Applied Symbolic Dynamics

From 1D to 2D to ODE

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The Unimodal or Logistic Map

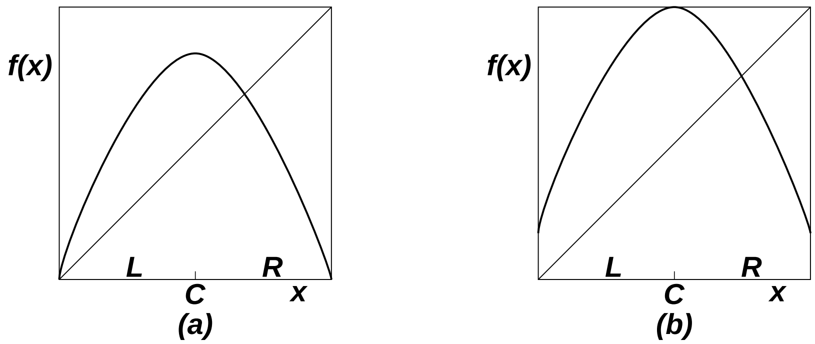


Figure 1: Examples of unimodal map. The "normalization" at the two end points of the interval I is not essential.

Natural order:

$$L \leq C \leq R$$

Doing Iteration on the Graph

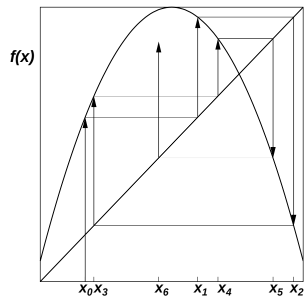


Figure 2: Graphic representation of iterations of a map.

$$\begin{aligned} x_1 &= f(x_0), \\ x_2 &= f(x_1), \\ x_3 &= f(x_2), \\ \cdots &\ \cdots \\ x_n &= f(x_{n-1}), \\ \cdots &\ \cdots \end{aligned}$$

Numerical trajectory or orbit:

$$x_0 x_1 x_2 \cdots x_n \cdots$$

$\Rightarrow$ Symbolic orbit:

$$\alpha_0 \alpha_1 \alpha_2 \cdots \alpha_n \cdots,$$

where $\alpha_i \in \{L, C, R\}$.

This may be a many-to-one correspondence which opens the possibility of classification.

Admissibility Condition

A numerical orbit can always be transformed into a symbolic orbit, but not the other way around. An arbitrarily given symbolic sequence may not be generated by the dynamics.

One needs Admissibility Conditions which are based on ordering of real numbers in an interval:

$$L \leq C \leq R$$

Unimodal Map

An unimodal function $f(x)$ has two monotonic branches:

$$y = f_L(x) \text{ and } y = f_R(x).$$

The inverse of these monotonic branches are also monotonic:

$$x = f_L^{-1}(y) \text{ and } x = f_R^{-1}(y).$$

In order to get concrete results in Applied Symbolic Dynamics we may take the nonlinear function

$$x_{n+1} = f(\mu, x_n), \quad n = 0, 1, 2, \cdots$$

To be:

$$y = f(x) = 1 - \mu x^2, \quad x \in (-1, +1), \quad \mu \in (0, 2)$$

Or:

$$y = f(x) = \mu - x^2, \quad x \in (-\sqrt{\mu}, +\sqrt{\mu}), \quad \mu \in (0, 2)$$

In the last case we have

$$L(y) \equiv f_L^{-1}(y) = -\sqrt{\mu - y},$$

and

$$R(y) \equiv f_R^{-1}(y) = \sqrt{\mu - y}$$

Parity of Monotonic Functions

A monotonically increasing function $f(x)$ preserves order:

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

A monotonically decreasing function $f(x)$ inverses order:

$$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$

1. **We assign a Positive Parity "+" to monotonically increasing f_L, $f_L^{-1} \equiv L(y)$, and the letter L.**

2. **We assign a Negative Parity "-" to monotonically decreasing f_R, $f_R^{-1} \equiv R(y)$, and the letter R.**

3. **For a differentiable function the parity is the sign of their derivative.**

4. **We may assign a parity "0" to the letter C.**

Nested Functions

A convenient notaion:

