Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Summer School

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Chaos:

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

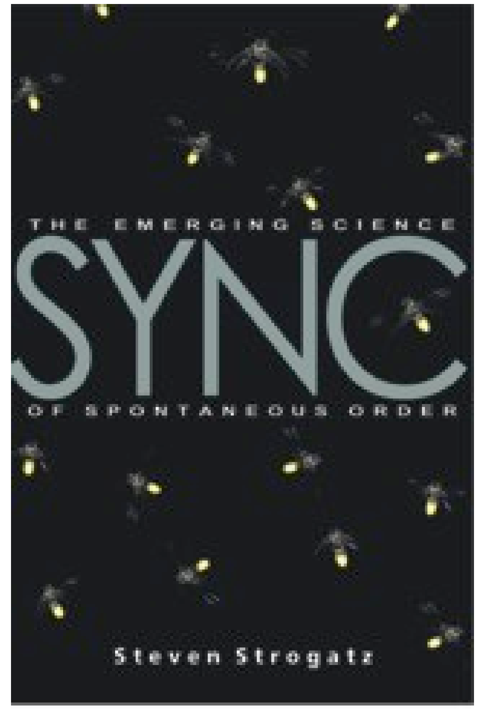
Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and "well-understood"

Where chaos turns up:

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers

Where chaos turns up:

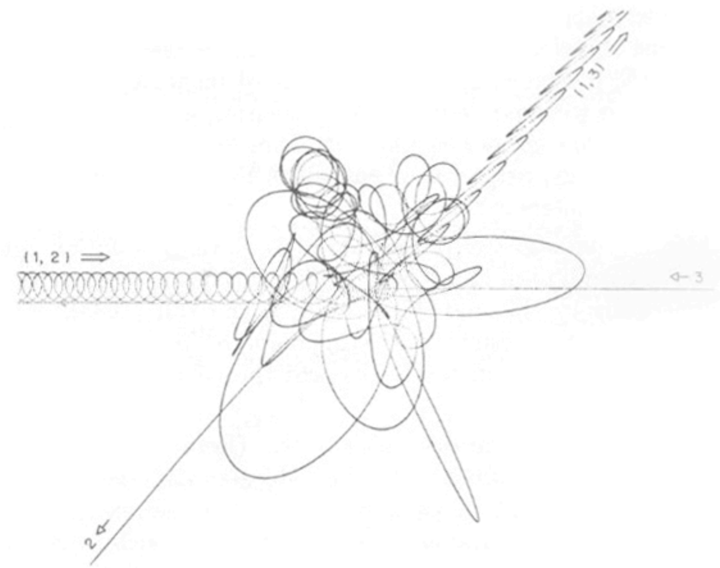
- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies



- and lots of other electronic, chemical, & biological systems

Where chaos turns up:

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….
- Protein folding
- Population biology
- And many, many other fields (including yours)



Hut & Bahcall
Ap.J. ***268***:*319*

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation

A useful graphical solution technique:

- “cobweb” diagram
- return map
- correlation plot

Bifurcations

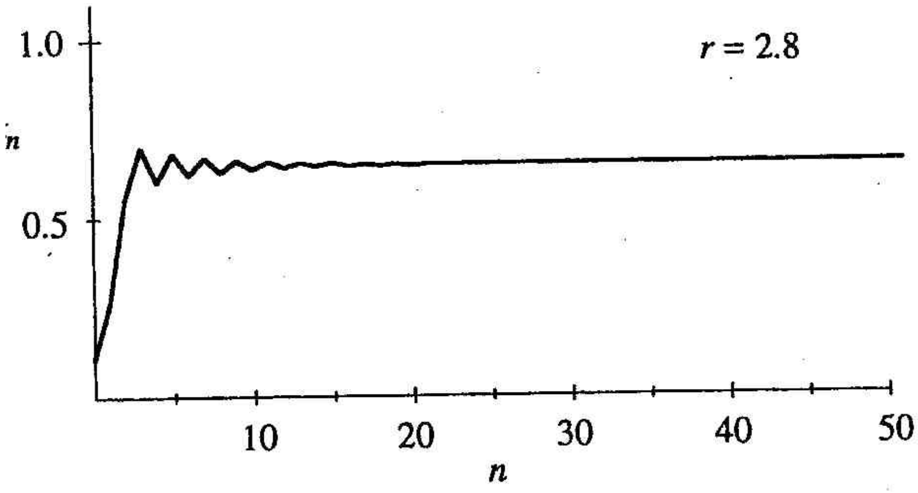
Qualitative changes in the dynamics caused by changes in *parameters*

Bifurcations

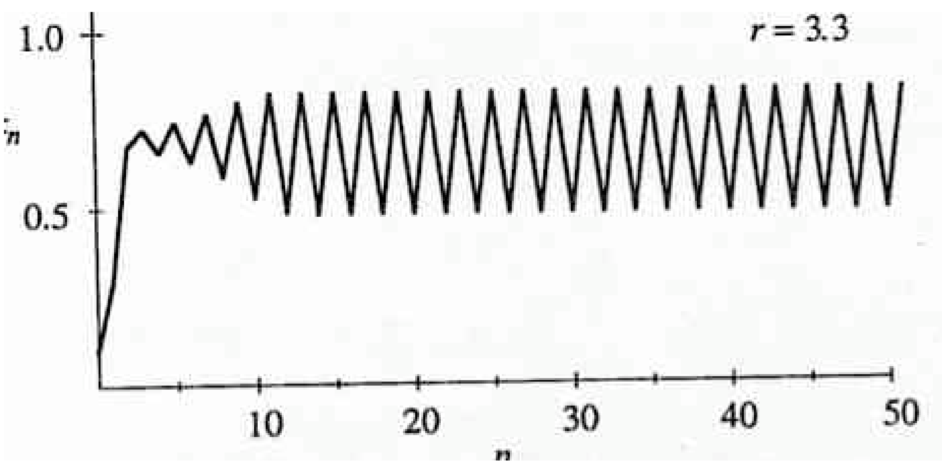
Qualitative changes in the dynamics caused by changes in *parameters:*

- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- etc. etc.

Bifurcations in the logistic map:

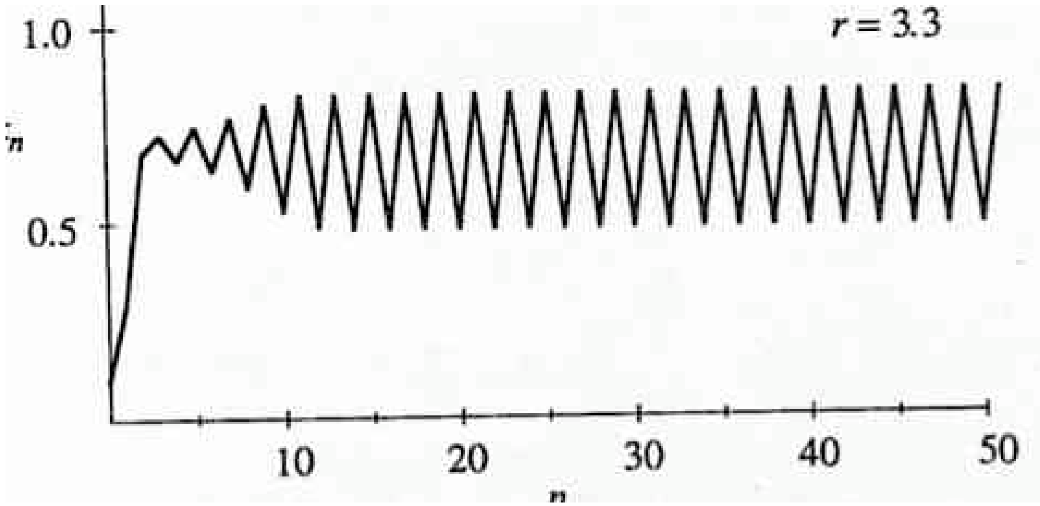


R=2.8

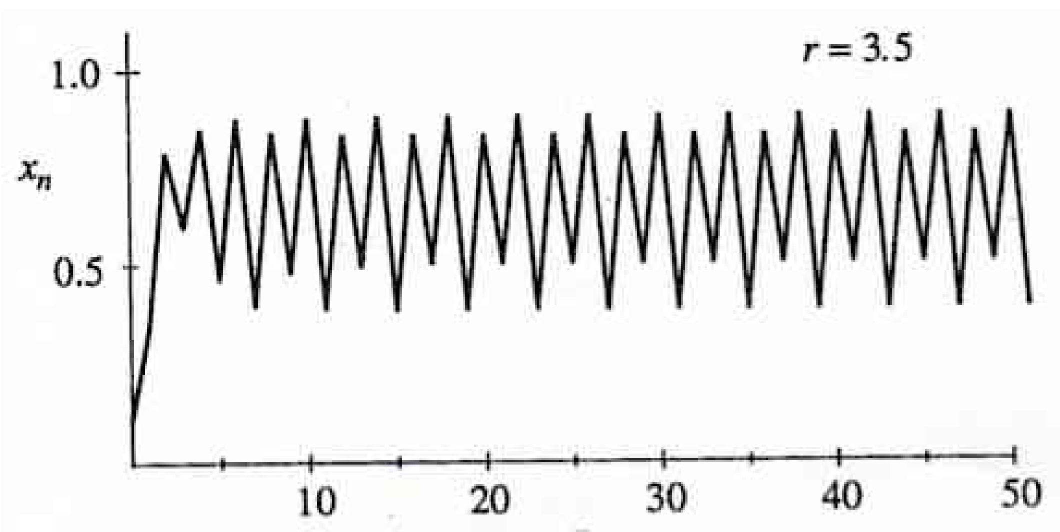


R=3.3

Discrete time: <u>should not connect dots!!</u>



R=3.3



R=3.5

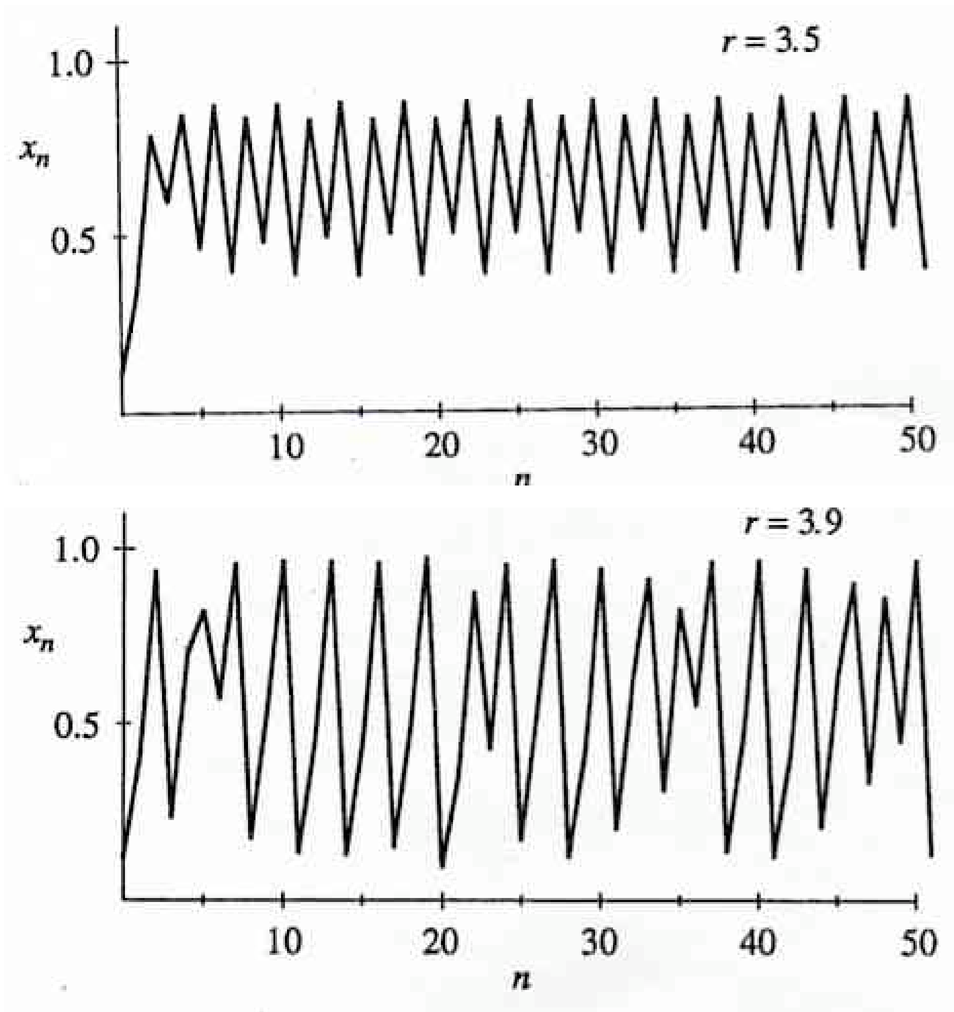
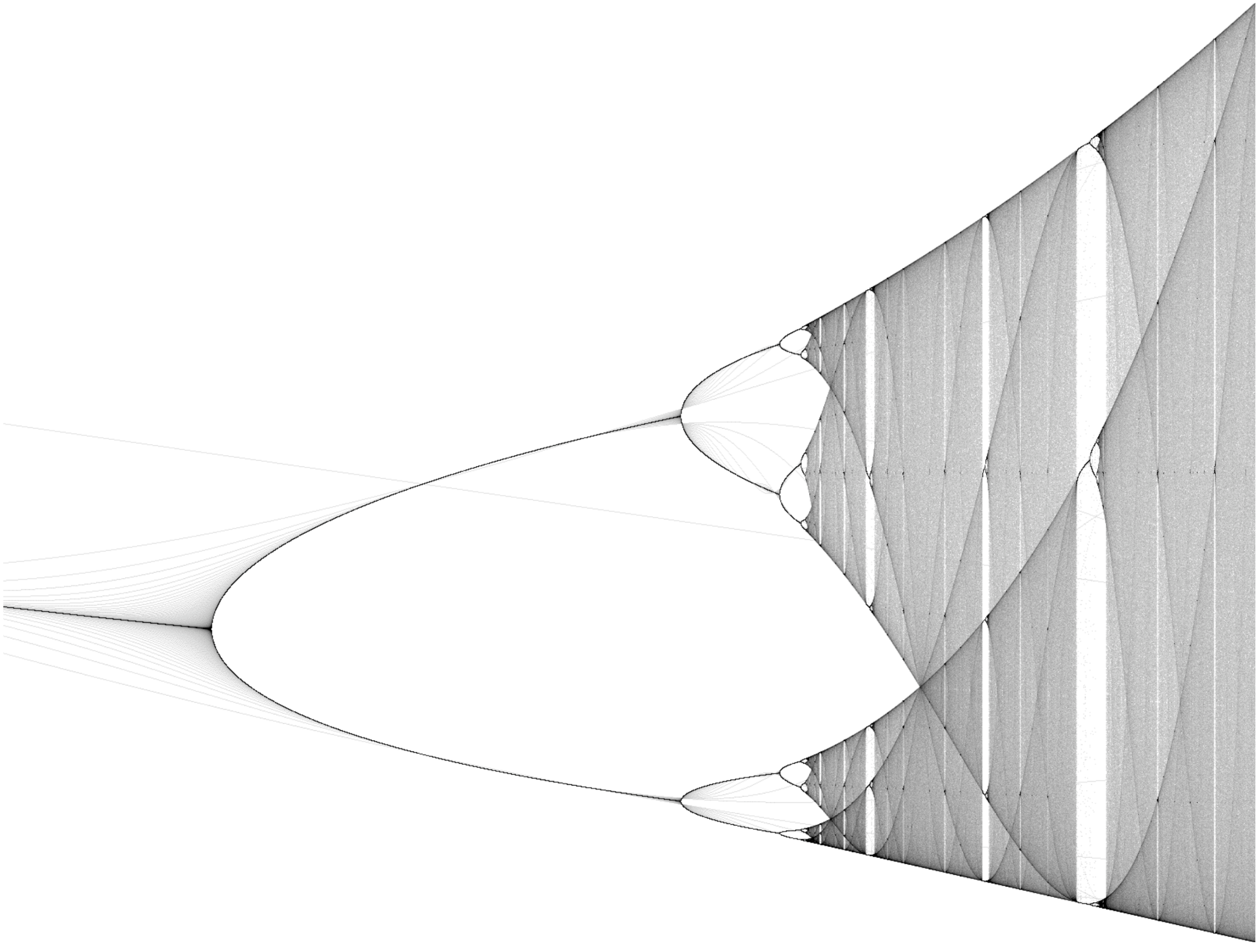


Figure 10.2.5

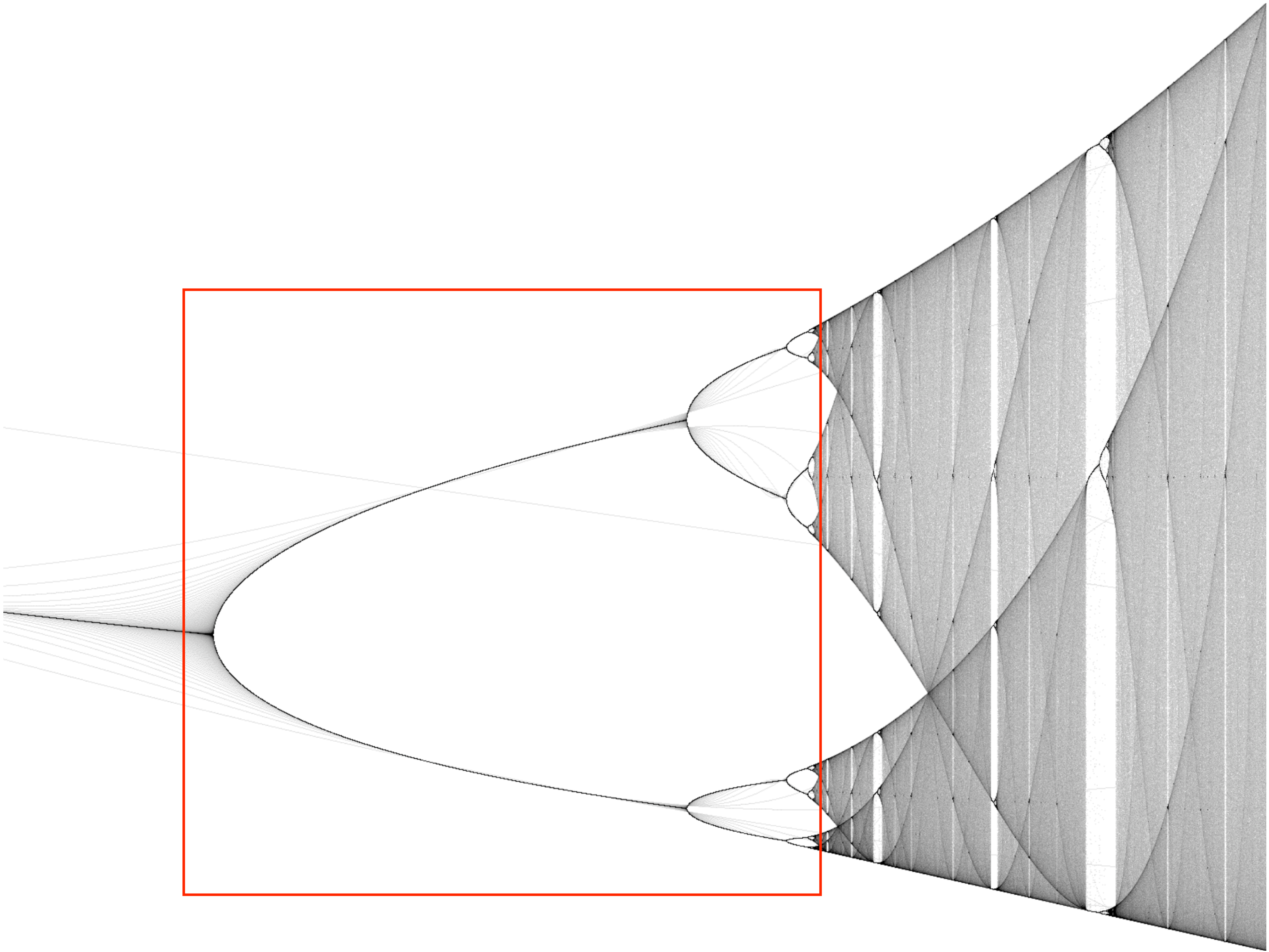
R=3.5

R=3.9

These plots stolen from *Strogatz*



- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)



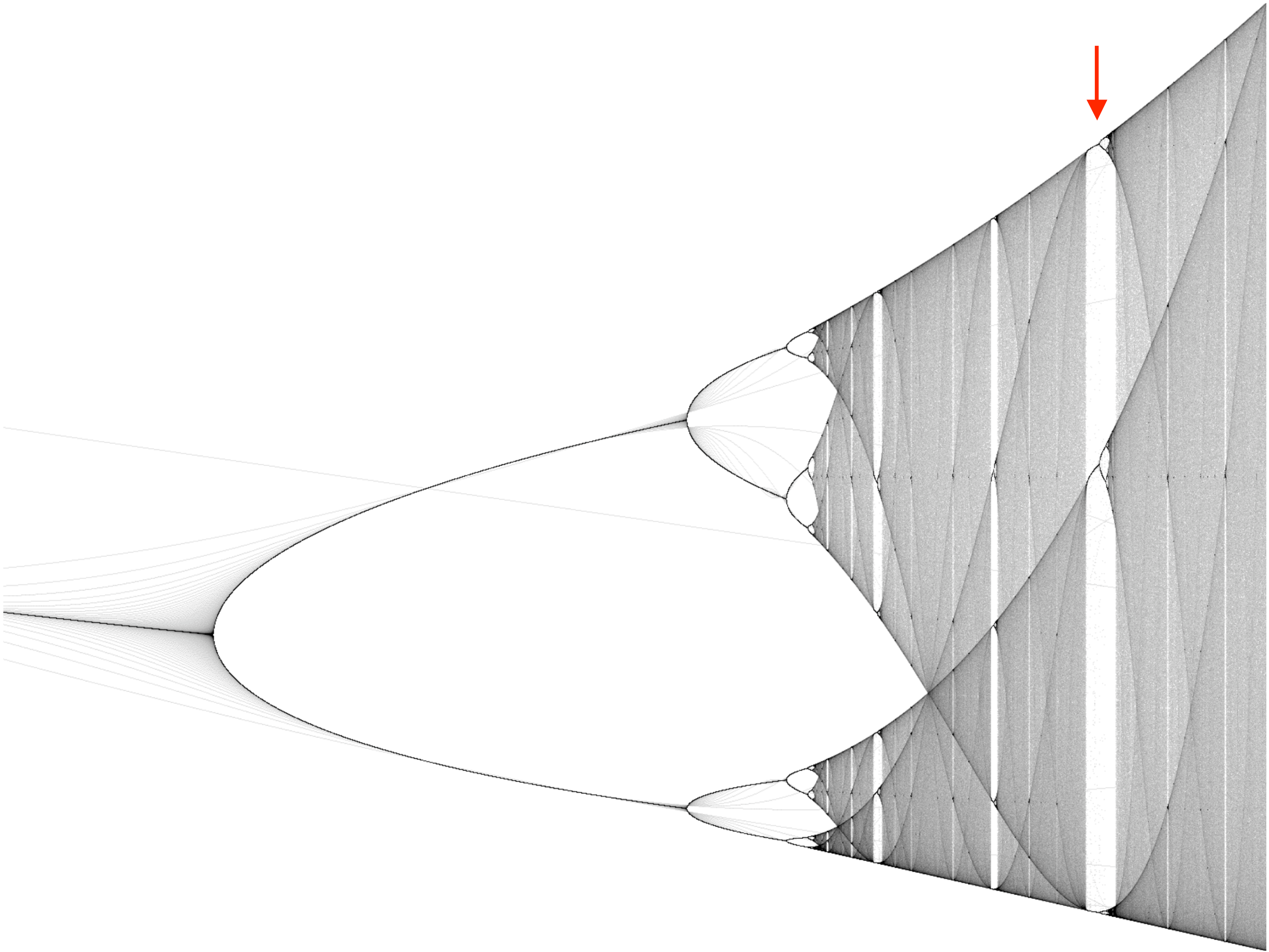
- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- *period-doubling cascade @ low R*

Universality!

