down
- Small islands of the minority state; as T increases, these islands grow
- Above critical temperature, islands=sea; at large scales, equal numbers of up and down
- When $T=T_c$, islands of all scales: system is scale-invariant!

Mean Field

- Ignore topology: forget lattice structure
- If a of the sites are up and $1–a$ are down, energy is $E = 2n^2\left(2a(1-a) - a^2 - (1-a)^2\right)$
- At any T, most-likely states have a=0 or a=1
- But the number of such states is $\binom{n}{an}$, which is tightly peaked around a=1/2.
- Total probability(a) = #states(a) Boltzmann(a)

Energy vs. Entropy



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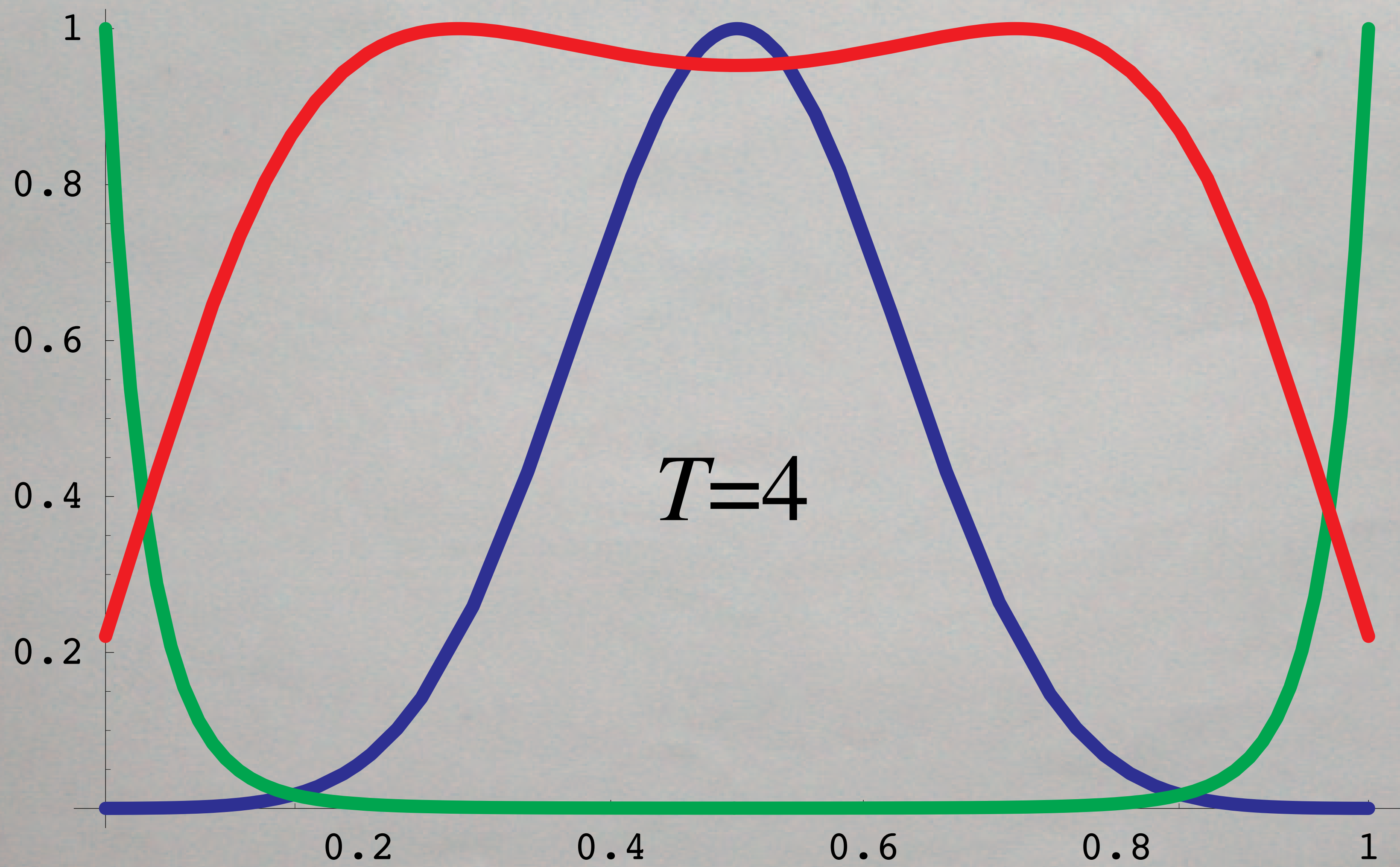
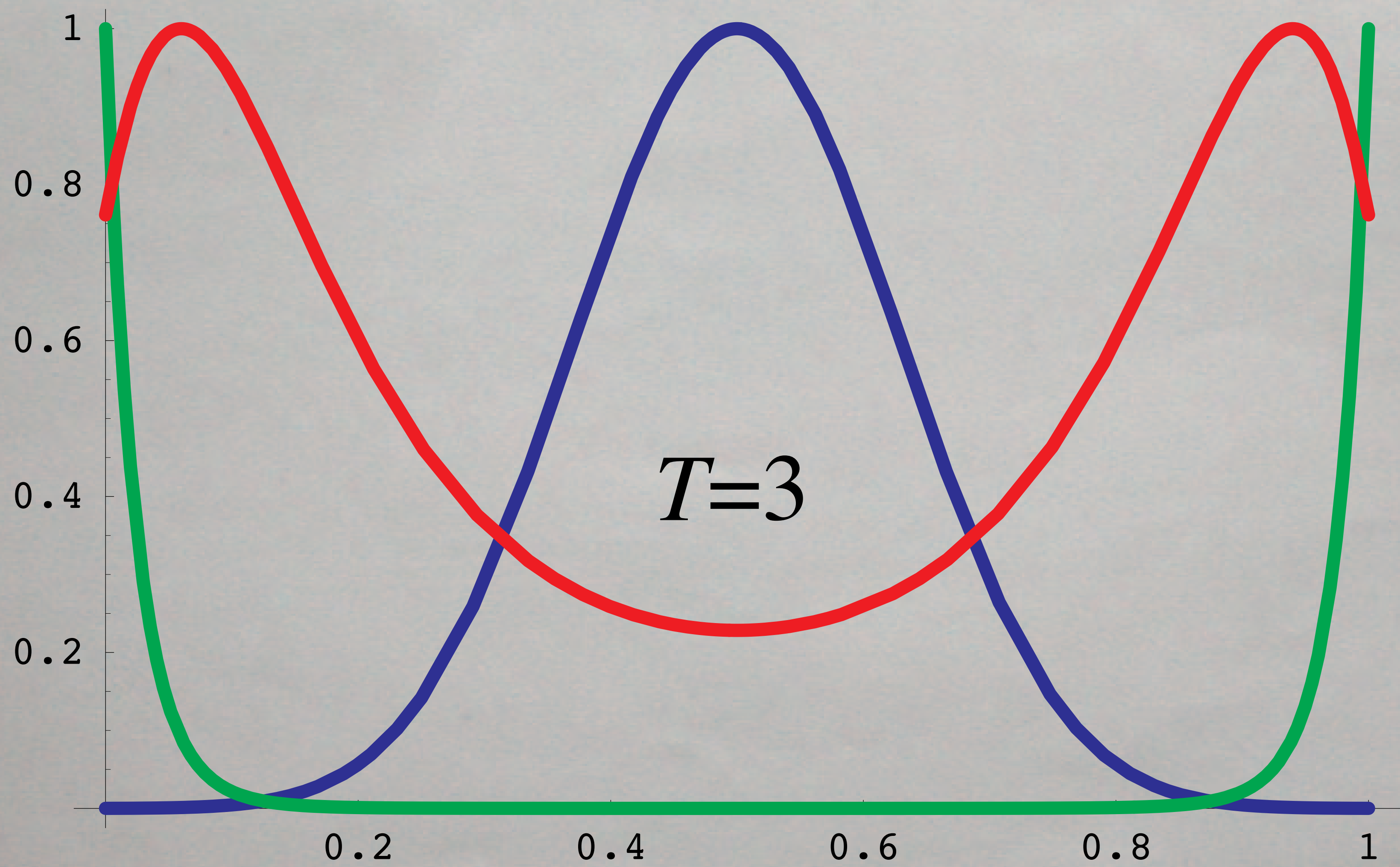