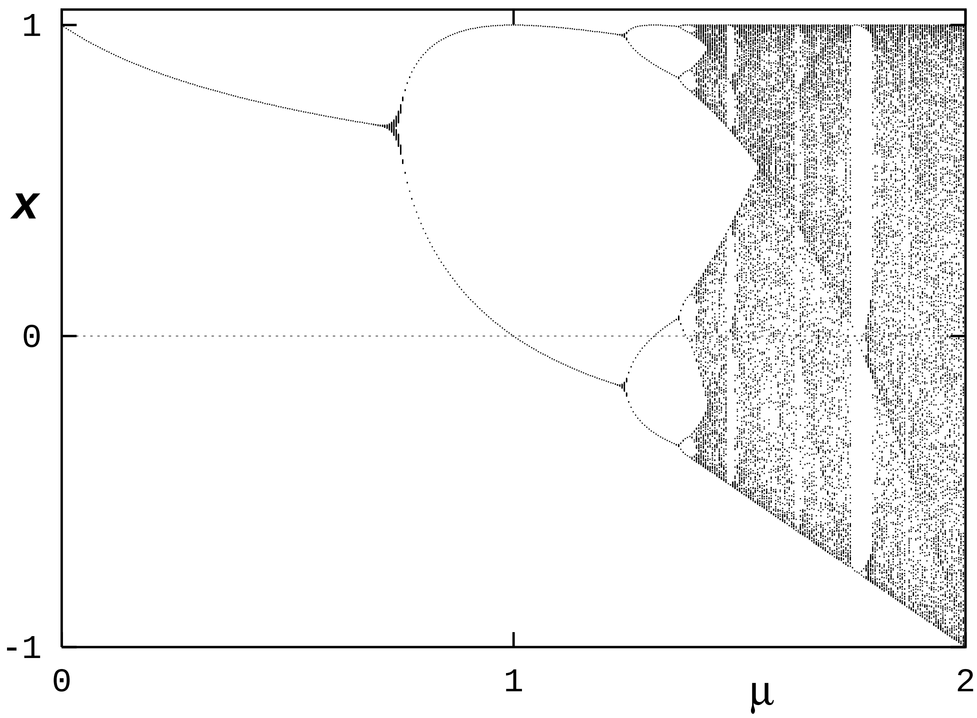
$$f(g(h(k(x)))) \equiv f \circ g \circ h \circ k(x)$$

If all f, g, h, and k are monotonic, then the nested function is also monotonic. Its parity is the product of parities of the constituent functions.

In particular,

$$F(n, x) = f \circ f \circ \cdots f \circ f(x)$$

A Bifurcation Diagram



How many questions one may ask by inspecting this diagram?

- Periodic windows. Their parameter values? How many of them? Say, how many different period 5 windows are there? How many period 97 windows?
- Dark lines. They seem to be quite smooth. Can we write down their equations?
- The tangencies and intersections of the dark lines. How to determine their parameter values?
- Why points near each bifurcation are more scattered? Critical slowing down.
- $\cdots\cdots$

Periodic Windows

Fixed Point:

$$\begin{array}{ccc}(L, & C, & R) \\ (+, & 0, & -)\end{array}$$

Period 2 Window:

$$\begin{array}{ccc}(RR, & RC, & RL) \\ (+, & 0, & -)\end{array}$$

Period 4 Window:

$$\begin{array}{ccc}(RLRL, & RLRC, & RLRR) \\ (+, & 0, & -)\end{array}$$

The infinite (Feigenbaum) limit by repeatedly applying the substitutions:

$$\begin{array}{lcr} R & \Rightarrow & RL \\ (C & \Rightarrow & RC) \\ L & \Rightarrow & RR \end{array}$$

These substitutions are:

1. **Order perserving;**
2. **Parity perserving.**

Periodic Window Theorem

A superstable period: $(\Sigma C)^\infty$

Can be extenmded to a periodic window:

$$((\Sigma C)_-,\ \Sigma C,\ (\Sigma C)_+)$$

where

$$\begin{aligned}(\Sigma C)_+ &\equiv \max\{\Sigma R, \Sigma L\}\\ (\Sigma C)_- &\equiv \min\{\Sigma R, \Sigma L\}\end{aligned}$$

$(\Sigma C)_+$: **Upper sequence of** ΣC

$(\Sigma C)_+$: **Lower sequence of** ΣC

Names of Unstable Periodic Orbits

Stable orbit may appear under someone else's name. It acquires own unique name after loss of stability.

Period	Words	Number
1	L, R	2
2	RL	1
3	RLC	2
4	$RLRR, RLLC$	3
5	RLR^2C, RL^2RC, RL^3C	6
6	$RLLRLR, RLR^3C, RL^2R^2C, RL^3RC, RL^4C$	9
7	$RLR^4C, RLR^2LRC, RL^2RLRC, RL^2R^3C$	
	$RL^2R^2LC, RL^3RLC, RL^3R^2C, RL^4RC, RL^5C$	18

C **may be either** L **or** R**.**

Criterion for a good symbolic dyanmics, inferred from experiments.

Example: Harry Swinney's study of chemical oscillations.

Generalized Composition Rule

To get longer admissible words from shorter ones

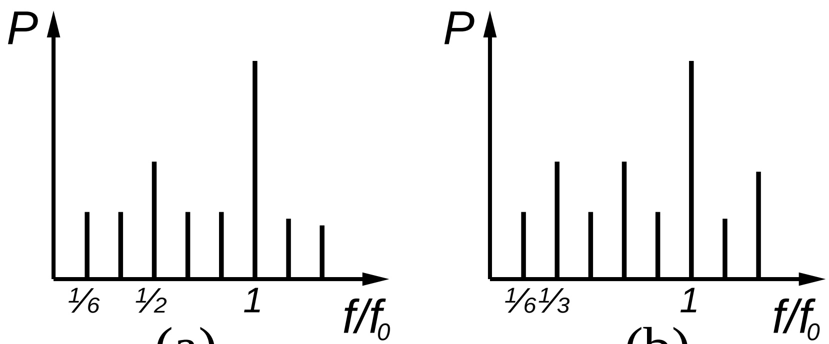
$$R \Rightarrow \rho$$
$$L \Rightarrow \lambda$$

These substitutions must be:

1. **Order perserving;**
2. **Parity perserving; and**
3. **Satisfying 3 more conditions.**

When ρ and λ are taken from the two sides of one and the same periodic window it reduces to so-called $*$-composition introduced by Pomeau and coworkers in 1978, hence the adjective "generalized".

