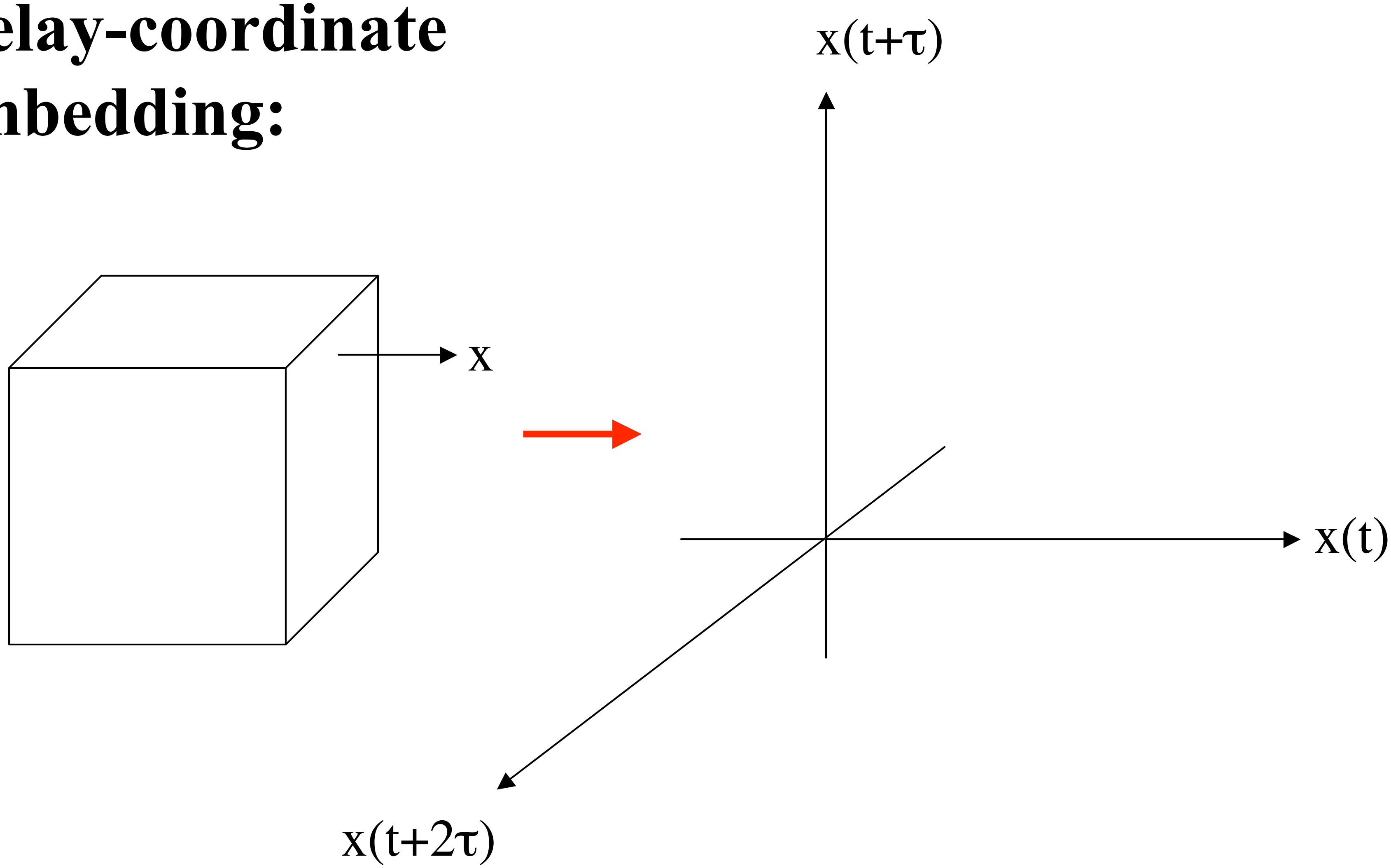
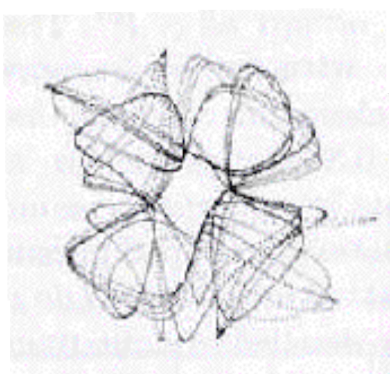
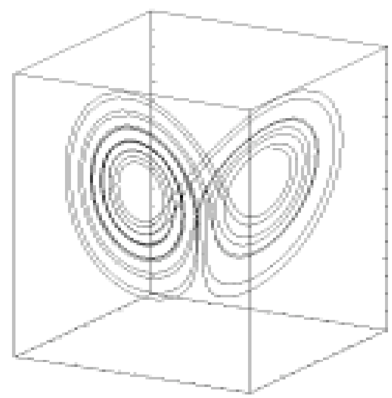
Delay-coordinate embedding:



Takens* theorem:

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (have same topology as) the original state-space dynamics.





If Δt is not uniform:

Theorem (Takens): for $\tau > 0$ and $m > 2d$, reconstructed trajectory is diffeomorphic to the true trajectory

Conditions: evenly sampled in time

Interspike interval embedding:

idea: lots of systems generate spikes — hearts, nerves, etc.

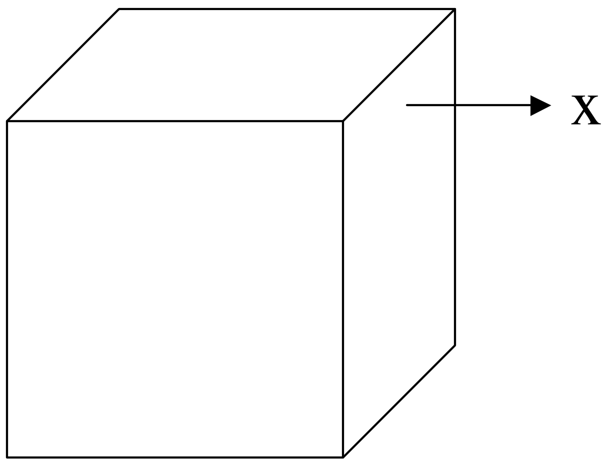
if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

in which case the Takens theorem still holds.

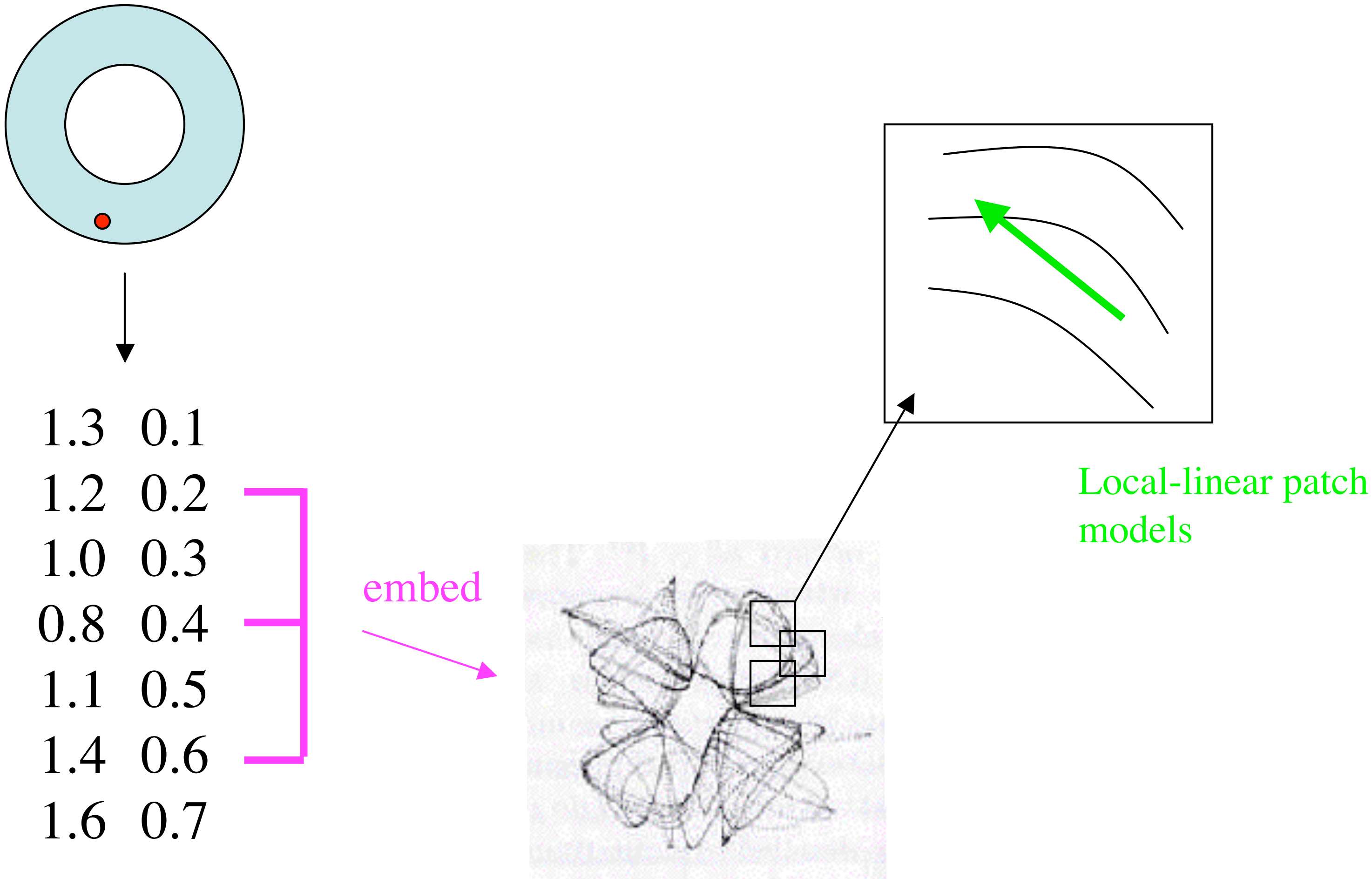
(with the Δts as state variables)

Sauer, *Chaos* **5**:127

What if that black box were a roulette wheel?

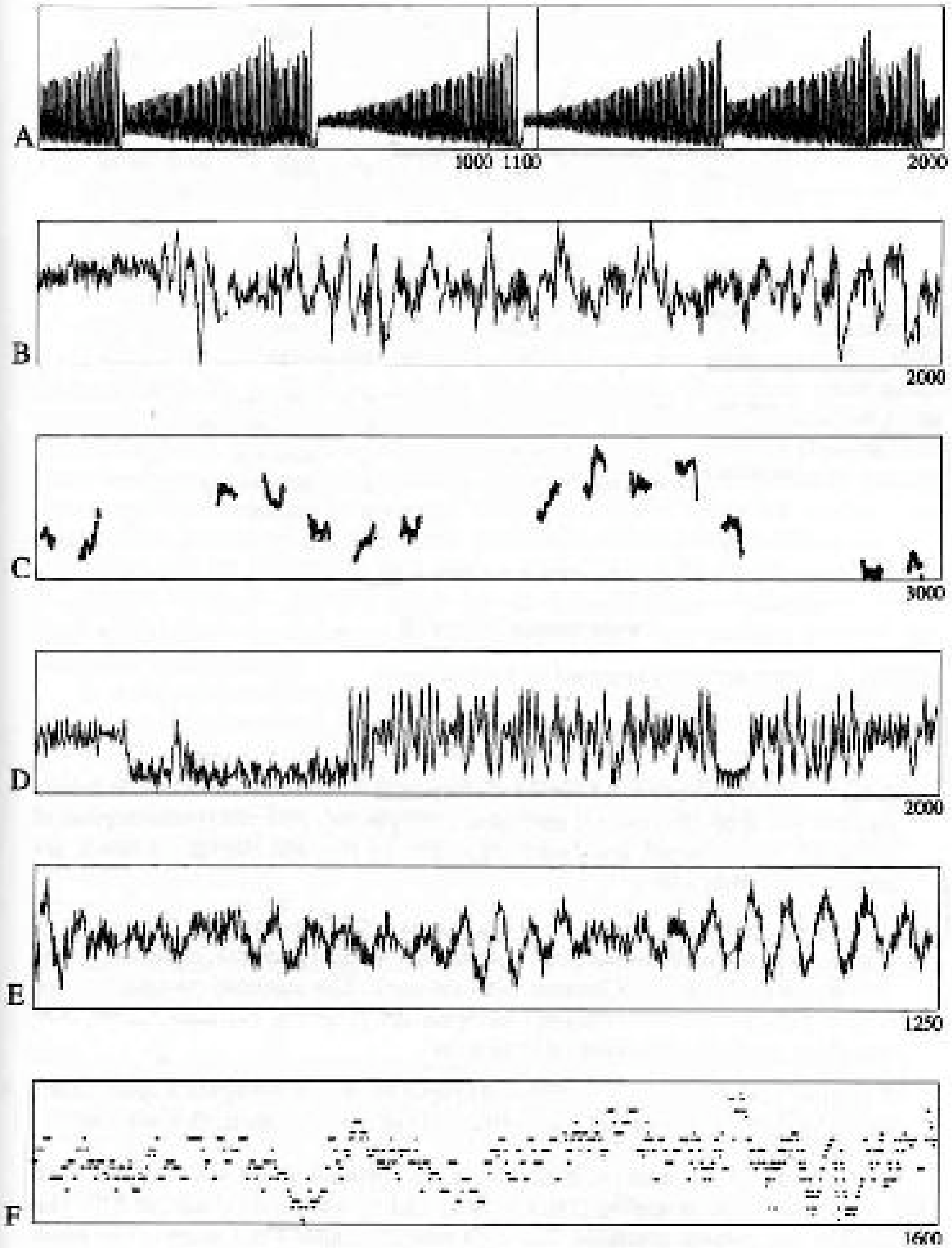


The Eudaemonic Pie
(or The Newtonian Casino)


1.3 0.1
1.2 0.2
1.0 0.3
0.8 0.4
1.1 0.5
1.4 0.6
1.6 0.7
embed
Local-linear patch models

The Santa Fe competition:

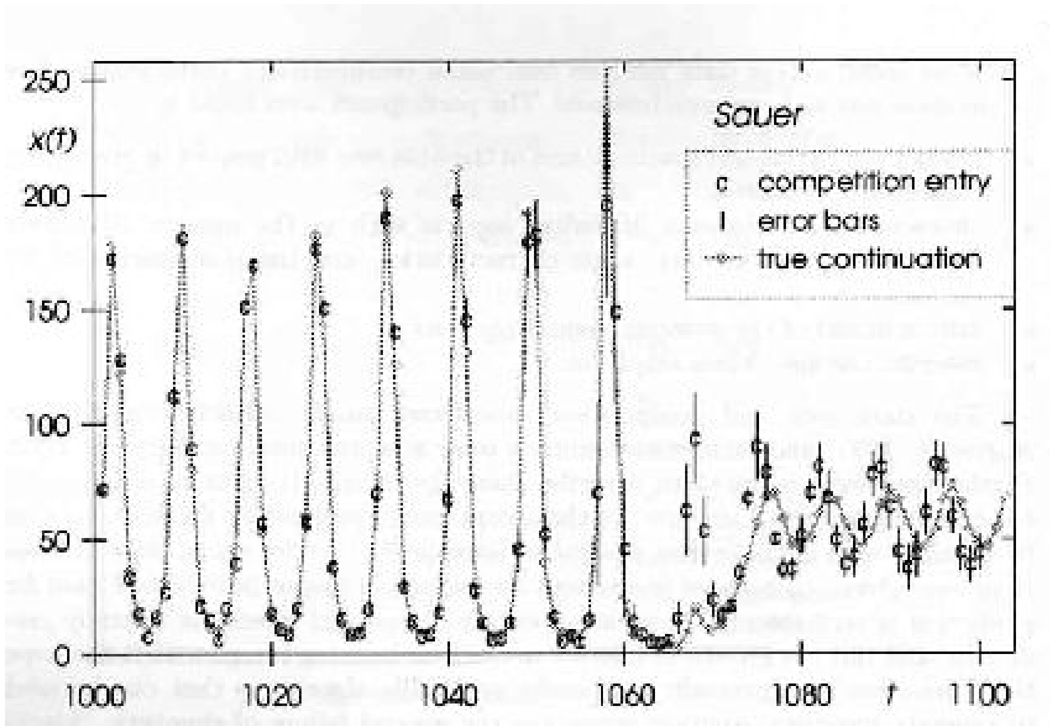
- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past,* Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)


A
1000 1100
2000
B
2000
C
3000
D
2000
E
1250
F
1600

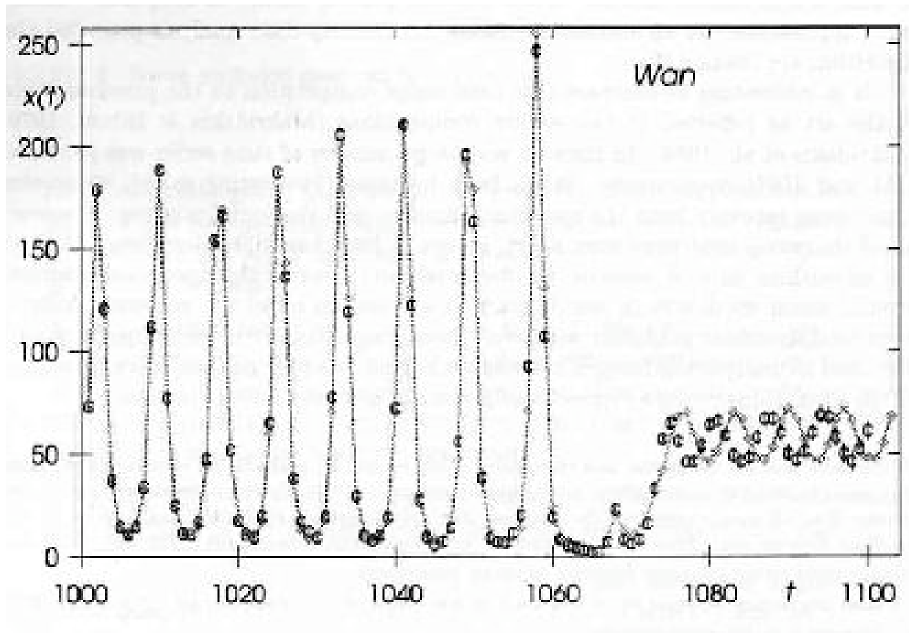
The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