Feigenbaum number and many other interesting chaotic/dynamical properties hold for any 1D map with a quadratic maximum.

Proof: renormalizations. See Strogatz §10.7

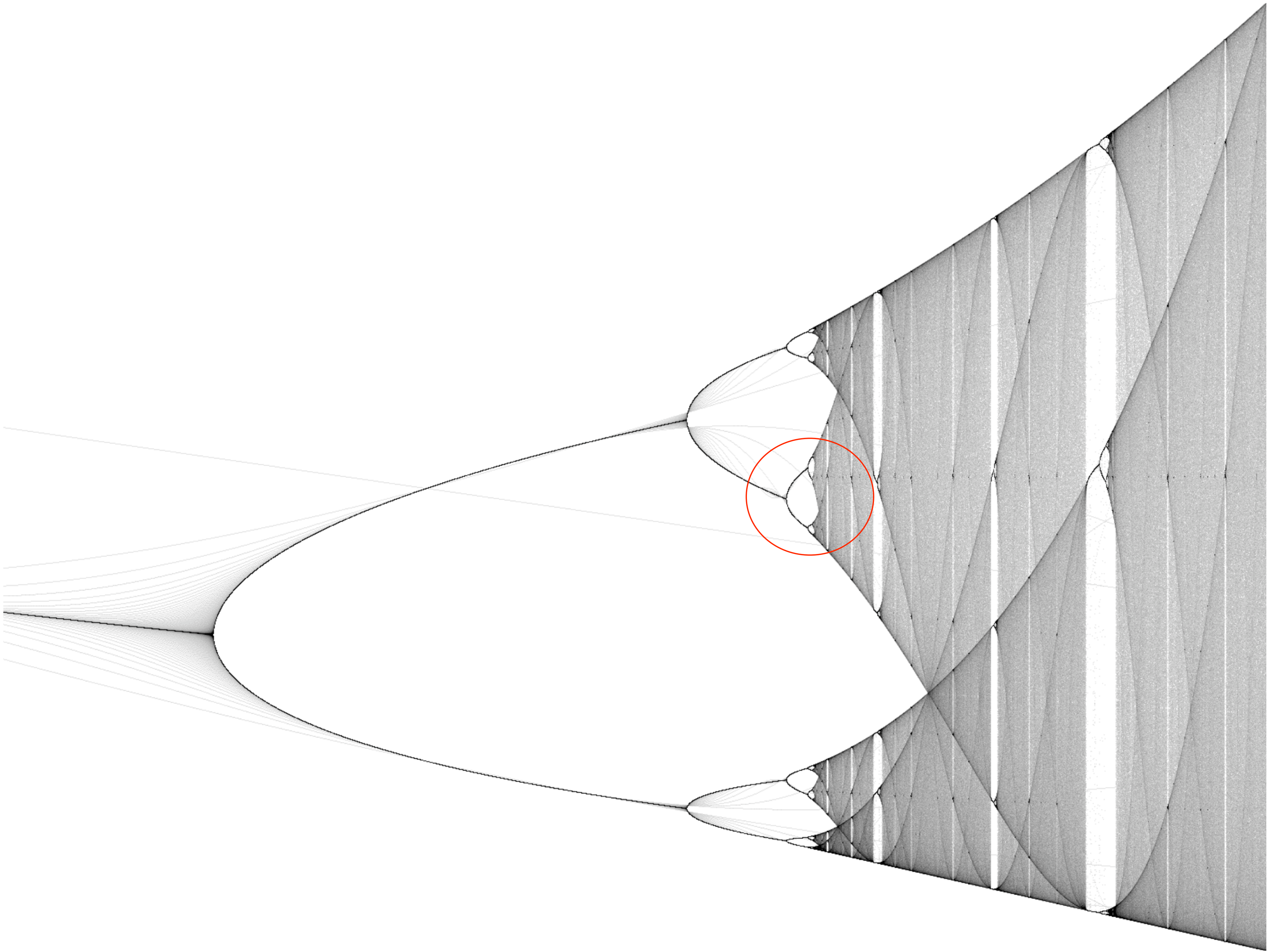
Don't take this too far, though…



- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- *windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)*

A bit more lore on periods and chaos:

- Sarkovskii (1964)
- Yorke (1975)
- Metropolis *et al.* (1973)



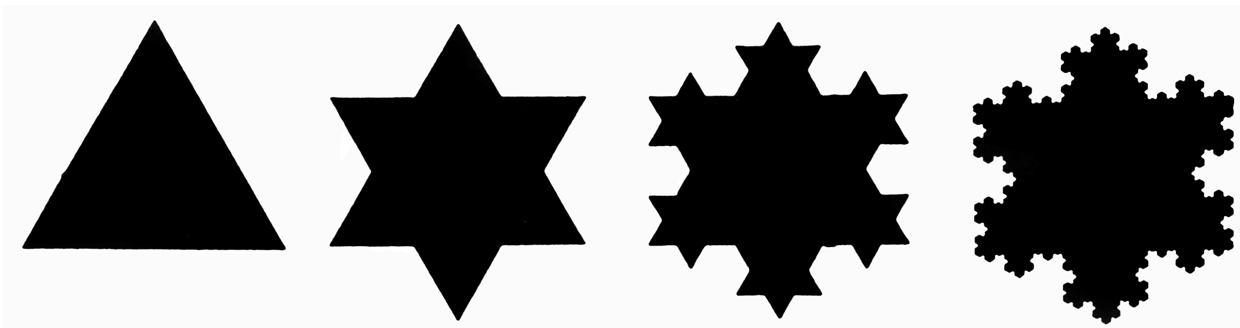
- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
- *small copies of object embedded in it (fractal)*

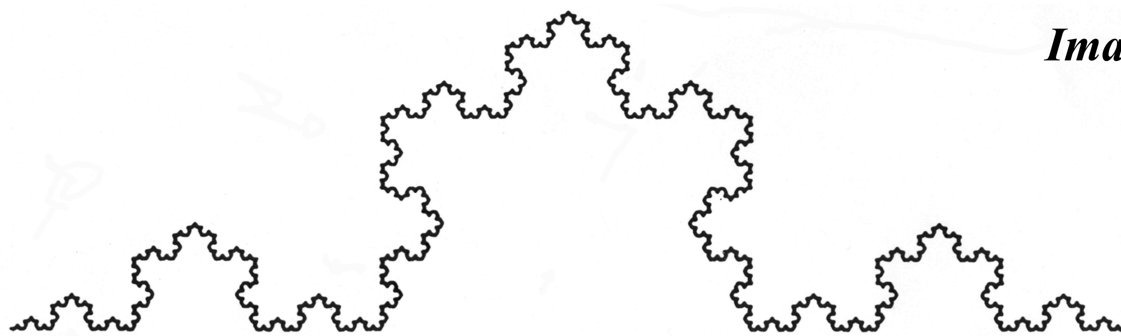
Fractals and Chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.

Fractals

- non-integer Hausdorff dimension
- self-similar

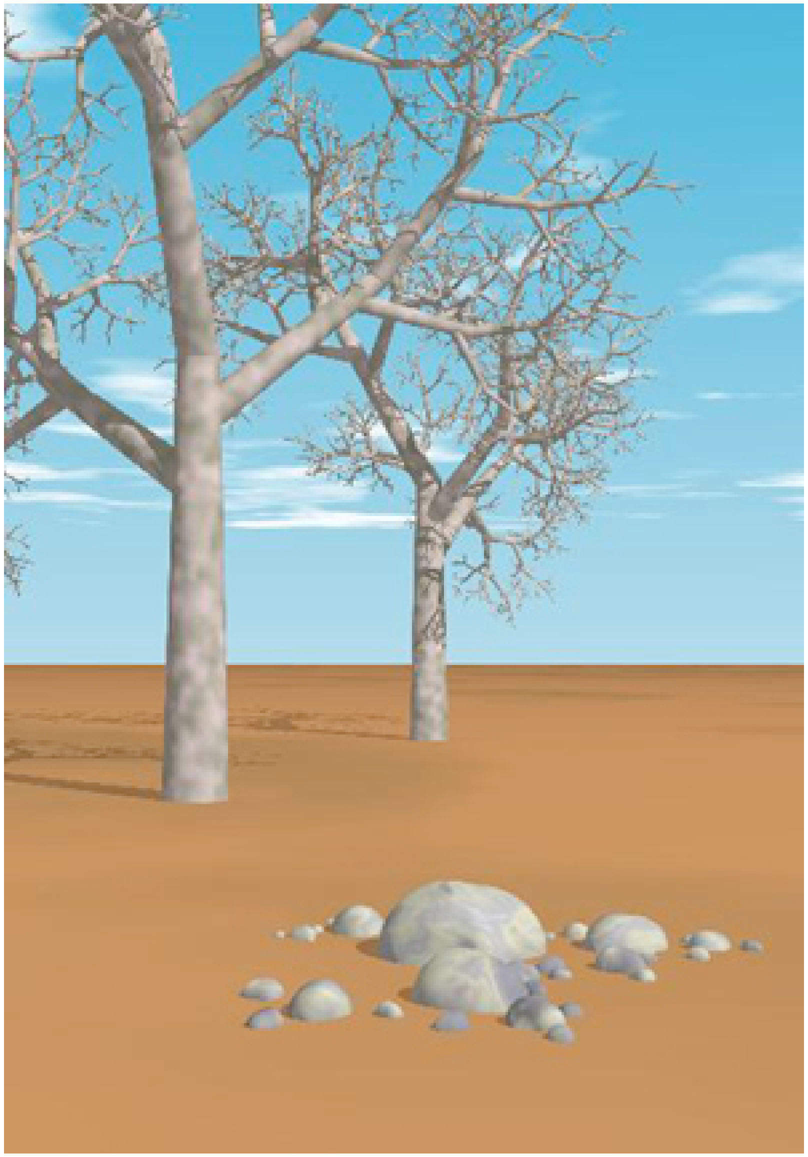




Images from Gleick.

Examples: Cantor set, coastlines, trees, lungs, clouds, drainage basins, …

In computer graphics…



Matthew Ward, WPI
http://davis.wpi.edu/~matt/courses/fractals/trees.html

In maps:

Newton's method on $x^4 - 1 = 0$

From Strogatz

That was all about *maps*.

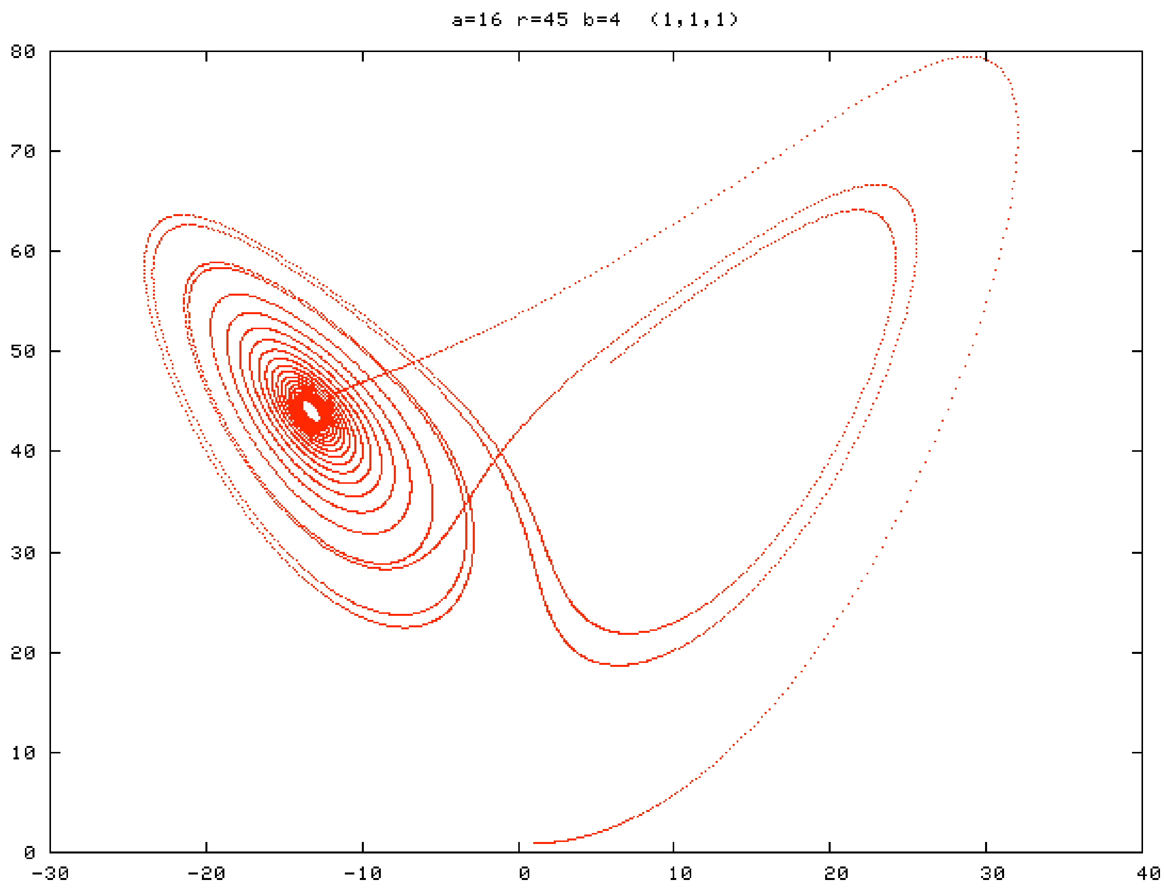
- discrete time systems:
    - time proceeds in clicks
    - “maps”
    - modeling tool: difference equation

Next: *flows.*

- continuous time systems:
    - time proceeds smoothly
    - “flows”
    - modeling tool: differential equations

Terminology…

- State variable
- State space
- Trajectory
- Initial condition
- Transient
- Attractor
- Fixed point (un/stable)
- Basin of attraction
- Bifurcation
- Parameter


a=16 r=45 b=4 (1,1,1)
80
70
60
50
40
30
20
10
0
-30
-20
-10
0
10
20
30
40

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

J. Atm. Sci. **20:**130

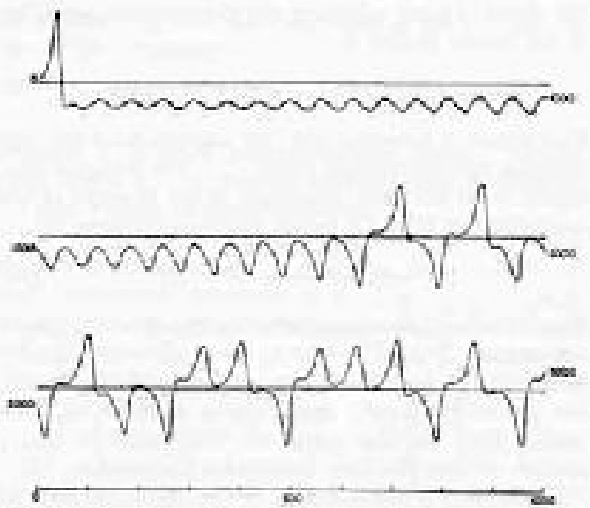


FIG. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

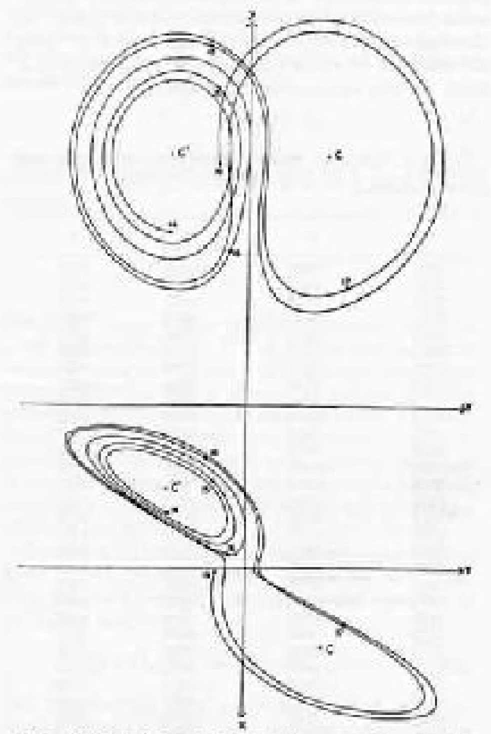


FIG. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

- Equations:

$$x' = a(y-x)$$

$$y' = rx - y - xz$$

$$z' = xy - bz$$

- State variables:
  - x convective intensity
  - y temperature
  - z deviation from linearity in the vertical convection profile

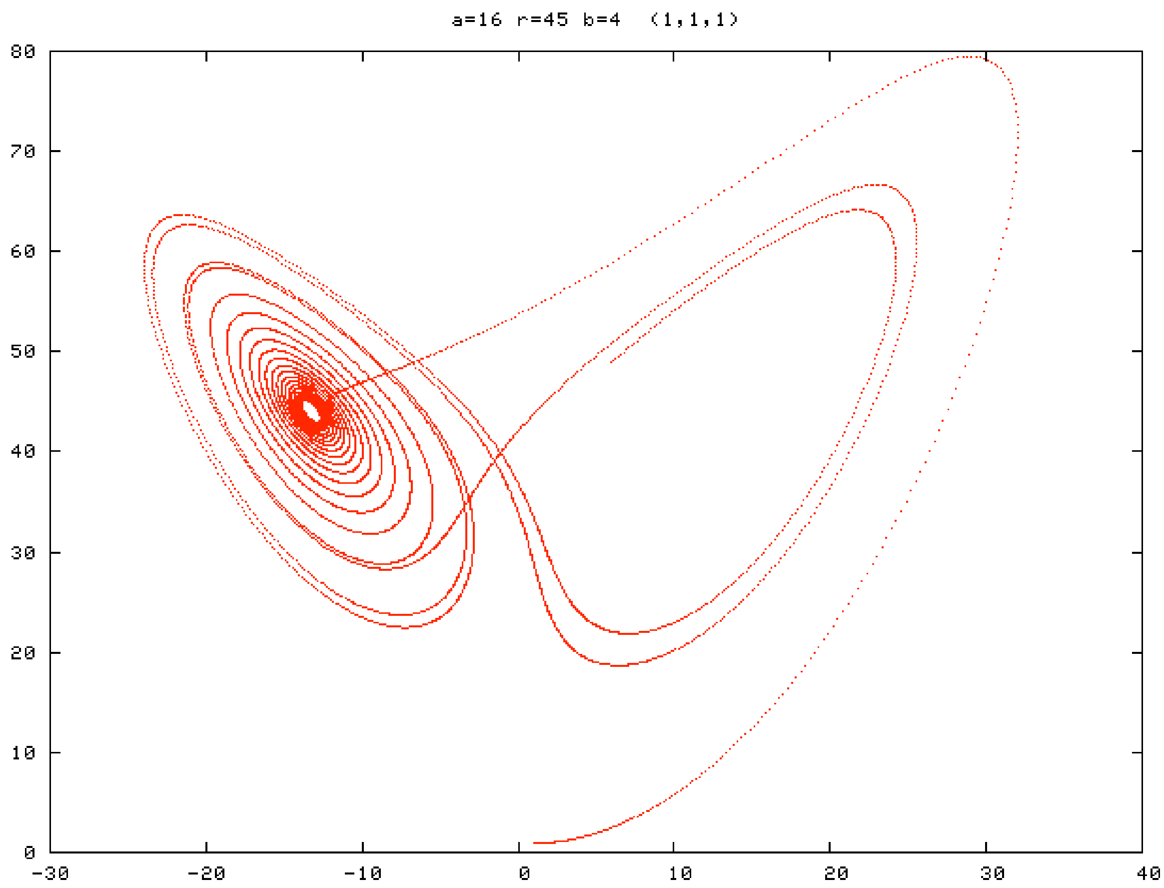
- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to ΔT
  - *b* aspect ratio of the fluid sheet

- Equations:

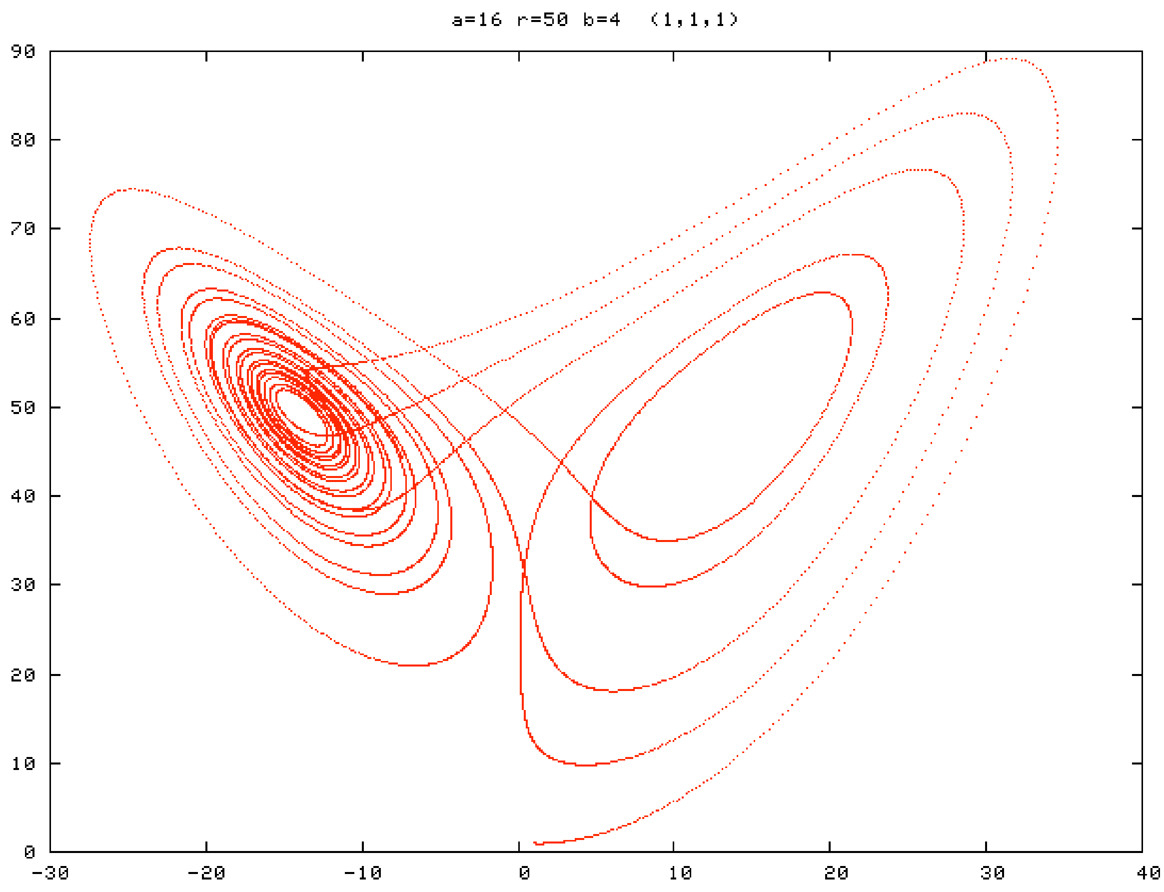
$$x' = a(y-x)$$

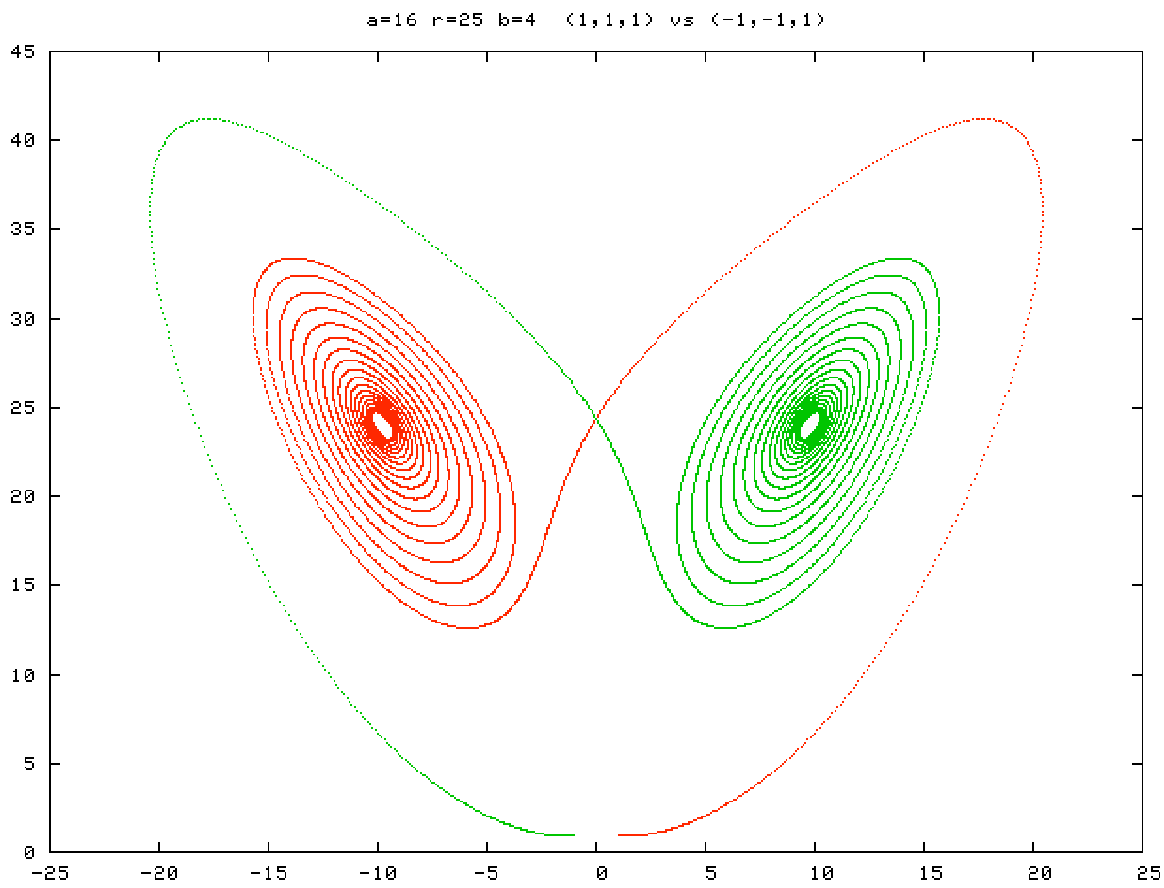
$$y' = rx - y - xz$$

$$z' = xy - bz$$

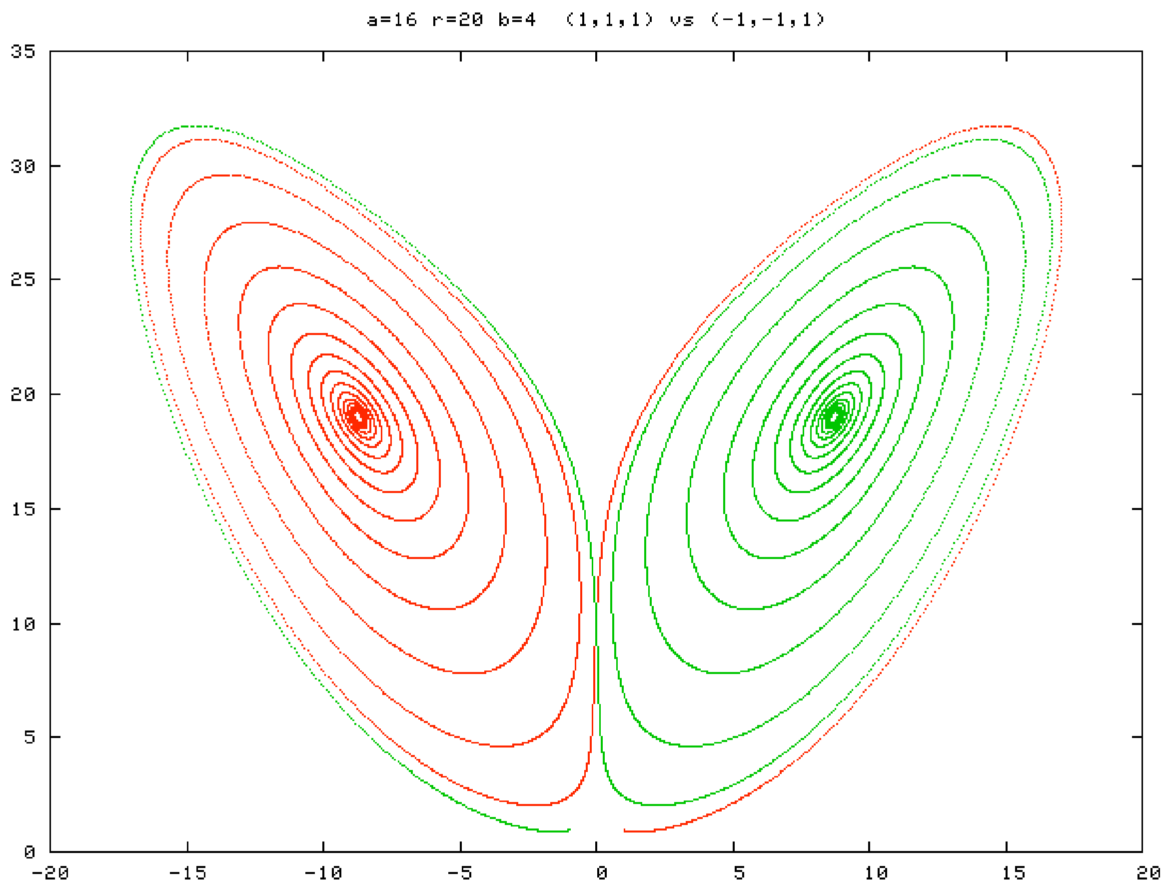


a=16 r=45 b=4 (1,1,1)
80
70
60
50
40
30
20
10
0
-30
-20
-10
0
10
20
30
40


a=16 r=50 b=4 (1,1,1)
90
80
70
60
50
40
30
20
10
0
-30
-20
-10
0
10
20
30
40



a=16 r=25 b=4 (1,1,1) vs (-1,-1,1)
45
40
35
30
25
20
15
10
5
0
-25
-20
-15
-10
-5
0
5
10
15
20
25


a=16 r=20 b=4 (1,1,1) vs (-1,-1,1)
35
30
25
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15
10
5
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-20
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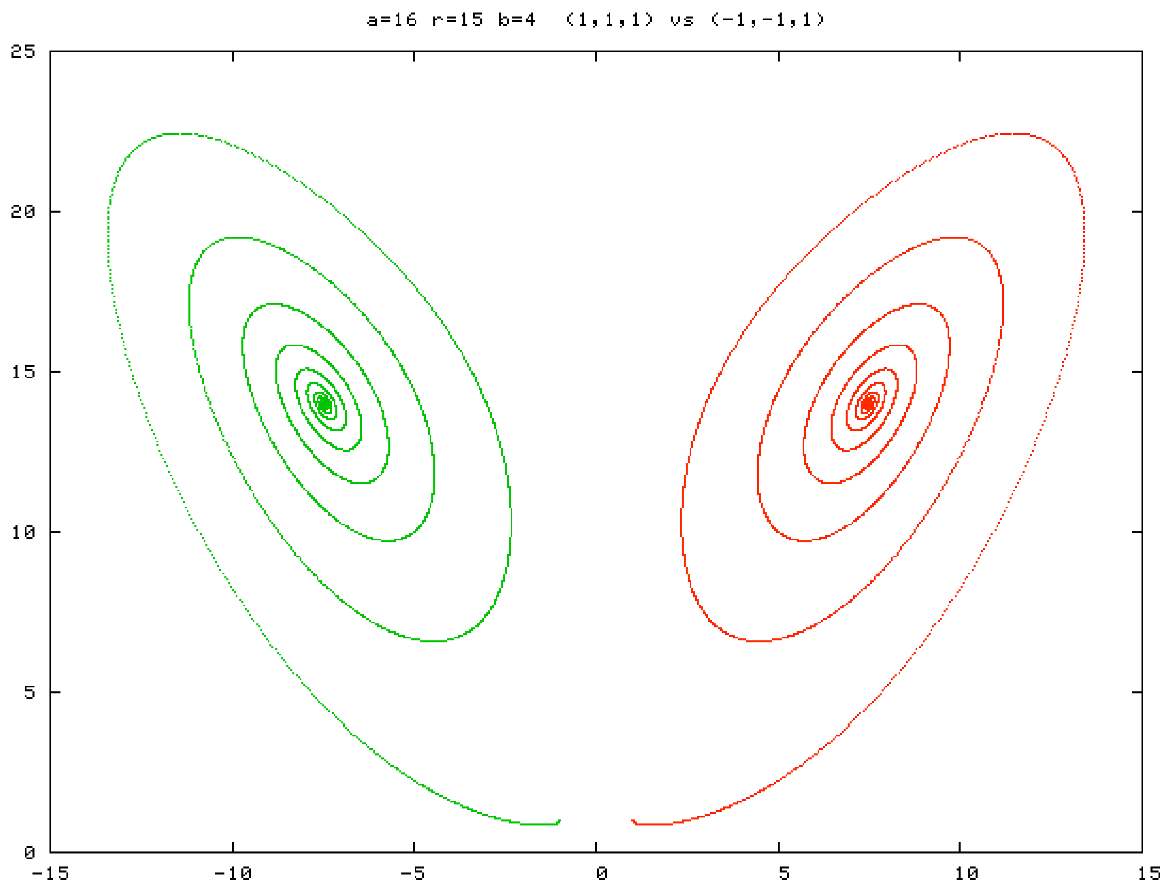
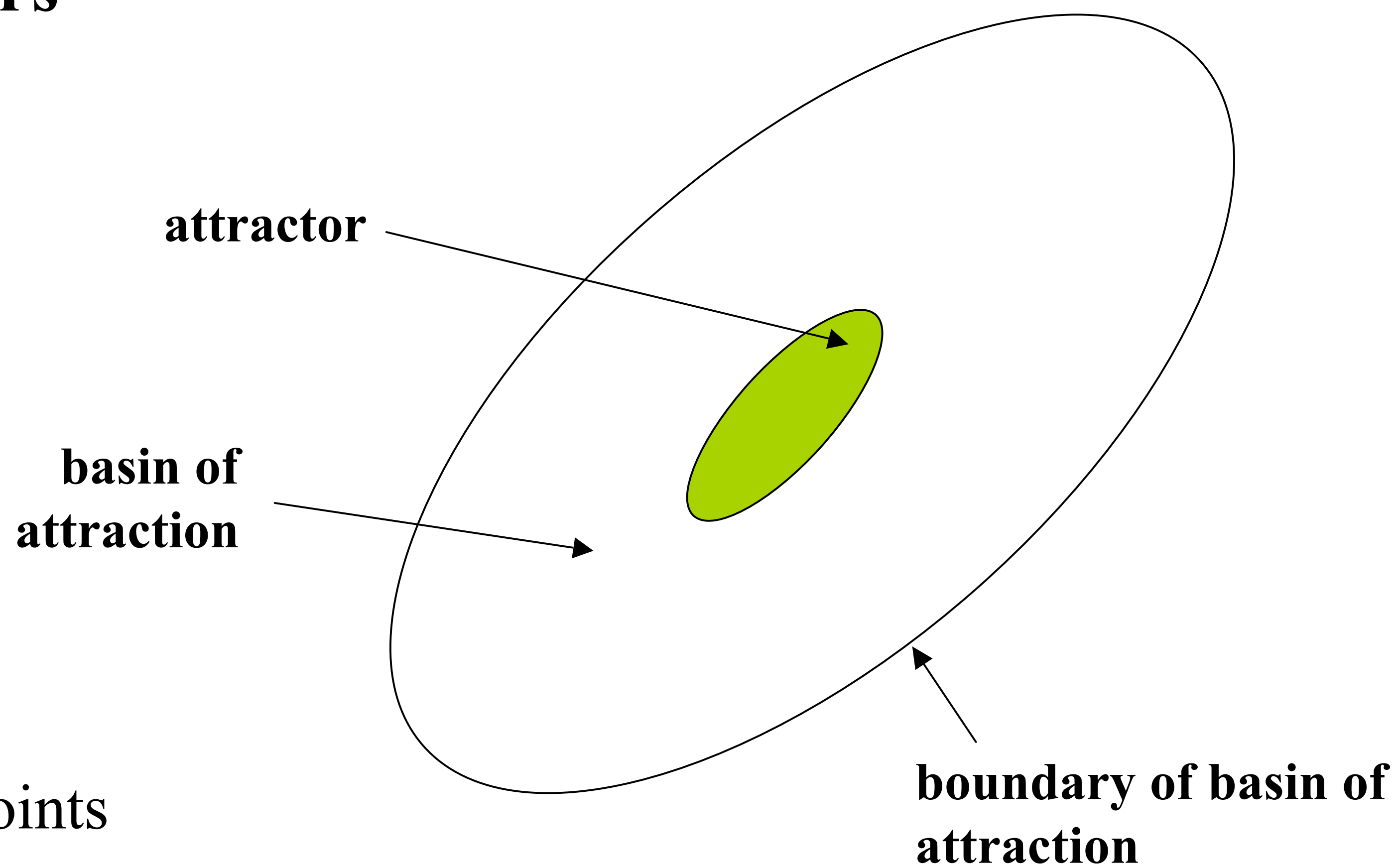
-10
-5
0
5
10
15
20


a=16 r=15 b=4 (1,1,1) vs (-1,-1,1)
25
20
15
10
5
0
-15
-10
-5
0
5
10
15

Attractors

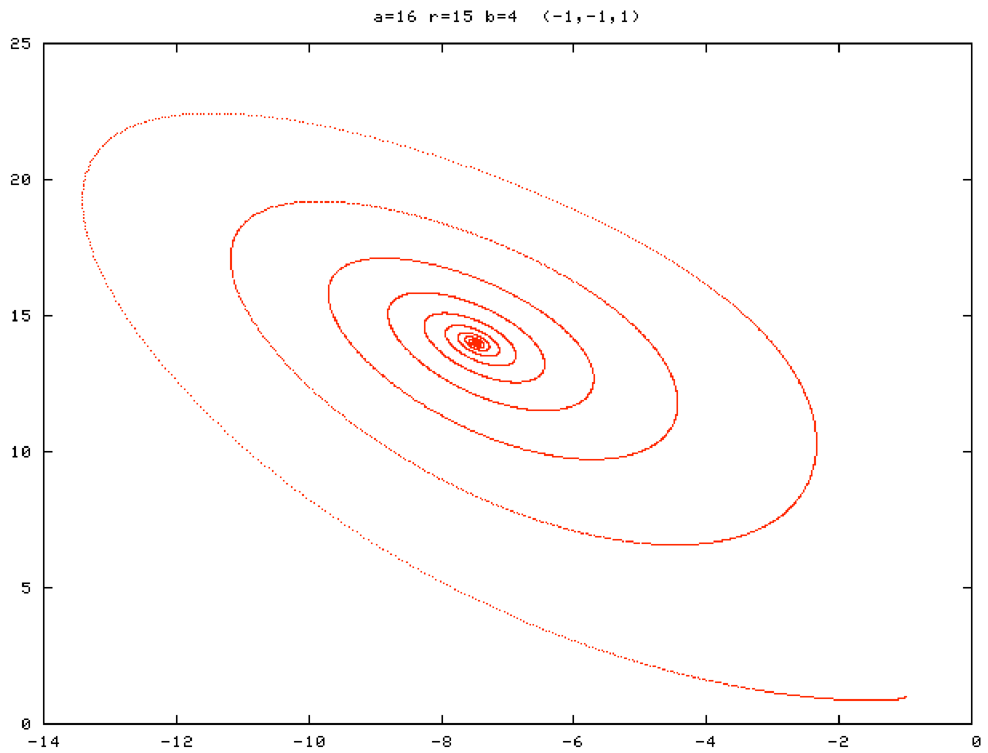


- fixed points
- limit cycles
- quasiperiodic orbits
- chaotic attractors

(dissipative systems only…)

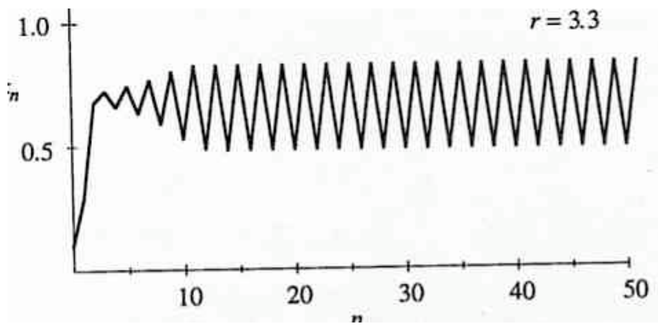
Attractors:

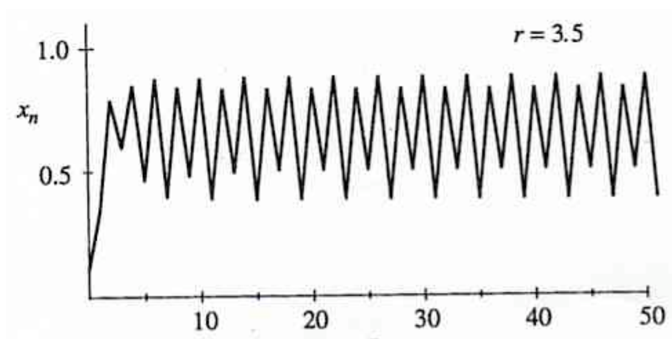
- Fixed point



Attractors:

- Limit cycle





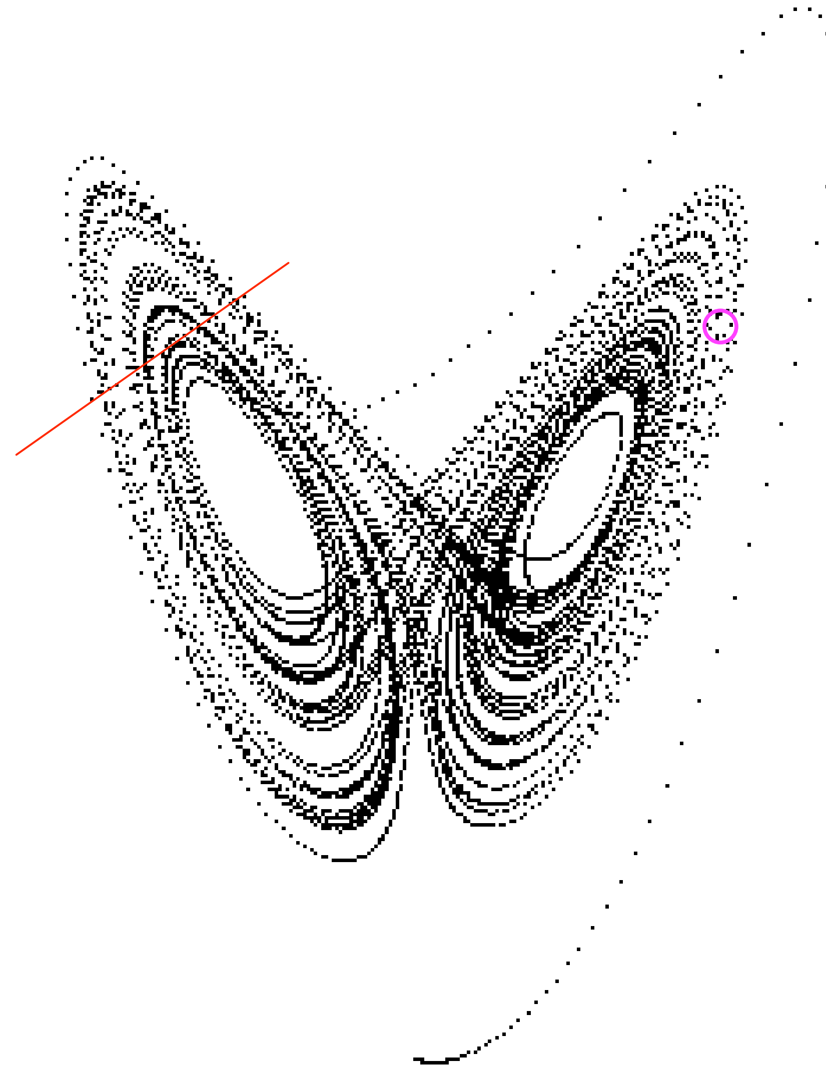
Strogatz

Also wall voltage!