Power Spectra and ∗-composition



(a) (b)

Fine structures of power spectra for $RL * RC$ (a) and $R * RLC$ (b), where 1 indicates the fundamental frequency of the system.

“Word-Lifting” Technique

There are 3 different period 5 windows: $(RLRRC)^\infty$, $(RLLRC)^\infty$, and $(RLLLC)^\infty$.

We “lift” each word to an equation:

$$RLRRC \Rightarrow f(C) = R \circ L \circ R \circ R \circ (C)$$
$$RLLRC \Rightarrow f(C) = R \circ L \circ L \circ R \circ (C)$$
$$RLLLC \Rightarrow f(C) = R \circ L \circ L \circ L \circ (C)$$

which determine the parameter of the corresponding superstable period 5 orbit separately.

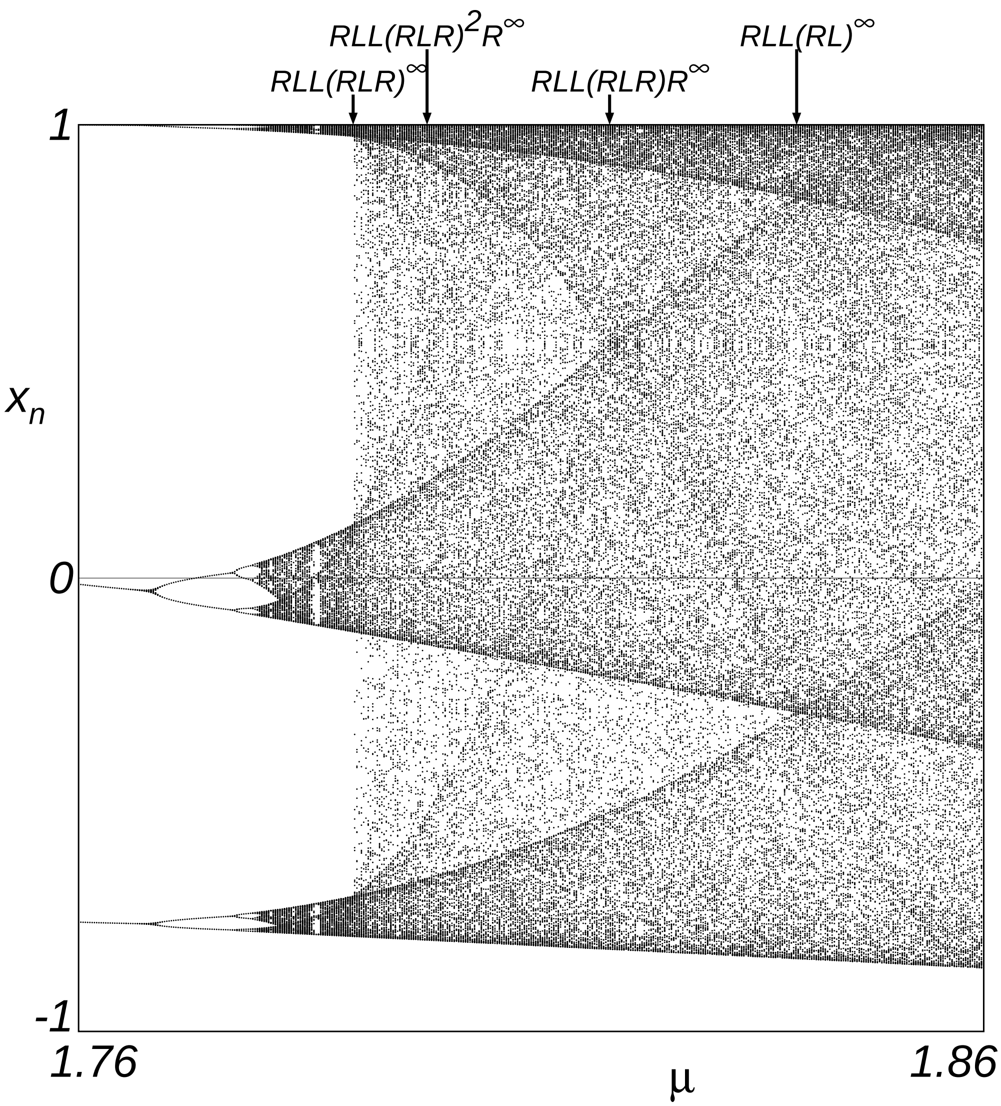
For the function $f(\mu, x) = \mu - x^2$ the first equation looks like:

$$\mu = \sqrt{\mu + \sqrt{\mu - \sqrt{\mu - \sqrt{\mu}}}}$$

Don’t try to get rid of the nested square roots. Instead, transform it into an iteration:

$$\mu_{n+1} = \sqrt{\mu_n + \sqrt{\mu_n - \sqrt{\mu_n - \sqrt{\mu_n}}}}$$

Bifurcation diagram near the period 3 window.