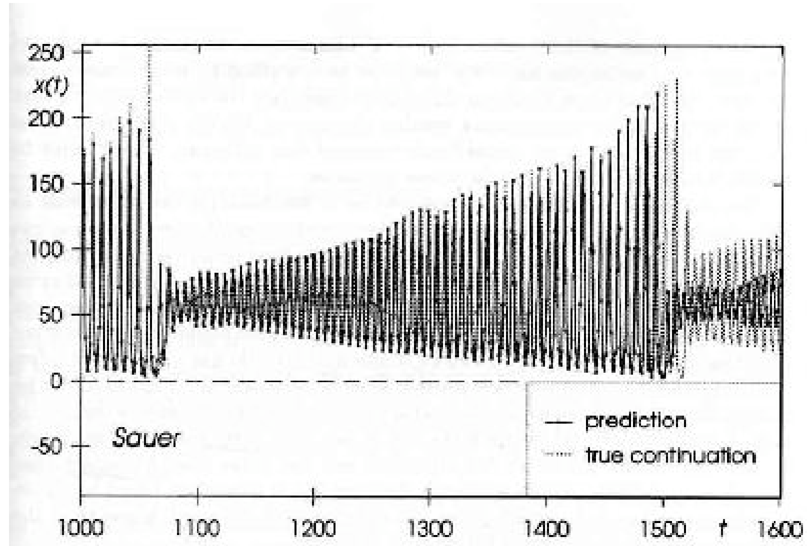
Embedding + patch models: (Sauer)



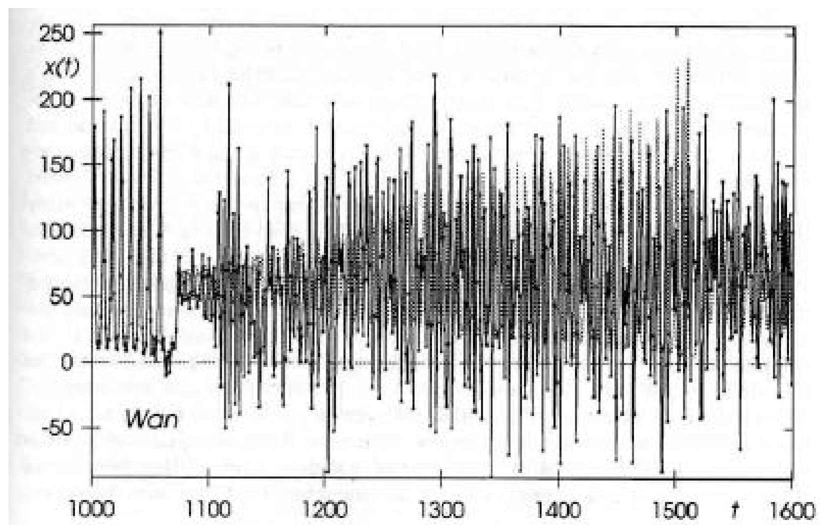
Neural net: (Wan)



Further out:



Sauer



Wan

Sauer's algorithm:

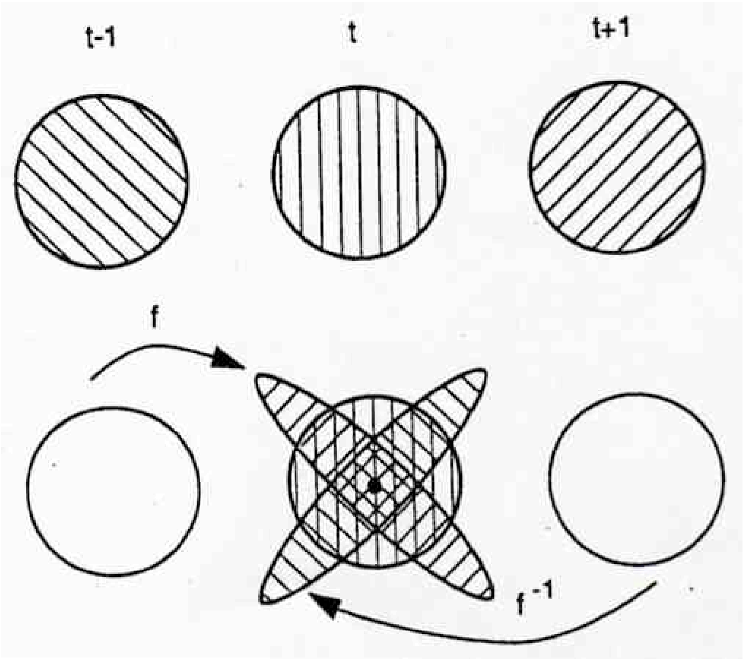
In his competition entry, shown in Figure 3, Sauer used a careful implementation of local-linear fitting that had five steps:

1. Low-pass embed the data to help remove measurement and quantization noise. This low-pass filtering produces a smoothed version of the original series. (We explained such filtered embedding at the end of Section 4.1.)
2. Generate more points in embedding space by (Fourier-) interpolating between the points obtained from Step 1. This is to increase the coverage in embedding space.
3. Find the k nearest neighbors to the point of prediction (the choice of k tries to balance the increasing bias and decreasing variance that come from using a larger neighborhood).
4. Use a local SVD to project (possibly very noisy) points onto the local surface. (Even if a point is very far away from the surface, this step forces the dynamics back on the reconstructed solution manifold.)
5. Regress a linear model for the neighborhood and use it to generate the forecast.

Filtering:

Linear: a bad idea if the system is chaotic

Nonlinear: use the stable and unstable manifold structure on a chaotic attractor…

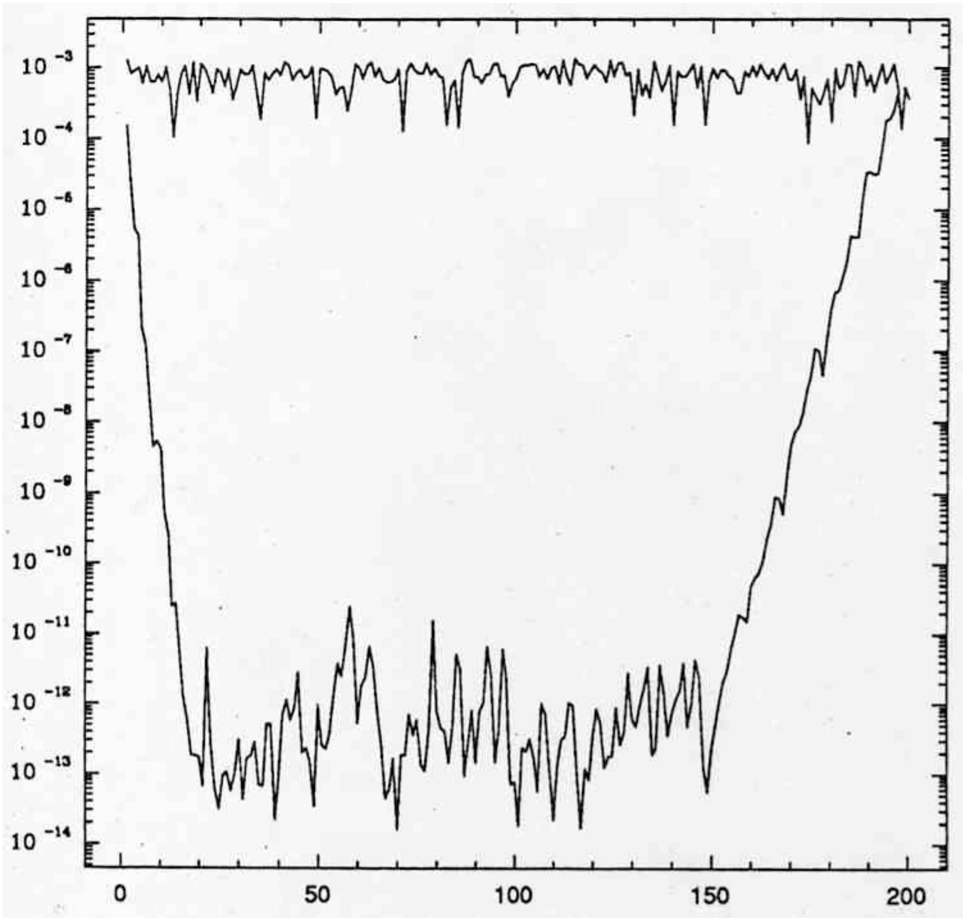


Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
- $\Rightarrow$ noise reduction!
- Works best if manifolds are perpendicular, but requires only transversality

Results:



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Computational Topology:

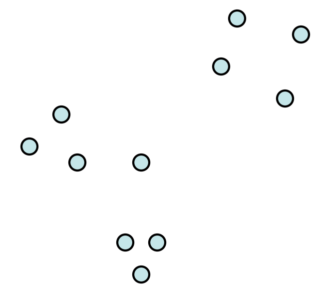
Why: this is the fundamental mathematics of shape. complements geometry.

What: compute topological properties *from finite data*

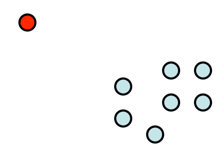
How:

- introduce resolution parameter
- count components and holes at different resolutions
- deduce topology from patterns therein

Connectedness: definitions

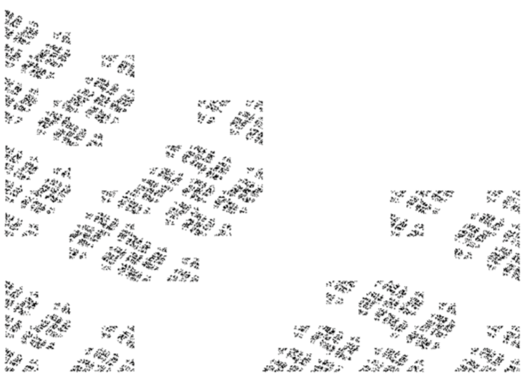


- how many “lumps” in a data set:
- ε-connectedness (after Cantor)
- ε-connected components
- ε-isolated points:

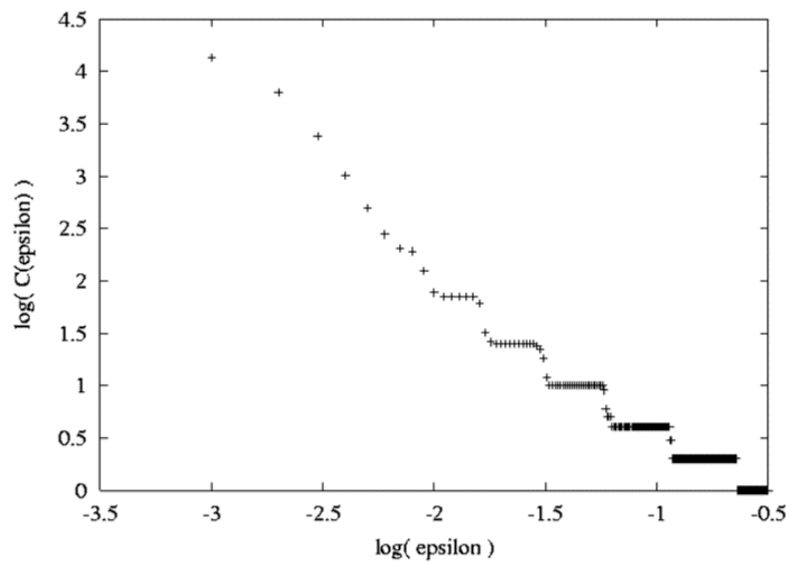


Connectedness: examples

If the data points are samples of a disconnected fractal like this:



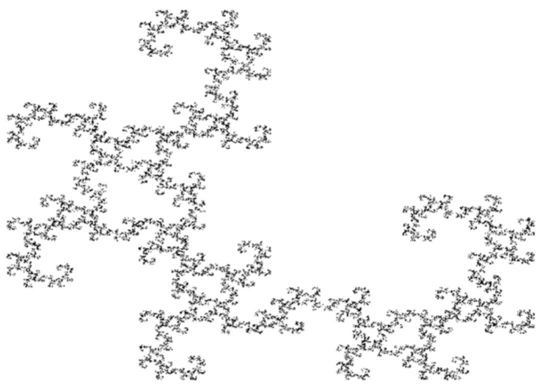
The number of connected components looks like this:



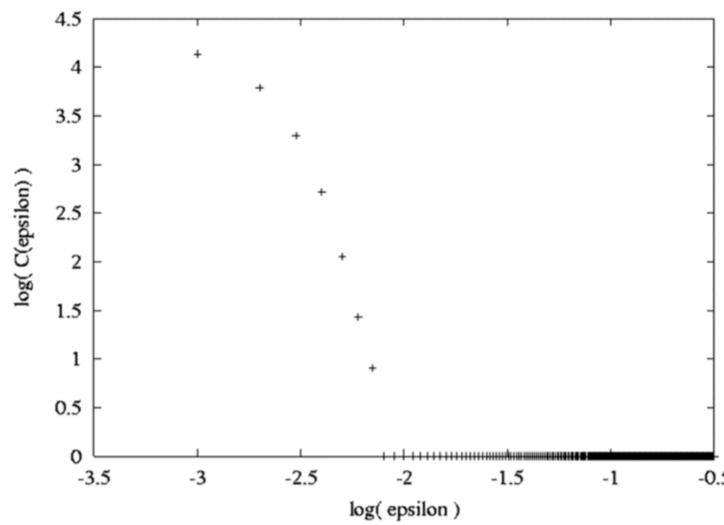
(note obvious tie-in to fractal dimension)

Connectedness: examples

If the data points are samples of a connected set like this:

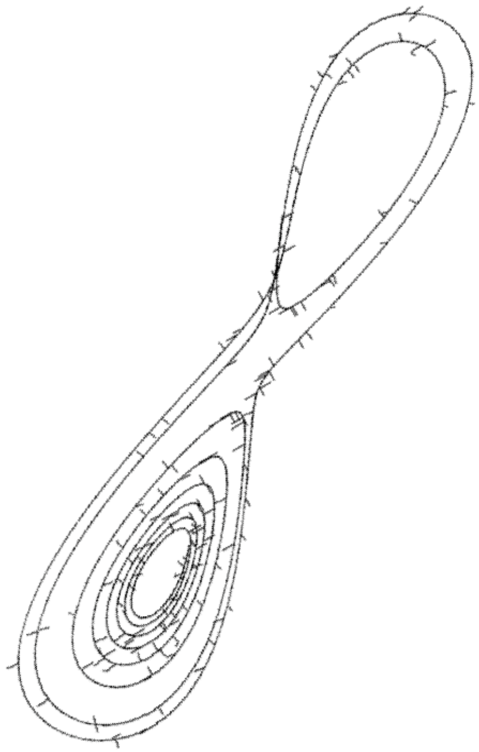


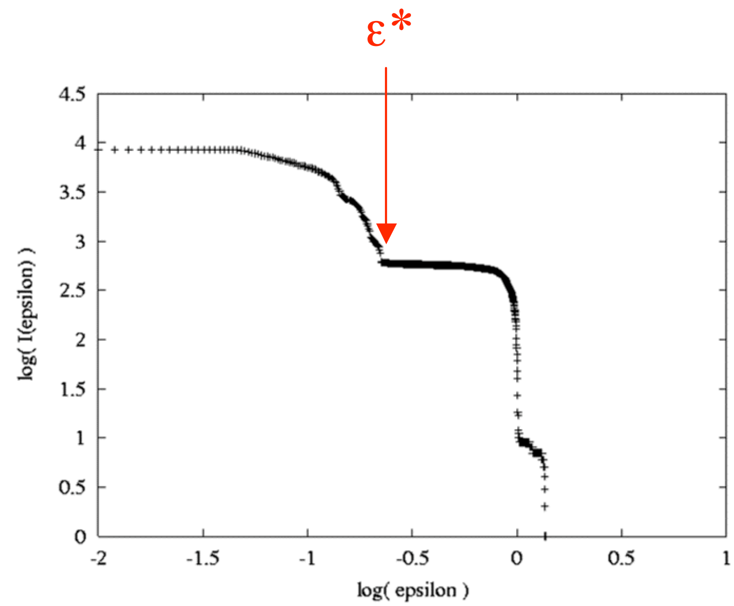
The number of connected components looks like this:



Connectedness: applications

The effects of noise are to add isolated points to the set and a shoulder to the C(e) curve:

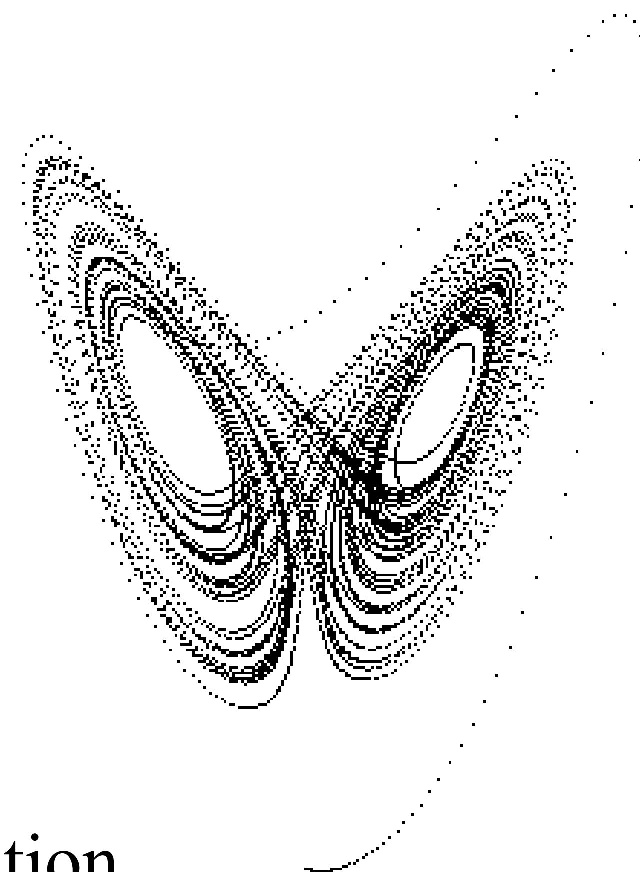




So if you know that the object is connected, you can reasonably assume that any isolated points are noisy, and remove them by pruning with ε = ε*

Robins, Bradley, Rooney

Chaos and control…

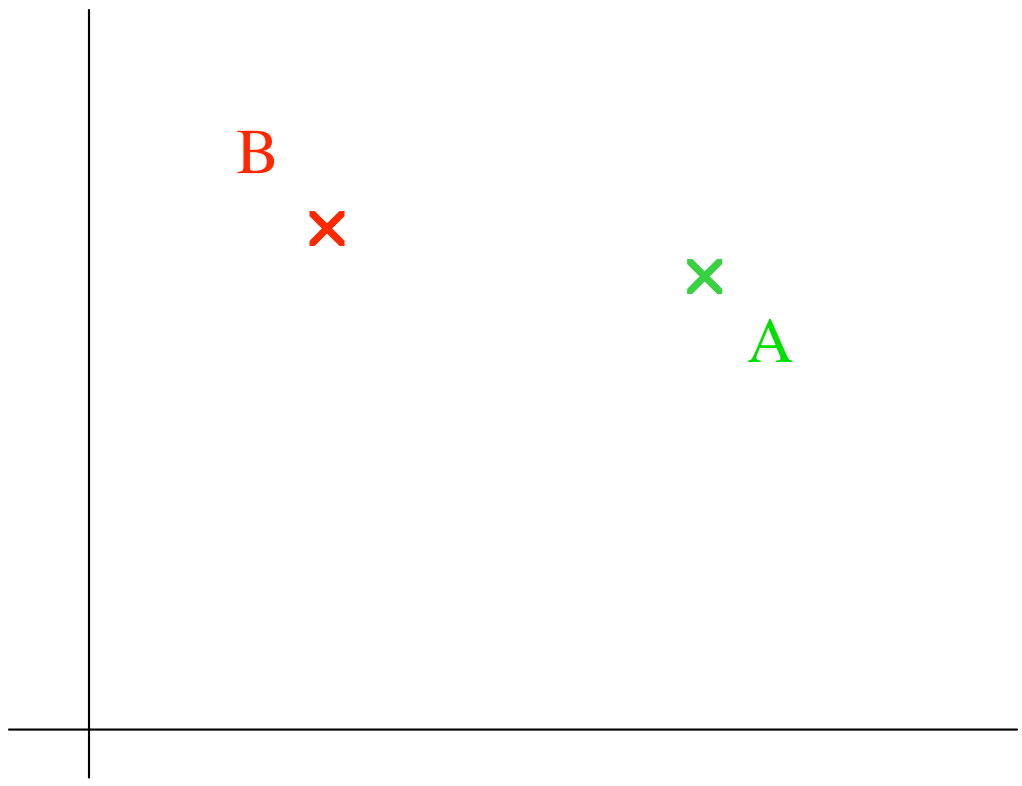


key concepts:

- dense attractor coverage
- exponential trajectory separation
- un/stable manifold structure
- local-linear control

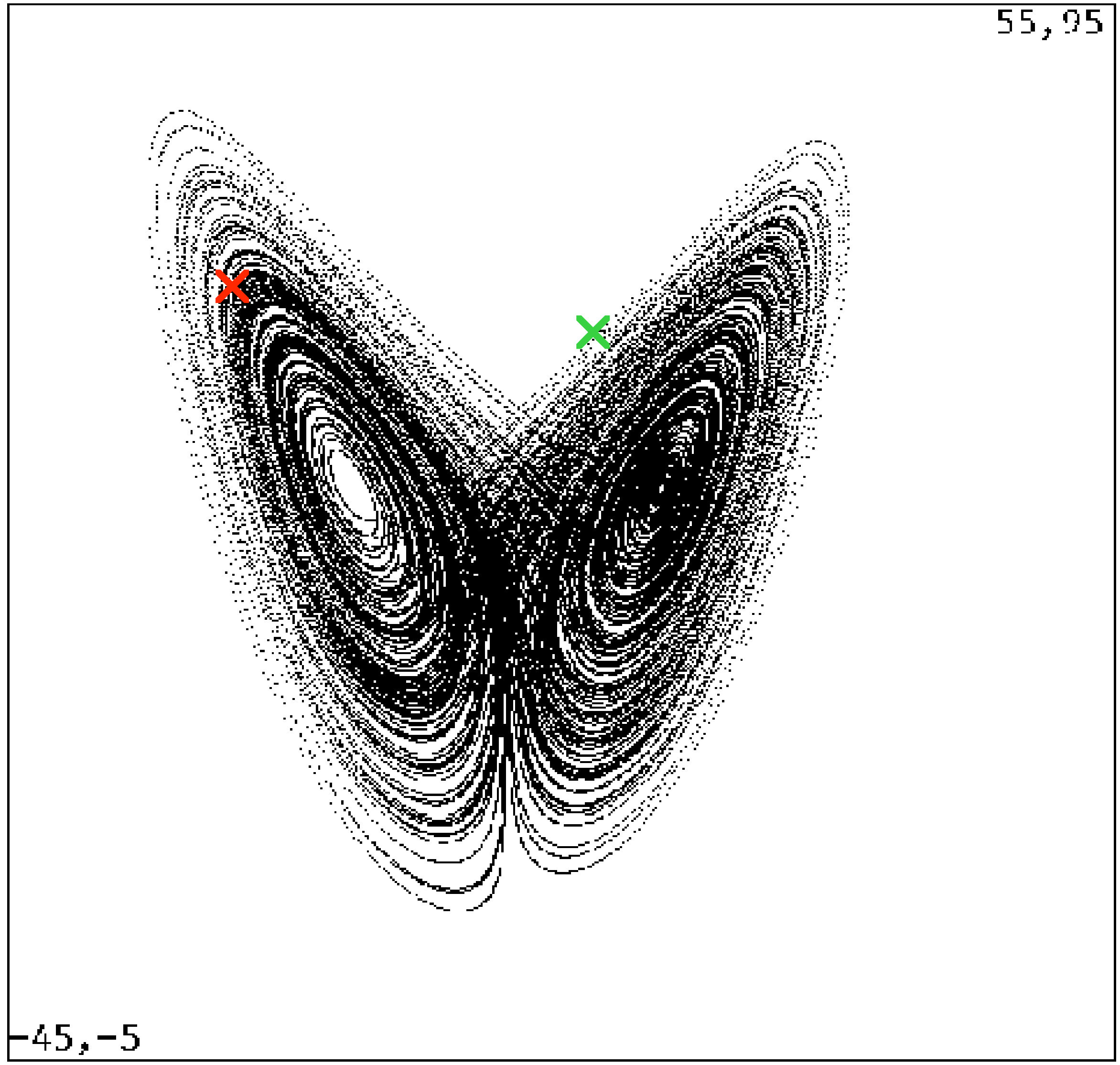
Control:

getting from A to B, minimizing some cost functional…



Lorenz System:

denseness, reachability, and control



R = 50

OYG control: taking advantage of the unique properties of chaos…

- dense attractor coverage → reachability
- un/stable manifold structure → controllability

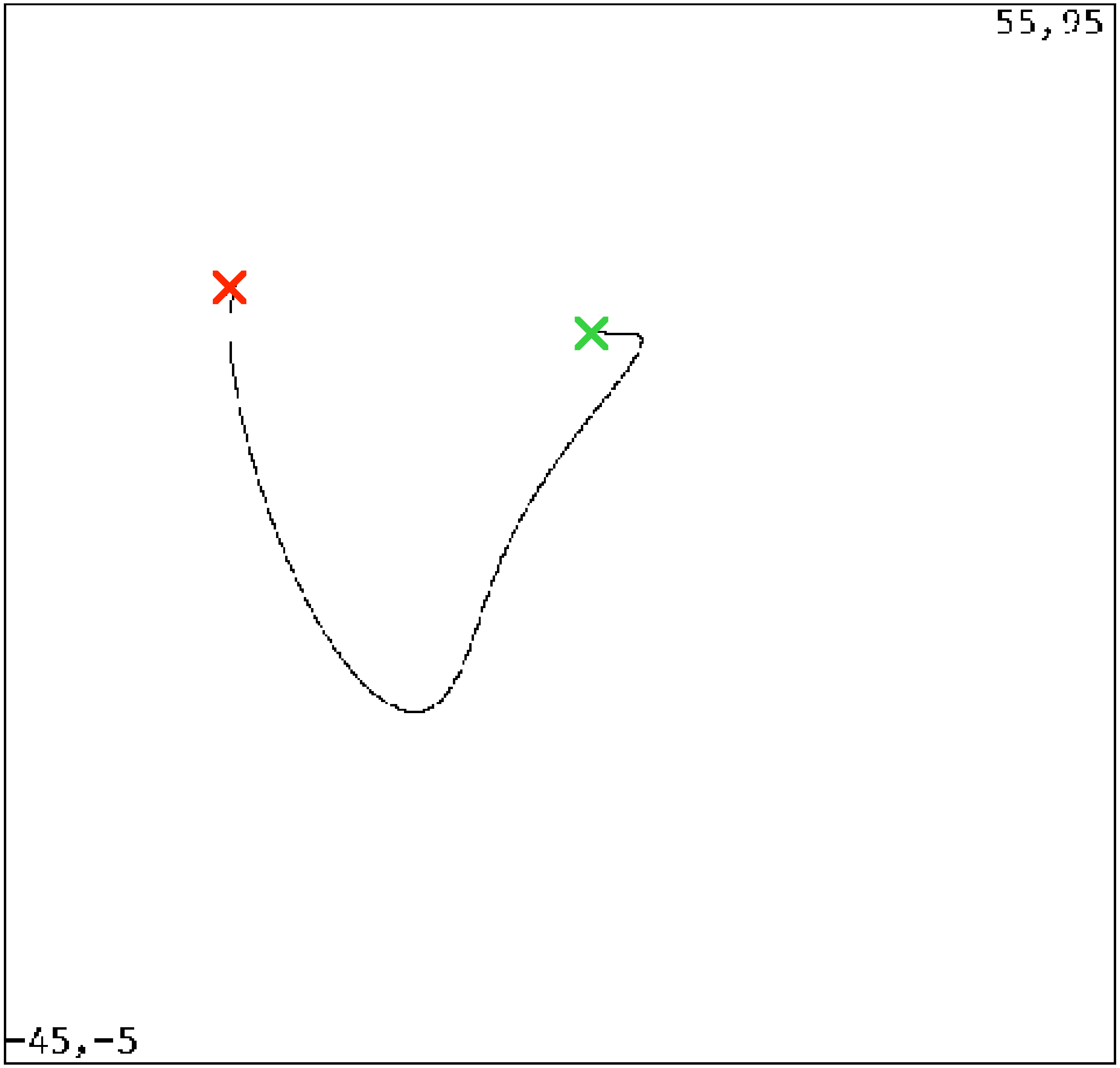
Ott *et al.*, *PRL* **64**:1196

- dense attractor coverage → reachability
- un/stable manifold structure → controllability
- exploit sensitive dependence, too???

⇨ “targeting”

Lorenz System:

SDOIC-based targeting



Four R switches; 240X faster

Shinbrot review paper: [47]

Program in Applied Mathematics

Erik Bollt

University of Colorado at Boulder
Boulder CO 80309-0526
(303) 492-4668

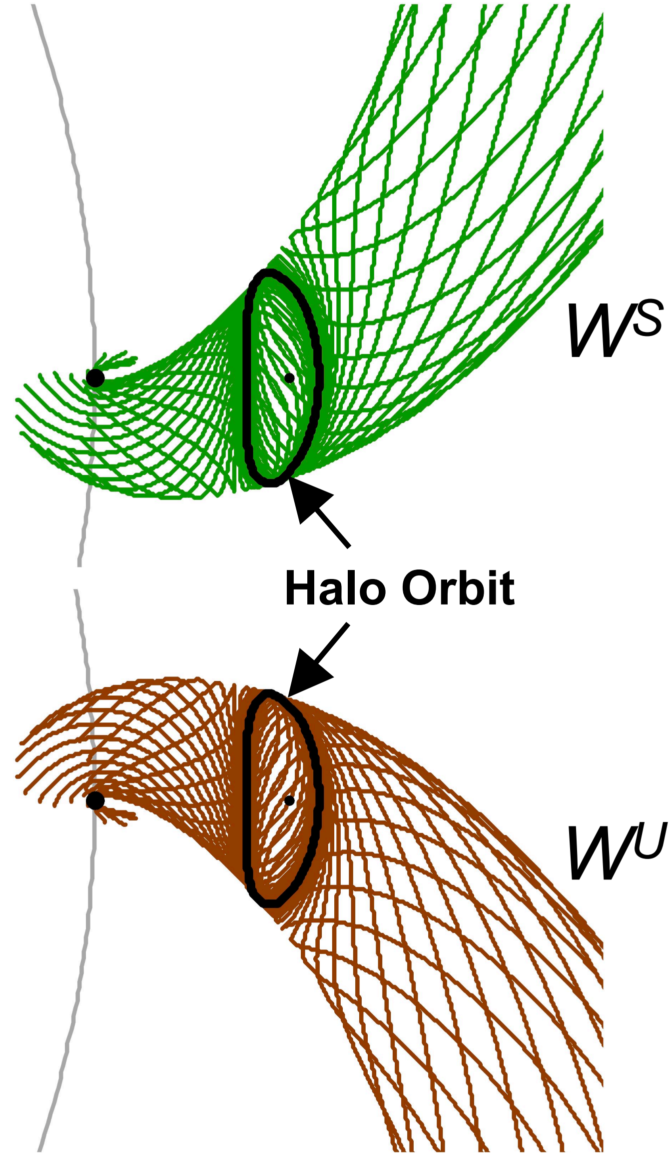
Jeff Parker

Using invariant manifolds…

Want to get a spacecraft onto a "halo orbit," which is a UPO of the dynamics.

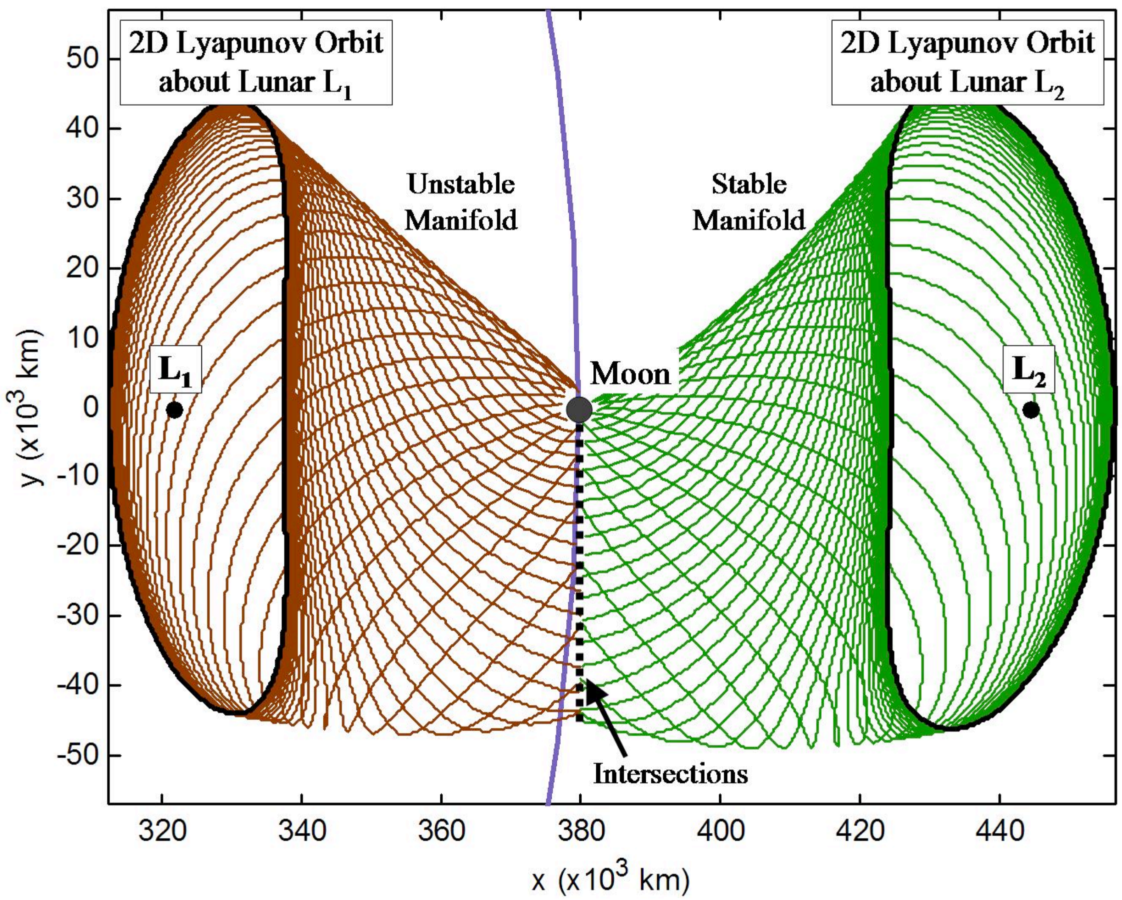
Unstable Periodic Orbits (UPOs) have invariant manifolds:

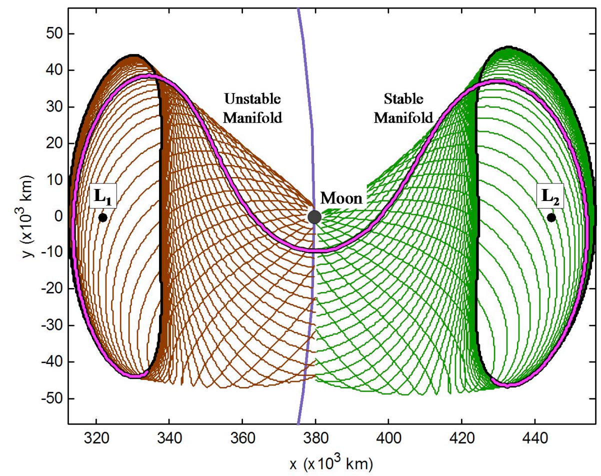
- Stable Invariant Manifold (W^S)
  - *The set of all trajectories a particle could use to arrive onto the UPO.*
- Unstable Invariant Manifold (W^U)
  - *The set of all trajectories a particle could take after a small perturbation from the UPO.*

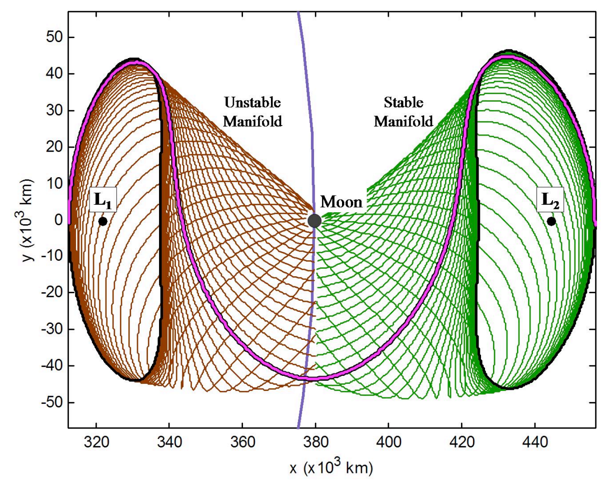


Low-energy (cheap) orbit transfers

- Depart along W^{U}_{L1} & arrive on W^{S}_{L2}

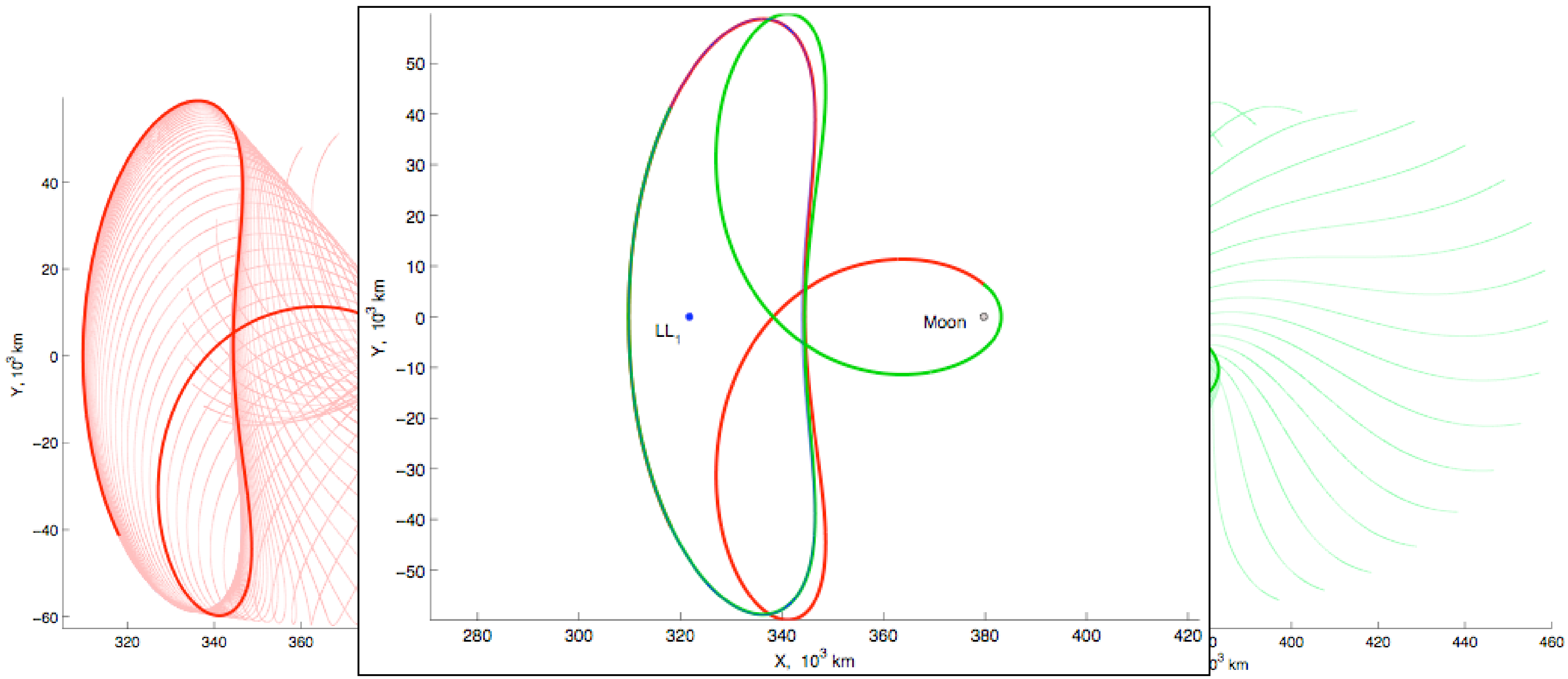






Homoclinic orbits - The best case

- If a trajectory in Stable and Unstable intersect ("homoclinic connection")



Unstable Manifold of an LL_1 Lyapunov Orbit

Stable Manifold of an LL_1 Lyapunov Orbit

Can we do any of that in spatially extended systems?

