Optimal Prediction

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Complex Systems Summer School

Notation etc.

Upper-case letters are random variables, lower-case their realizations

Stochastic process $\ldots, X_{-1}, X_0, X_1, X_2, \ldots$

$X_s^t = (X_s, X_{s+1}, \ldots X_{t-1}, X_t)$

Past up to and including t is $X_{-\infty}^t$, future is X_{t+1}^{∞}

Making a Prediction

Look at $X_{-\infty}^{t}$, make a guess about X_{t+1}^{∞}
Most general guess is a probability distribution
Only ever attend to selected aspects of $X_{-\infty}^{t}$

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Look at $X_{-\infty}^{t}$, make a guess about X_{t+1}^{∞}
Most general guess is a probability distribution
Only ever attend to selected aspects of $X_{-\infty}^{t}$ mean, variance, phase of 1st three Fourier modes
$\therefore$ guess is a *function* or **statistic** of $X_{-\infty}^{t}$
What's a good statistic to use?

Predictive Sufficiency

For any statistic σ,

$$I[X_{t+1}^{\infty}; X_{-\infty}^{t}] \geq I[X_{t+1}^{\infty}; \sigma(X_{-\infty}^{t})]$$

σ is **sufficient** iff

$$I[X_{t+1}^{\infty}; X_{-\infty}^{t}] = I[X_{t+1}^{\infty}; \sigma(X_{-\infty}^{t})]$$

Sufficient statistics retain all predictive information in the data (need information theory to be precise about this)

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Excuse for not worrying about particular loss functions

Causal States

Crutchfield and Young (1989)
Histories a and b are equivalent iff

$$\Pr\left(X_{t+1}^{\infty} | X_{-\infty}^{t} = a\right) = \Pr\left(X_{t+1}^{\infty} | X_{-\infty}^{t} = b\right)$$

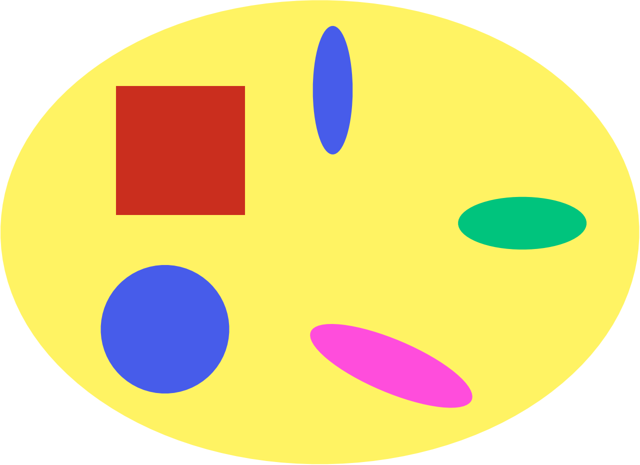
$[a] \equiv$ all histories equivalent to a
The statistic of interest, the **causal state**, is

$$\epsilon(x_{-\infty}^{t}) = [x_{-\infty}^{t}] = s_t$$

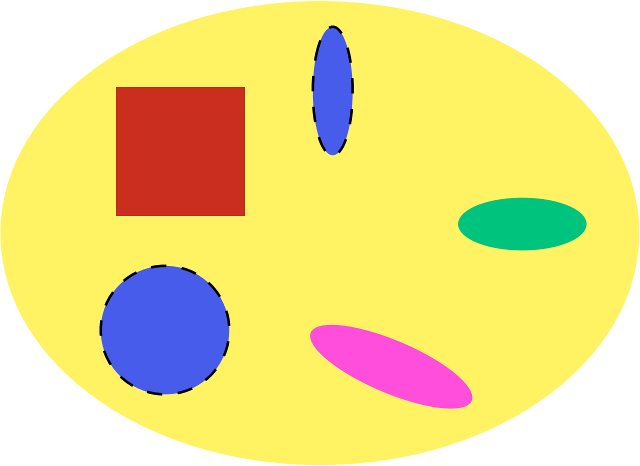
Each state is an equivalence class of histories
Each state is a conditional distribution over future events
IID = 1 state, periodic = p states

About "Causal"