Attractors:

- Quasi-periodic orbit…

“Strange” or chaotic attractors:



• *often* fractal

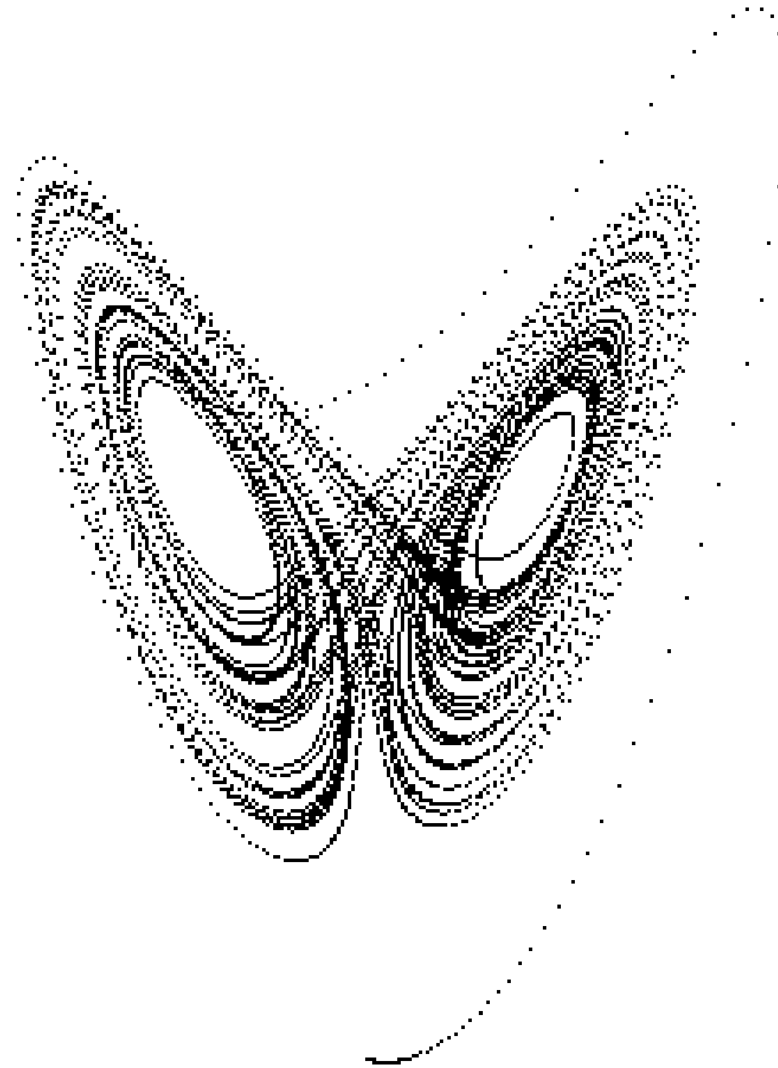
• covered densely by trajectories

• exponential divergence of neighboring trajectories

Lyapunov exponents:

- *positive λ is a signature of chaos*
- negative λ compress state space; positive λ stretch it
- nonlinear analogs of eigenvalues: one λ for each dimension
- $\Sigma\lambda < 0$ for dissipative systems
- long-term average in definition; biggest one dominates as t –> infinity
- λ are same for all ICs in one basin

“Strange” or chaotic attractors:



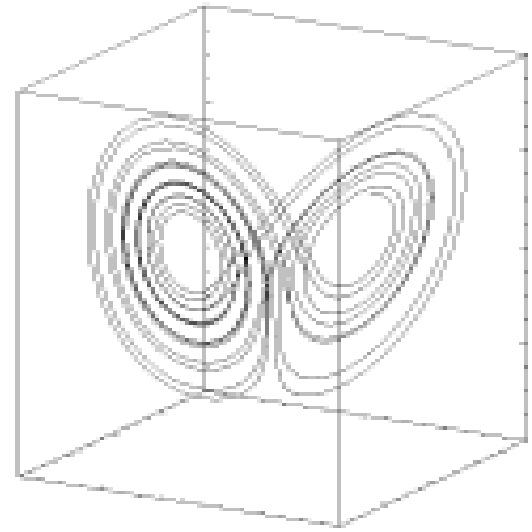
• exponential divergence of neighboring trajectories

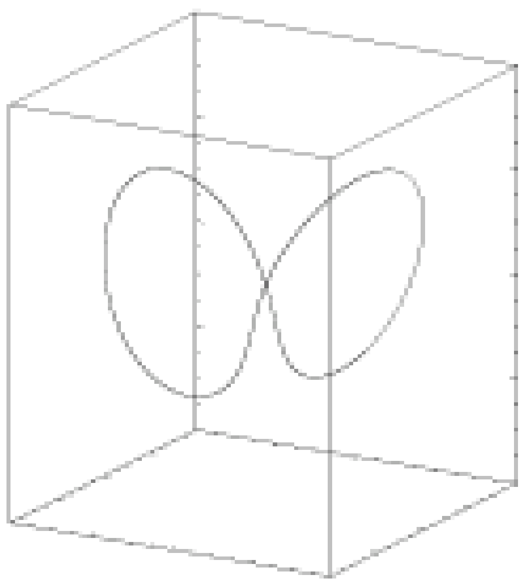
• *often* fractal

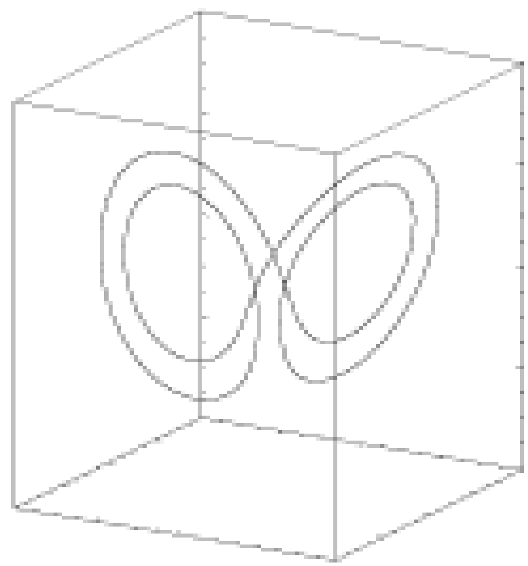
• covered densely by trajectories

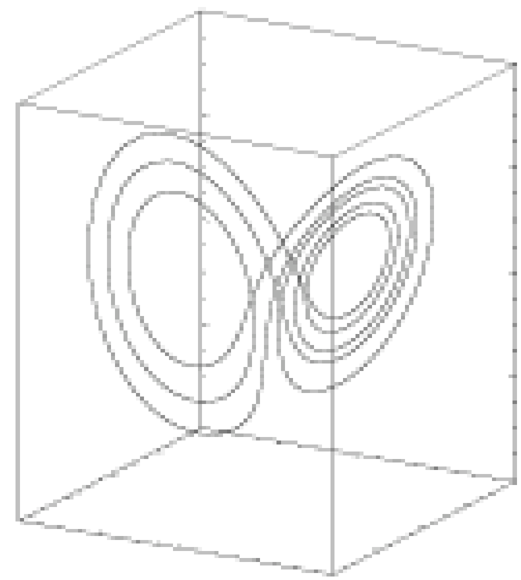
• contain an infinite number of “unstable periodic orbits”…

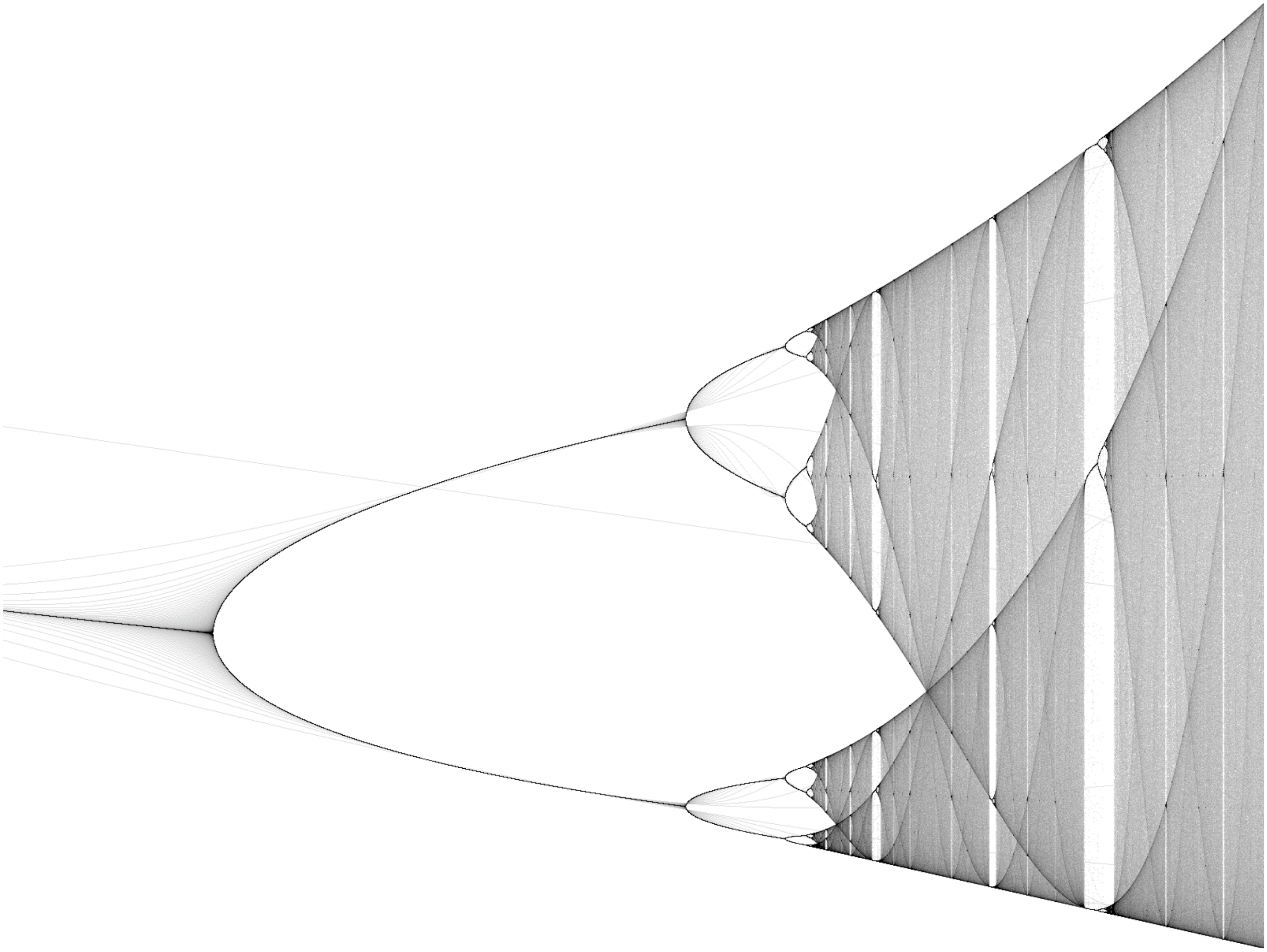
Unstable periodic orbits (UPOs):



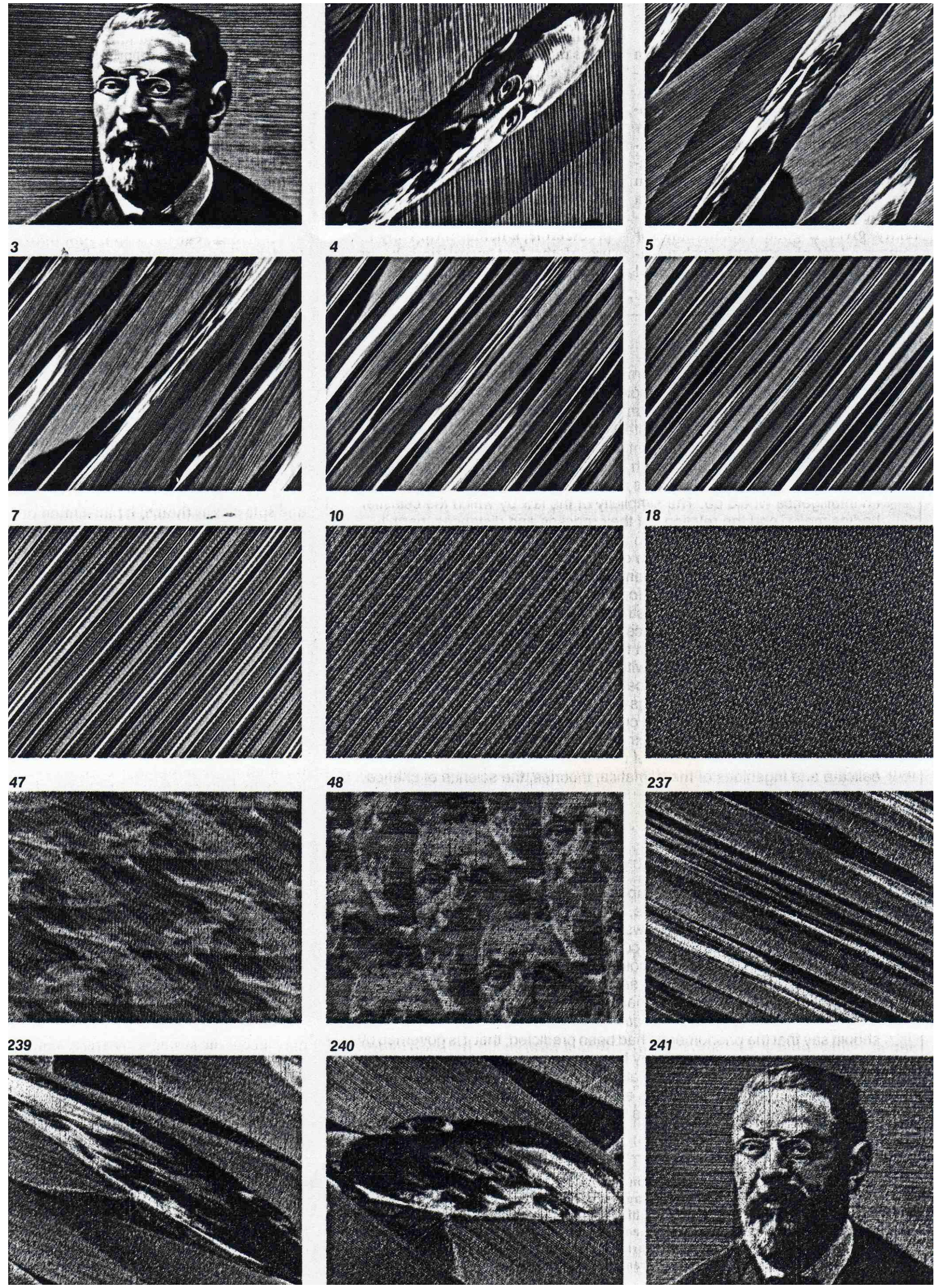




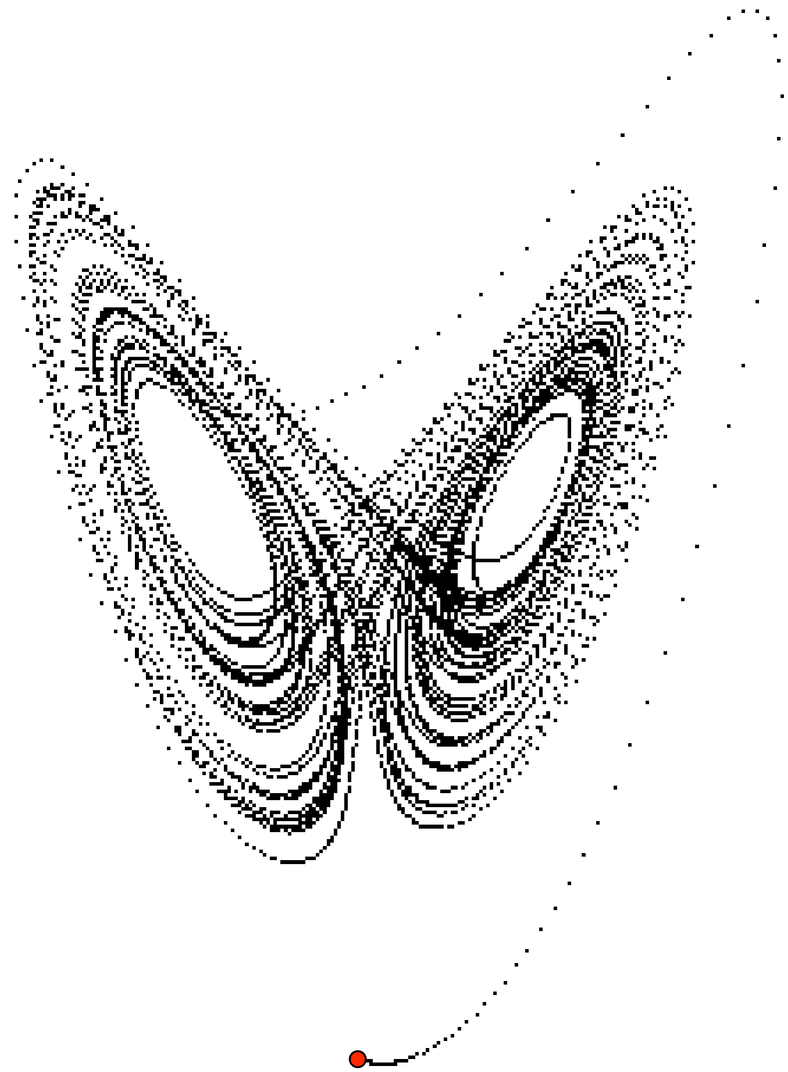




Poincare recurrence



Crutchfield *et al*.
Chaos **255:**46



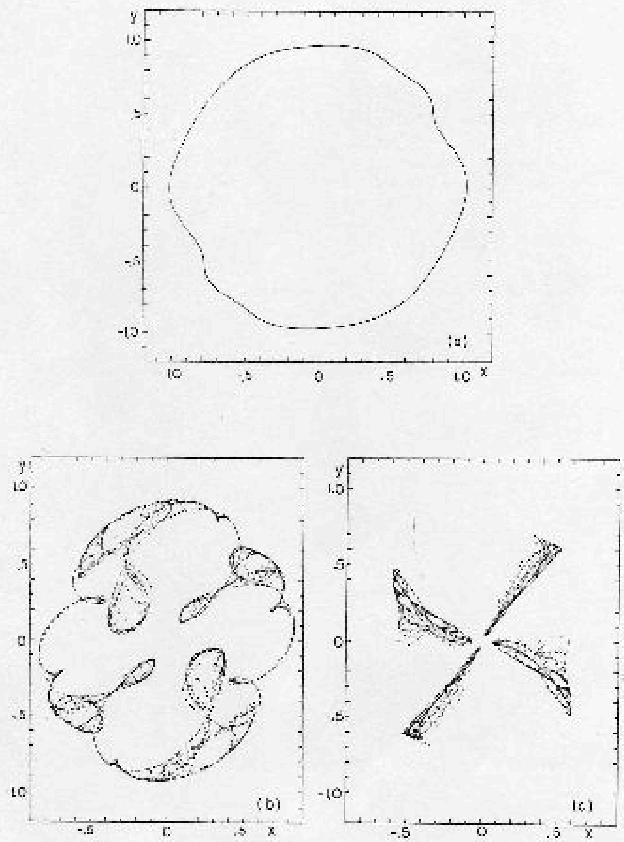
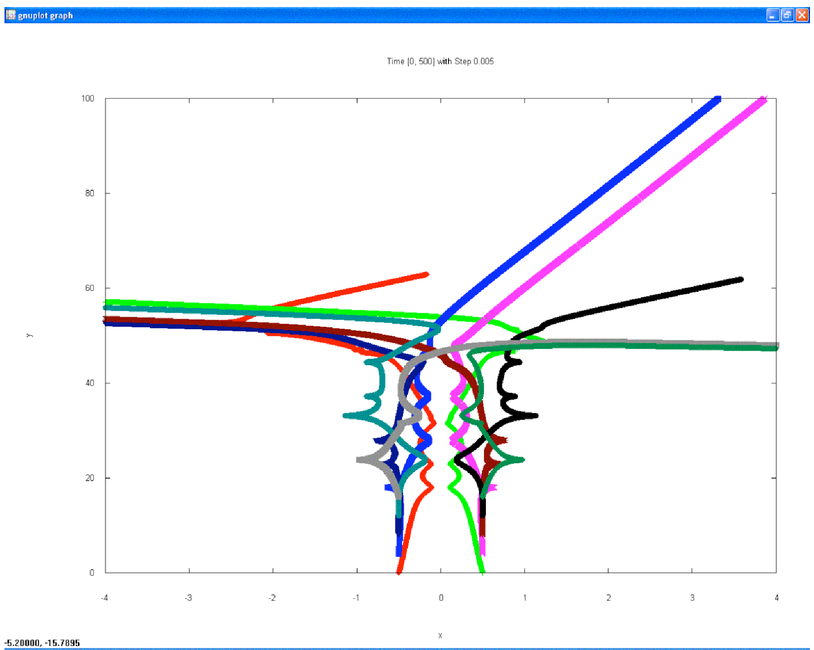
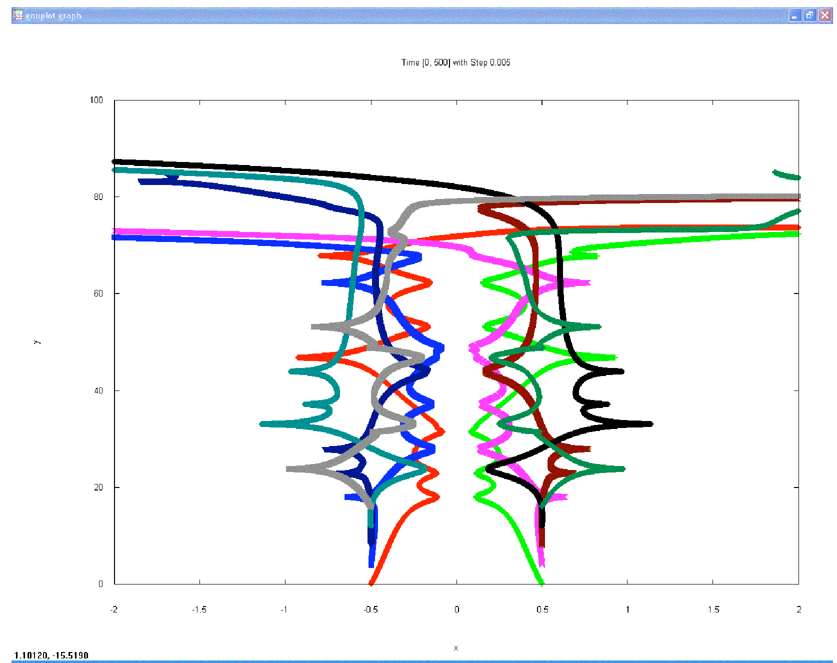


Fig. 15. (a) The attractor A when a fourth-order Runge-Kutta scheme is applied to eqs. (22), with $\tau = 0.91$. (b) The same as (a) except that $\tau = 1.5$. (c) The same as (a), except that $\tau = 1.7$.