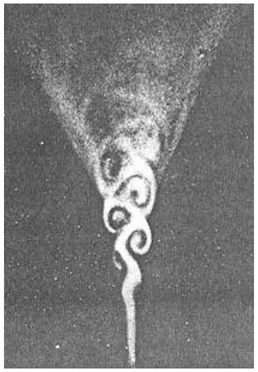
(i.e. harness the butterfly effect, exploit un/stable manifold geometry?)

- Sensitive flames (1856 – 1930's)

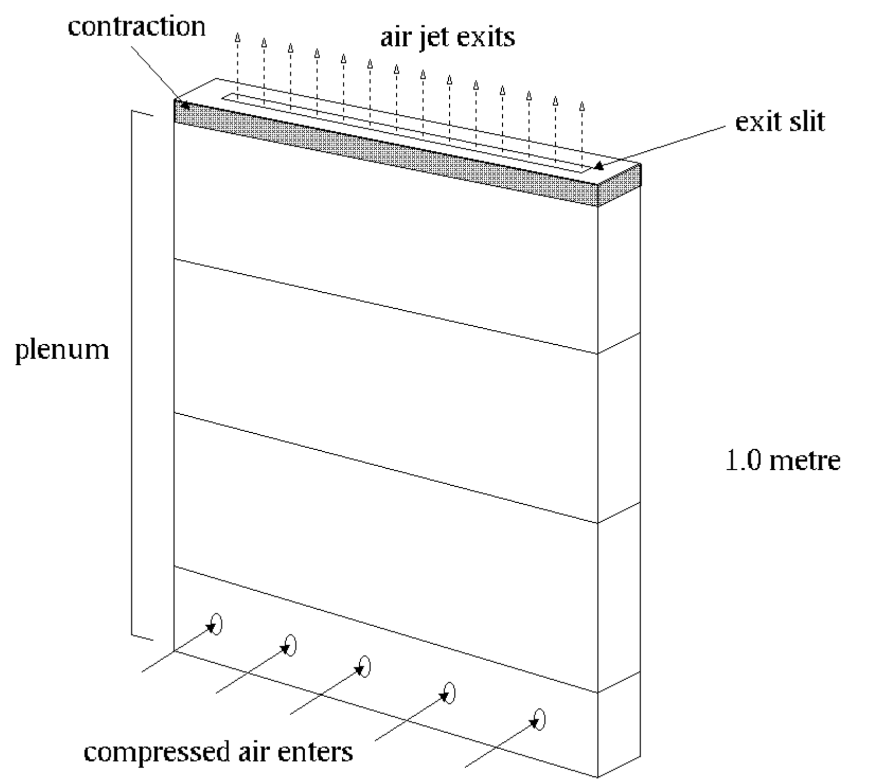
I repeat a passage from Spenser:

"Her ivory forehead full of bounty brave,
  Like a broad table did itself dispread;
For love his lofty triumphs to engrave,
  And write the battles of his great godhead.
All truth and goodness might therein be read,
  For there their dwelling was, and when she spake,
Sweet words, like dropping honey she did shed;
  And through the pearls and rubies softly brake
A silver sound, which heavenly music seemed to make."

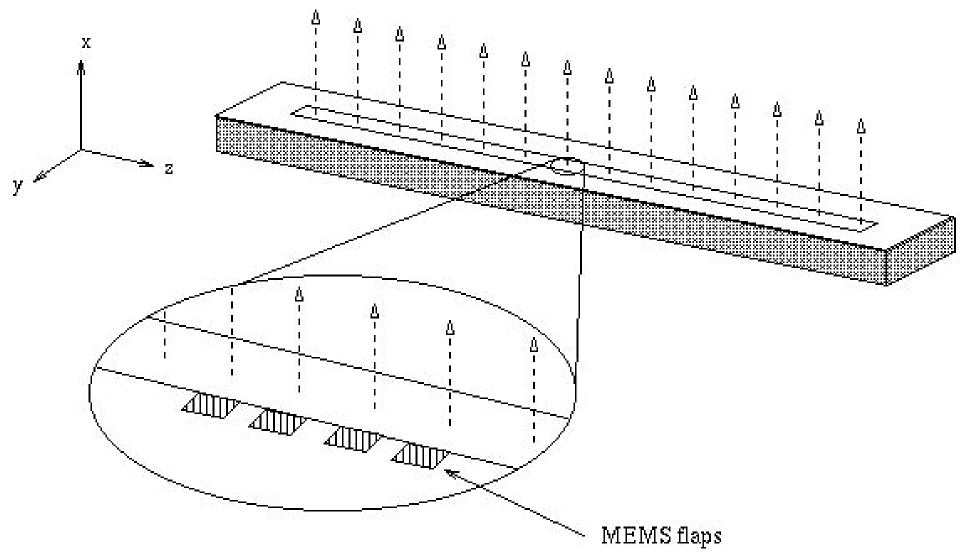
The flame selects from the sounds those to which it can respond. It notices some by the slightest nod, to others it bows more distinctly, to some its obeisance is very profound, while to many sounds it turns an entirely deaf ear.



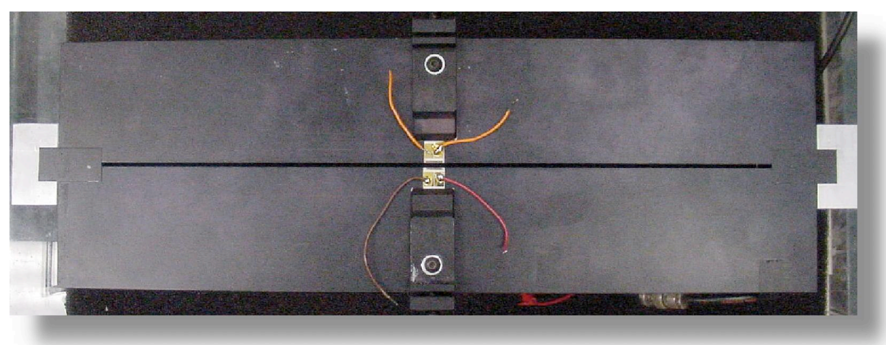
2D jet apparatus



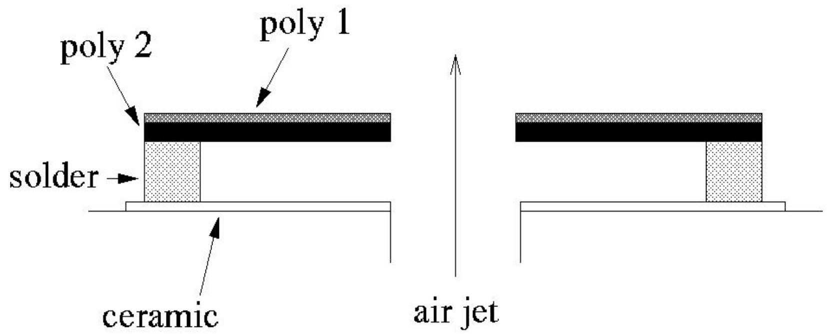
Forcing the jet flow

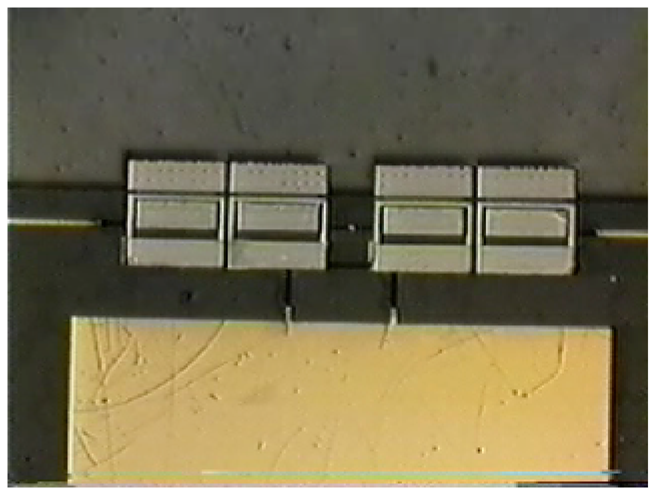


Slit: 2.5 X 400 mm



MEMS actuators

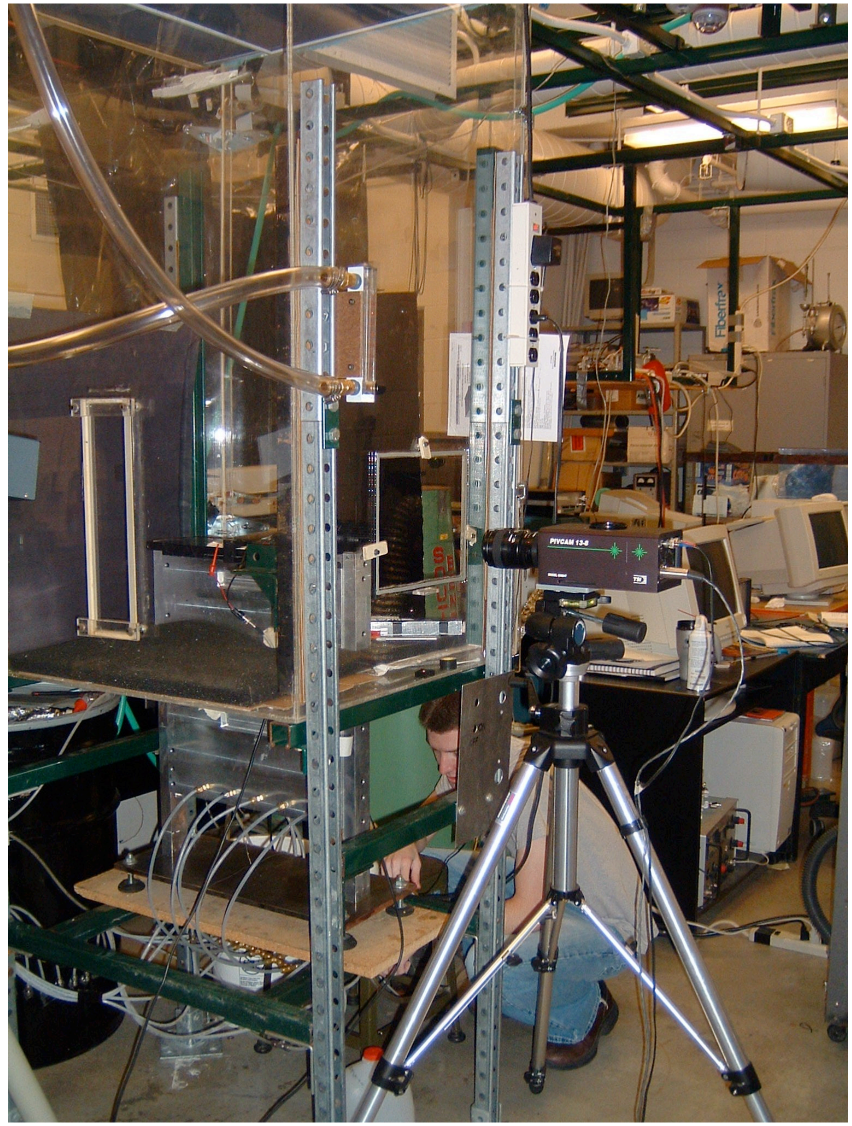




Video : overhead view at 2Hz, 10V

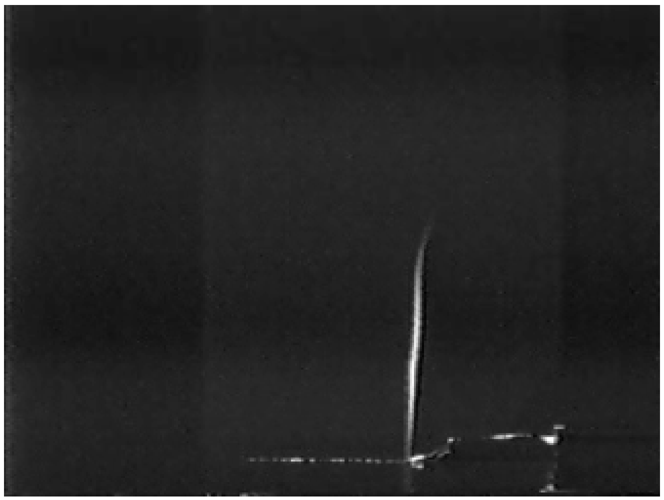
Area of individual flap is 1.0 x 0.25mm

Measurement & Isolation:



The Butterfly effect in action…

no forcing

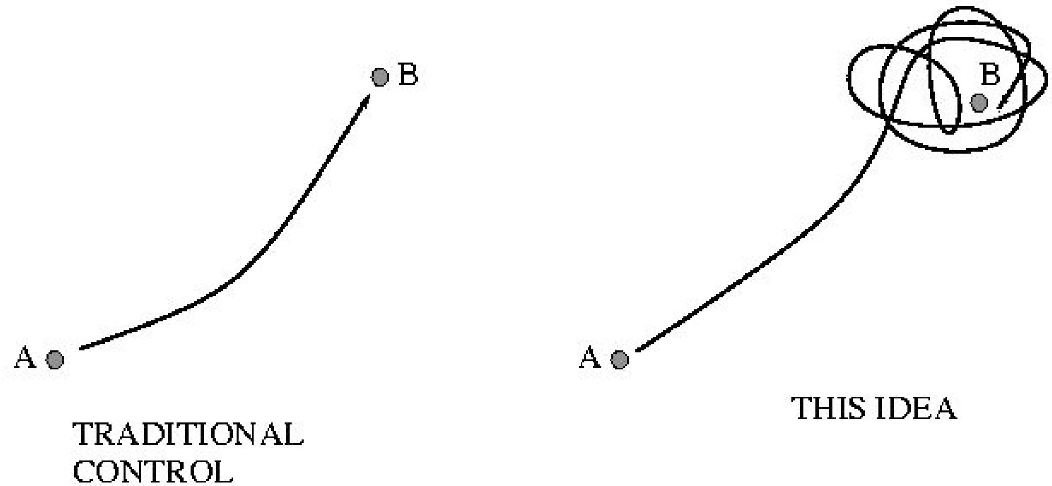


6Hz forcing

Forcing generates coherent structures that enhance entrainment and mixing

With Tom Peacock, Jean Hertzberg

"Chaos-enhanced reachability"



- can control position/volume/density of attractor *–within limits*
- possibly not reachable any other way
- nondeterministic – not for time-critical applications

808 IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—I: FUNDAMENTAL THEORY AND APPLICATIONS, VOL. 40, NO. 11, NOVEMBER 1993

Using Chaos to Broaden the Capture Range of a Phase-Locked Loop

Elizabeth Bradley, *Member, IEEE*

Communication and chaos:

- Two coupled Lorenz systems will synchronize
- Robust w.r.t. a small amount of noise
- Use this to transmit & receive information

$$x' = a(y-x) \qquad\qquad x' = a(y-x)$$

$$y' = rx - y - xz \quad \longrightarrow \quad y' = r(x+\varepsilon x) - y - xz$$

$$z' = xy - bz \qquad\qquad z' = xy - bz$$

- Chaotic carrier wave, so hard to intercept or jam

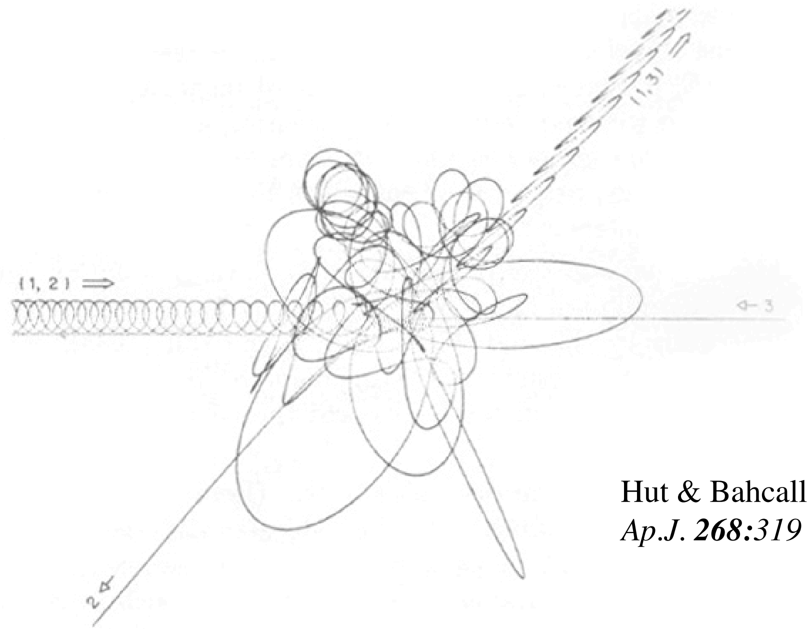
Pecora & Carroll

Another interesting application: chaos in the solar system

- orbits of Pluto, Mars
- Kirkwood gaps
- rotation of Hyperion & other satellites
- …

Solar system stability:

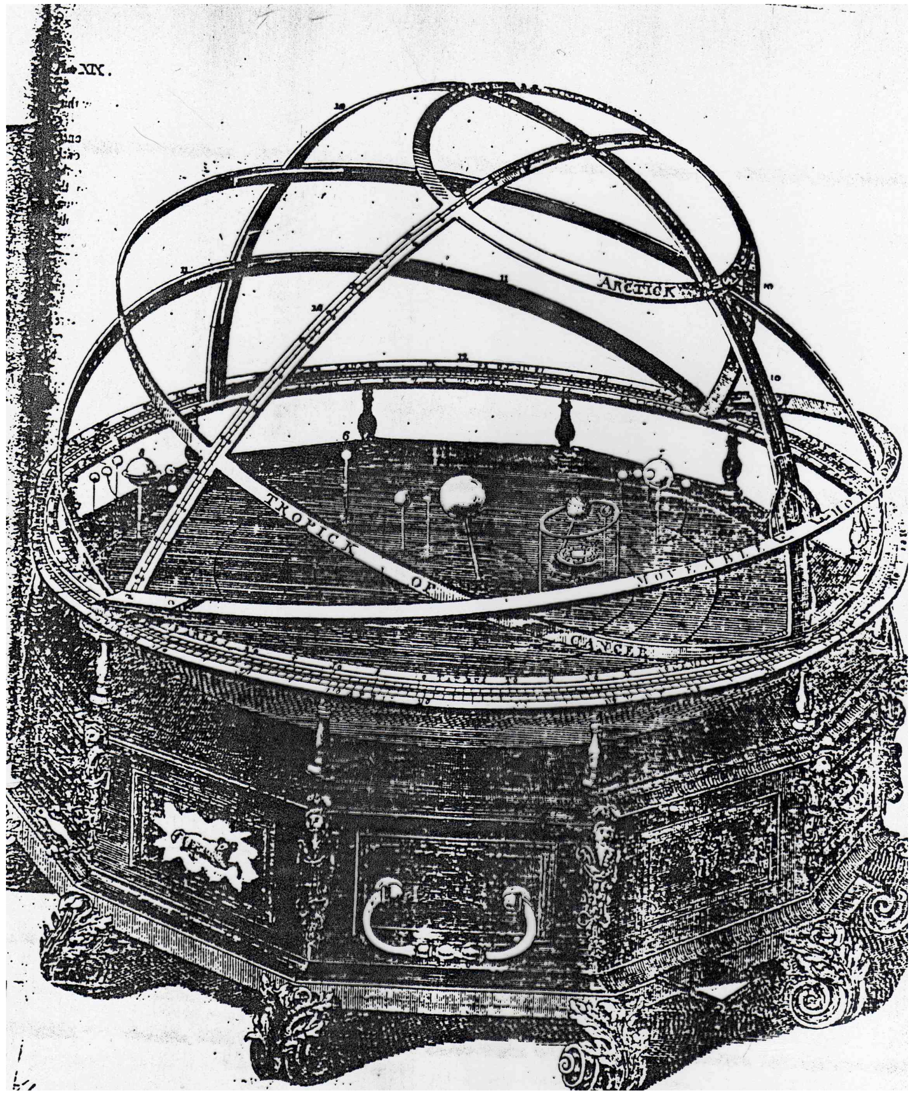
- recall: two-body problem not chaotic
- but three (or more) can be…



Hut & Bahcall
Ap.J. ***268:****319*

Exploring that issue, circa 1880:

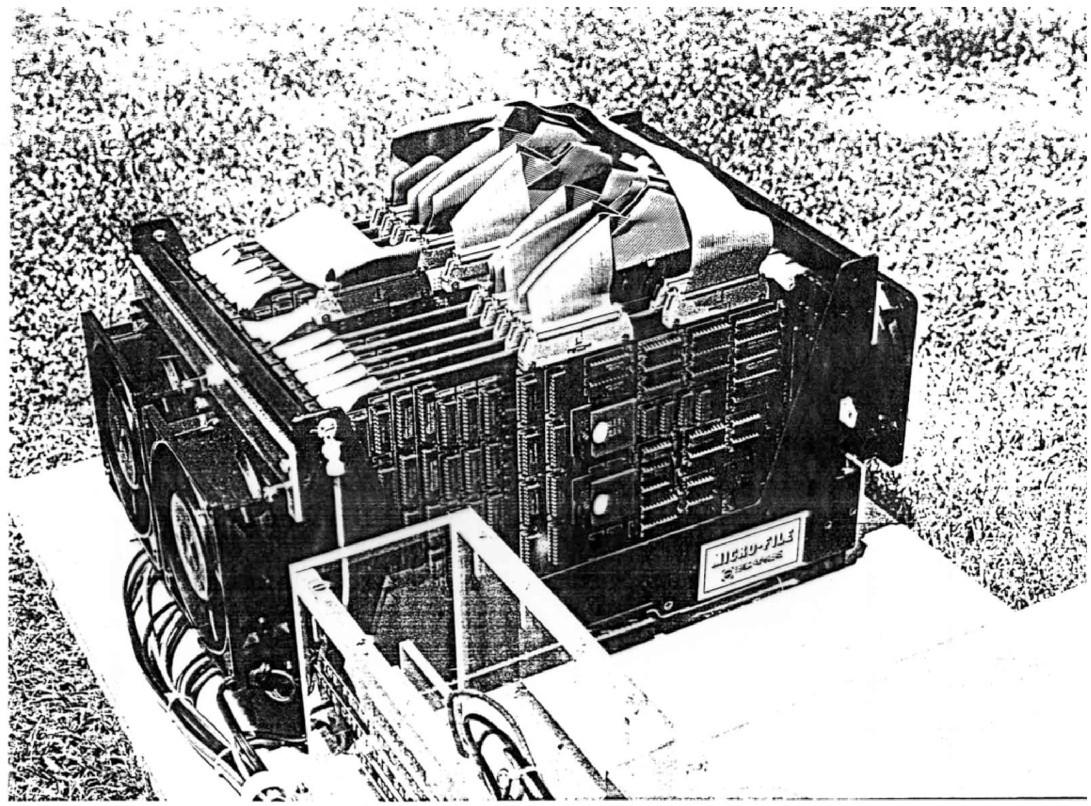
- an *orrery*



Exploring that issue, circa 1980:

- write the *n*-body equations for the solar system
- solve them on a special-purpose computer

The *digital orrery*
(Wisdom & Sussman)



ONR N00014-86-K-0180

Numerical Evidence That the Motion of Pluto Is Chaotic

GERALD JAY SUSSMAN AND JACK WISDOM

The Digital Orrery has been used to perform an integration of the motion of the outer planets for 845 million years. This integration indicates that the long-term motion of the planet Pluto is chaotic. Nearby trajectories diverge exponentially with an *e*-folding time of only about 20 million years.

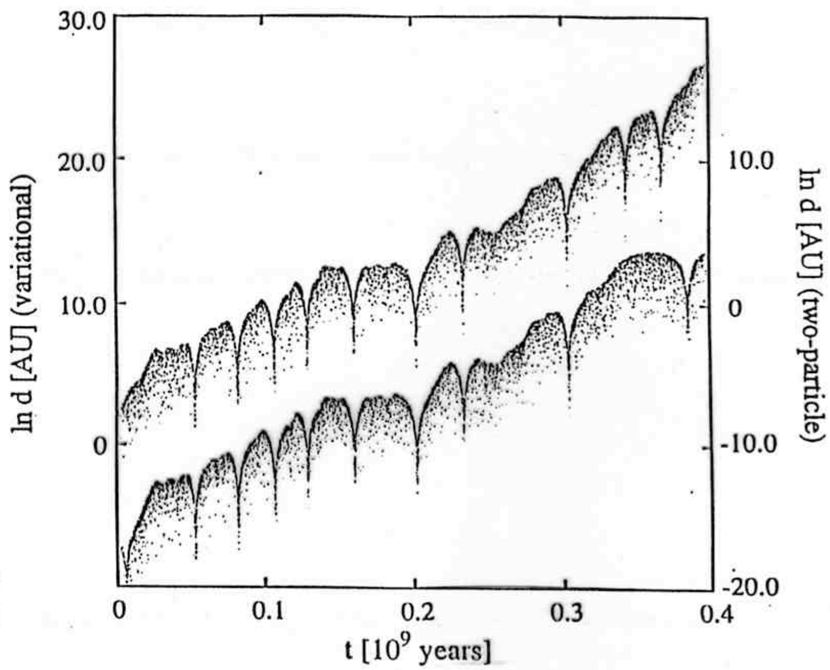
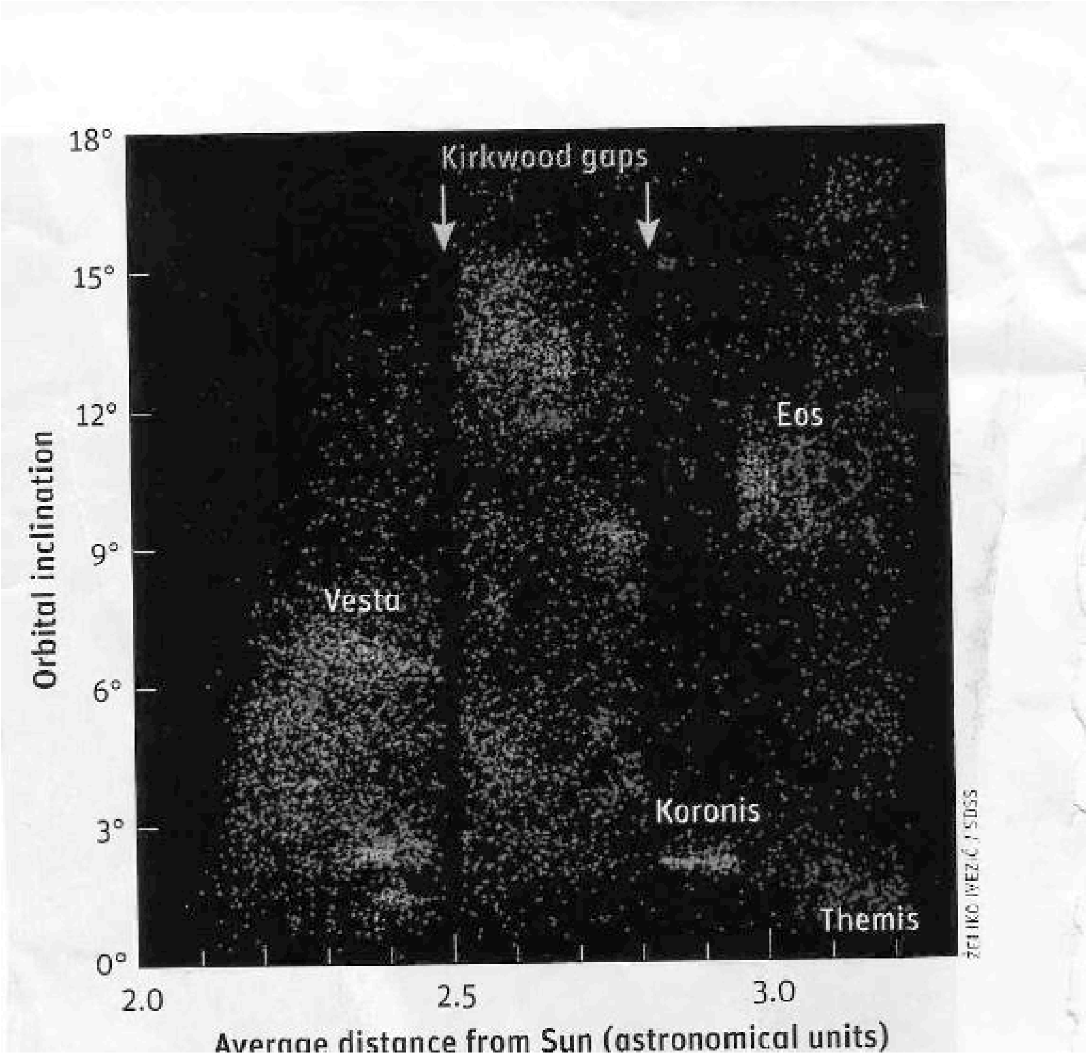


Figure 3: The exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithms of the distance measures as a function of time. In the upper trace we see the growth of the variational distance around a reference trajectory. In the lower trace we see how two Plutos diverge with time. The distance saturates near 80AU when the Plutos are on opposite sides of the Sun. The variational method of studying neighboring trajectories does not have the problem of saturation. Note that the two methods are in excellent agreement until the two-trajectory method has nearly saturated.

Should we worry?

- No.

Kirkwood gaps:



Sky & Telescope

Chaos and the Kirkwood gaps

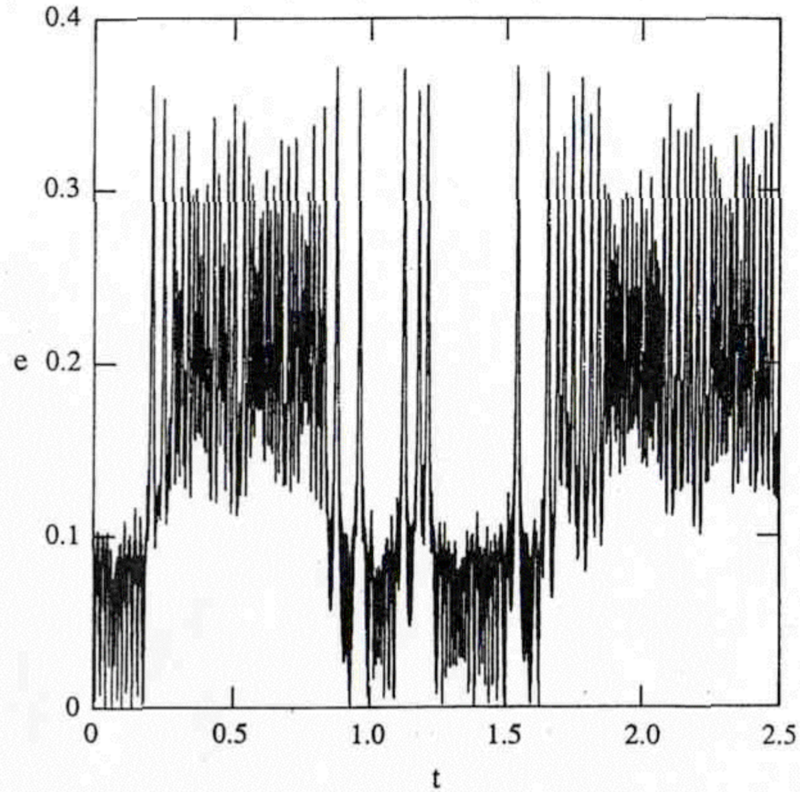
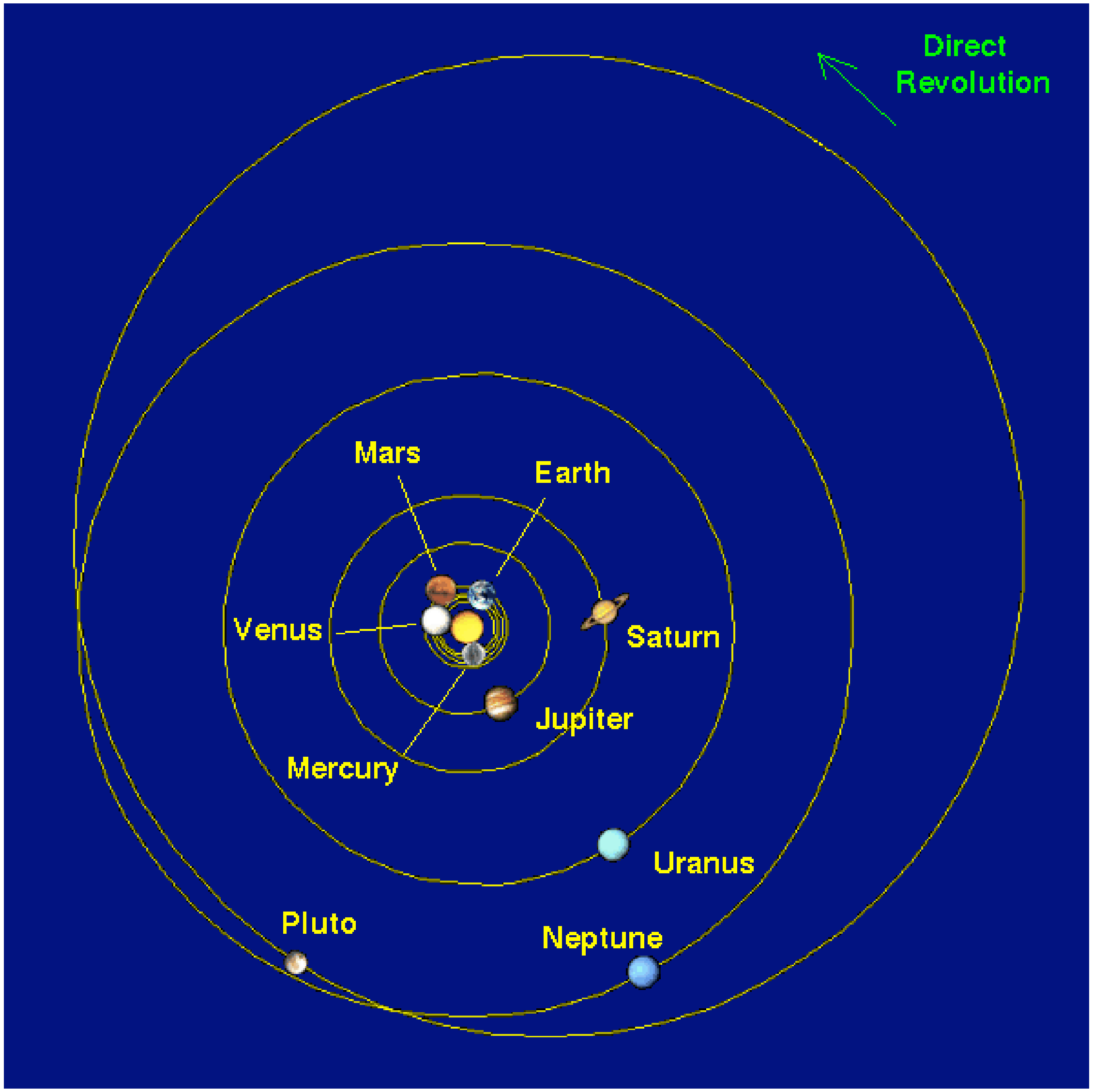


FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. the time is now measured in millions of years. Bursts of high eccentricity behavior are interspersed with intervals of irregular low eccentricity behavior, broken by occasional spikes.