Energy vs. Entropy



Correlations

- $C(r)$ = correlation between two sites r apart
- If $T > T_c$, correlations decay exponentially:

$$C(r) \sim e^{-r/\ell}$$

- Correlation length ℓ decreases as T grows
- As we approach T_c, correlation length diverges
- At T_c, power-law correlations (scale-free):

$$C(r) \sim \ell^{-\alpha}$$

Percolation

- Fill a fraction p of the sites in a lattice
- When $p < p_c$, small islands, whose size is exponentially distributed:

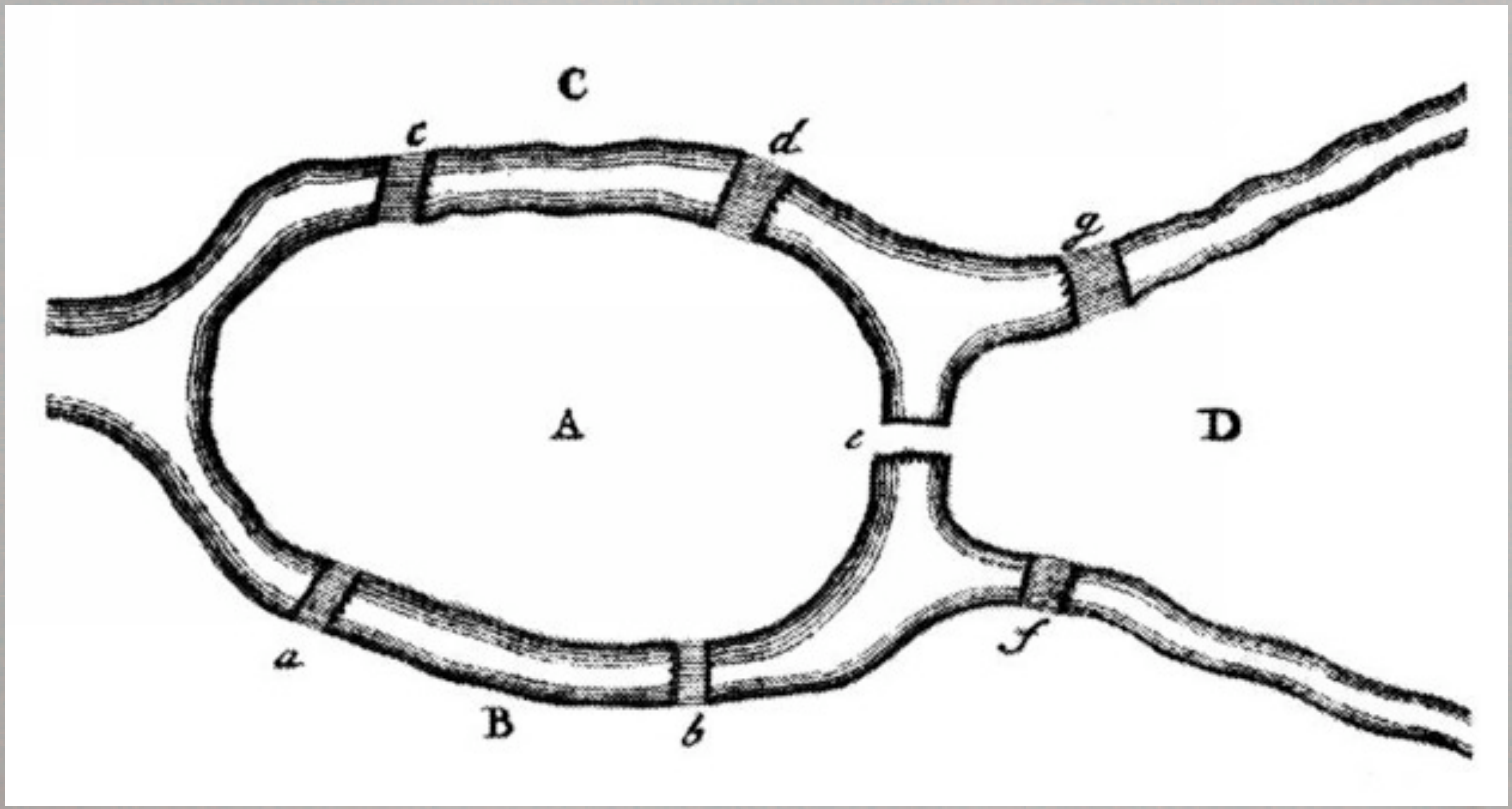
$$P(s) \sim e^{-s/\overline{s}}$$

- When $p > p_c$, "giant cluster" appears
- At p_c, power-law distribution of cluster sizes:

$$P(s) \sim s^{-\alpha}$$

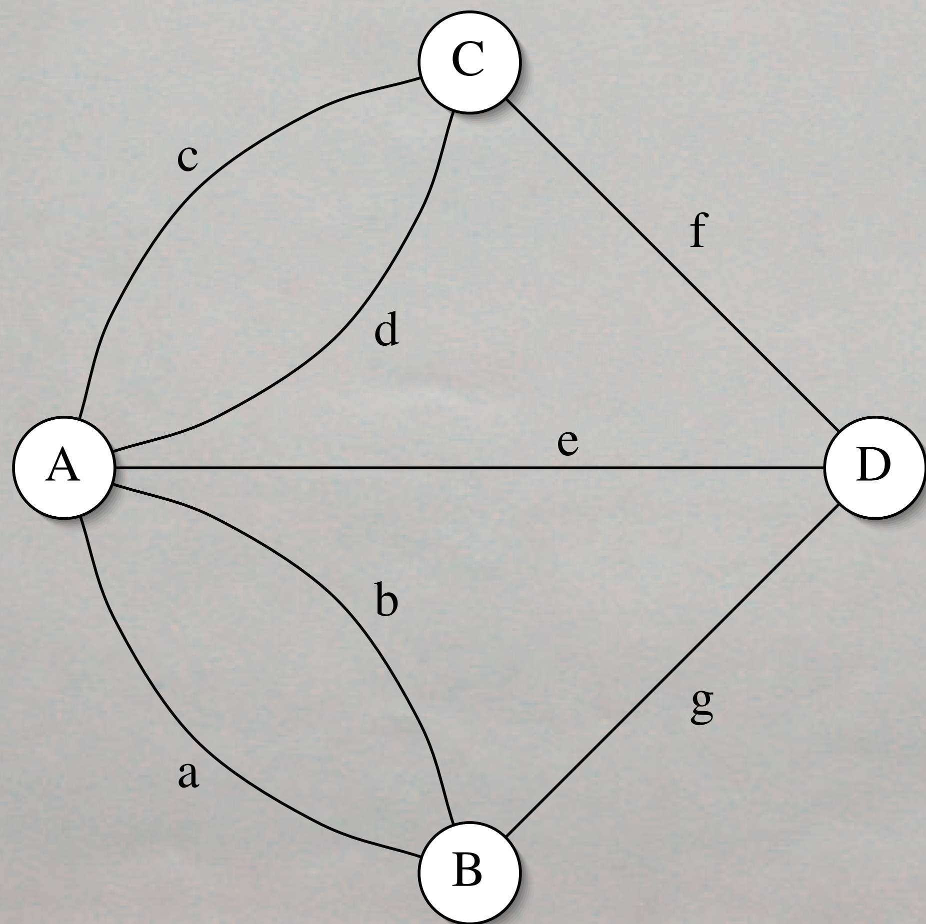
Computational Complexity

- Why are some problems qualitatively harder than others?



