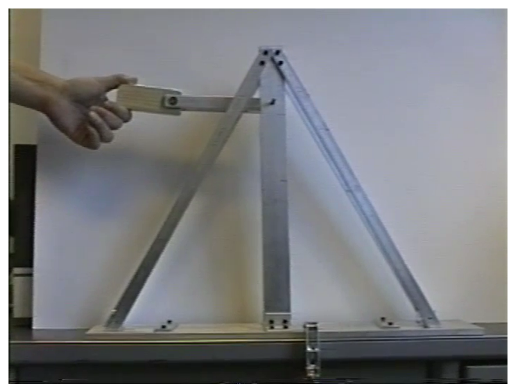
So far: mostly about *maps.*

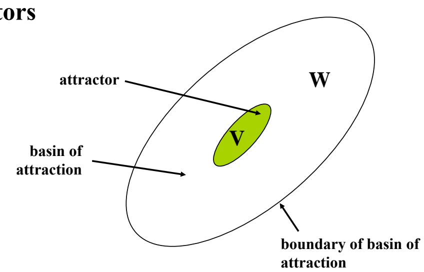
- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: differ***ence*** equation

Next up: *flows*

- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differ***ential*** equations



Attractors



- Attractors exist only in dissipative systems!
- Dissipation ⟺ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

Courtesy of Allison Brown
Best Poster prize, Experimental Chaos Conference, 2012

Conditions for chaos in continuous-time systems

Necessary:

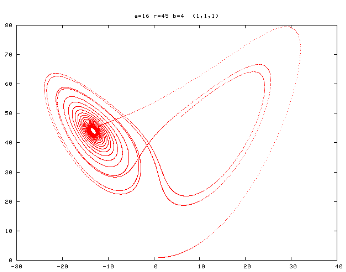
- Nonlinear
- At least three state-space dimensions (NB: only one needed in maps)

Necessary and sufficient:

- Cannot be solved in closed form ("nonintegrable," in Hamiltonian parlance)

Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter



Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

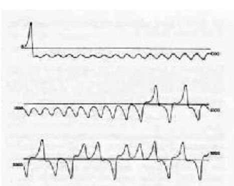


FIG. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

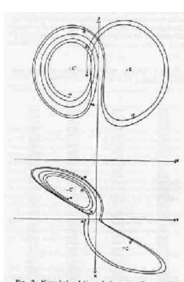


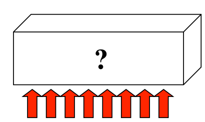
FIG. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

- Equations:

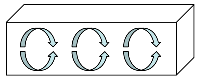
$x' = a(y - x)$

$y' = rx - y - xz$

$z' = xy - bz$



(first three terms of a Fourier expansion of the Navier-Stokes eqns)



- State variables:
  - *x* convective intensity
  - *y* temperature
  - *z* deviation from linearity in the vertical convection profile

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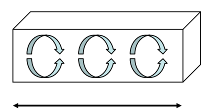
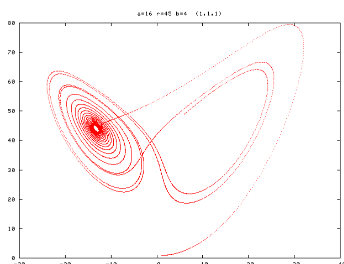
- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to ΔT 
  - *b* aspect ratio of the fluid sheet