Wisdom, *Nuclear Phys. B* **2:***391*



http://csep10.phys.utk.edu/astr161/lect/solarsys/revolution.html

Evidence in favor of the conjecture:

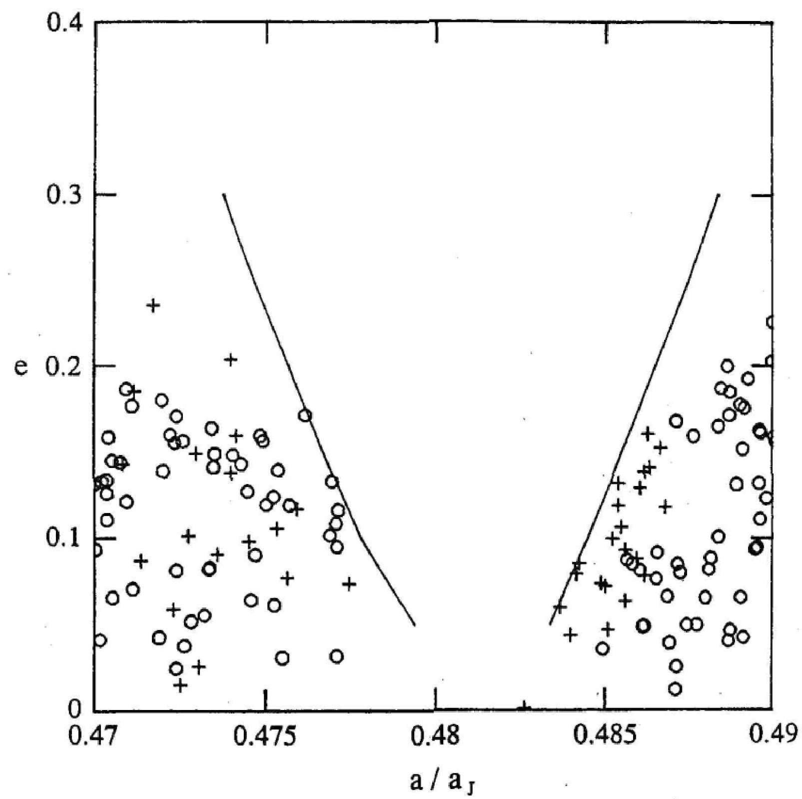


FIGURE 9. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and quasiperiodic region in the gap, but trajectories of both types are planet crossing.

Wisdom, *Nuclear Phys. B* **2:***391*

Chaotic tumbling of satellites:

Voyager and Galileo saw this…

Ap. J. **97**:570
Ap. J. **98**:1855

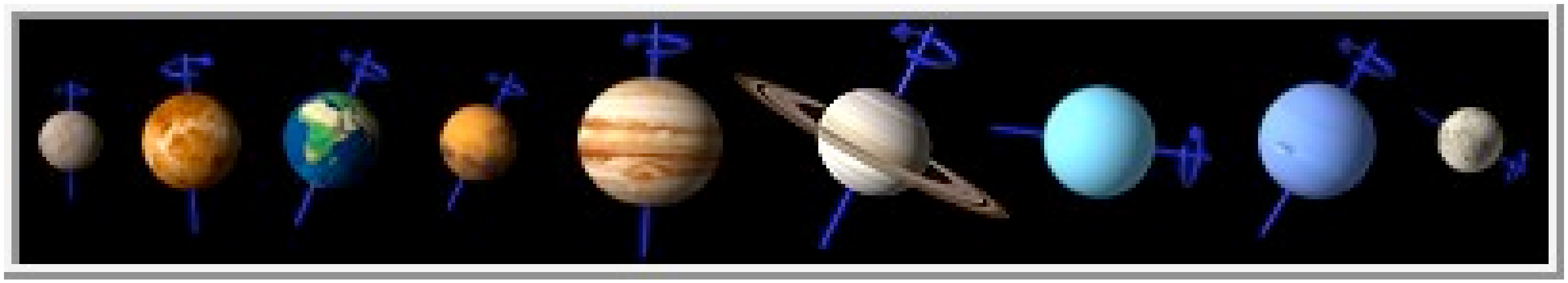
…and it happens for all satellites at some point in their history, unless they are perfectly spherical and in perfectly circular orbits (pf: KAM theorem; see [53] on syllabus.)

NASA movie of Hyperion tumbling

http://www.nasa.gov/mission_pages/cassini/multimedia/pia06243.html

More chaos in the solar system:

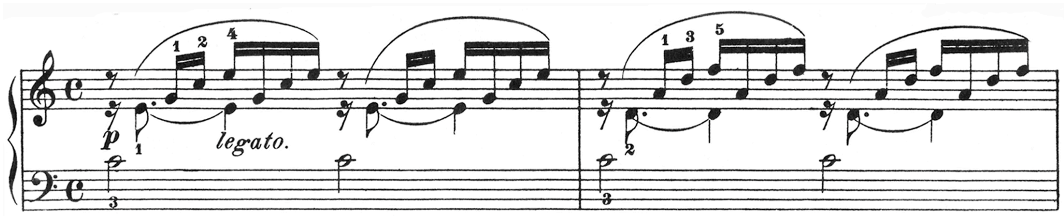
• obliquity of Mars (Touma & Wisdom, *Science* **259:***1294)*



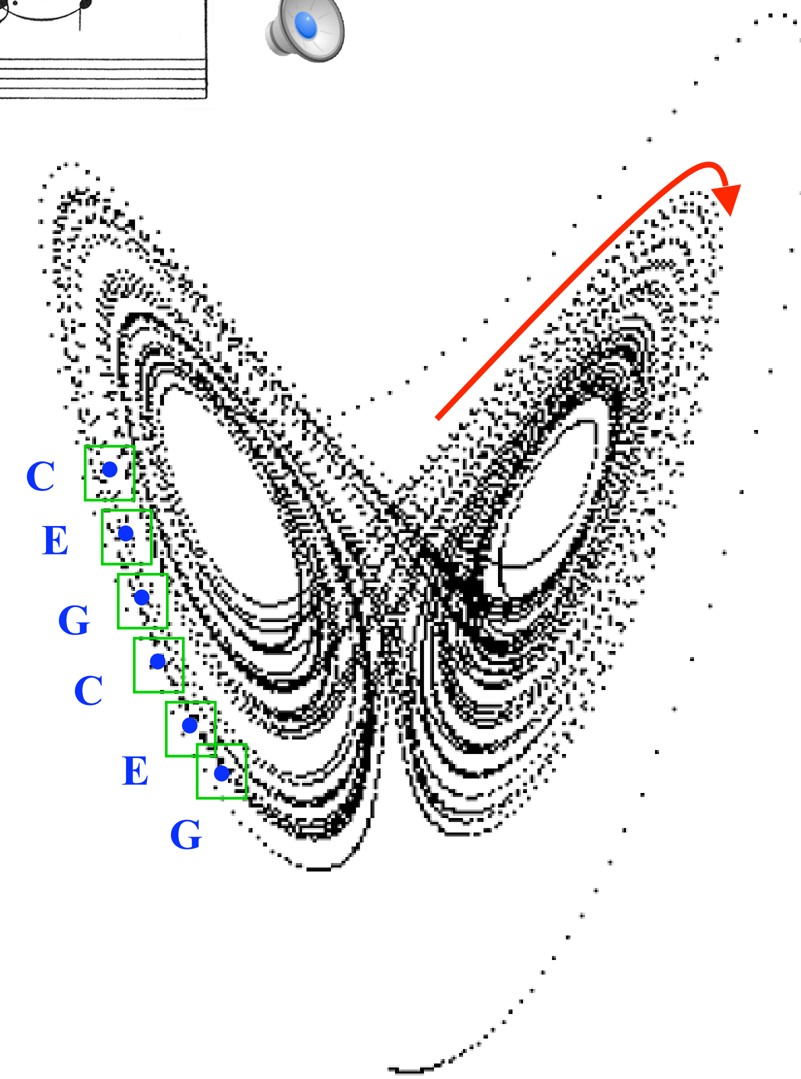
www.solarviews.com

• etc.

Musical Variations from a Chaotic Mapping



Pitch sequence:
C, E, G, C, E, G, C, E…

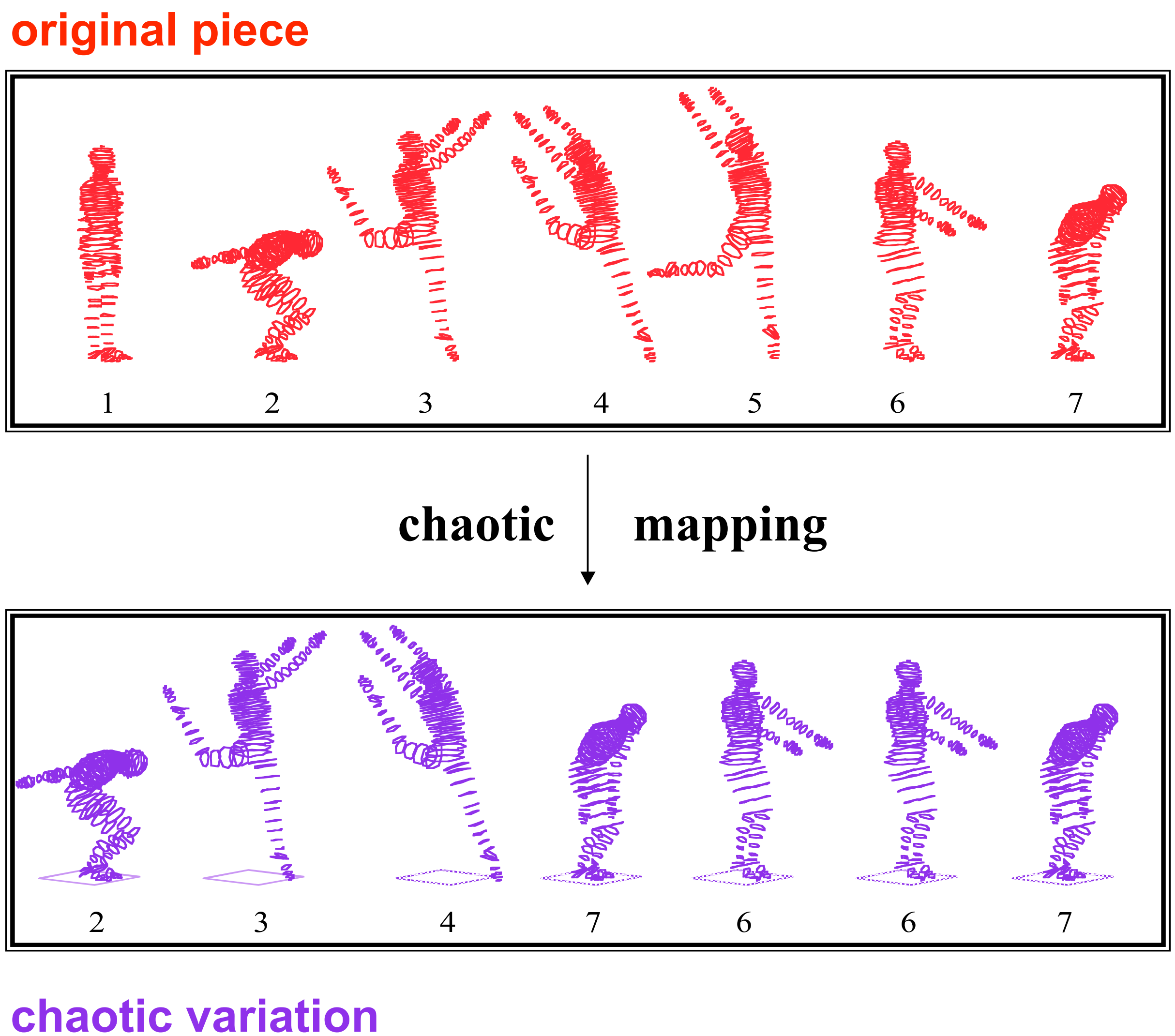


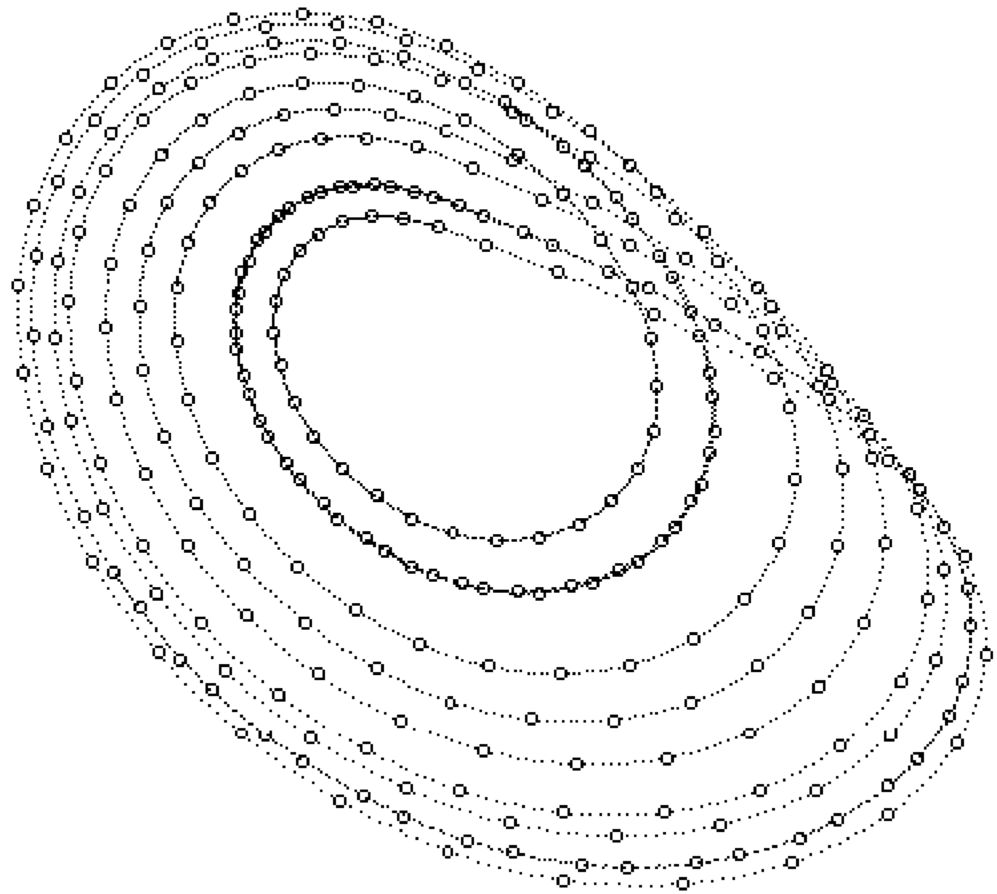
C **symbol dynamics**

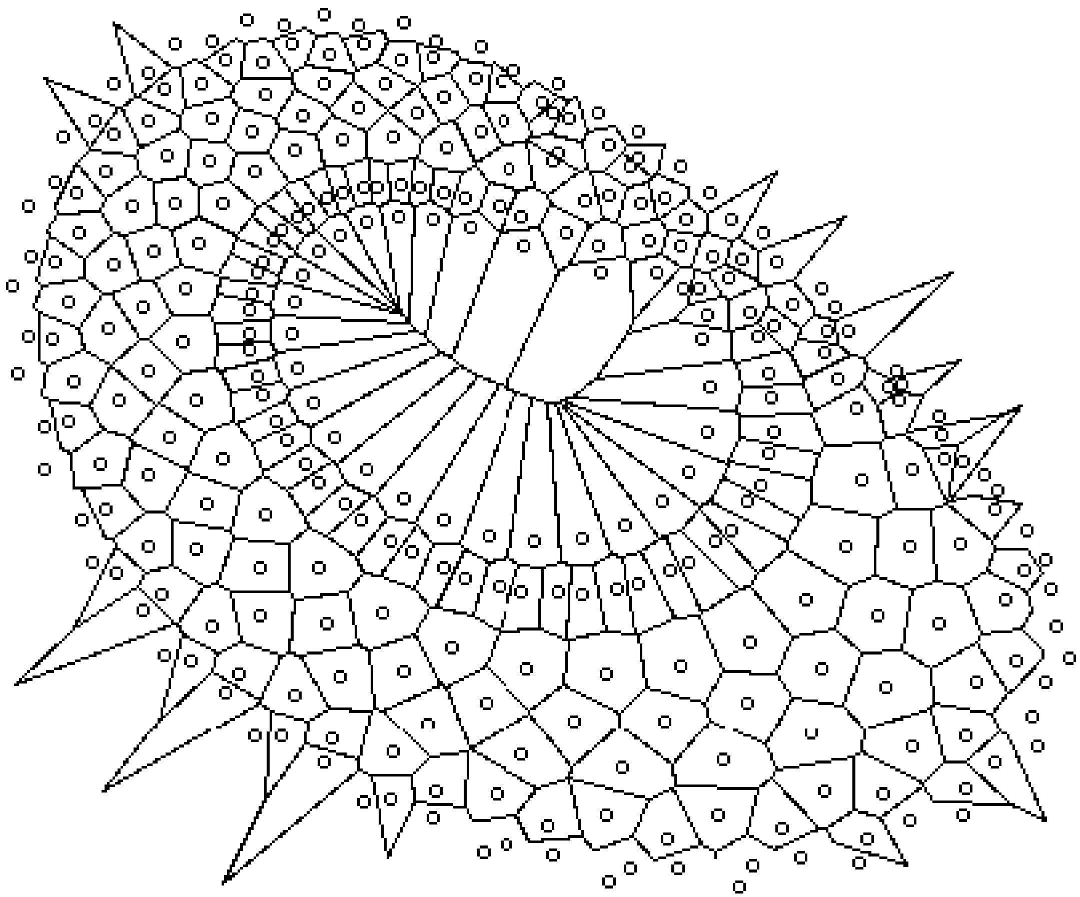
variation!

