So far: mostly about maps.

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation

Next up: flows

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations

Attractors

- Attractors exist only in dissipative systems!
- Dissipation $\Rightarrow$ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation... just not chaotic attractors

Courtesy of Allison Brown
Best Poster prize, Experimental Chaos Conference, 2012
Conditions for chaos in continuous-time systems

**Necessary:**
- Nonlinear
- At least three state-space dimensions (NB: only one needed in maps)

**Necessary and sufficient:**
- Cannot be solved in closed form (“nonintegrable,” in Hamiltonian parlance)

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Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter

---

**Equations:**

\[
\begin{align*}
x' &= a(y - x) \\
y' &= rx - y - xz \\
z' &= xy - bz
\end{align*}
\]

(first three terms of a Fourier expansion of the Navier-Stokes eqns)
• State variables:
  - $x$: convective intensity
  - $y$: temperature
  - $z$: deviation from linearity in the vertical convection profile

• Parameters:
  - $a$: Prandtl number - fluids property
  - $r$: Rayleigh number - related to $\Delta T$
  - $b$: aspect ratio of the fluid sheet

$x' = 16(y - x)$
$y' = 45x - y - xz$
$z' = xy - 4z$

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Before we leave Lorenz...

Attractors

Four types:

• fixed points
• limit cycles (aka periodic orbits)
• quasiperiodic orbits
• chaotic attractors

A nonlinear system can have any number of attractors, of all types, sprinkled around its state space.

Their basins of attraction (plus the basin boundaries) partition the state space.

And there’s no way, a priori, to know where they are, how many there are, what types, etc.
Attractors

• Fixed point

Attractors

• Limit cycle

Attractors

• Quasi-periodic orbit…

“Strange” or chaotic attractors

• often fractal
• covered densely by trajectories
• exponential divergence of neighboring trajectories…

Lyapunov exponents

• nonlinear analogs of eigenvalues: one $\lambda$ for each dimension

Lyapunov exponents: summary

• nonlinear analogs of eigenvalues: one $\lambda$ for each dimension
• negative $\lambda_i$ compress state space; positive $\lambda_i$ stretch it
• $\Sigma \lambda_i < 0$ for dissipative systems
• $\lambda_i$ are same for all ICs in one basin
• long-term average in definition; biggest one ($\lambda_1$) dominates as $t \to \infty$
• positive $\lambda_1$ is a signature of chaos
“Strange” or chaotic attractors:

- exponential divergence of neighboring trajectories
- often fractal
- covered densely by trajectories
- contain an infinite number of “unstable periodic orbits”…

Unstable periodic orbits (UPOs)

“Attractor bones”…

Poincare recurrence