Term introduced by Crutchfield and Young (1989)
For statistics, "causal" $\approx$ conditional independence *under manipulation* (Spirtes *et al.*, 2001; Pearl, 2009)
These states give us conditional independence but no guarantees about counterfactuals; *candidates* for causal models (Shalizi and Moore, 2003)



set of histories, color-coded by conditional distribution of futures



Partitioning histories into causal states

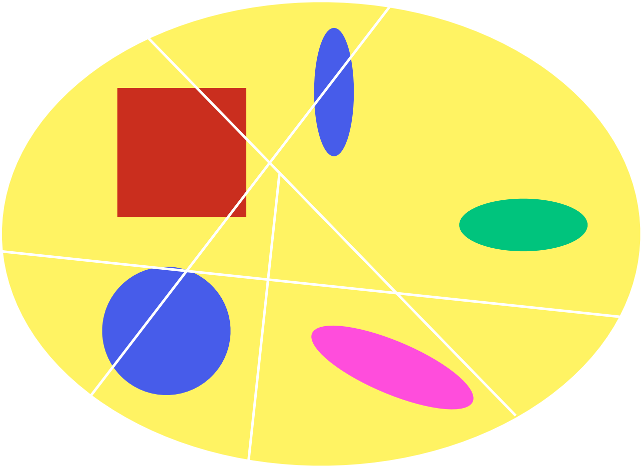
Sufficiency

Shalizi and Crutchfield (2001)

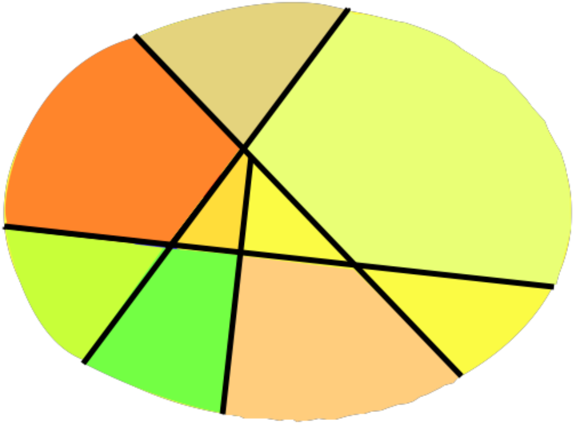
$$I[X_{t+1}^{\infty}; X_{-\infty}^{t}] = I[X_{t+1}^{\infty}; \epsilon(X_{-\infty}^{t})]$$

because

$$\begin{aligned}
\Pr\left(X_{t+1}^{\infty} | S_t = \epsilon(x_{-\infty}^{t})\right) \\
&= \int_{y \in [x_{-\infty}^{t}]} \Pr\left(X_{t+1}^{\infty} | X_{-\infty}^{t} = y\right) \Pr\left(X_{-\infty}^{t} = y | S_t = \epsilon(x_{-\infty}^{t})\right) dy \\
&= \Pr\left(X_{t+1}^{\infty} | X_{-\infty}^{t} = x_{-\infty}^{t}\right)
\end{aligned}$$



A non-sufficient partition of histories



Effect of insufficiency on predictive distributions

Markov Properties

Future observations are independent of the past given the causal state:

$$X_{t+1}^{\infty} \perp\!\!\!\perp X_{-\infty}^{t} | S_t$$

because of sufficiency:

$$\begin{aligned} & \Pr\left(X_{t+1}^{\infty} | X_{-\infty}^{t} = x_{-\infty}^{t}, S_t = \epsilon(x_{-\infty}^{t})\right) \\ = & \Pr\left(X_{t+1}^{\infty} | X_{-\infty}^{t} = x_{-\infty}^{t}\right) \\ = & \Pr\left(X_{t+1}^{\infty} | S_t = \epsilon(x_{-\infty}^{t})\right) \end{aligned}$$

Recursive Updating/Deterministic Transitions

Recursive transitions for states:

$$\epsilon(x_{-\infty}^{t+1}) = T(\epsilon(x_{-\infty}^{t}), x_{t+1})$$

Automata theory: “deterministic transitions” (even though there are probabilities)

