Cooperation networks

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Nodes
Nodes and edges.
Nodes and edges.

Why edges, structure, cooperation?
**Node:** something alive.
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H. Morowitz:

the disordering and degradative influences of thermal energy as reflected in the second law of thermodynamics. Living systems self-replicate, that is, they give rise to organisms like themselves, thus accounting for the continuity of life. Living organisms must
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Simplest model: noisy multiplicative growth, *e.g.* of biomass, group size, wealth *etc.*
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Edge: resource-pooling, support, sharing.
...but

- Expectation value of resources unaffected by pooling (conservation law).
- Richer entity loses when resources are pooled.
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**Standard answer:** new function.
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  - What’s “function”?
  - Small/simple/early networks have no new function.
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**Key insight**

Growth is not ergodic, *i.e.* expectation value does not reflect what happens over time.

Not in expectation, but over time, pooling leads to faster growth.
Game

Heads: win 50%. 
Tails: lose 40%.
Toss coin once a minute

Time in minutes

money in $

0 100 200 300 400 500 600

1 2 3 4 5
One sequence
Fermat and Pascal 1654:
Imagine all possibilities and average over them.
Call this average the “expectation value.”
Fermat and Pascal 1654: 10 sequences
Fermat and Pascal 1654:
20 sequences

money in $ vs. Time in minutes
Fermat and Pascal 1654:
Average of 20 sequences
Fermat and Pascal 1654:
Average of 1000 sequences

money in $
Time in minutes

$\begin{align*}
\text{money in $} \\
\text{Time in minutes}
\end{align*}$
Fermat and Pascal 1654:
Average of 1,000,000 sequences

<table>
<thead>
<tr>
<th>Time in minutes</th>
<th>Money in $</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>30</td>
<td>100</td>
</tr>
<tr>
<td>40</td>
<td>1000</td>
</tr>
<tr>
<td>50</td>
<td>10000</td>
</tr>
</tbody>
</table>

The graph shows a linear increase in money over time.
Conclusion
Game worth playing, on average.
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Puzzle
But people won’t play – why?
Conclusion
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Solution: psychology, irrationality, utility theory.
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Alternative: What average?
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Solution: psychology, irrationality, utility theory.

Alternative: What average?
Game worth playing if averaging across parallel universes.
**Boltzmann 1872:**

"Is expectation value also average over time?"
Boltzmann 1872:

“Is expectation value also average over time?”

→ Play for a long time, and see what happens.
BOLTZMANN 1872:
Play for one hour...

![Graph showing money in dollars over time in minutes.](image-url)
BOLTZMANN 1872:
..continue one day (note scales)...

[Graph showing a decay curve with logarithmic scales for money in $ and time in hours.]
Boltzmann 1872:
..continue one week (note scales)...

![Graph showing the decay of money over time in days](image)
BOLTZMANN 1872:
...continue one year (note scales)...

![Graph showing money in dollars over time in months.](image-url)
Ensemble perspective

Time in minutes

1654

Money in $

10000

1000

100

10

1

10 20 30 40 50 60

Time perspective

Time in months

1872

Money in $

10^{10,000}

100

10^{-10,000}

3 6 9 12
Ensemble perspective

Non-ergodic

Time perspective

1654

1872

money in $
... people won’t play – why?
... people won’t play – why?

Because they would lose over time.
... people won’t play – why?

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We care about the future (time), not the multiverse (expectation values).
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**Shameless ad**

Formal economics built on expectation values.

My program

Re-develop all of economics.
Two living things, following noisy exponential growth

\[ dx_i = x(\mu dt + \sigma dW_i) \]
Expectation value (Fermat 1654)

$$\langle x(t) \rangle = \exp(\mu t)$$

Expectation-value growth rate

$$g_\langle \rangle = \frac{1}{t} \ln \left( \frac{\langle x(t) \rangle}{x(0)} \right) = \mu$$
The diagram illustrates the growth of wealth over rounds for two variables, $x_1$ and $x_2$, along with an expectation value line. The wealth is plotted on a logarithmic scale, showing exponential growth.
Time-average growth rate (OP 2011)

\[ g_t = \lim_{t \to \infty} \frac{1}{t} \ln \left( \frac{x(t)}{x(0)} \right) = \mu - \frac{\sigma^2}{2} \]

(building on Whitworth 1870, Itô 1944, Kelly 1956 etc.)
Ole Peters

1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

round

10
-50
10
0
10
50
10
100
10
150
10
200
10
250

wealth

x1
x2
expectation value
long-time growth

wealth
x 1
x 2
expectation value
long-time growth
Fluctuations reduce growth.

- reducing fluctuations increases growth (risk management).
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💡 cooperation reduces fluctuations 💡
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coop. reduces fluctuations  coop. increases growth.
Fluctuations reduce growth.

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  cooperation reduces fluctuations

→ cooperation increases growth.

→ cooperators outcompete non-cooperators.
$y_1(t) + \Delta y_1 + y_2(t) + \Delta y_2$

$y_1(t) + \Delta y_1$

$y_2(t) + \Delta y_2$

$y_1(t + \Delta t)$

$y_2(t + \Delta t)$
No cooperation:

\[ dx = x(\mu dt + \sigma dW) \]
\[ g_t(x) = \mu - \frac{\sigma^2}{2} \]

Cooperation reduces \( \sigma \):

\[ dy = y(\mu dt + \frac{\sigma}{\sqrt{2}} dW) \]
\[ g_t(y) = \mu - \frac{\sigma^2}{4} \]
Big thoughts

- Emergence: cooperating whole (network) is more than sum of non-cooperating parts (nodes).
- In multiplicative growth, pooling is not just adding – different mathematics.
  Conservation law broken by growth $\rightarrow$ non-linear.
- Much of biology is risk management.
Explained the link – atom of network design.

**Extension to network design**

- Cooperating with inferior entity, $\mu_1 > \mu_2$, still beneficial, if
  \[ \mu_1 - \frac{\sigma_1^2}{2} > \frac{\mu_1 + \mu_2}{2} - \frac{\sigma_1^2 + \sigma_2^2}{8}. \]

- Fully connected graph with partial sharing (weighted edges): taxation
  \[ dx_i = x_i[(\mu - \tau)dt + \sigma dW] + \langle x \rangle_N \tau dt. \]

- $n$ cooperators grow at $\mu - \frac{\sigma^2}{2n}$. 
Explained the link – atom of network design.

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- Cooperating with inferior entity, \( \mu_1 > \mu_2 \), still beneficial, if
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  dx_i = x_i[(\mu - \tau)dt + \sigma dW] + \langle x \rangle_N \tau dt.
  \]
  \[
  = x_i[\mu dt + \sigma dW] - (x_i - \langle x \rangle_N)\tau dt.
  \]

- \( n \) cooperators grow at \( \mu - \frac{\sigma^2}{2n} \).
Conditions

- Important for early-growth networks, prior to new function (Geoff West’s economies of scale, specialization etc.).
- Correlation between cooperators reduces cooperation benefit.
- Fluctuations increase cooperation benefit.
Business world

- Used for big financial risk management systems.
Thank you.