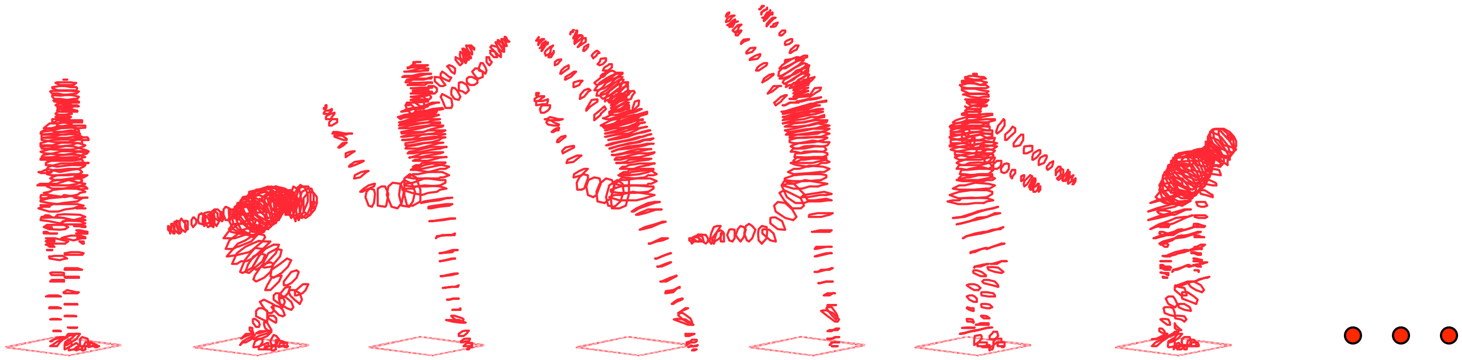
Dabby *Chaos* **6**:95

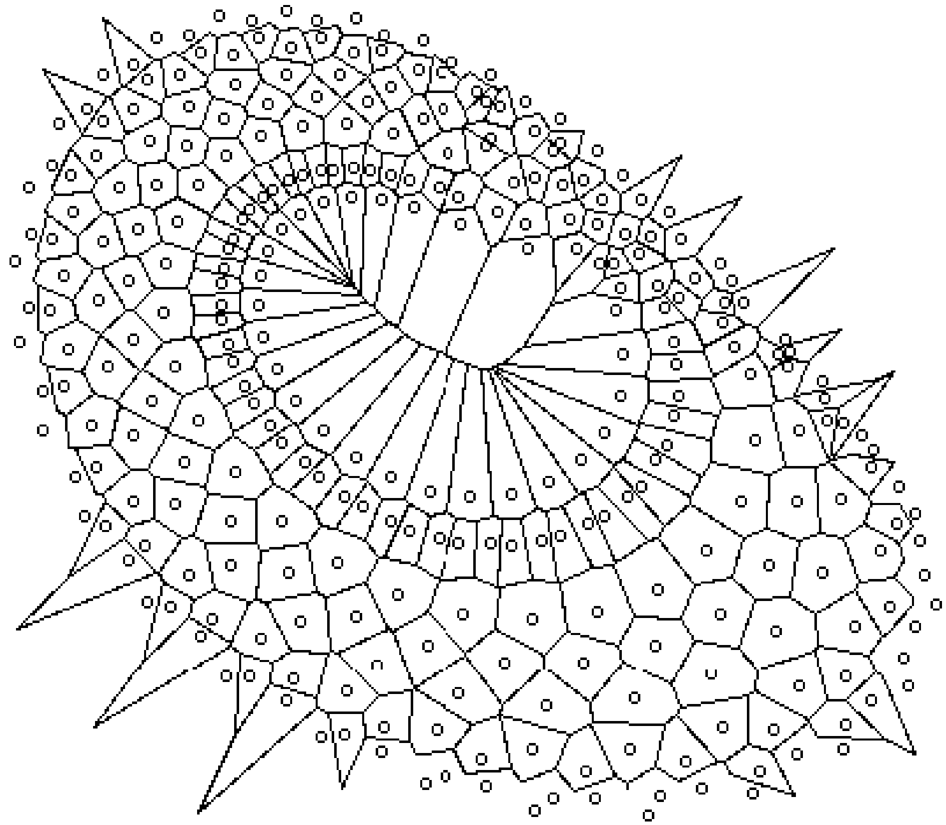
Chaotic variations on movement sequences