Eventually Periodic Orbits

$$\rho\lambda^{\infty}$$

Their parameters may be determined by "Word-Lifting" as well:

$$\begin{aligned} f(C) &= \rho(\nu) \\ \nu &= \lambda(\nu) \end{aligned}$$

For example, the parameter of eventually periodic orbit RL^{∞} is determined from:

$$\begin{aligned} f(C) &= R(\nu) \\ \nu &= L(\nu) \end{aligned}$$

For the unimodal map $f(x) = \mu - x^2$ we have:

$$\begin{aligned} \mu &= \sqrt{\mu - \nu} \\ \nu &= -\sqrt{\mu - \nu} \end{aligned}$$

Which leads to $\mu = 2$, the most chaotic or random map.

Equations of all dark lines: another "lifting"

- **Iteration of numbers starting from x_0:**

$$x_{n+1} = f(\mu, x_n), \quad n = 0, 1, 2, \cdots \;\; x_0 \text{ given}$$

$$\Downarrow$$

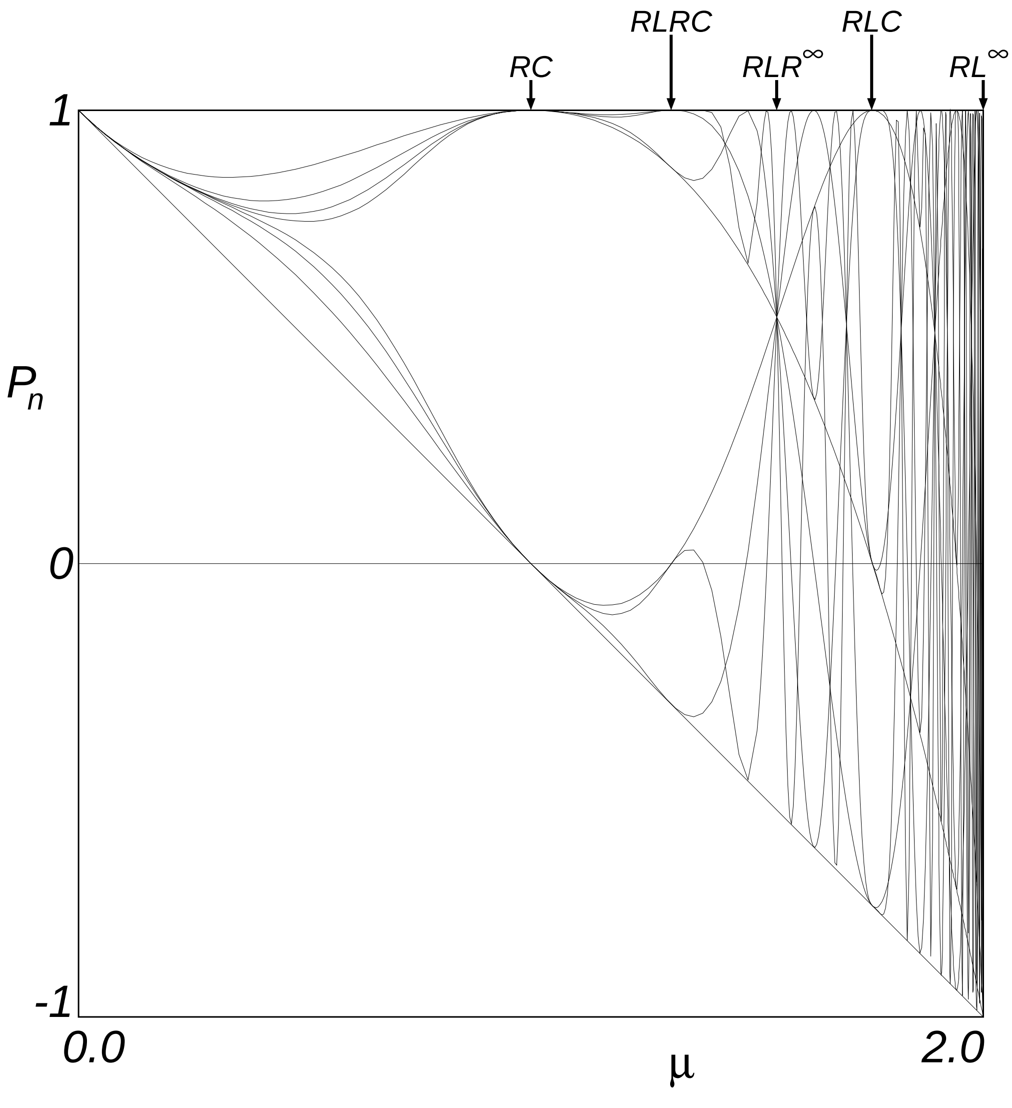
- **Iterations of functions starting from a constant:**

$$P_0(\mu) \equiv f(\mu, C)$$

$$P_{n+1}(\mu) = f(\mu, P_n(\mu))$$

Then these $P_n(\mu)$, $n = 0, 1, 2, \cdots$ describe all the dark lines and sharp edges of the chaotic bands.

Dark Lines in Bifurcation Diagram



Can we write down the equations of all these dark lines?