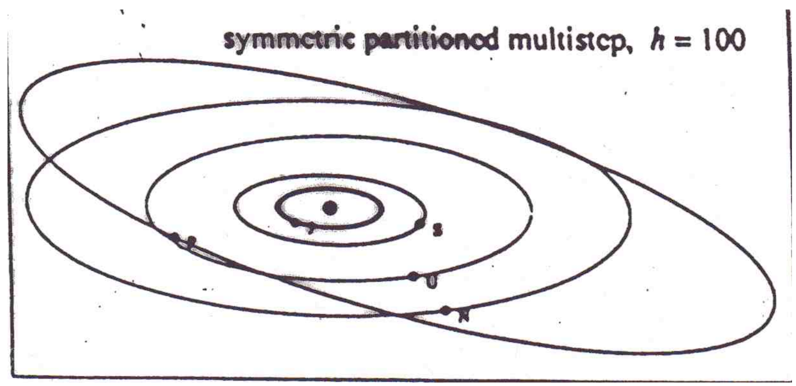
Different timestep

Lorenz, *Physica D* **35:**229



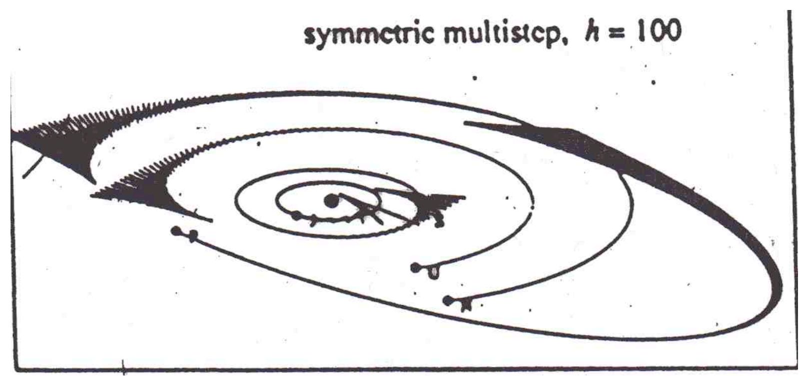


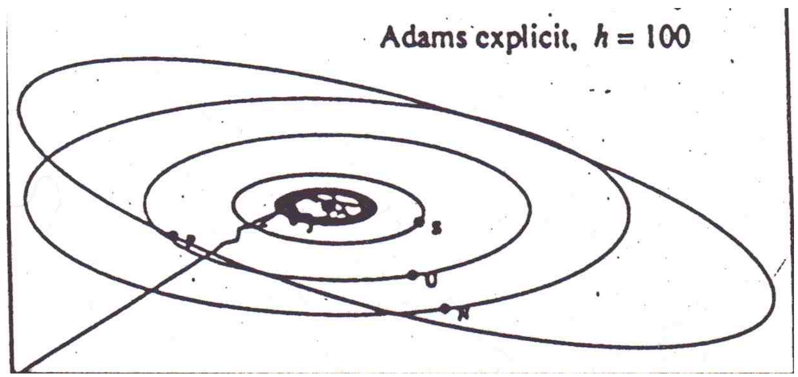
Different arithmetic



Different solver algorithm…

Need to look up what paper these came from…


symmetric multistep, $h = 100$


Adams cxplicit, h = 100
S
U
N

Moral: numerical methods can run amok in "interesting" ways…

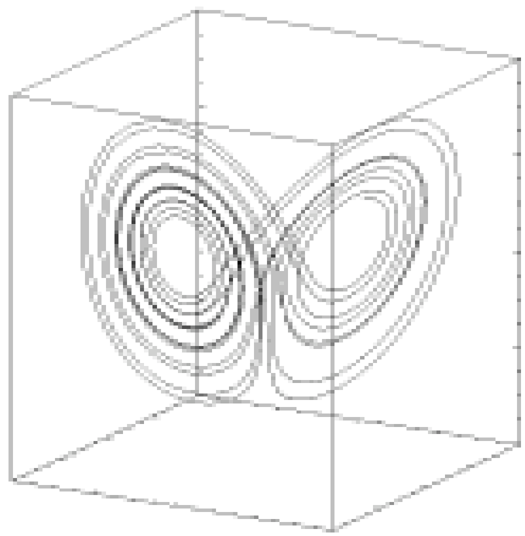
- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?
    - *change the timestep*
    - *change the method*
    - *change the arithmetic*

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….



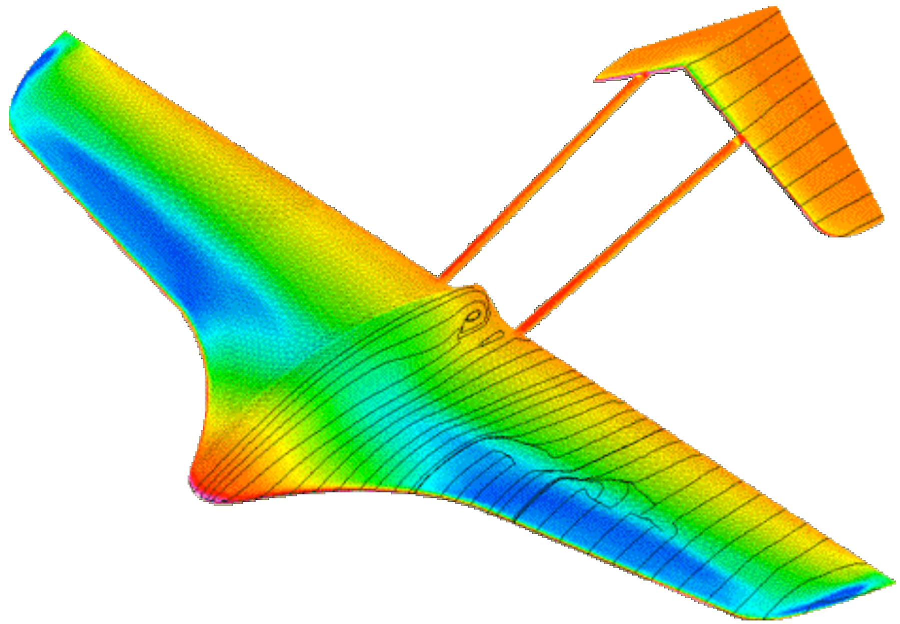
…??!?

Shadowing lemma:

Every noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

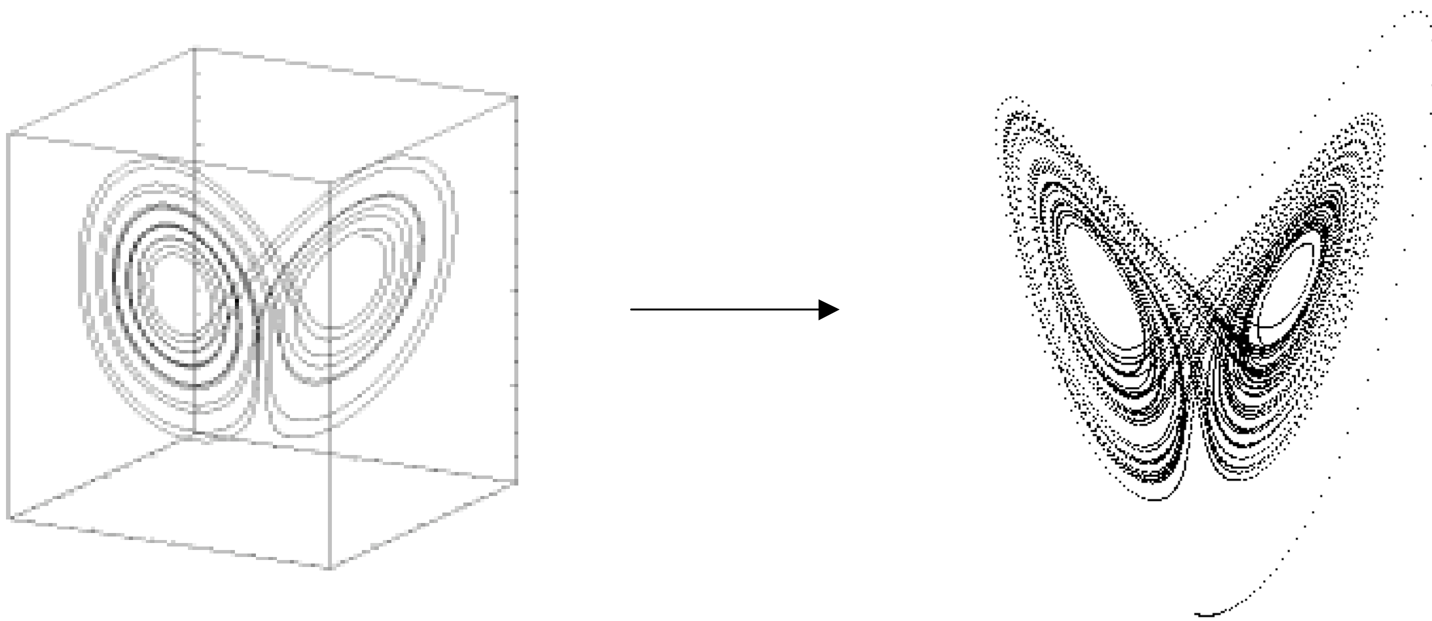
Important: this is for *state* noise, not *parameter* noise.

Solving *PDEs*

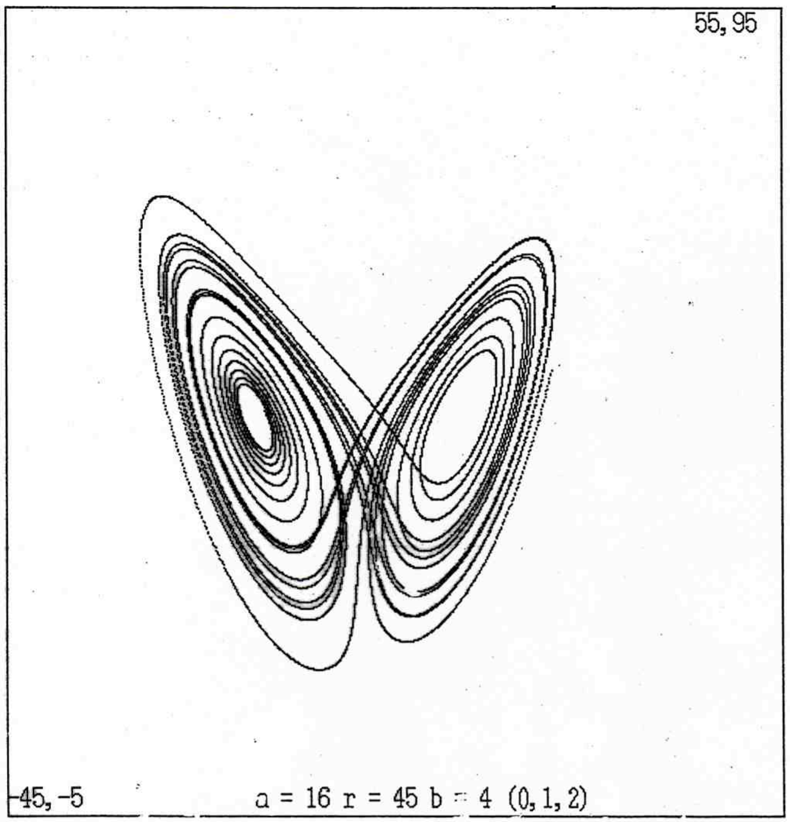


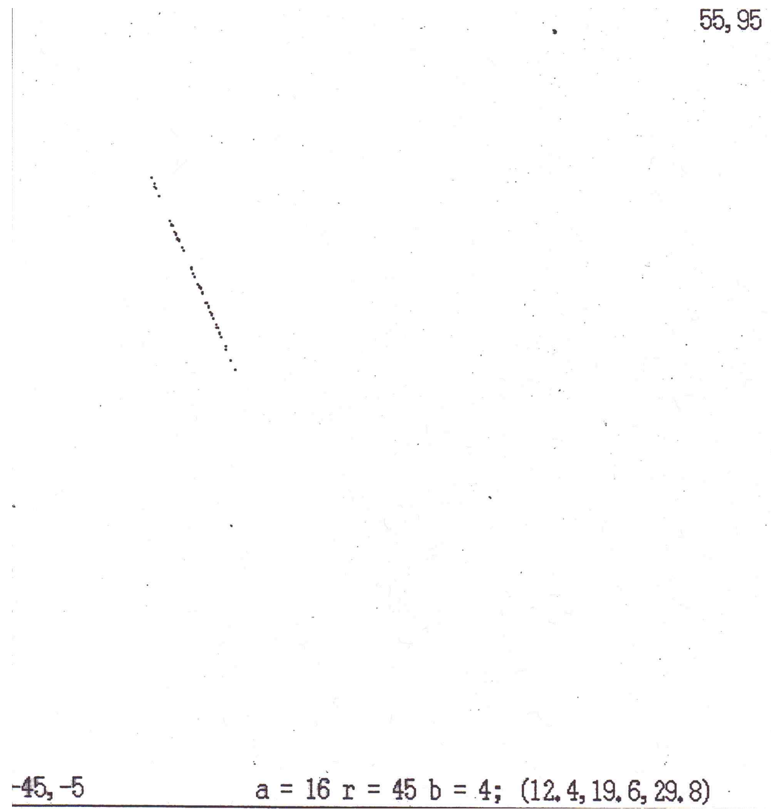
www.tecplot.com

Projection:

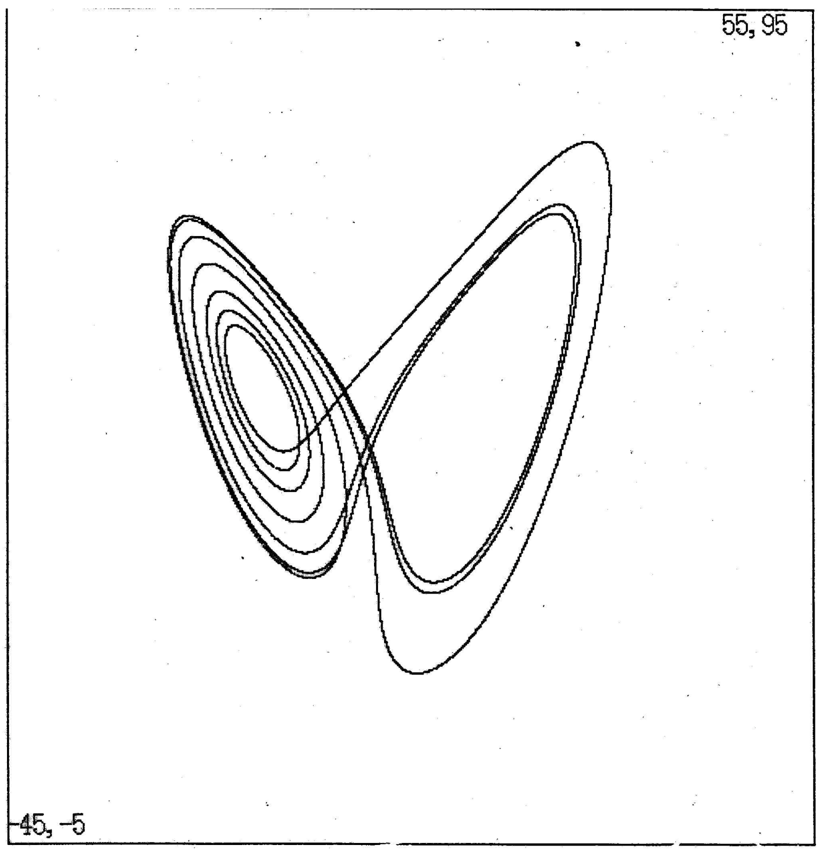


Section:

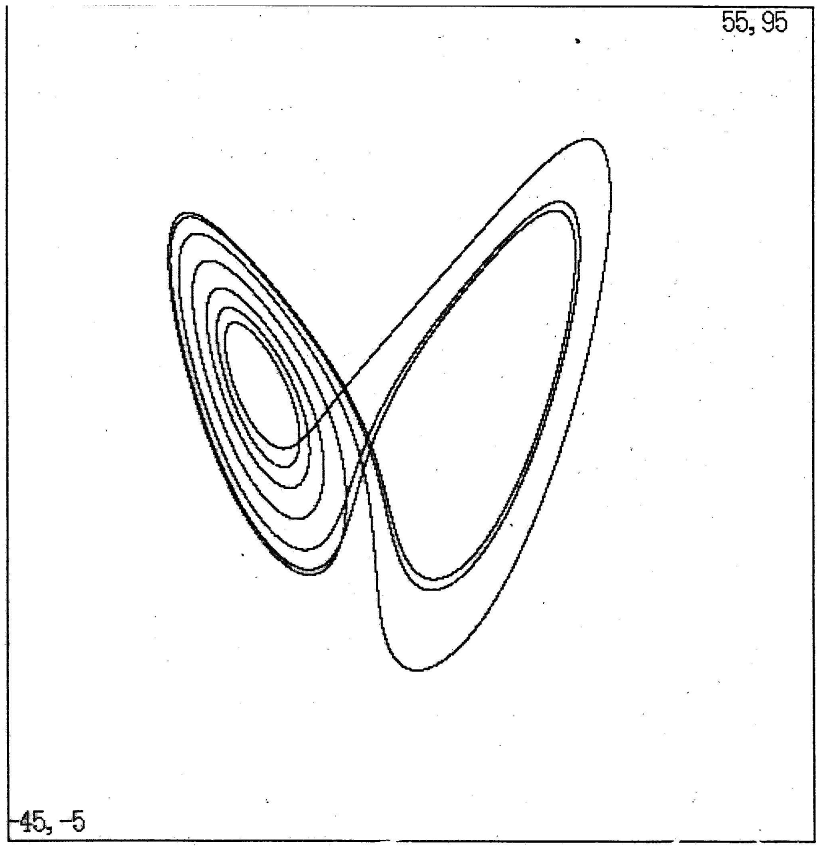


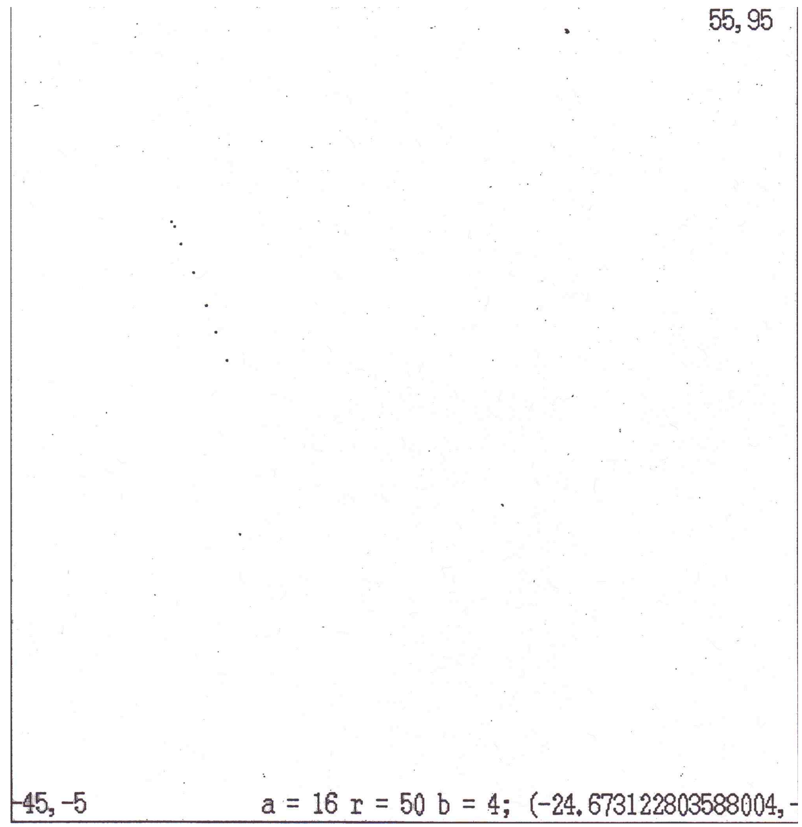


Section of a UPO:

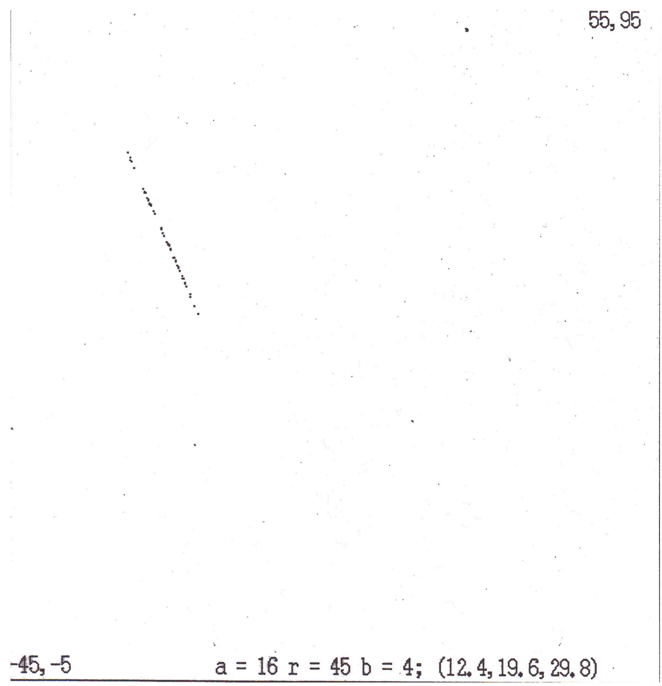


?


55, 95
-45, -5


55, 95
-45, -5
a = 16 r = 50 b = 4; (-24.673122803588004,-

Aside: finding UPOs



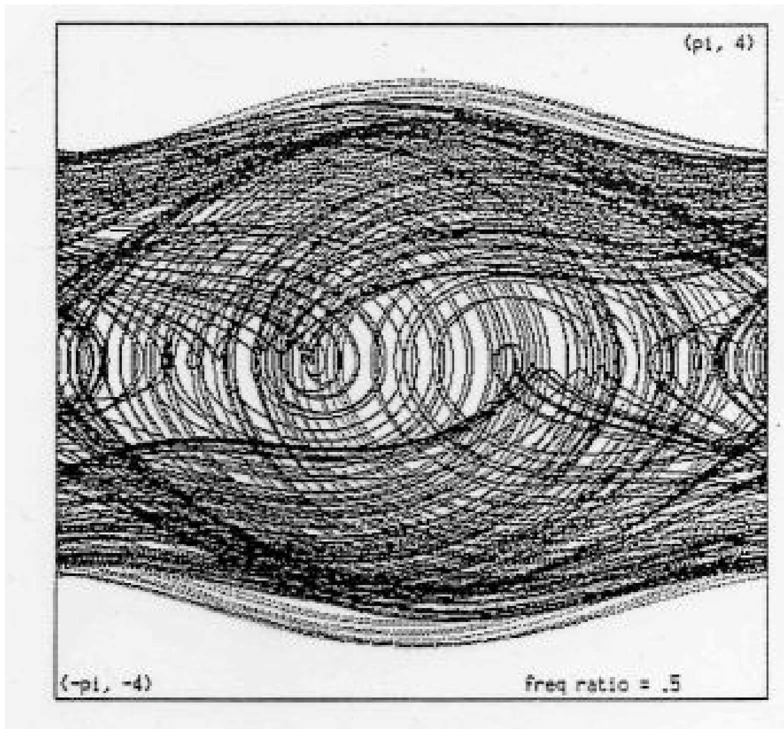
- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers

Back to sections…*time-slice* ones now.