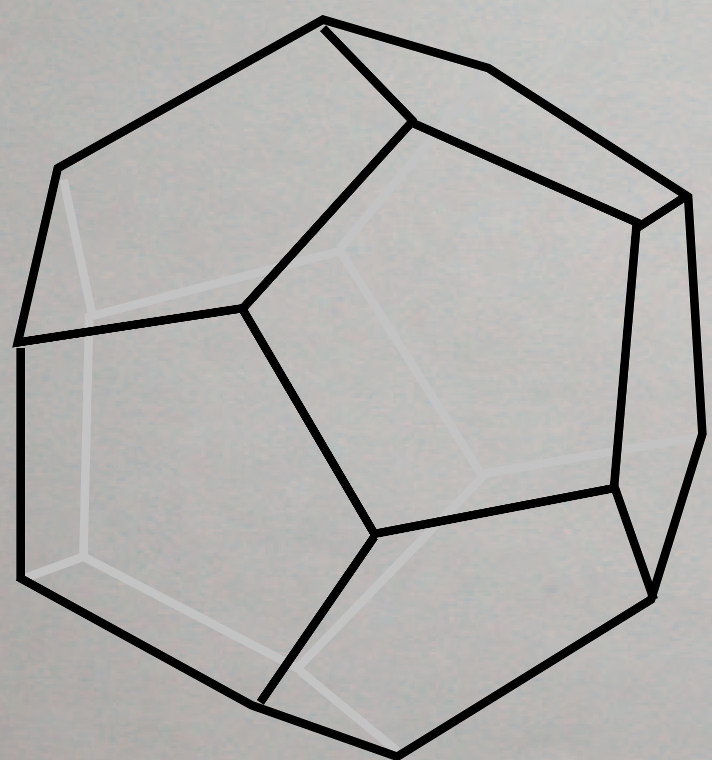
Computational Complexity

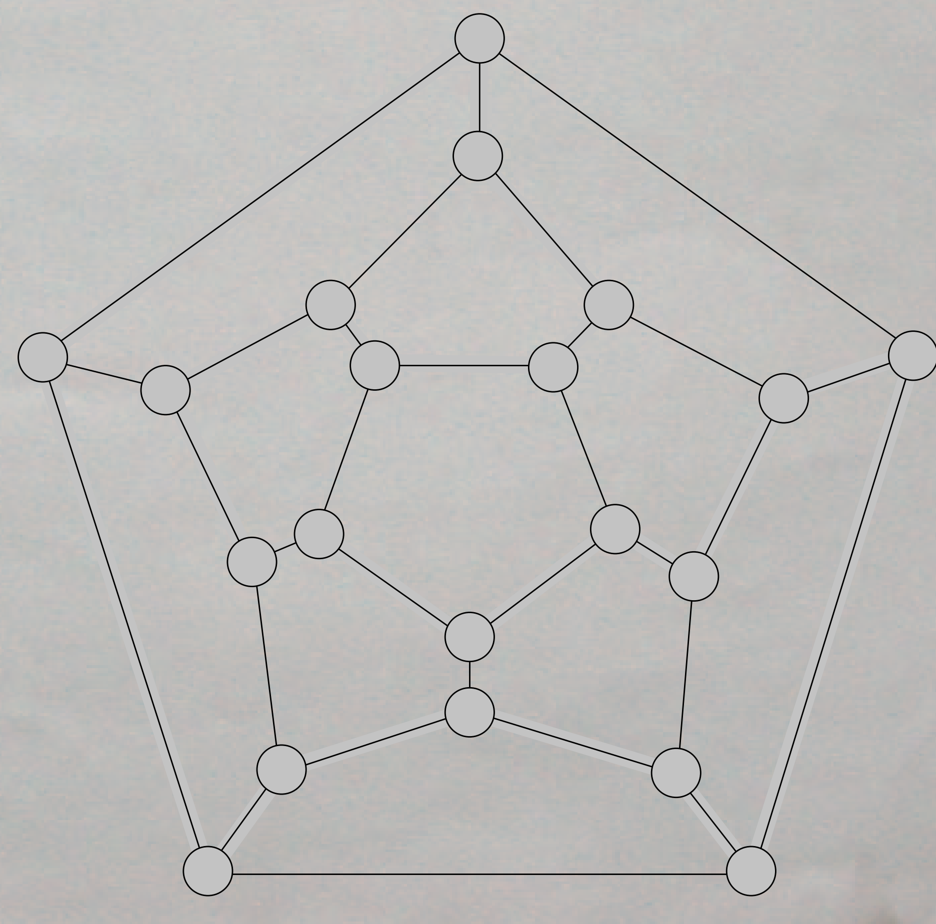
- A simple insight: at most 2 vertices can have odd