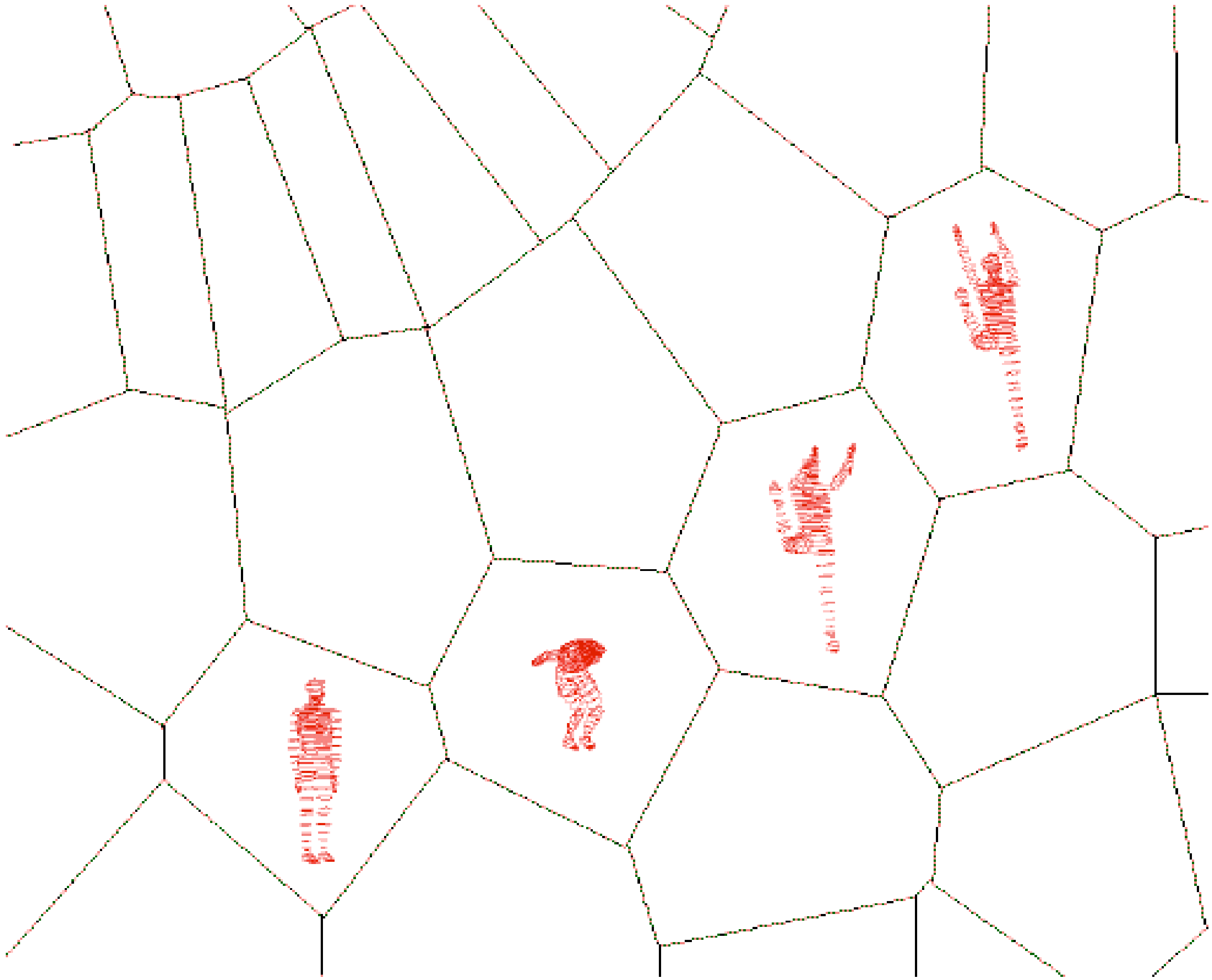


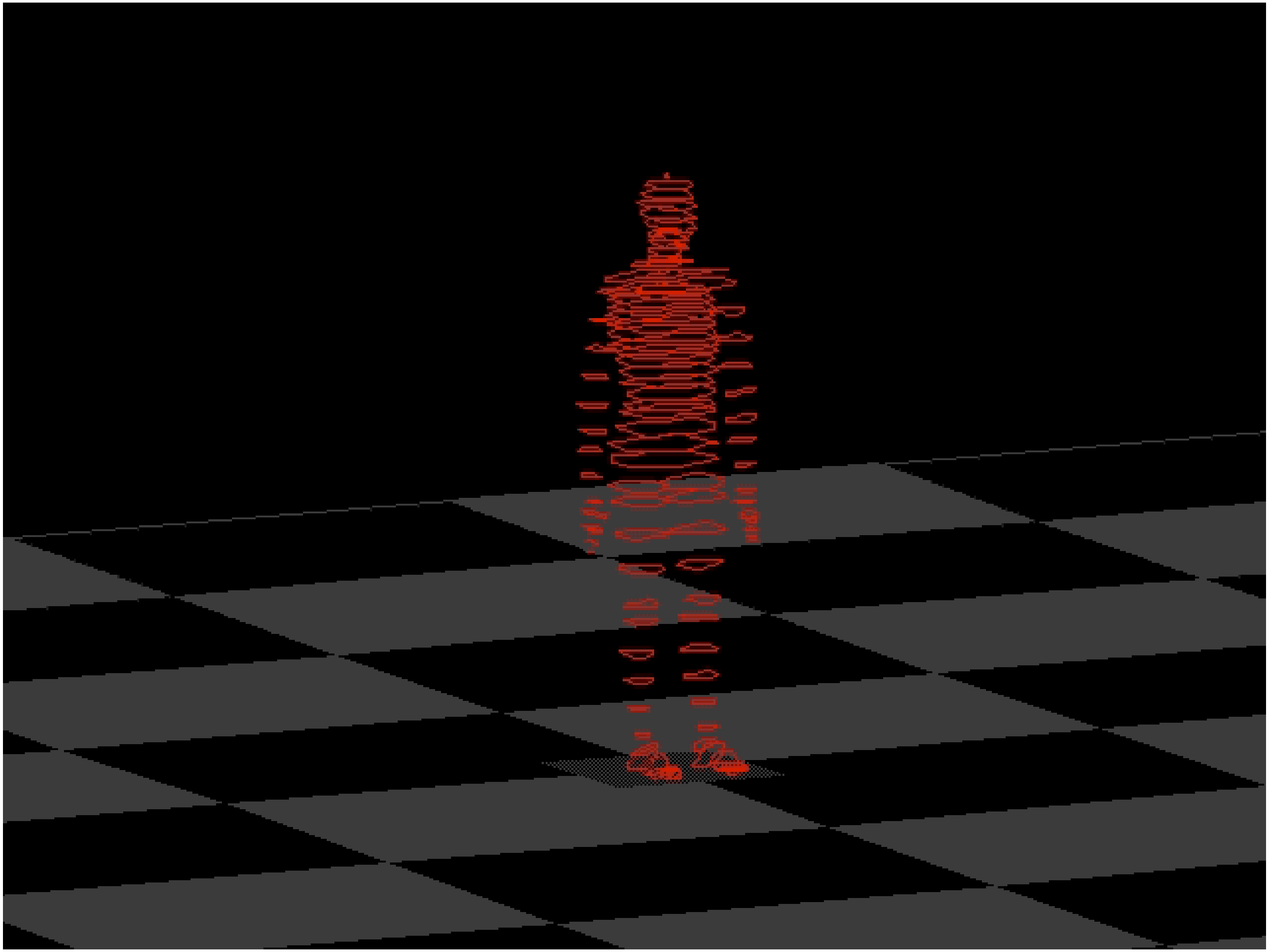


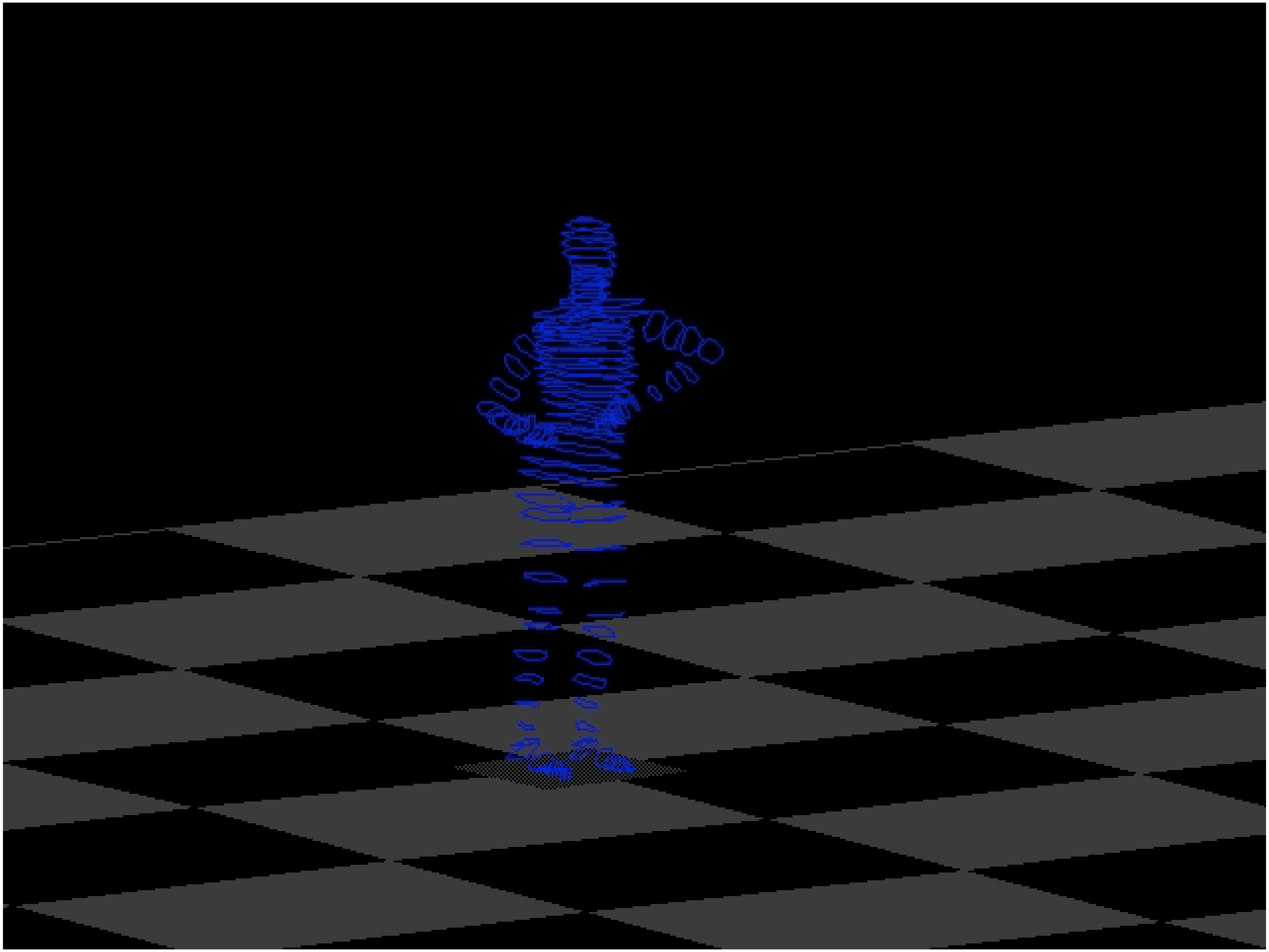




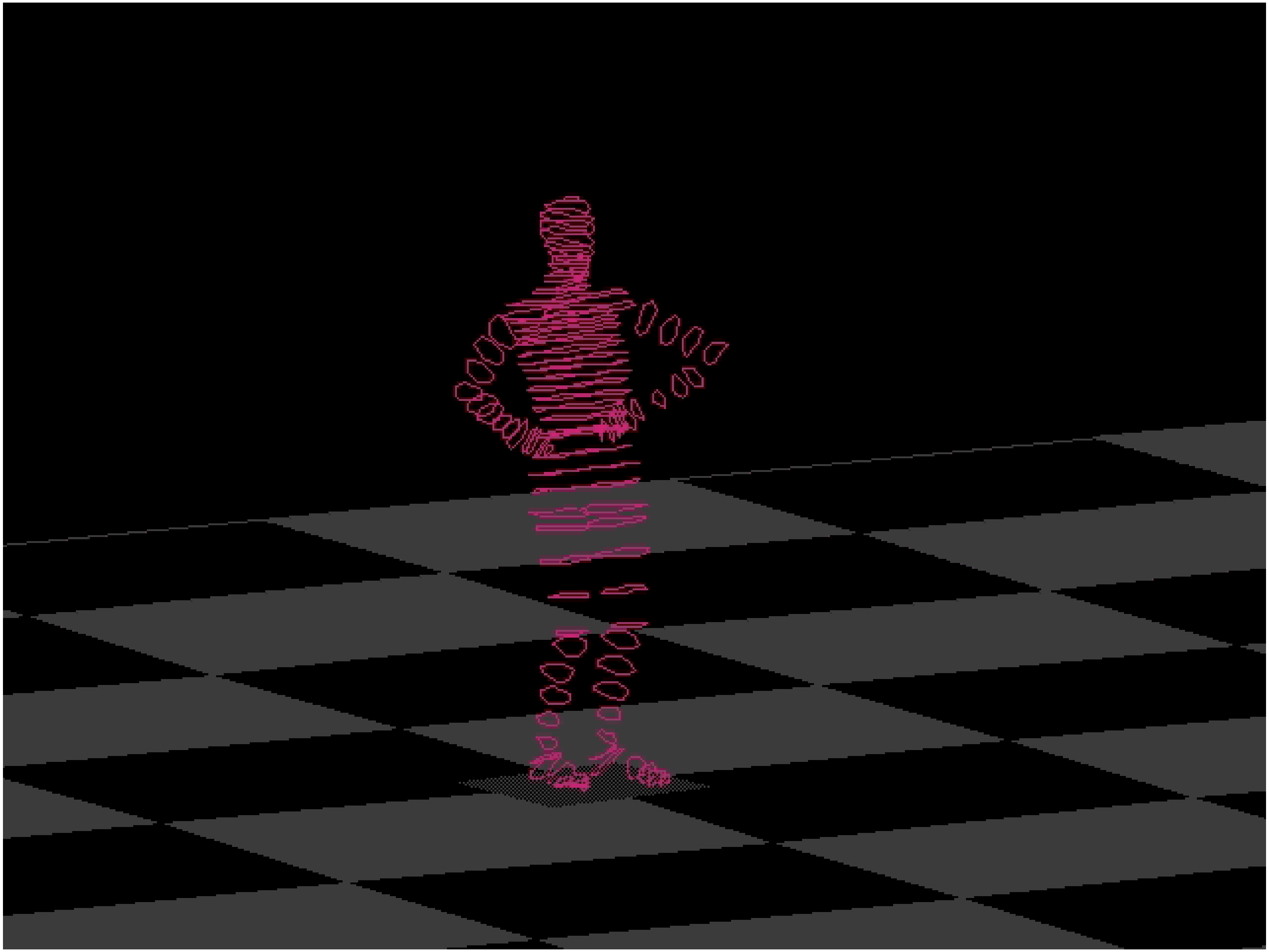




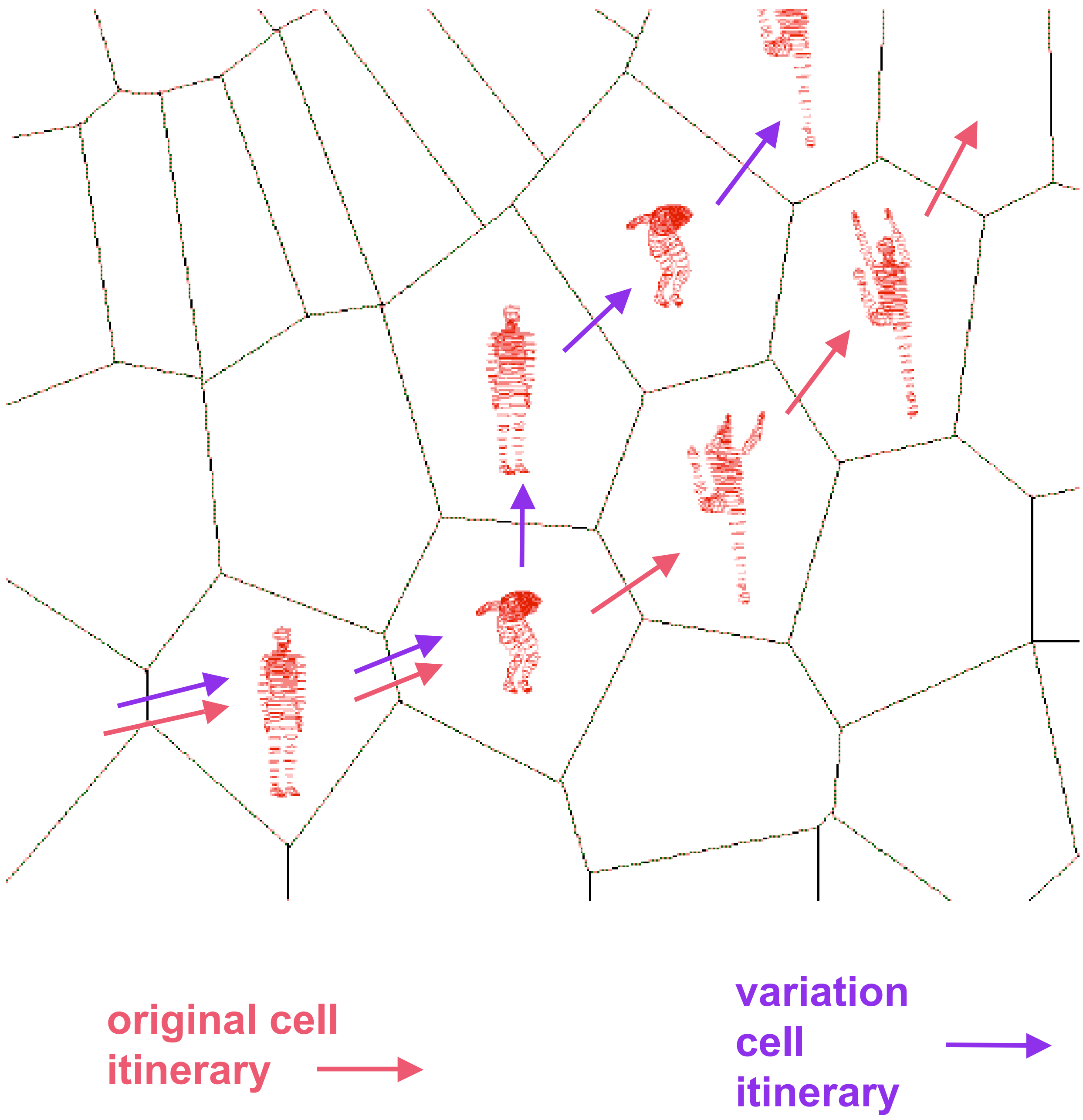


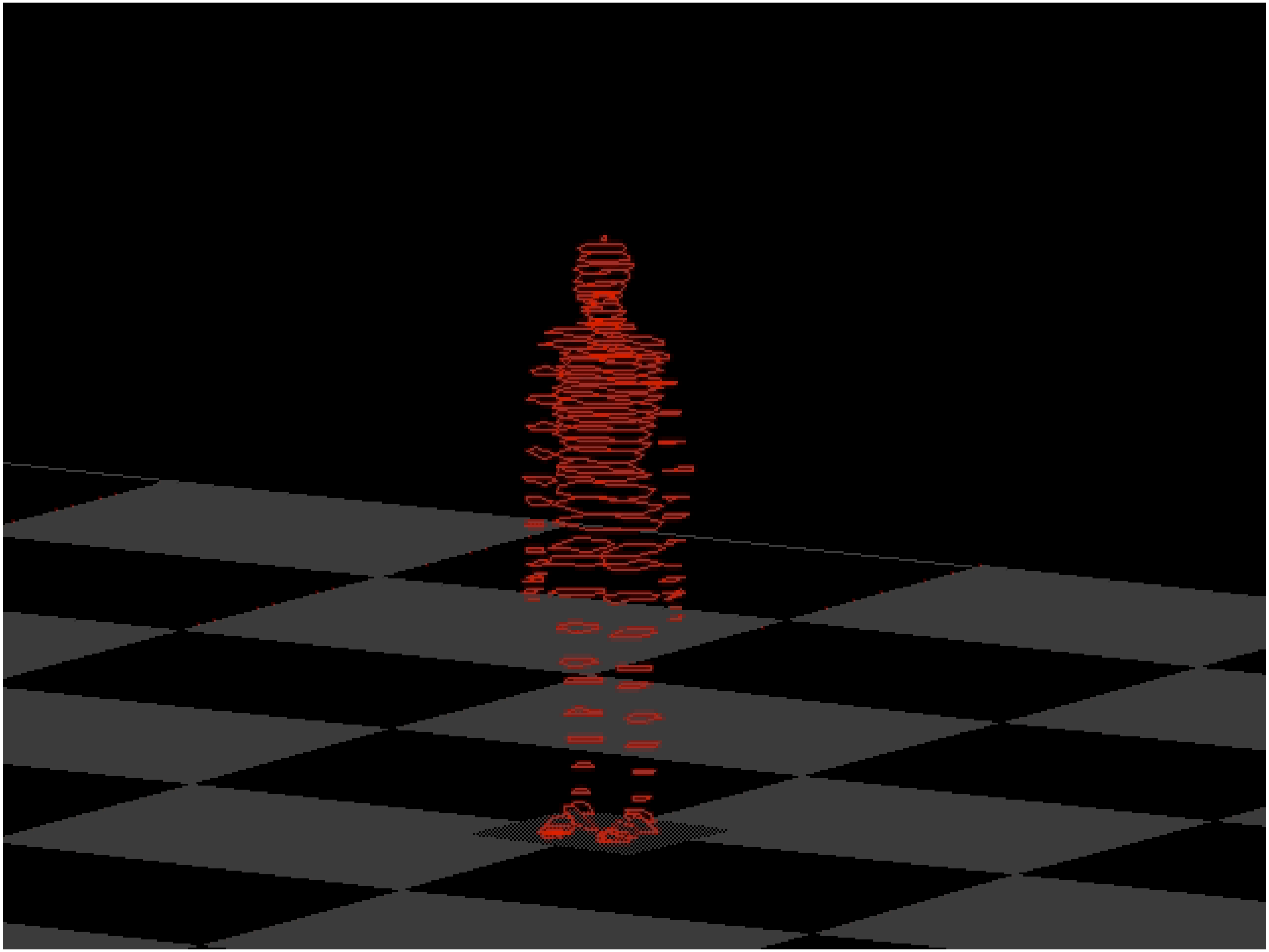


Lorenz

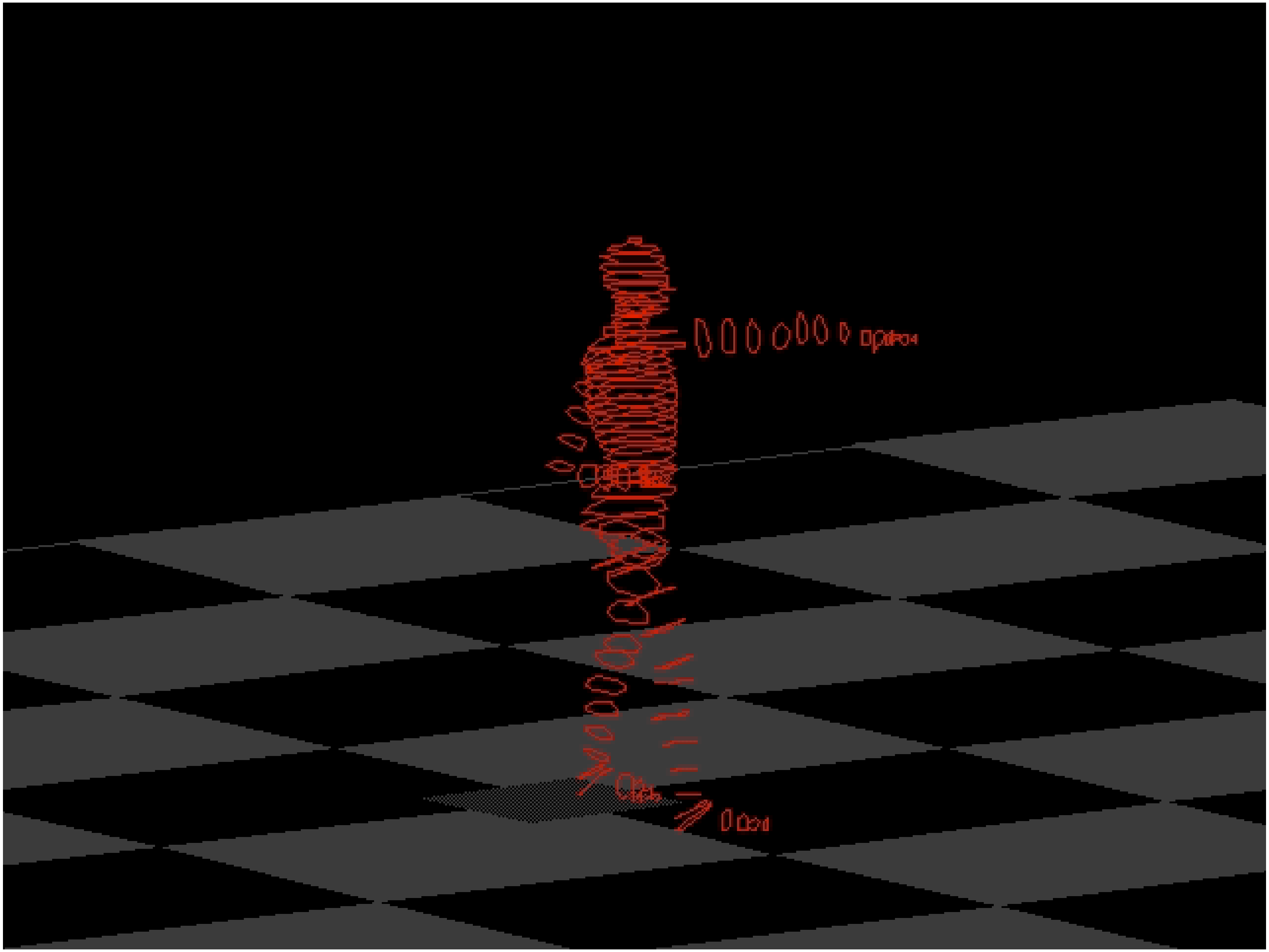


Rossler

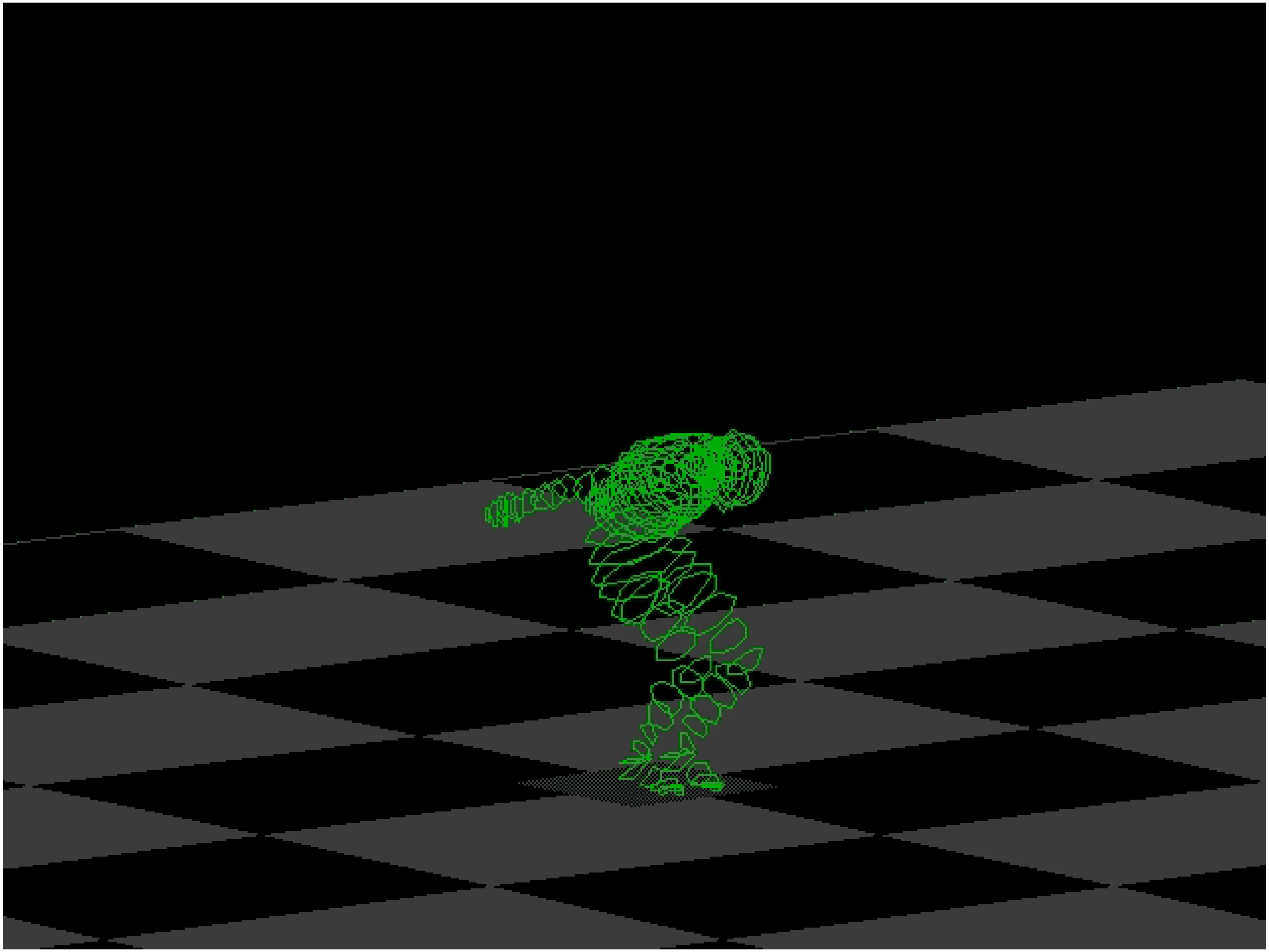

original cell
itinerary
variation
cell
itinerary



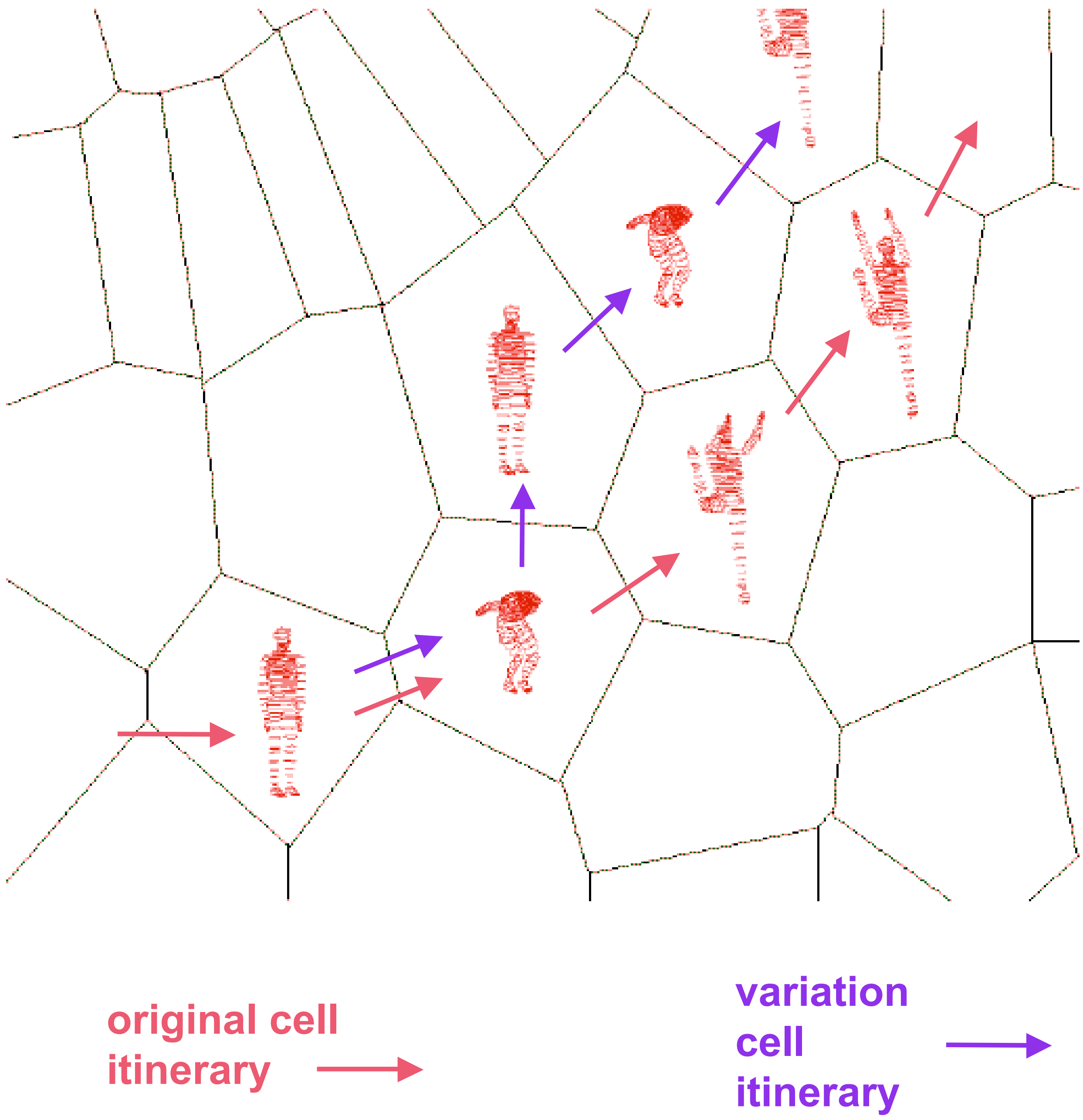
medley

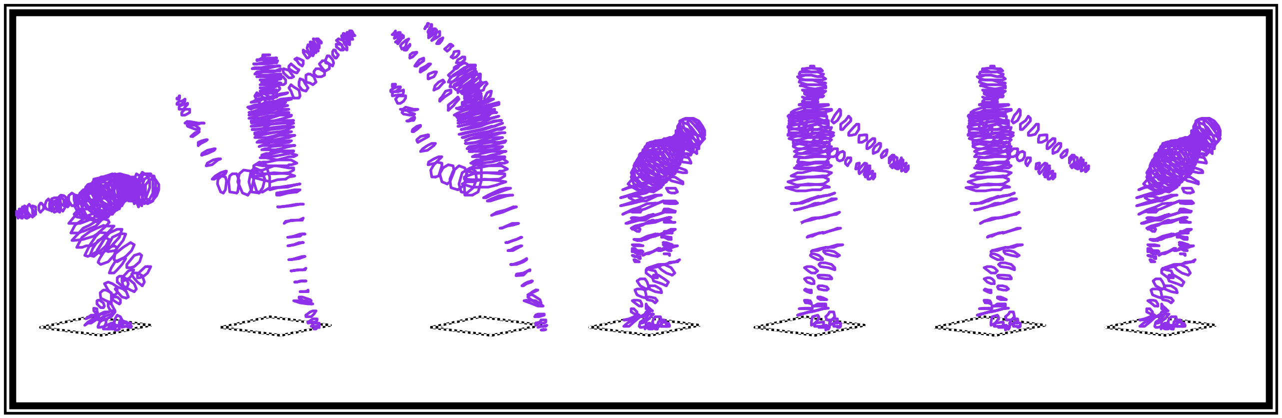


Rossler medley



random medley


original cell itinerary
variation cell itinerary



abrupt transition