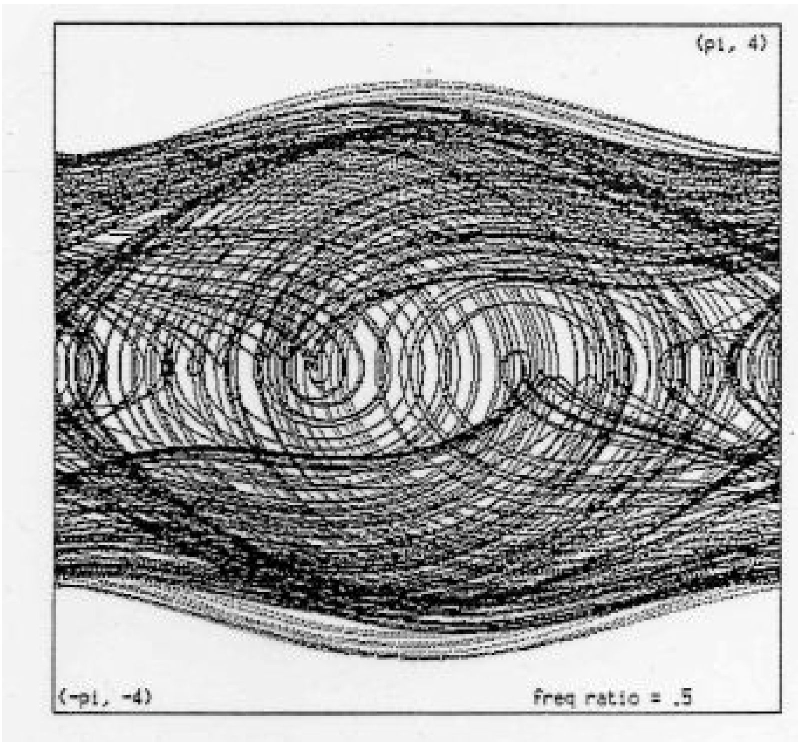
Time-slice sections of periodic orbits: some thought experiments

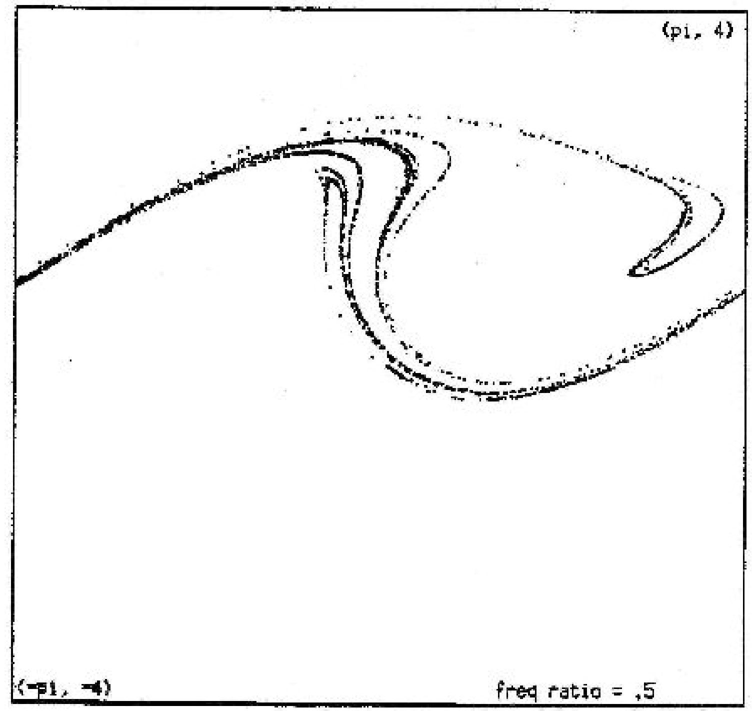
- pendulum rotating @ 1 Hz and strobe @ 1 Hz?
- pendulum rotating @ 1 Hz and strobe @ 2 Hz?
- pendulum rotating @ 1 Hz and strobe @ 3 Hz?
- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
- pendulum rotating @ 1 Hz and strobe @ π Hz? (or some other irrational)

When this becomes really useful:

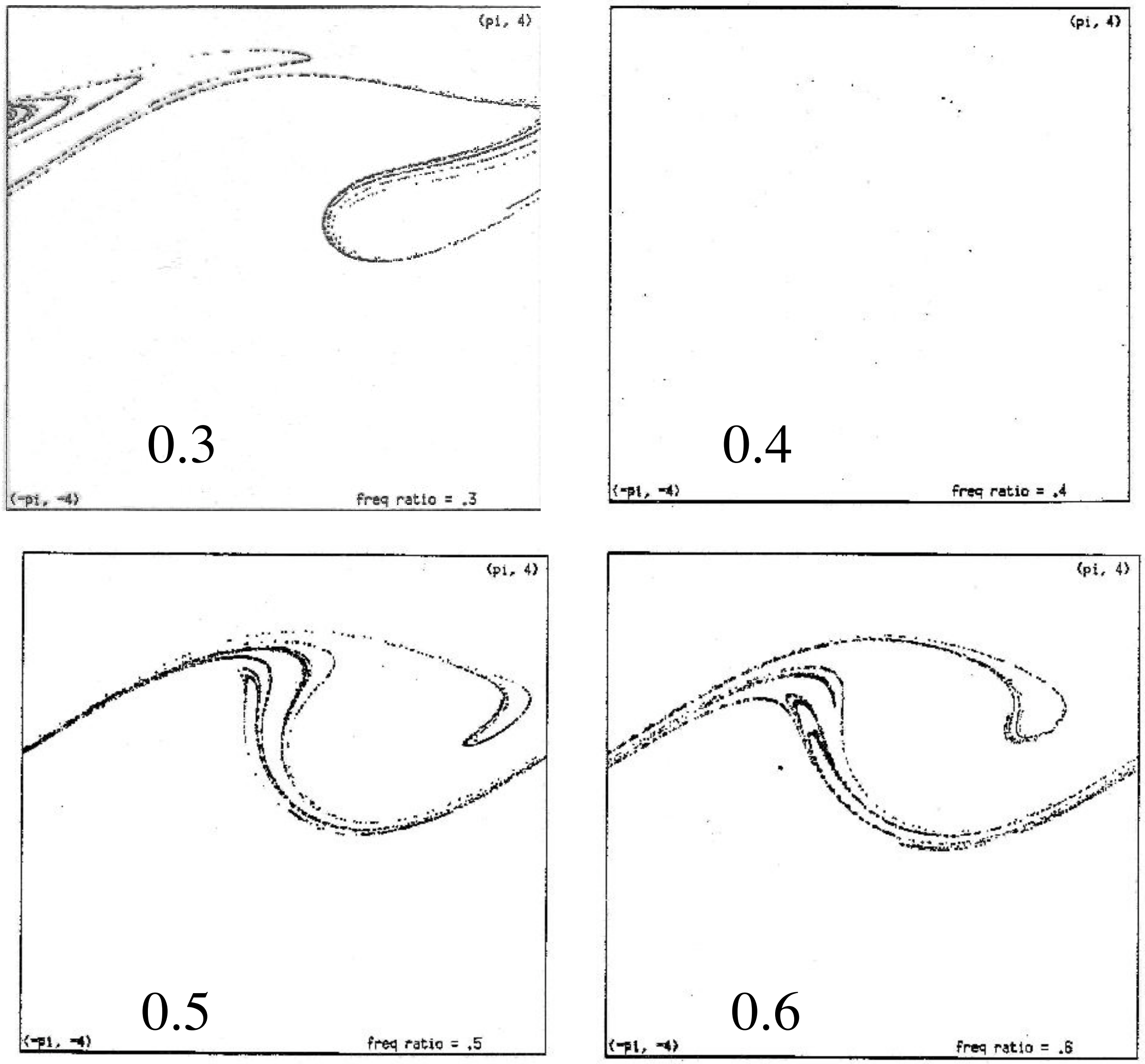


Poincare section:





What bifurcations look like on a Poincare section:

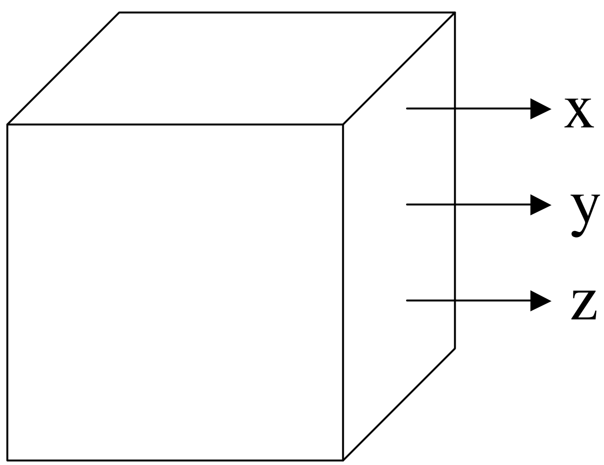


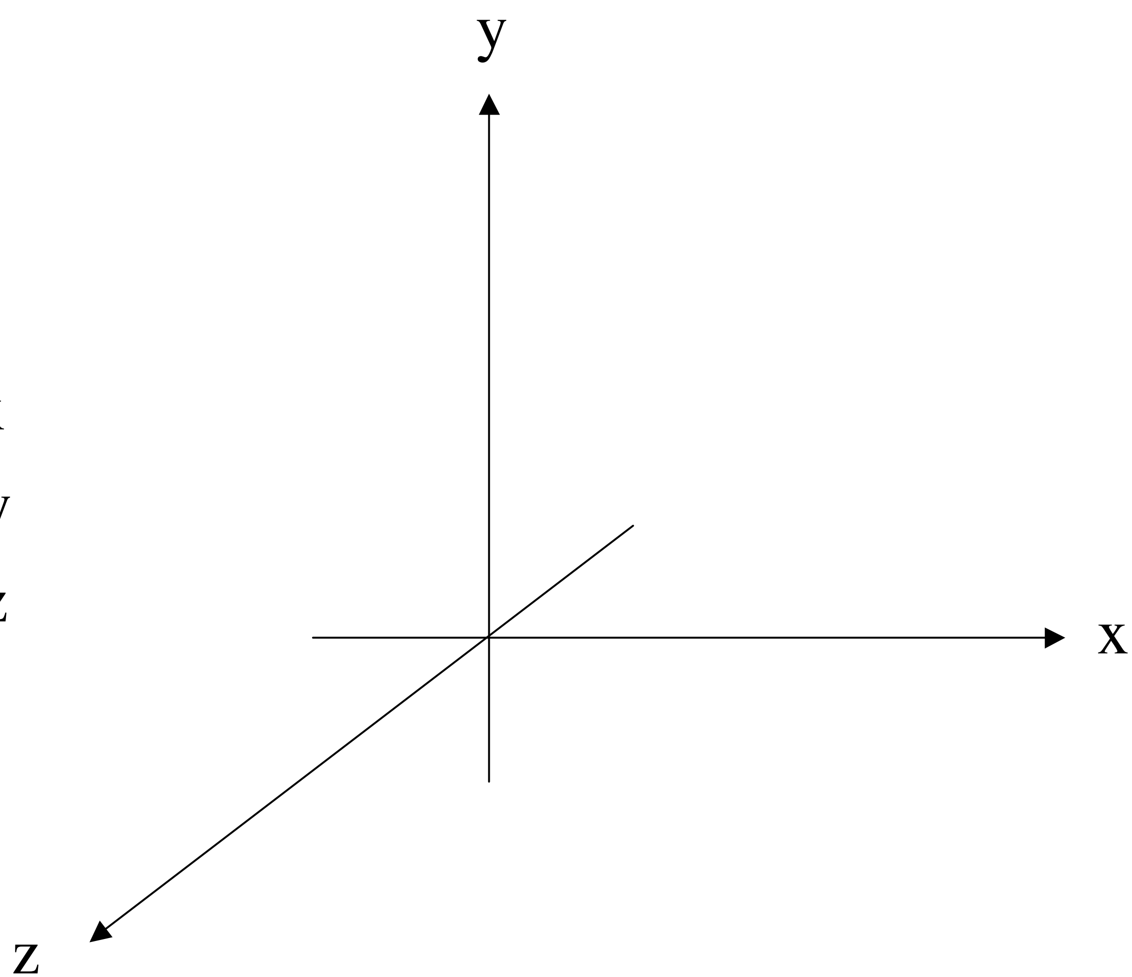
What this looks like in real life…

Computing sections:

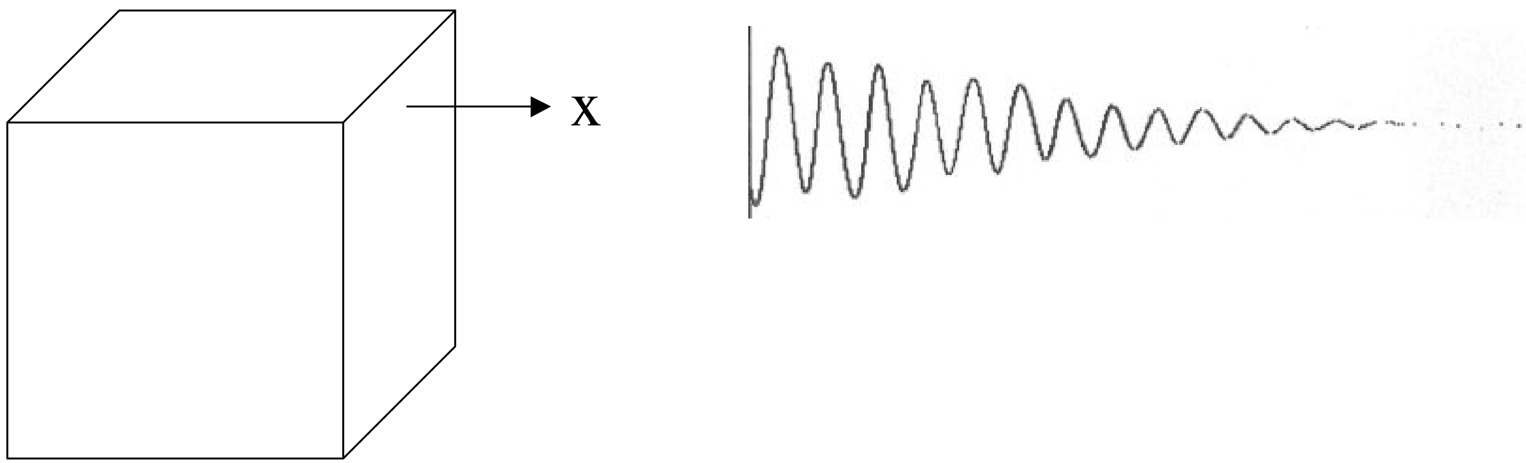
- Space-slice
- Time-slice

We’ve been assuming that we can measure all the state variables:

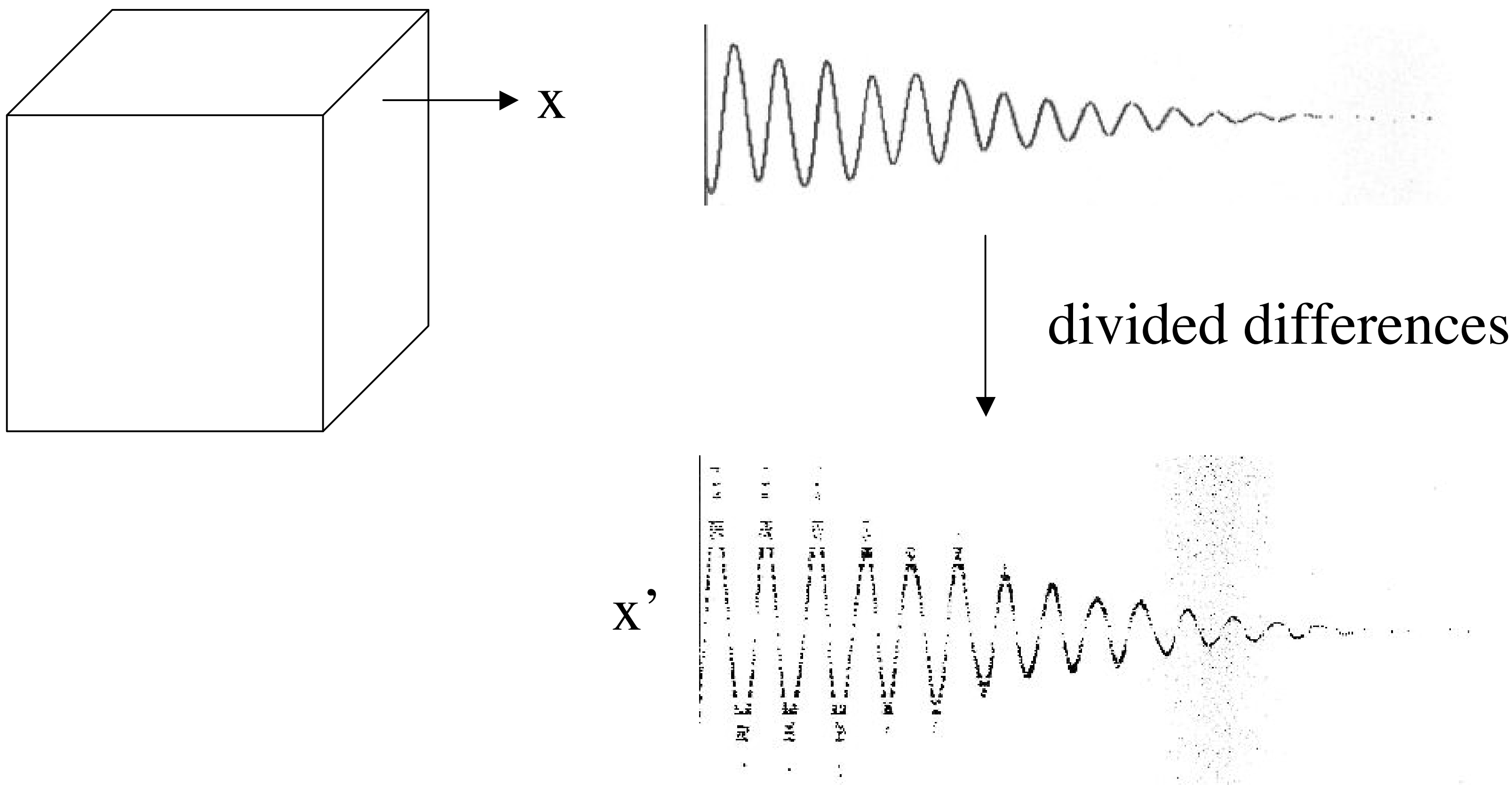




But often you can't:

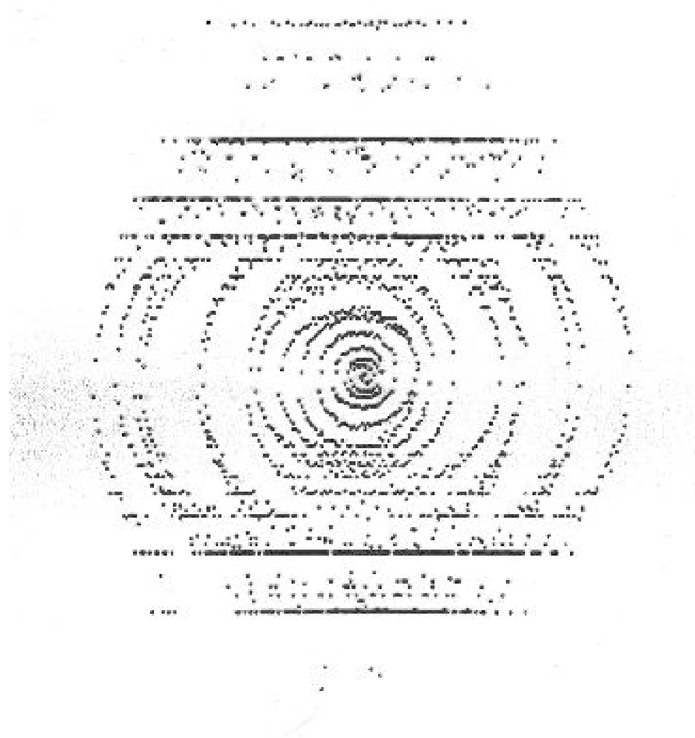


How to reconstruct the other state vars?



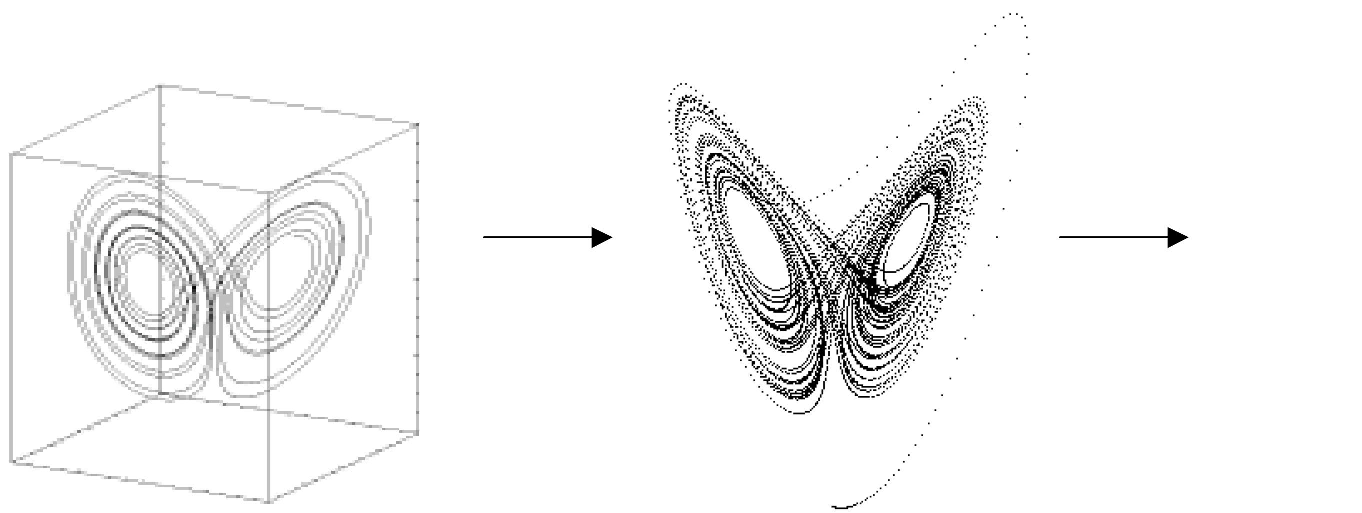
derivatives magnify noise!

What this looks like in the state space:



This is not useful for computation.