degree, so no tour is possible!



Computational Complexity

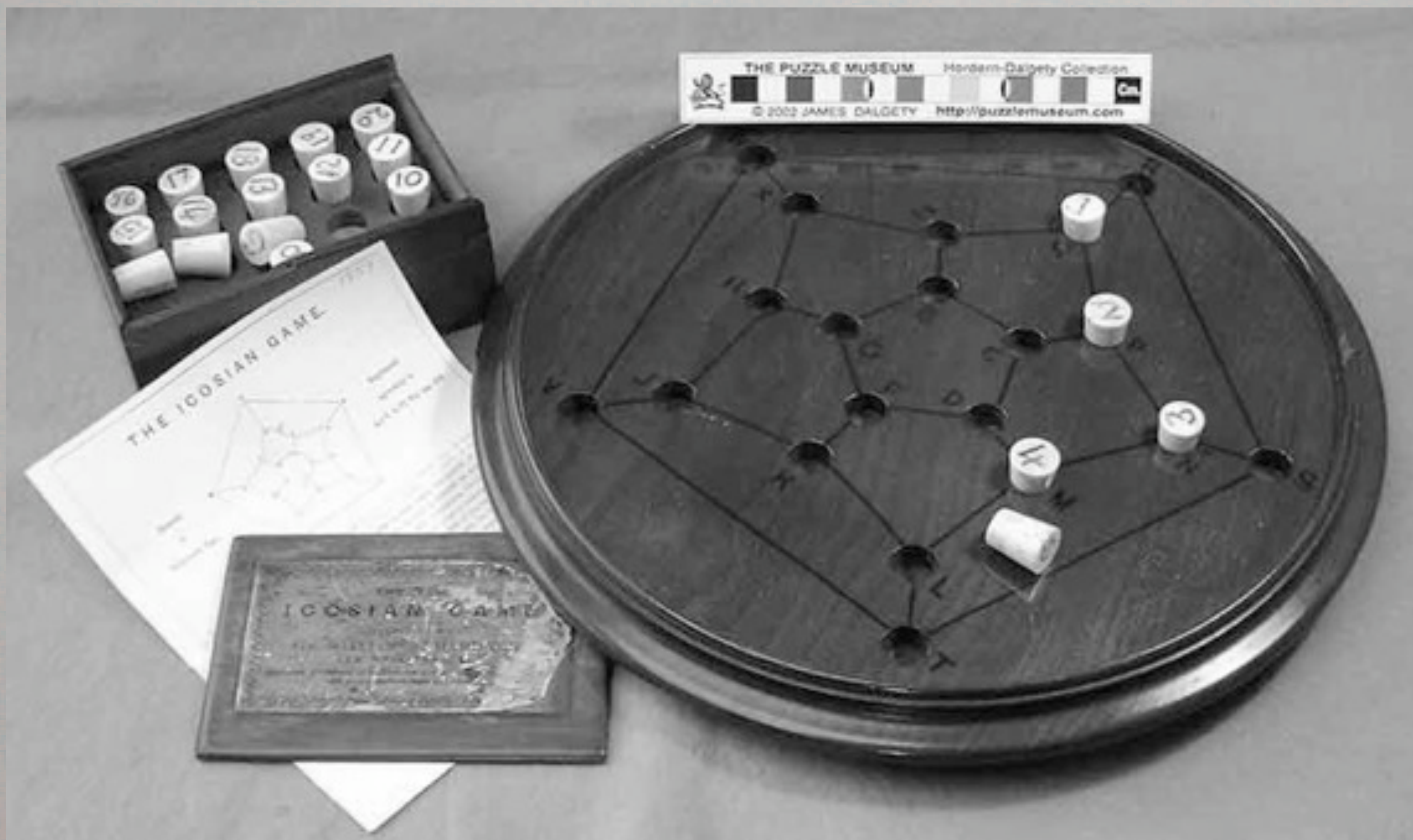
- What if we want to visit every vertex, instead of every edge?