Dark Lines within the Period 3 Window

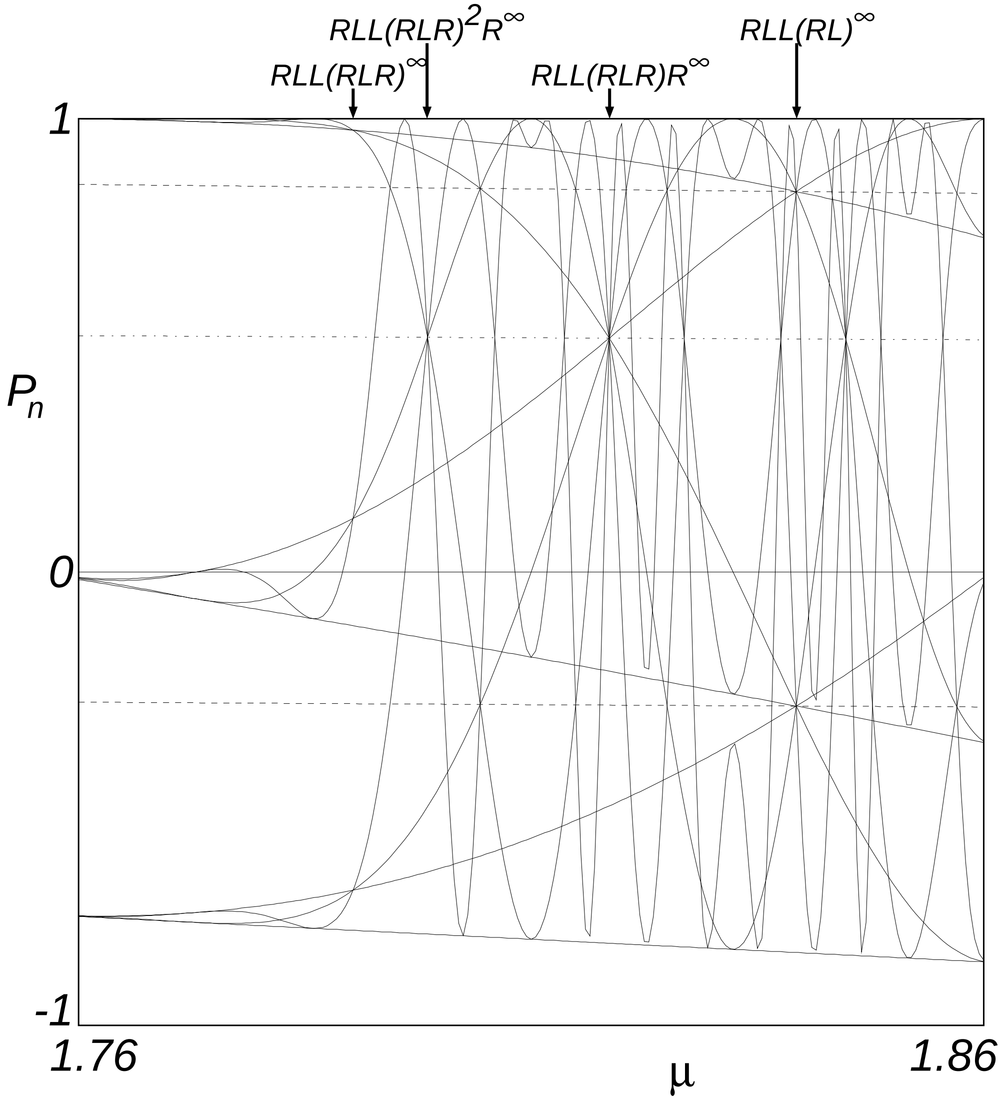


Figure 3: Approximately from RLC to RL^2RC.

Symmetry Breaking and Restoration

- **A general phenomenon in systems with discrete symmetry: anti-symmetric cubic map, Lorenz model, etc.**

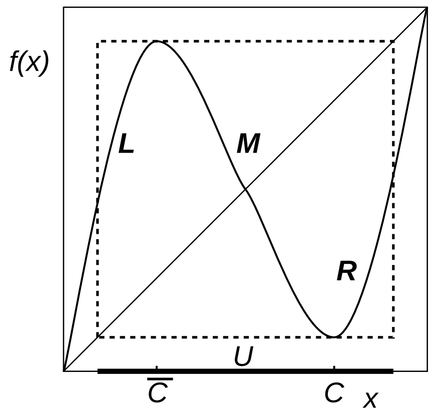
- **Symmetry-breaking always precedes period-doubling. Called "precursor" to period-doubling, etc.**

- **Only even period may undergo symmetry-breaking, but not all even periods can do so. Symbolic dynamics provides the selection rule.**

- **Symmetry restoration always happens in chaotic regime. Symbolic dynamics may help to determine the parameter where this happens.**