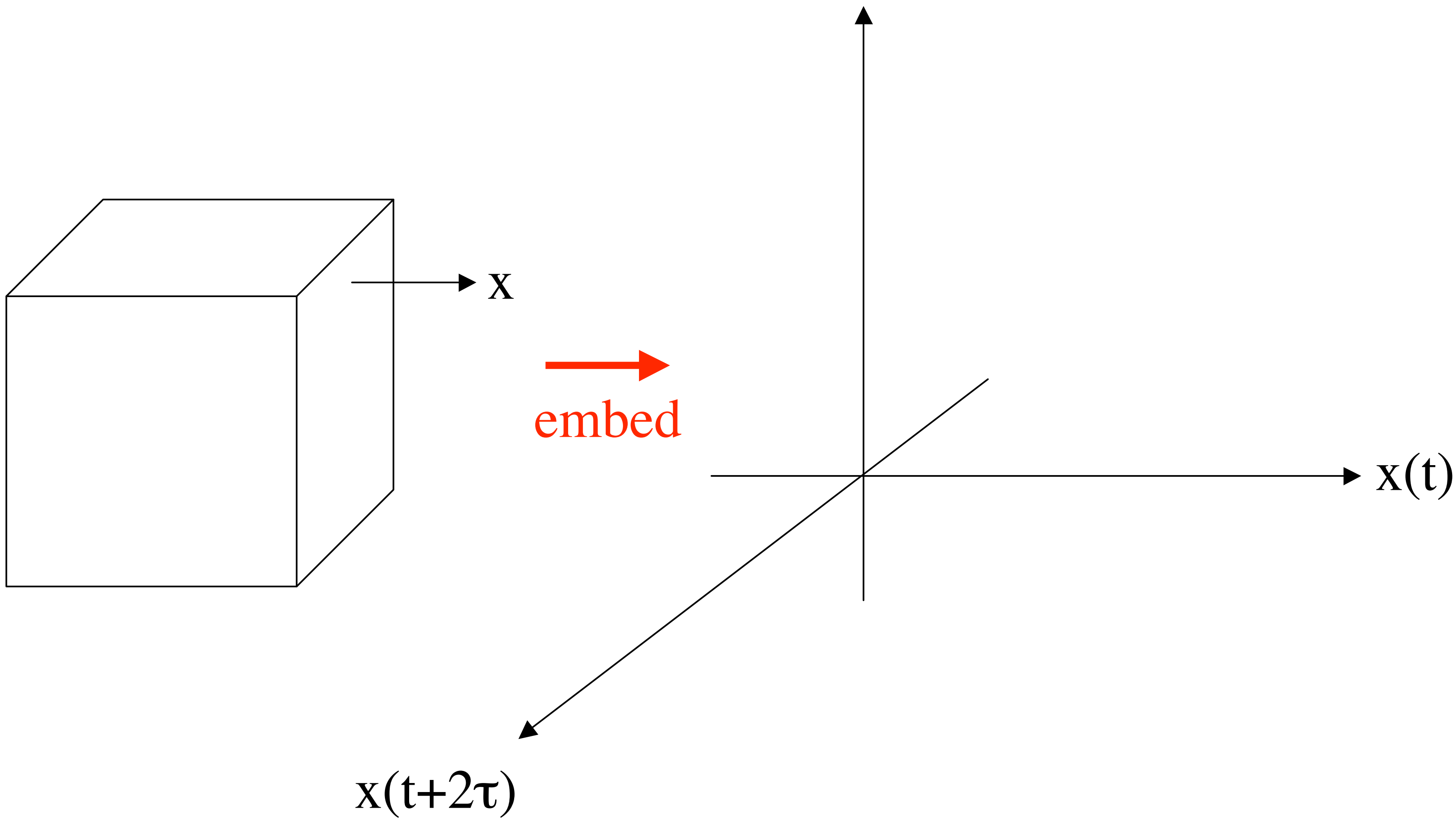
What we want here is to undo a projection:



Delay-coordinate embedding:

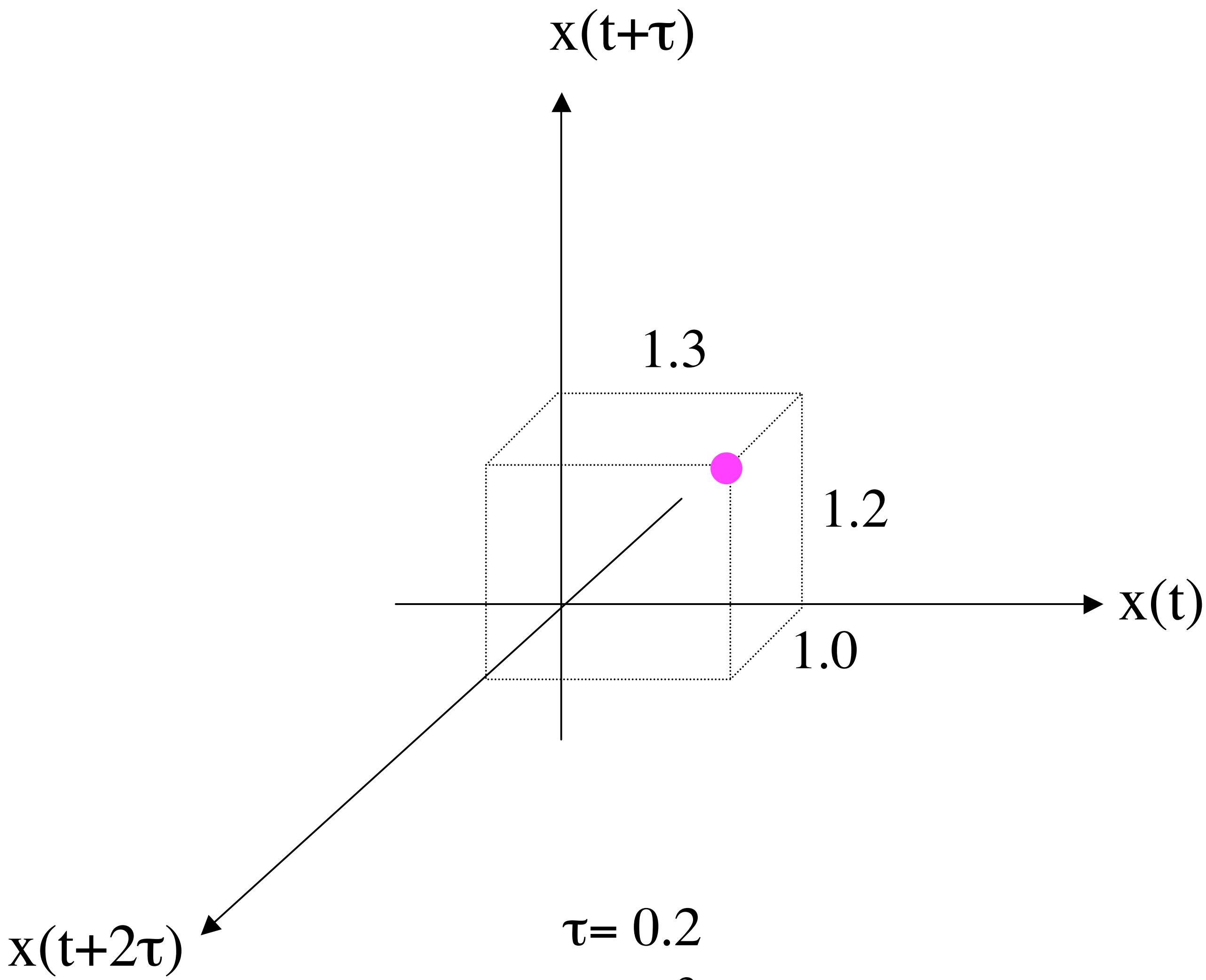
“reinflate” that squashed data to get a *topologically identical* copy of the original thing.

Reconstruction space:



Mechanics:

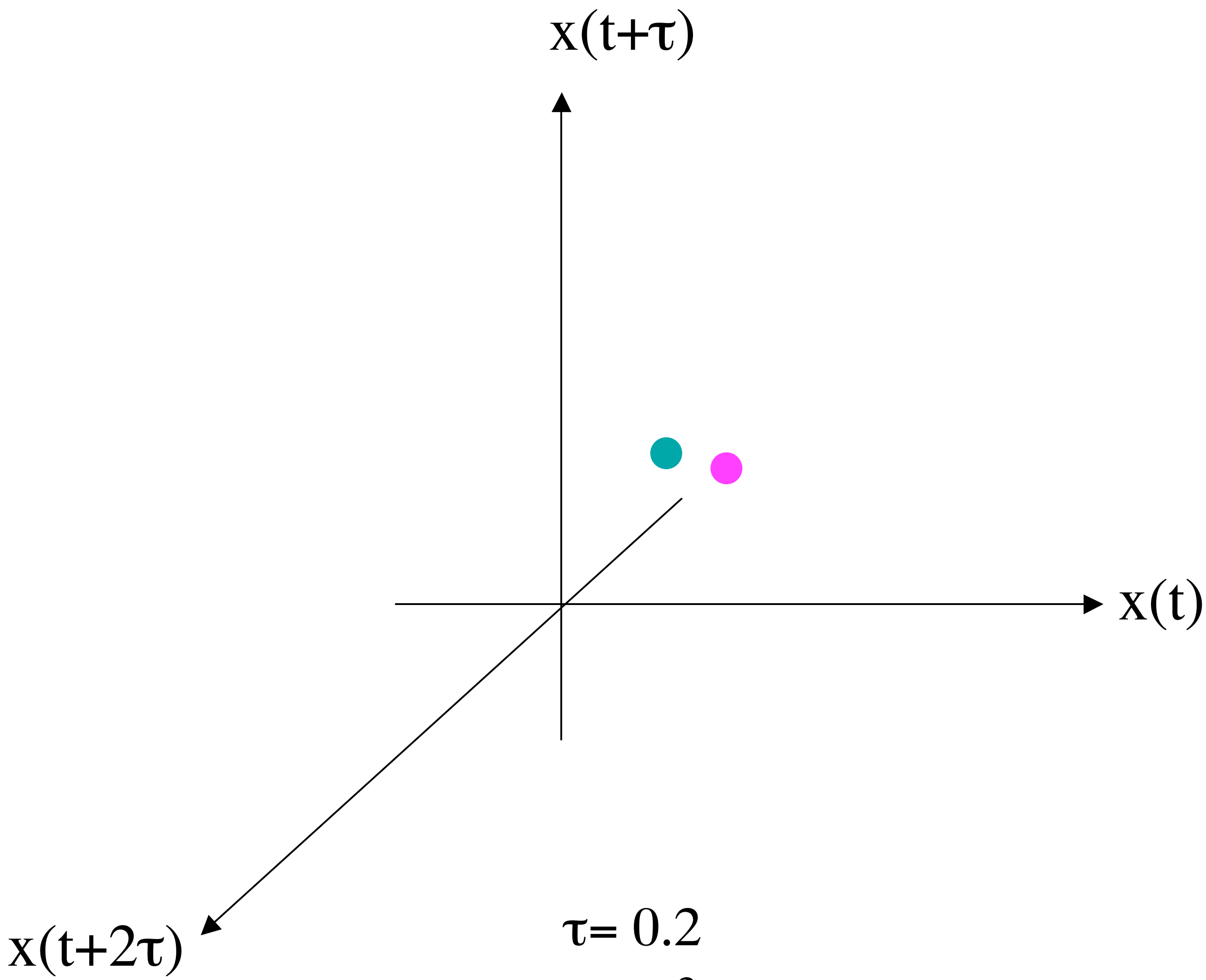
x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



$\tau = 0.2$

$m = 3$

x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7

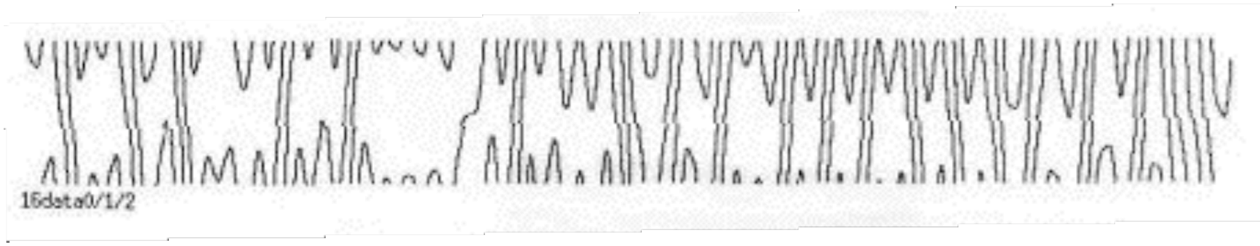


$\tau = 0.2$

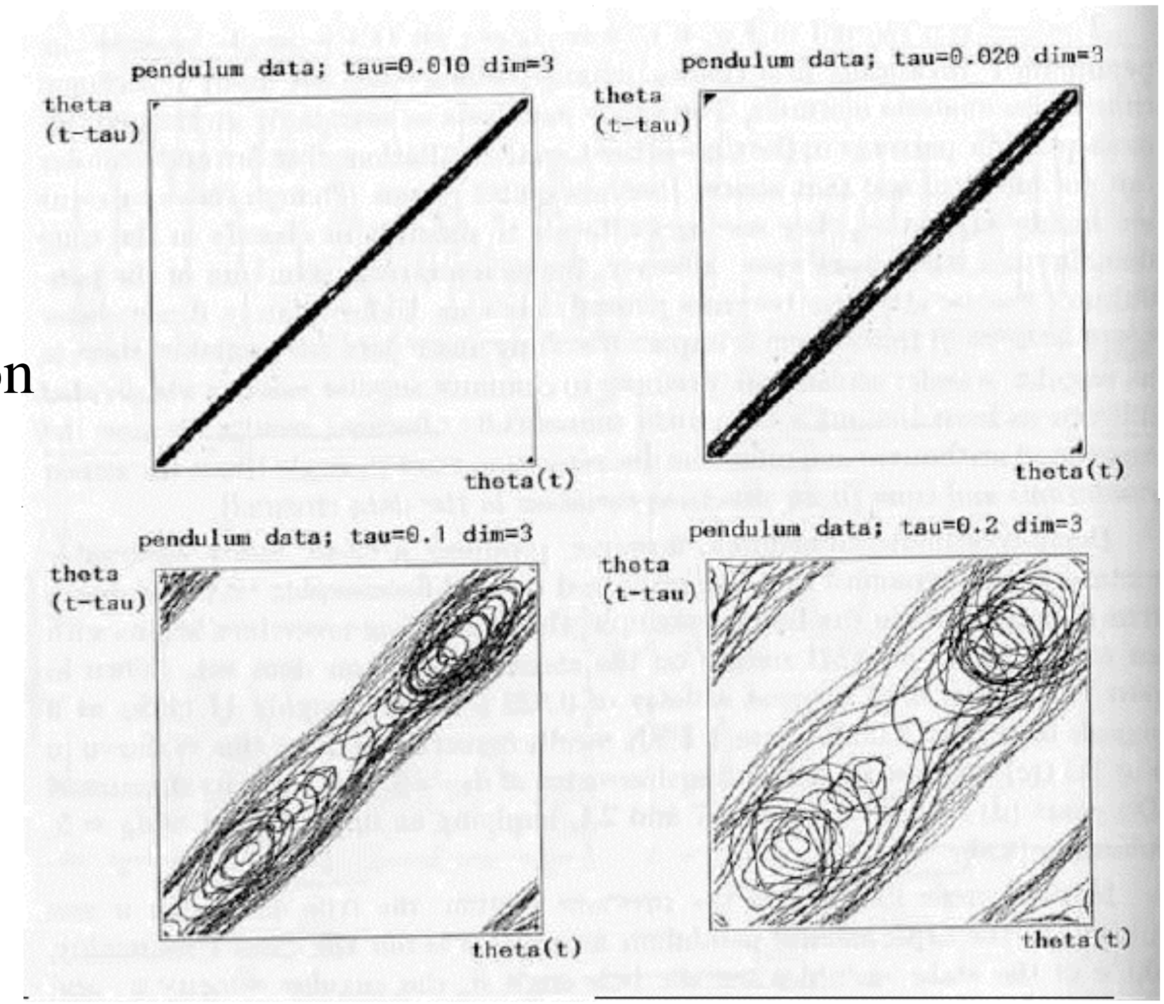
$m = 3$

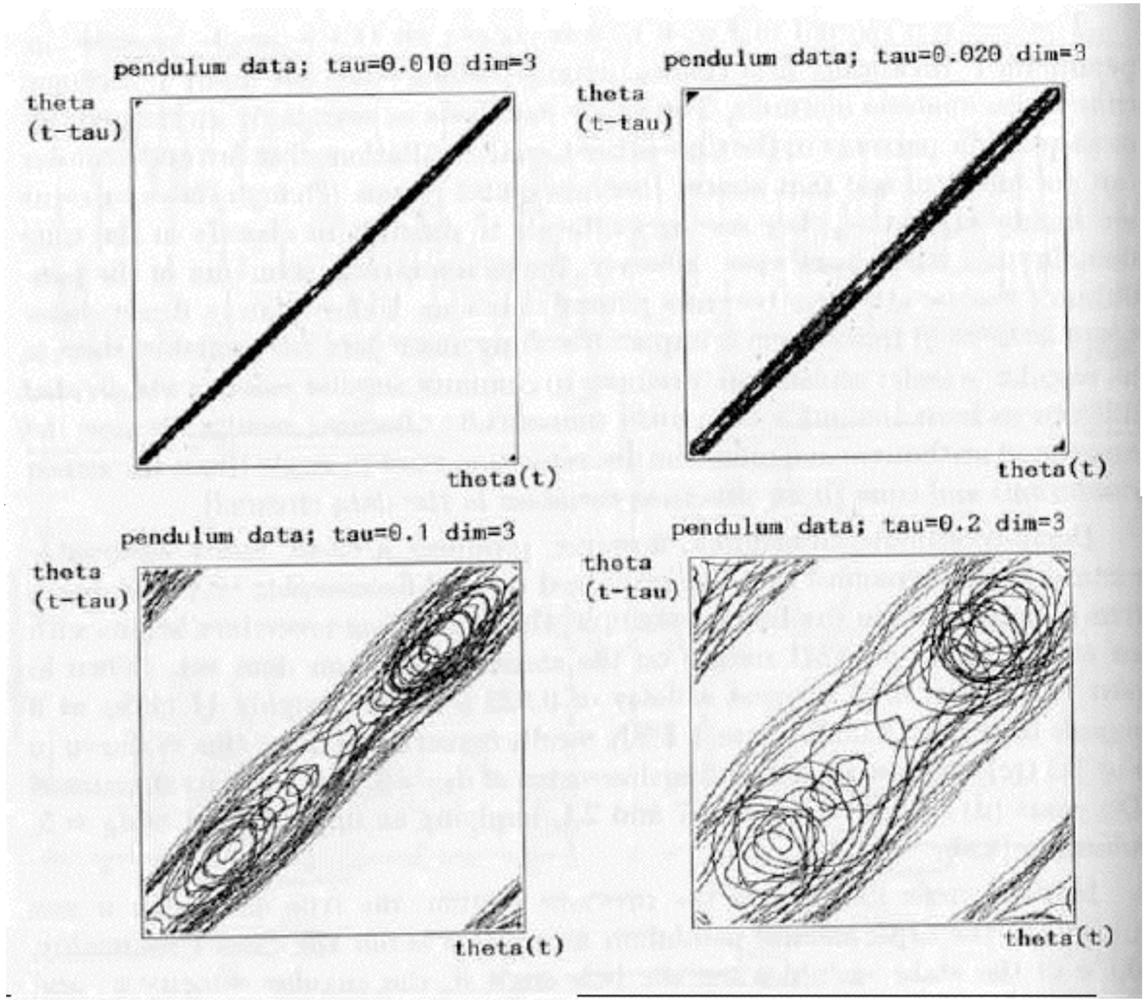
What this looks like:

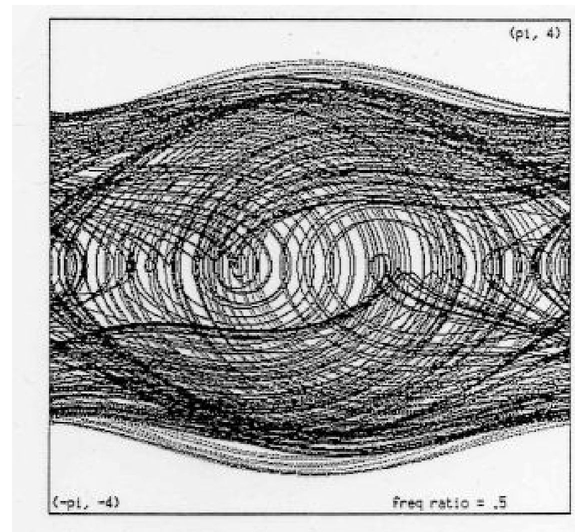
Data:



Reconstruction space:




pendulum data; tau=0.010 dim=3
theta
(t-tau)
theta(t)
pendulum data; tau=0.020 dim=3
theta
(t-tau)
theta(t)
pendulum data; tau=0.1 dim=3
theta
(t-tau)
theta(t)
pendulum data; tau=0.2 dim=3
theta
(t-tau)
theta(t)


(pi, 4)
(-pi, -4)
freq ratio = .5

Takens* theorem:

For the right τ and enough dimensions, the dynamics in this *reconstruction space* are diffeomorphic to the original state-space dynamics.

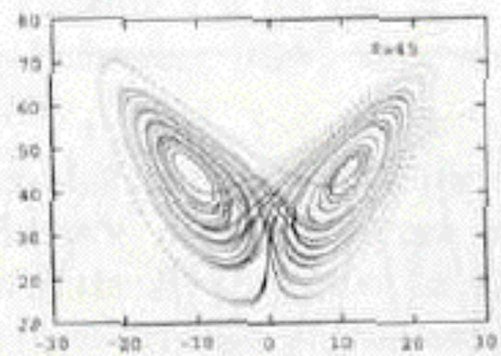
* Whitney, Mane, …

Diffeomorphisms and topology:

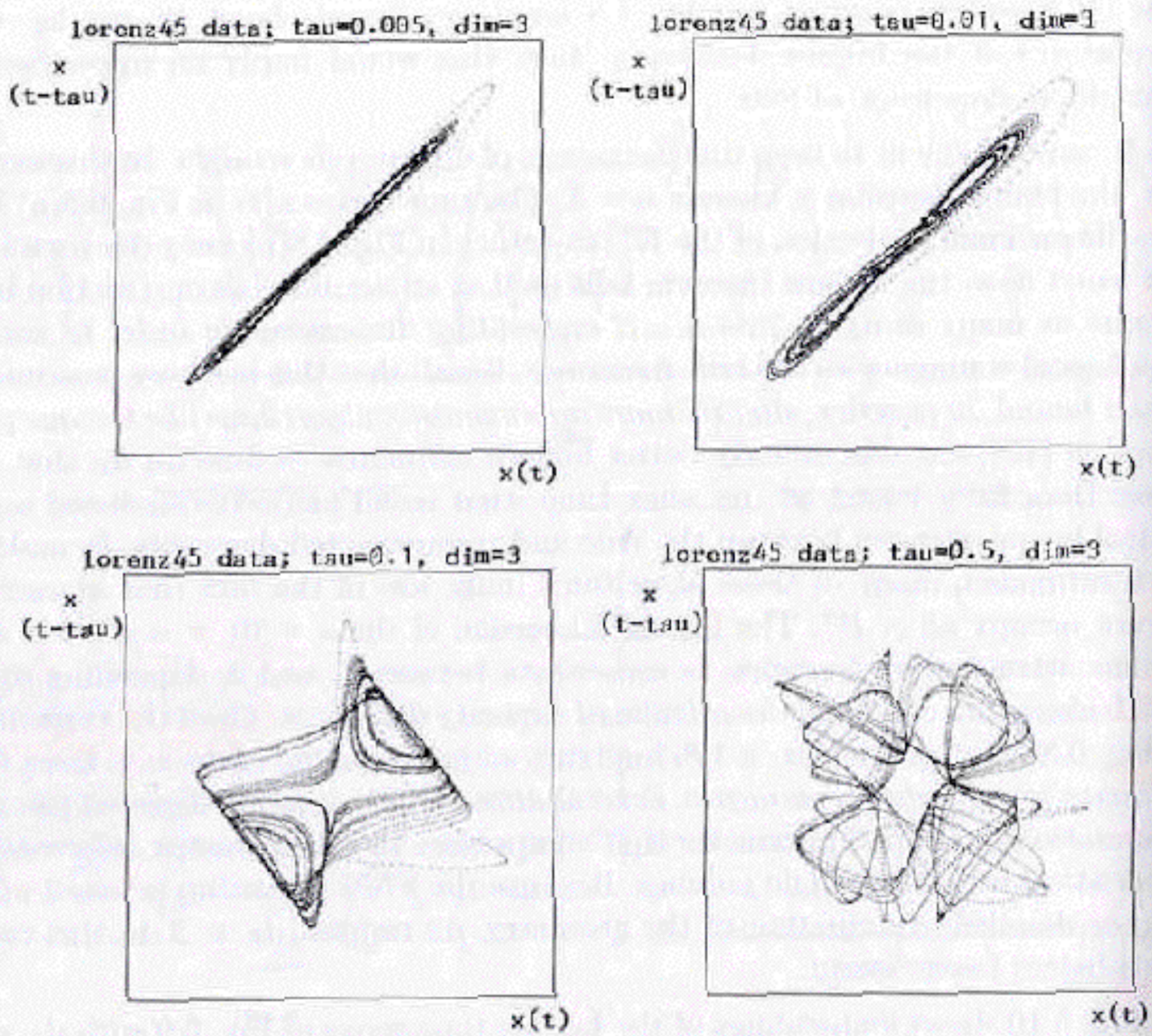
Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape
- have same dynamical invariants (e.g., λ)