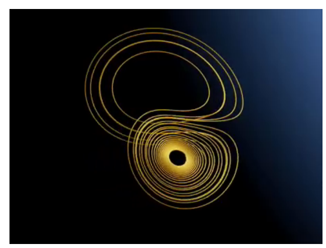


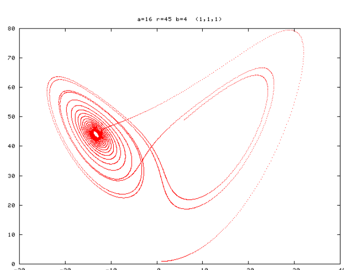


$x' = 16(y - x)$

$y' = 45x - y - xz$

$z' = xy - 4z$





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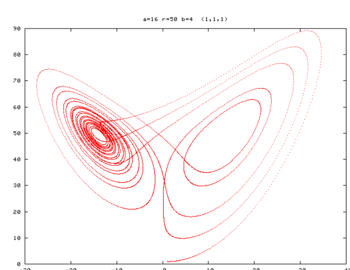
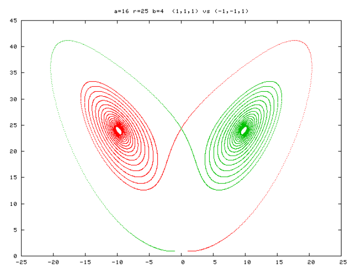
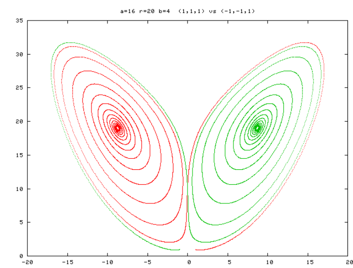


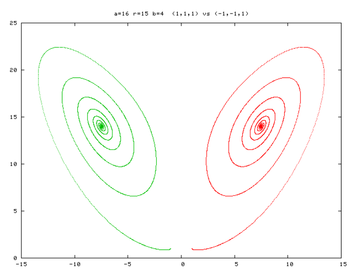
$x' = 16(y - x)$

$y' = \mathbf{50}x - y - xz$

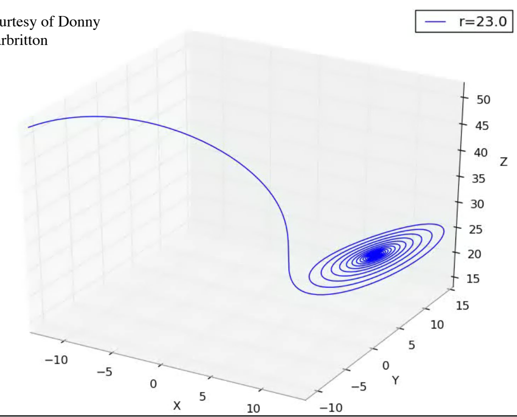
$z' = xy - 4z$



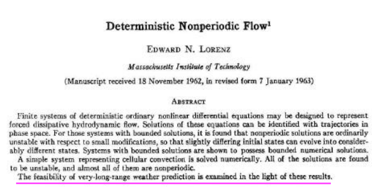




Courtesy of Donny Warbritton



Before we leave Lorenz…



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The feasibility of very-long-range weather prediction is examined in the light of these results.

Attractors

Four types:

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

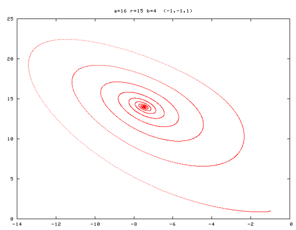
A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction (plus the basin boundaries) *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc.

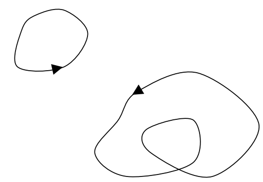
Attractors

• Fixed point



Attractors

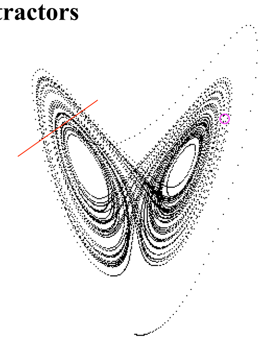
• Limit cycle



Attractors

• Quasi-periodic orbit…

"Strange" or chaotic attractors



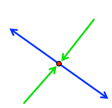
• *often* fractal

• covered densely by trajectories

• exponential divergence of neighboring trajectories…

Lyapunov exponents

• nonlinear analogs of eigenvalues: one λ for each dimension



Lyapunov exponents: summary

• nonlinear analogs of eigenvalues: one λ for each dimension

• negative λ_i compress state space; positive λ_i stretch it

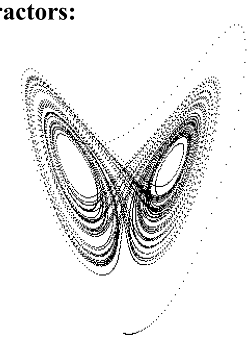
• $\Sigma\lambda_i < 0$ for dissipative systems