The Anti-symmetric Cubic map

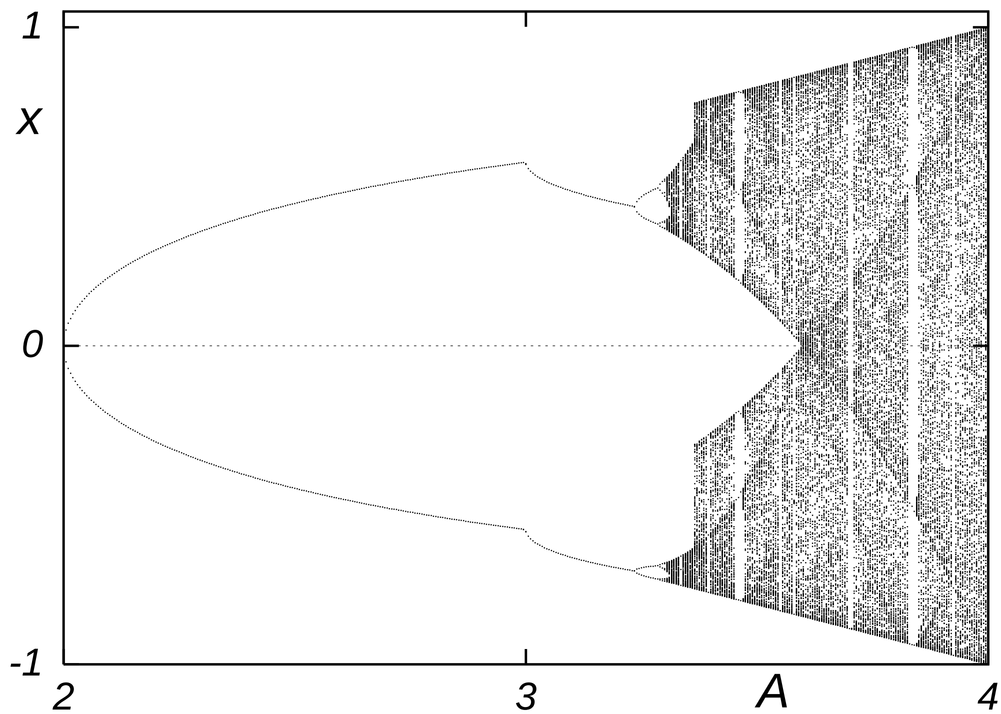
$$x_{n+1} = Ax_n^3 + (1 - A)x_n$$



Symmetry transformation:

$$R \Leftrightarrow L, \;\; C \Leftrightarrow \overline{C}, \;\; M \Leftrightarrow M$$

Bifurcation Diagram of the Antisymmetric Cubic Map



Note the breaking of the symmetric period 2 and the restoration of symmetry in the chaotic band.

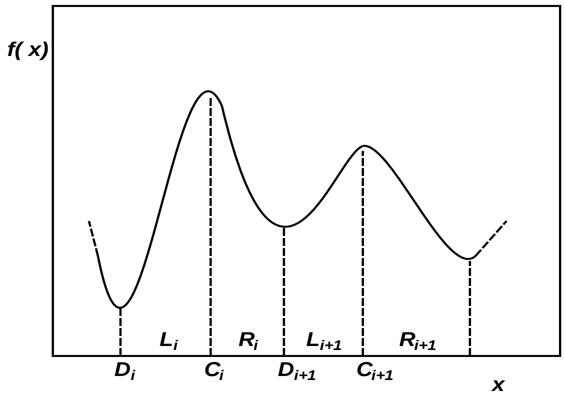
Selection Rule for Symmetry-Breaking

Only even periods can undergo symmetry-breaking, but not all even periods can do so.

List of even periods that are capable to undergo symmetry-breaking:

1. **The only Period 2:** $C\overline{C}$
2. **The only one among five period 5 orbits:** $RCL\overline{C}$
3. **Two among 32 period 6 orbits:** $RMCLM\overline{C}$, $RRCLL\overline{C}$**.**
4. **In general, even periods of the form** $\Sigma C\overline{\Sigma C}$**.**

1D Map with Multiple Critical Points



- **Ordering rule and parities remain the same.**
- **Admissibility conditions may be formulated easily.**
- **Word-lifting works, but leads to curves or surfaces in the parameter space.**

Number of Periodic Orbits

- **The counting problems solved completely by using several different methods (1984-1994).**
- **Number of (unstable) periodic orbits are topological invariants of a dynamical system.**
- **They are useful in comparing theoretical models with experiments.**
- **Complete versus Incomplete Invariants.**

Circle Mappings

Example: the sine-circle map:

$$\theta_{n+1} = \theta_n + a + \frac{b}{2\pi}\sin(2\pi\theta_n) \pmod 1$$

- **In a sense, they are intermediate between 1D and 2D maps.**
- **They are models for coupled oscillator systems.**
- **New phenomena: quasiperiodic orbits, frequency-locking, new routes to chaos, etc.**
- **Quite specific symbolic dynamics**

Two-Dimensional Mappings

$$\begin{aligned} x_{n+1} &= f(a, x_n) + by_n \\ y_{n+1} &= x_n \end{aligned}$$

$$f(a, x) = \begin{cases} 1 - ax^2 & \text{Henon} \\ 1 - a|x| & \text{Lozi} \\ ax - sgn(x) & \text{Tel} \end{cases}$$

Importance of piecewise-linear models:

1. **Simple nonlinearity.**
2. **Possibility of analytical treatment.**

Dynamical Foliations

1D maps are irreversible, leading to semi-infinite symbolic sequences when coarse-graining.