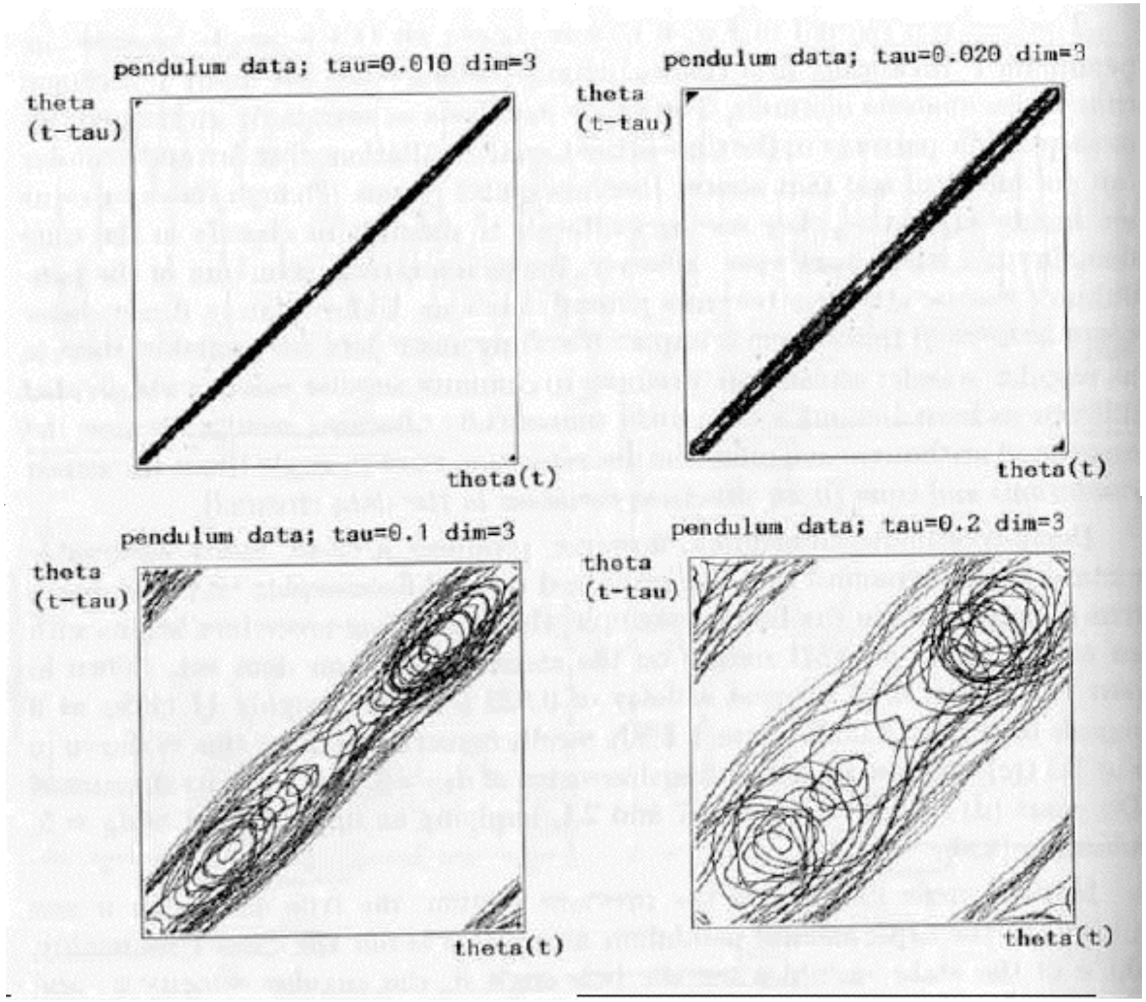


(b)



lorenz45 data; tau=0.005, dim=3
x
(t-tau)
x(t)
lorenz45 data; tau=0.01, dim=3
x
(t-tau)
x(t)
lorenz45 data; tau=0.1, dim=3
x
(t-tau)
x(t)
lorenz45 data; tau=0.5, dim=3
x
(t-tau)
x(t)

Picking τ:


pendulum data; tau=0.010 dim=3
theta (t-tau)
theta(t)
pendulum data; tau=0.020 dim=3
theta (t-tau)
theta(t)
pendulum data; tau=0.1 dim=3
theta (t-tau)
theta(t)
pendulum data; tau=0.2 dim=3
theta (t-tau)
theta(t)

Picking *m*:

m > 2d: sufficient to ensure no crossings in reconstruction space:

…may be overkill.

"Embedology" paper: *m* > 2 dc
(box-counting dimension)

If Δt is not uniform:

Theorem (Takens): for $\tau>0$ and $m > 2d$, reconstructed trajectory is diffeomorphic to the true trajectory

Conditions: evenly sampled in time

Interspike interval embedding:

idea: lots of systems generate spikes — hearts, nerves, etc.

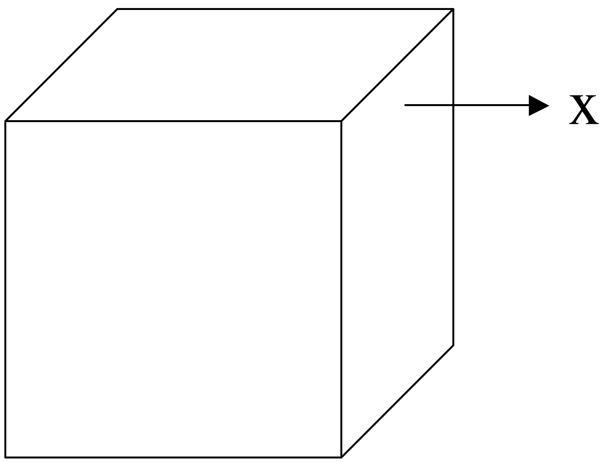
if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

in which case the Takens theorem still holds.

(with the Δts as state variables)

Sauer, *Chaos* **5**:127

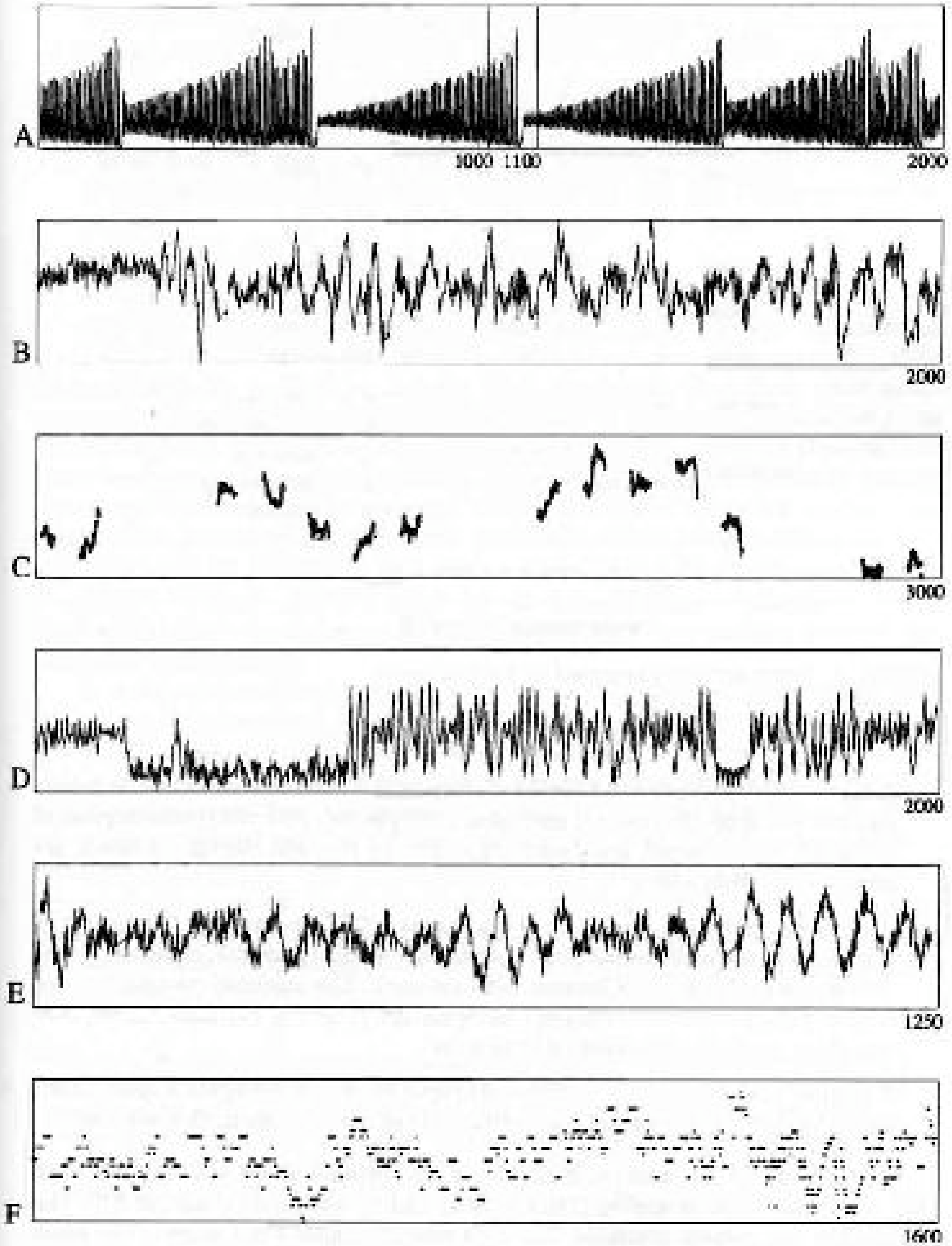
What if that black box were a roulette wheel?



The Eudaemonic Pie
(or The Newtonian Casino)

The Santa Fe competition:

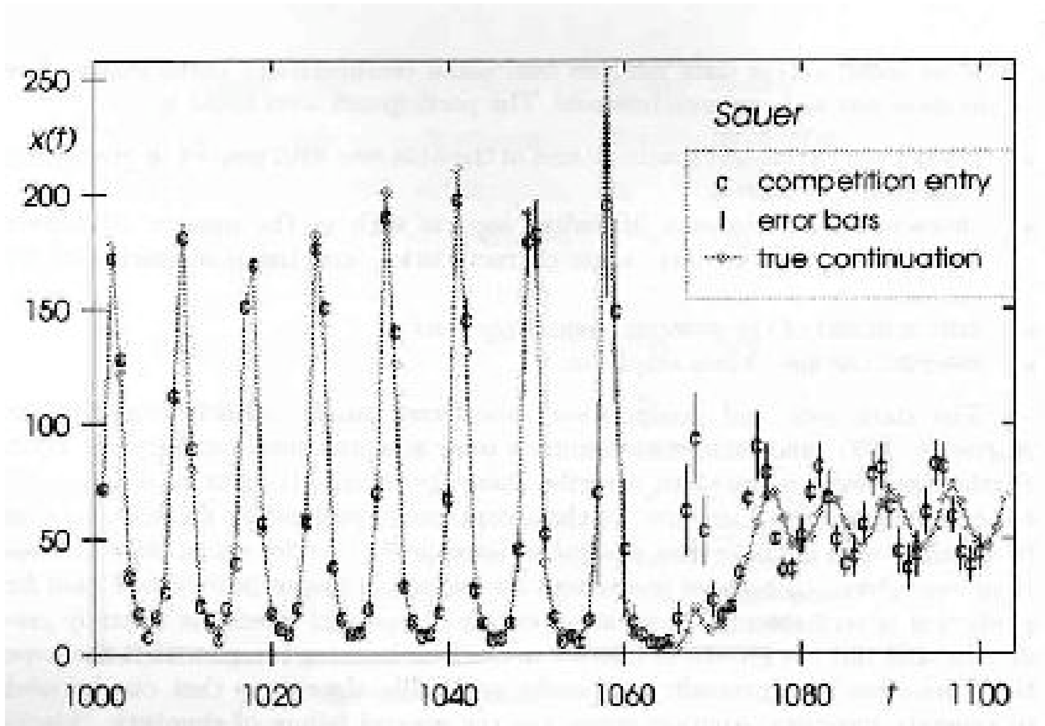
- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past,* Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)


A
1000
1100
2000
B
2000
C
3000
D
2000
E
1250
F
1600

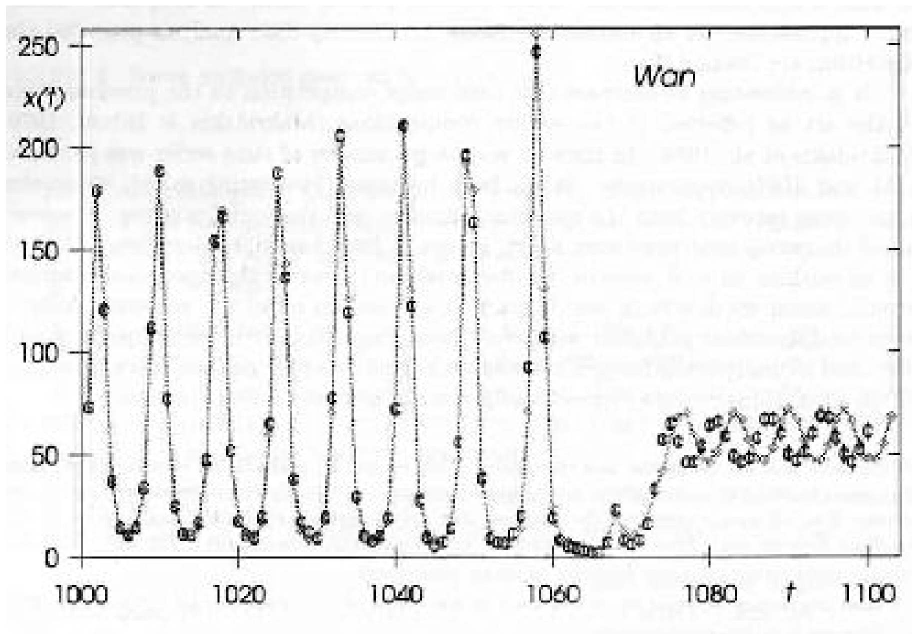
The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

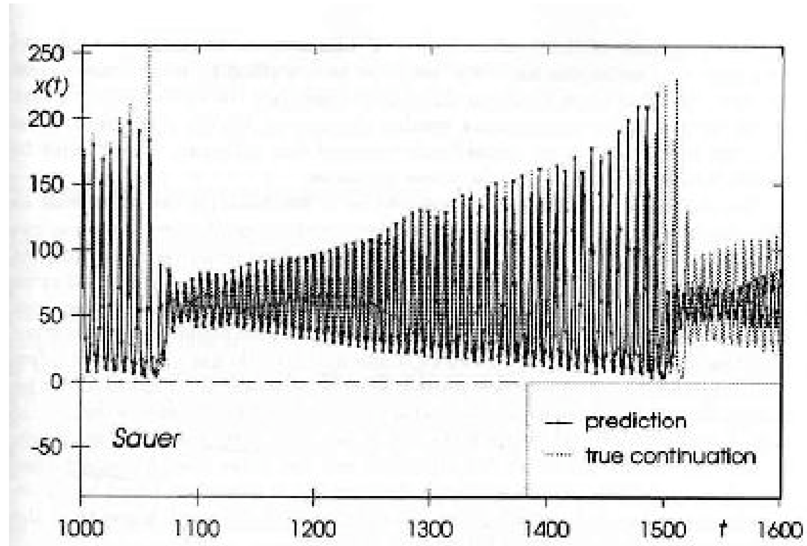
Embedding + patch models: (Sauer)



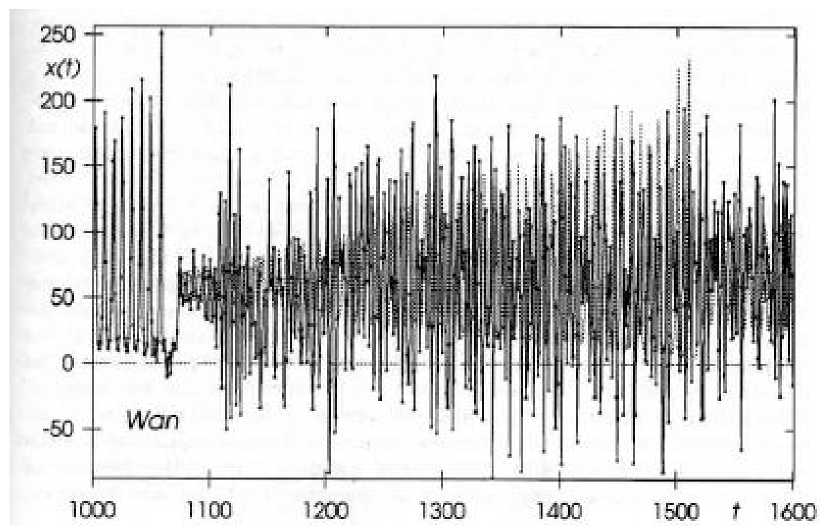
Neural net: (Wan)



Further out:



Sauer



Wan

Sauer's algorithm:

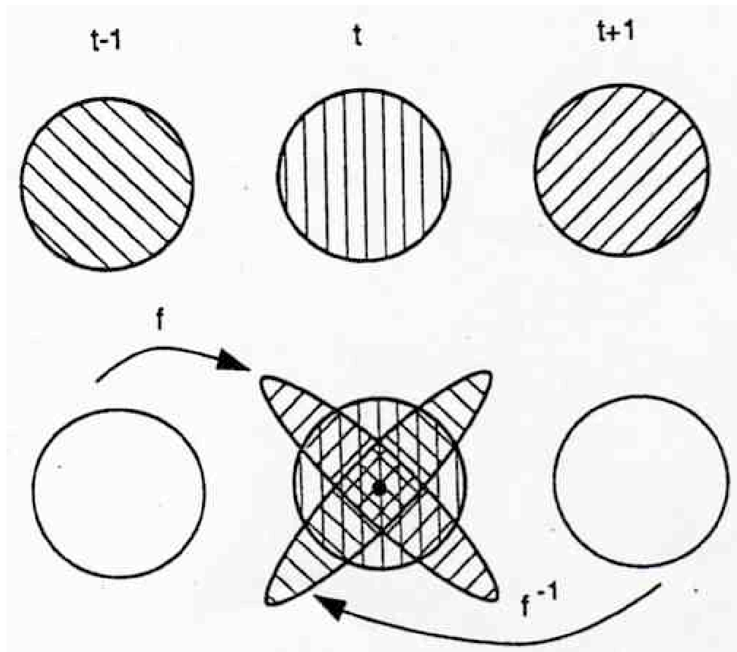
In his competition entry, shown in Figure 3, Sauer used a careful implementation of local-linear fitting that had five steps:

1. Low-pass embed the data to help remove measurement and quantization noise. This low-pass filtering produces a smoothed version of the original series. (We explained such filtered embedding at the end of Section 4.1.)
2. Generate more points in embedding space by (Fourier-) interpolating between the points obtained from Step 1. This is to increase the coverage in embedding space.
3. Find the k nearest neighbors to the point of prediction (the choice of k tries to balance the increasing bias and decreasing variance that come from using a larger neighborhood).
4. Use a local SVD to project (possibly very noisy) points onto the local surface. (Even if a point is very far away from the surface, this step forces the dynamics back on the reconstructed solution manifold.)
5. Regress a linear model for the neighborhood and use it to generate the forecast.

Filtering:

Linear: a bad idea if the system is chaotic

Nonlinear: use the stable and unstable manifold structure on a chaotic attractor…

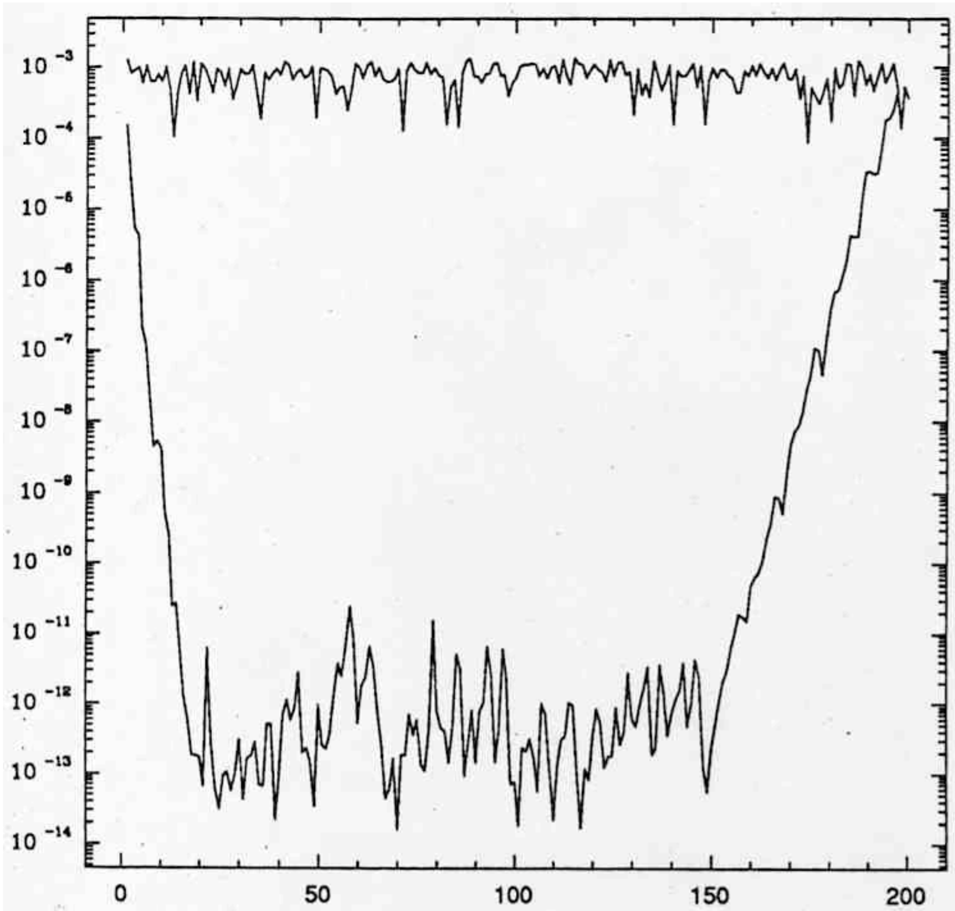


Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
- ⇒ noise reduction!
- Works best if manifolds are perpendicular, but requires only transversality

Results:



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983