Computational Complexity

- As far as we know, the only way to solve this problem is (essentially) exhaustive search!



Needles in Haystacks

- **P**: we can find a solution efficiently
- **NP**: we can *check* a solution efficiently

Complexity Classes

NP

Hamiltonian Path

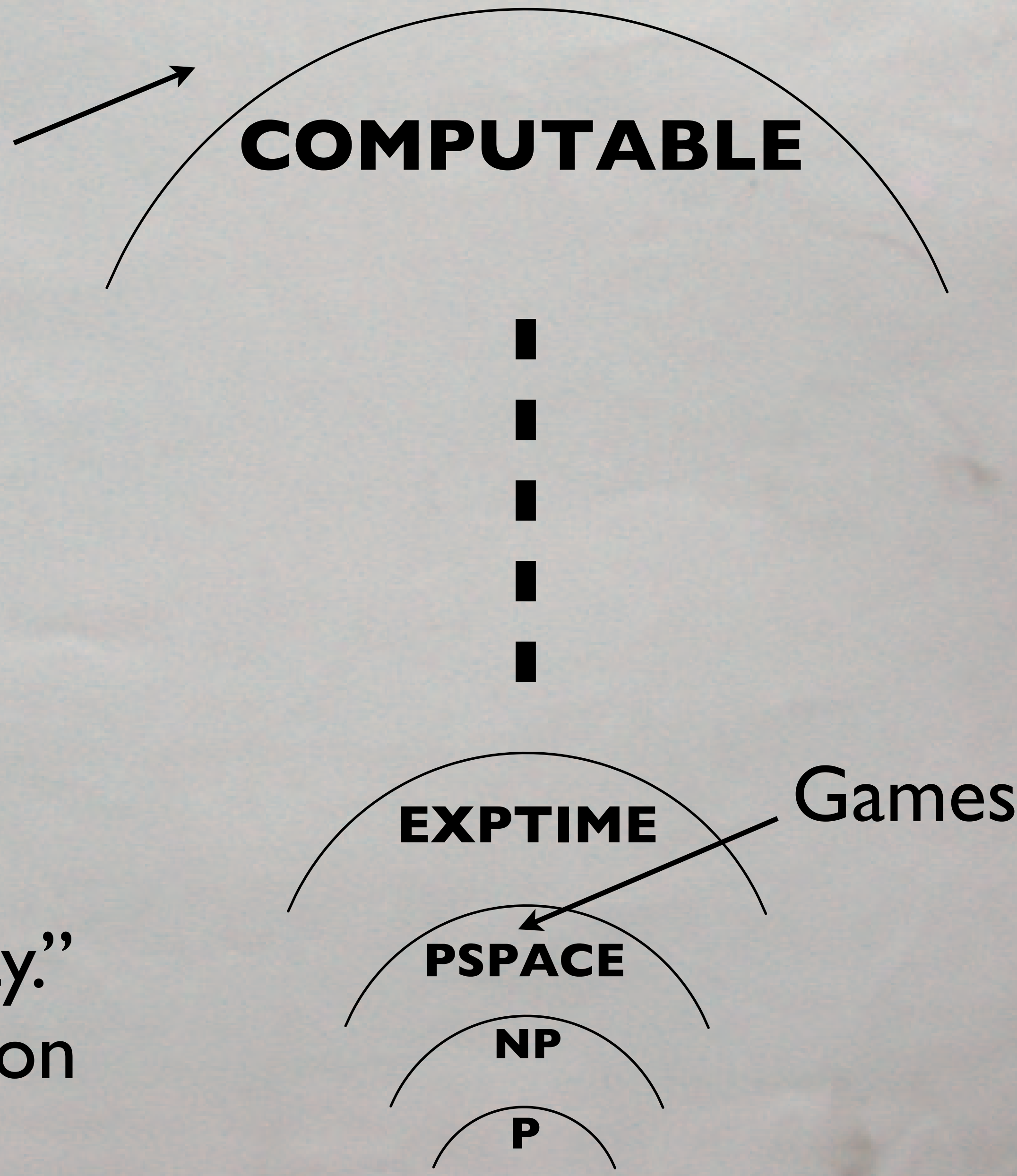
P

Eulerian Path

Multiplication

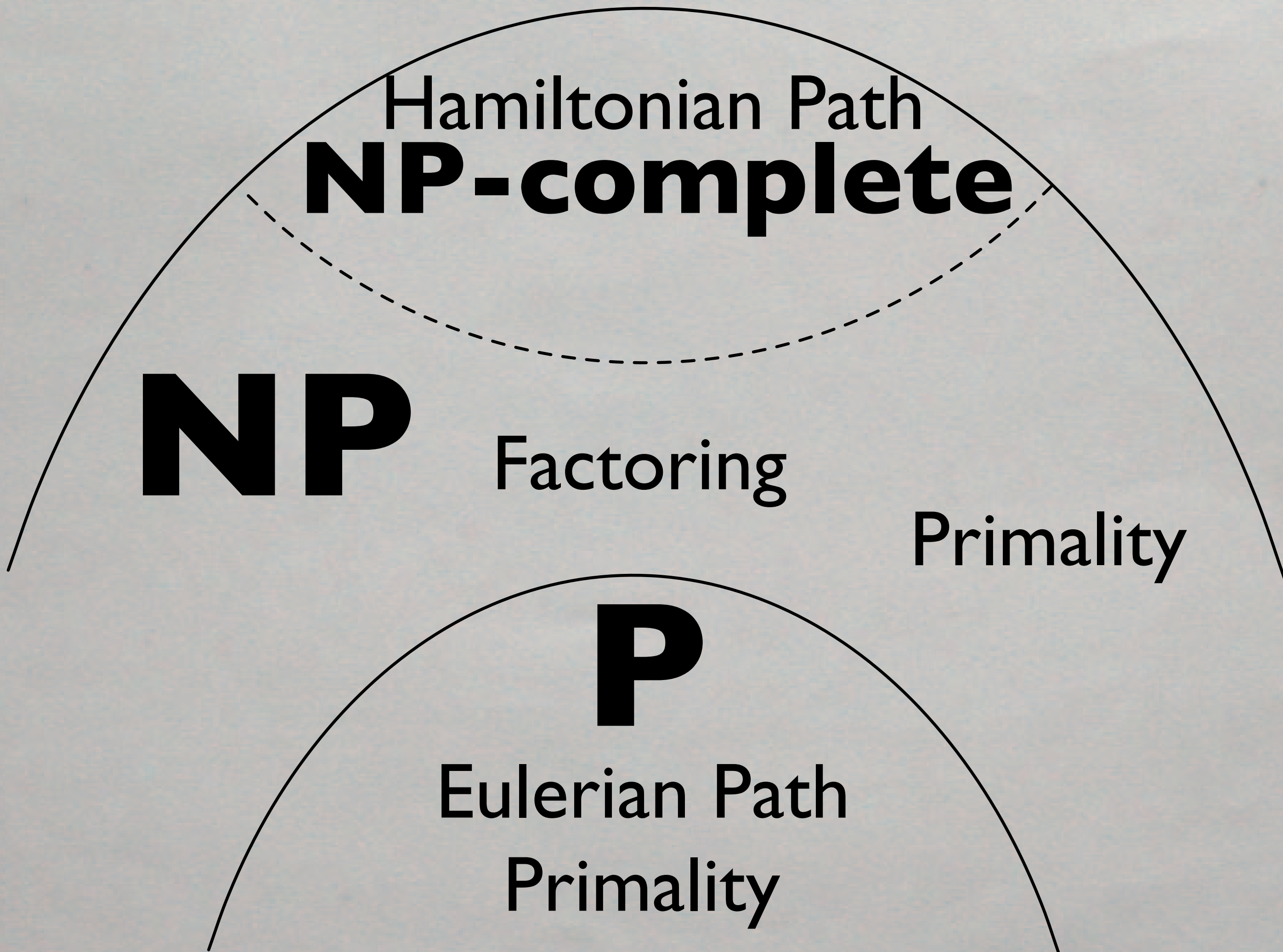
An Infinite Hierarchy

Turing's Halting Problem



"Computers play the same role in complexity that clocks, trains and elevators play in relativity."
– Scott Aaronson

The Hardest of Them All



Some problems in **NP** capture the entire class!

If we can solve any of them efficiently, then **P=NP.**

The Adversary

...designs problems that are as diabolically hard as possible, forcing us to solve them in the worst case. (Hated and feared by computer scientists.)