2D maps are reversible: dynamics goes in both directions of time. Forward and Backward symbolic sequences.

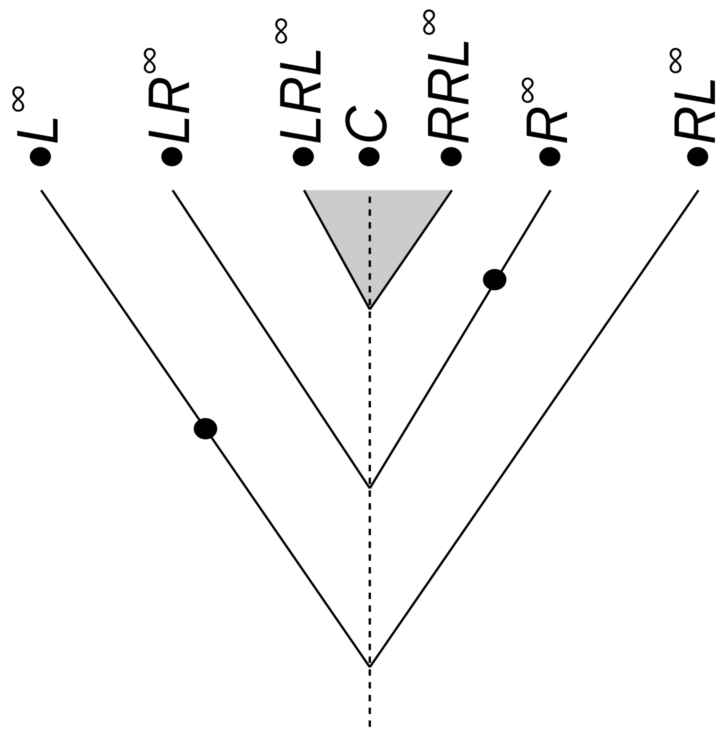
$$\cdots \alpha_{-2}\alpha_{-1}\alpha_0 \bullet \beta_0\beta_1\beta_2 \cdots$$

Main problem in constructing 2D symbolic dynamics: no ordering in 2D.

Solution: use the Forward Contracting Foliations (FCFs) and the Backward Contrracting Foliations (BCFs) generated by the dynamics itself.

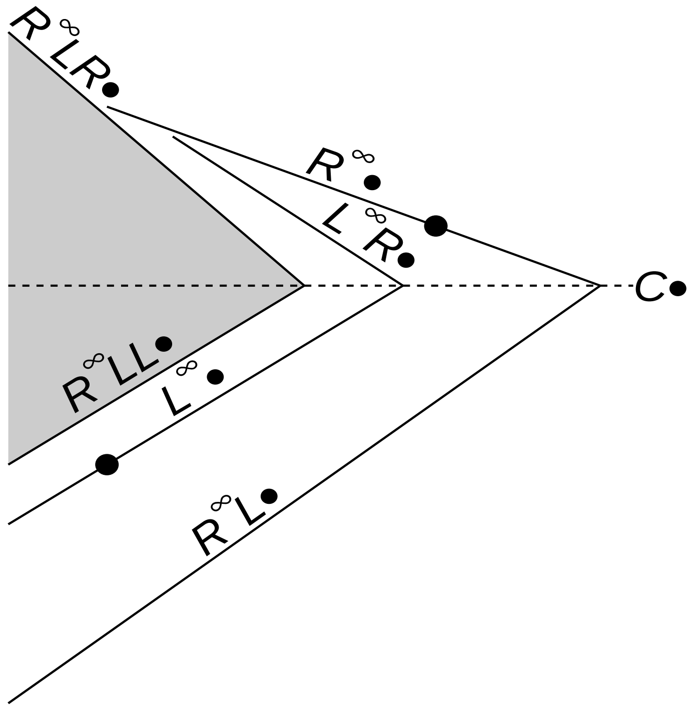
Ordering FCFs along a BCF; ordering BCFs along a FCF.

Forward-Contracting Foliations of Lozi Map



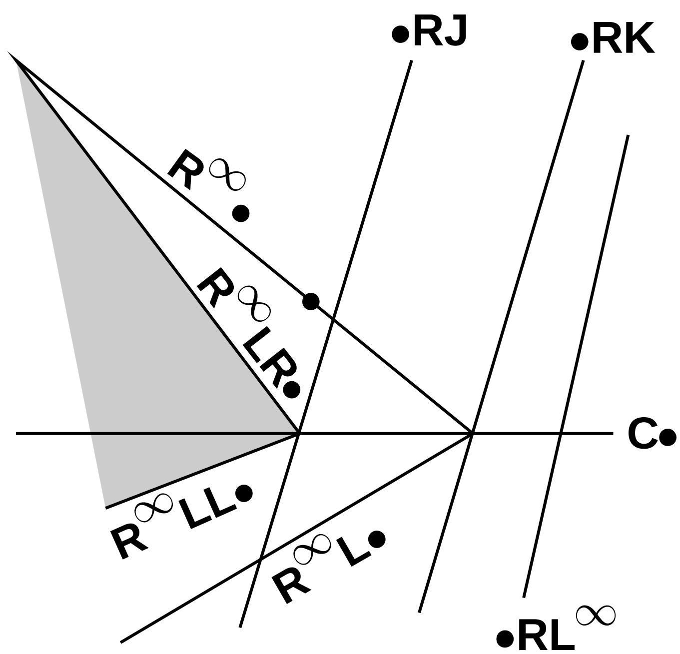
Note: The two • on foliation lines indicate fixed points.