If $a \sim b$, any future event F, and single observation f

$$\begin{aligned}
\Pr\left(X_{t+1}^{\infty} \in fF | X_{-\infty}^{t} = a\right) &= \Pr\left(X_{t+1}^{\infty} \in fF | X_{-\infty}^{t} = b\right) \\
\Pr\left(X_{t+1} = f, X_{t+2}^{\infty} \in F | X_{-\infty}^{t} = a\right) &= \Pr\left(X_{t+1} = f, X_{t+2}^{\infty} \in F | X_{-\infty}^{t} = b\right) \\
& \ldots \\
\Pr\left(X_{t+2}^{\infty} \in F | X_{-\infty}^{t} = a, X_{t+1}^{\infty} = f\right) &= \Pr\left(X_{t+2}^{\infty} \in F | X_{-\infty}^{t} = b, X_{t+1}^{\infty} = f\right) \\
\Pr\left(X_{t+2}^{\infty} \in F | X_{-\infty}^{t+1} = af\right) &= \Pr\left(X_{t+2}^{\infty} \in F | X_{-\infty}^{t+1} = bf\right) \\
af &\sim bf
\end{aligned}$$

EXERCISE: Filling in the missing step

Causal States are Markovian

$$S_{t+1}^{\infty} \perp\!\!\!\perp S_{-\infty}^{t-1} | S_t$$

because
S_{t+1}^{∞} is a function of S_t and X_{t+1}^{∞}
and
X_{t+1}^{∞} is independent of *all* of the past given S_t

Markovian Representation

The observed process (X_t) is non-Markovian and ugly
But it is generated from a homogeneous Markov process (S_t)
Not the usual sort of hidden Markov model because of the deterministic transitions
(An advantage, HMMs need complicated calculations to estimate distributions over their states)

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$=$ can be computed from any other sufficient statistic

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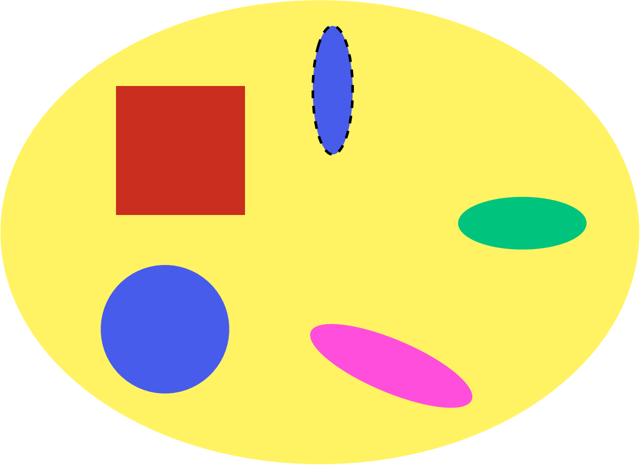
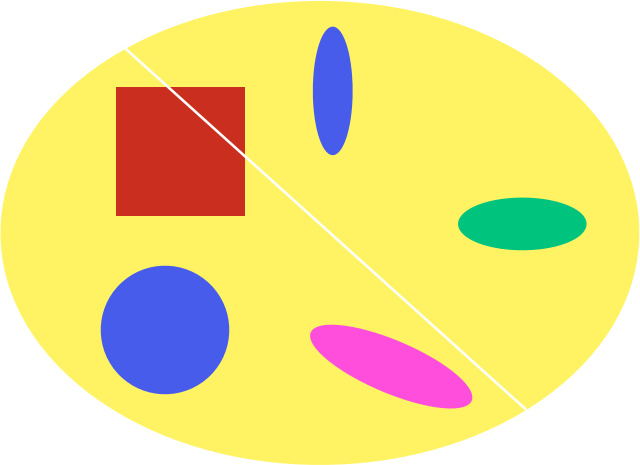
Therefore, if η is sufficient

$$I[\epsilon(X^t_{-\infty}); X^t_{-\infty}] \leq I[\eta(X^t_{-\infty}); X^t_{-\infty}]$$



Sufficient, but not minimal, partition of histories



Coarser than the causal states, but not sufficient

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but $\epsilon = g(\eta)$ so

$$\begin{aligned} g(h(\epsilon)) &= \epsilon \\ h(g(\eta)) &= \eta \end{aligned}$$

$g = h^{-1}$ and ϵ and η partition histories in the same way

Minimal stochasticity

If $R_t = \eta(X^t_{-\infty})$ is also sufficient, then

$$H[R_{t+1}|R_t] \geq H[S_{t+1}|S_t]$$

$\therefore$ the causal states are the closest we get to a deterministic model, without losing predictive ability

Entropy Rate

Recall $h_1 = \lim_{n\to\infty} H[X_n|X_1^{n-1}]$

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so knowing the causal states lets us calculate the entropy rate

History and Aliases

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- Sufficient posterior representation (Langford *et al.*, 2009)

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$=$ log(geometric mean(recurrence time)) for stationary processes
$=$ information about microstate in macroscopic observations (sometimes)

Can We Find Causal State Models?