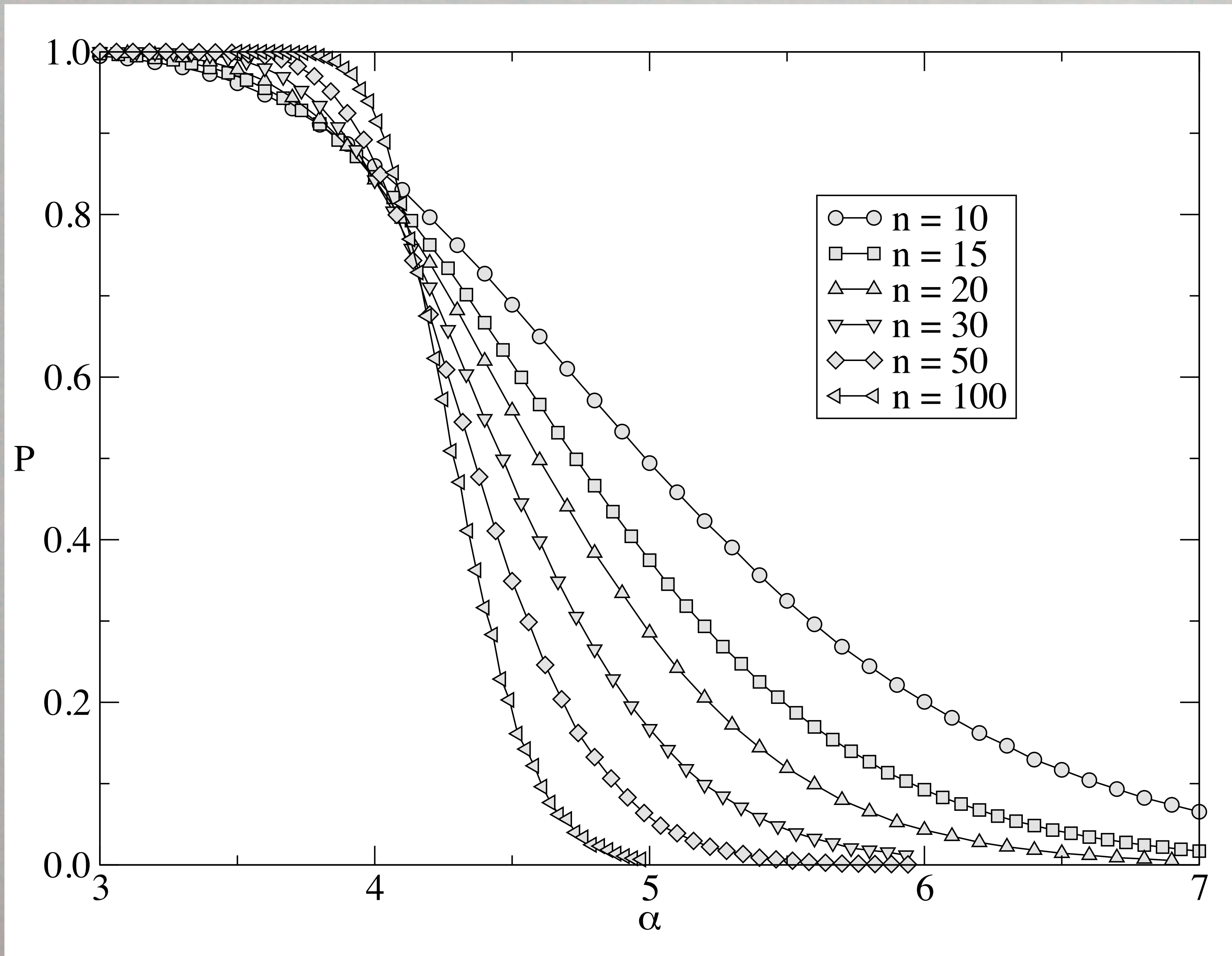
La Dame Nature

...asks questions whose answers are simpler and more beautiful than we have any right to imagine. (Worshipped by physicists.)

Random NP Problems

- A 3-SAT formula with n variables, m clauses
- Choose each clause randomly: $\binom{n}{3}$ possible triplets, negate each one with probability 1/2
- Precedents:
  - Random Graphs (Erdős-Rényi)
  - Statistical Physics: ensembles of disordered systems, e.g. spin glasses
- Sparse Case: $m = \alpha n$ for some density α

A Phase Transition



The Threshold Conjecture

- We believe that for each $k \geq 3$, there is a critical clause density α_k such that

$$\lim_{n \to \infty} \Pr\left[F_k(n, m = \alpha n) \text{ is satisfiable}\right]$$

$$= \begin{cases} 1 & \text{if } \alpha < \alpha_k \\ 0 & \text{if } \alpha > \alpha_k \end{cases}$$

- So far, only known rigorously for $k = 2$

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