Backward-Contracting Foliations of Lozi Map

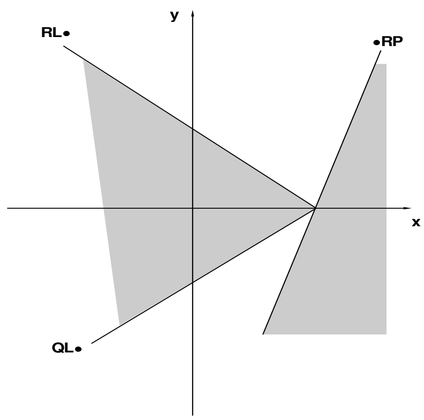


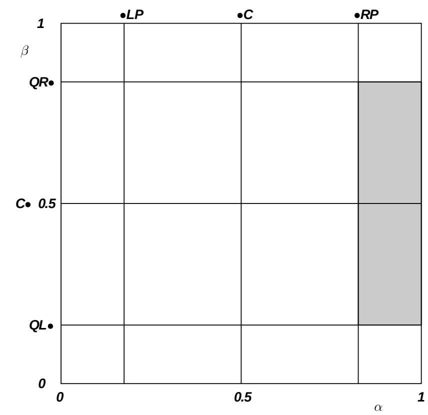
Note: the two • on foliation lines indicate fixed points.

A Tangency Between FCF and BCF



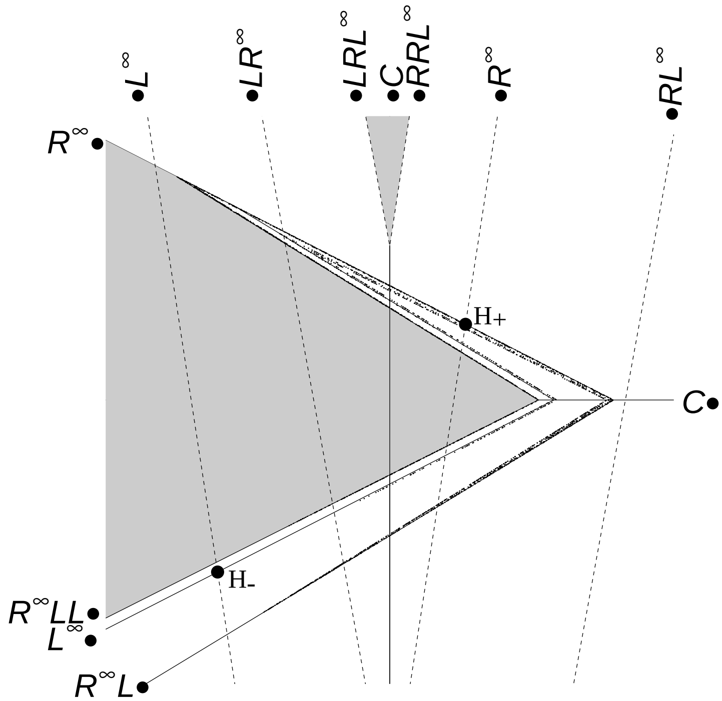
Tangency between $QR\bullet$, $QL\bullet$ with $\bullet P$





Left: phase plane; Right: symbolic plane

Dynamical Foliations of Lozi Map



Parameters: $a = 1.7$**,** $b = 0.5$

ODE: Ordinary Differential Equations

The soul of an ODE consists in the order of the highest derivative.

- **First order ODE: increasing or decreasing behaiour**

$$\frac{dx}{dt} = f(x)$$

- **2nd order ODEs: oscillations, limit cycles**

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

Problem: maximal number of limit cycles in a planar system of ODEs.

- **3rd order ODEs: quasiperiodic orbits, chaotic orbits**

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, z) \\ \frac{dy}{dt} &= g(x, y, z) \\ \frac{dz}{dt} &= h(x, y, z) \end{aligned}$$

- **Autonomuos ODEs vs. Non-Autonomous ODEs, in particular, periodically driven systems**

- **A notorious case: time-delayed ODE. Even a seemingly 1D equation is essentially infinite-dimensional.**

$$\frac{dx(t)}{dt} = f(x(t-a)), \quad a = \text{ constant}$$

Let $t = \tau/a$**, then we have:**

$$\frac{1}{a}\frac{d\tilde{x}}{d\tau} = f(\tilde{x}(\tau - 1))$$

Symbolic Dynamics of ODEs

Poincaré section $\Rightarrow$ 2D Poincaré map.

Construction of FCFs and BCFs numerically.

The problem to answer: number of periodic orbits for periods less than a given number, say, 7.

Detailed case studies:

1. **The periodically forced Brusselator**
2. **The Lorenz equations**
3. **The driven two-well Duffing equations**
4. **The NMR-laser model**
5. $\cdots\cdots$

Numerics under the guidance of topology