• λ_i are same for all ICs in one basin

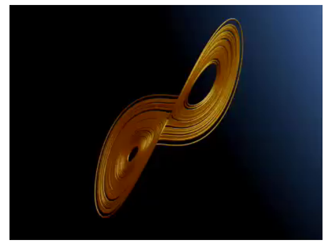
• long-term average in definition; biggest one (λ_1) dominates as t $\rightarrow \infty$

• *positive* λ_1 *is a signature of chaos*

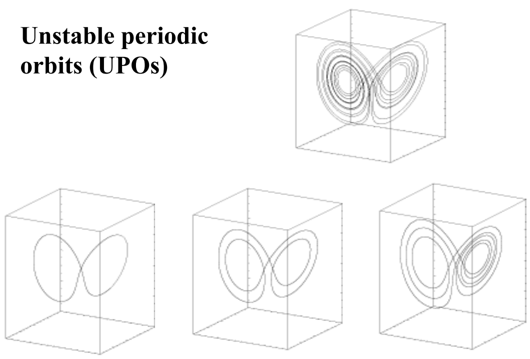
“Strange” or chaotic attractors:



- exponential divergence of neighboring trajectories
- *often* fractal
- covered densely by trajectories
- contain an infinite number of “unstable periodic orbits”…

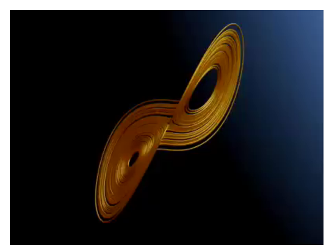


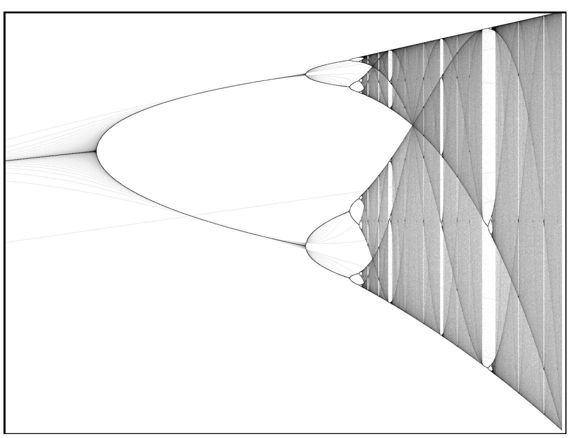
Unstable periodic orbits (UPOs)



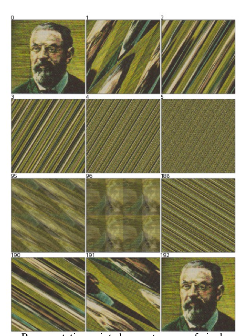
Bradley/Mantilla, *Chaos* **12**:596

“Attractor bones”…

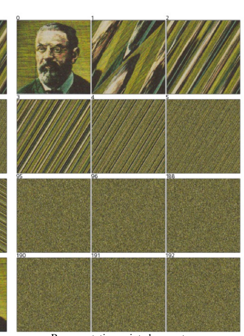




Poincare recurrence



Representative point chosen at corner of pixel



Representative point chosen at a random place in pixel

Crutchfield *et al*. *Chaos* **255**:46 and
http://www.mpipks-dresden.mpg.de/mpi-doc/kantzgruppe/wiki/projects/Recurrence.html