Depends on the meaning of “find”

- Parameter estimation with known structure (“learning”)
  - curved exponential families
  - maximum likelihood estimation is simple, consistent and efficient
- Reconstruct the structure from observed behavior (“discovery”)

CSSR: Causal State Splitting Reconstruction

Key observation: Recursion + one-step-ahead predictive sufficiency $\Rightarrow$ general predictive sufficiency

- Get next-step distribution right
- Then make states recursive

Assumes discrete observations, discrete time, finite causal states

Paper: Shalizi and Klinkner (2004); C++ code, `http://bactra.org/CSSR/`

One-Step Ahead Prediction

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Given current partition of histories into states, test whether going one step further back into the past changes the next-step conditional distribution

Use a real hypothesis test to control false positive rate

If yes, split that cell of the partition, but see if it matches an existing distribution

Must allow this merging or else lose minimality

If no match, add new cell to the partition

Stop when no more divisions can be made or a maximum history length Λ is reached

For consistency, $\Lambda < \frac{\log n}{h_1 + \iota}$ for some ι (from AEP)

Ensuring Recursive Transitions

Need to determinize a probabilistic automaton
Several ways of doing this; technical and not worth going into here
Trickiest part of the algorithm and can influence the finite-sample behavior

Convergence

$\mathcal{S}$ = true causal state structure
$\widehat{\mathcal{S}}_n$ = structure reconstructed from n data points
Assume: finite # of states, every state has a finite history, using long enough histories, technicalities:

$$\Pr\left(\widehat{\mathcal{S}}_n \neq \mathcal{S}\right) \rightarrow 0$$

$\mathcal{D}$ = true distribution, $\widehat{\mathcal{D}}_n$ = inferred
Error (in L_1/total variation) scales like independent samples

$$\mathbf{E}\left[|\widehat{\mathcal{D}}_n - \mathcal{D}|\right] = O(n^{-1/2})$$

Handwaving

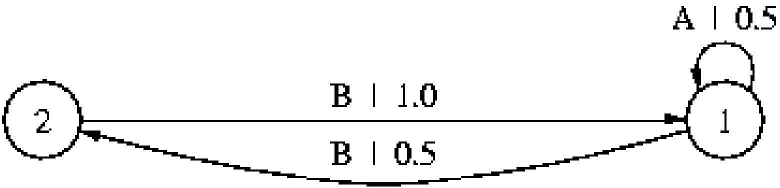
Empirical conditional distributions for histories converge (large deviations principle for Markov chains)

Histories in the same state become harder to accidentally separate

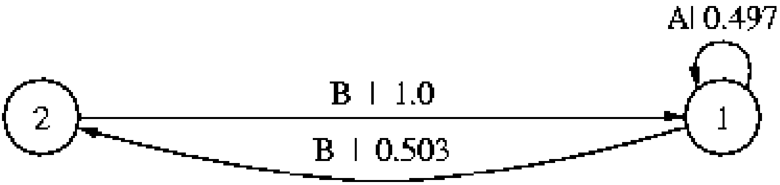
Histories in different states become harder to confuse

Each state's predictive distribution converges $O(n^{-1/2})$, from LDP again, take mixture

Example: The Even Process



Blocks of As of any length, separated by even-length blocks of Bs
Infinite-range correlation (not Markov at any order)



reconstruction with $\Lambda = 3$, $n = 1000$

Some Uses

Geomagnetic fluctuations (Clarke *et al.*, 2003)
Natural language processing (Padró and Padró, 2005a,c,b, 2007a,b)
Anomaly detection (Friedlander *et al.*, 2003a,b; Ray, 2004)
Information sharing in networks (Klinkner *et al.*, 2006; Shalizi *et al.*, 2007)
Social media propagation (Cointet *et al.*, 2007)
Neural spike train analysis (Haslinger *et al.*, 2010)
Spatio-temporal applications: next lecture!

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