Different timestep

Lorenz, Physica D 35: 229

Different arithmetic


Different solver algorithm…
Moral: numerical methods can run amok in “interesting” ways…

• can cause distortions, bifurcations, etc.
• and these look a lot like real, physical dynamics…
• source: algorithms, arithmetic system, timestep, etc.
• Q: what could you do to diagnose whether your results included spurious numerical dynamics?

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….

Shadowing lemma

Every* noise-added trajectory on a chaotic attractor is shadowed by a true trajectory.

Important: this is for state noise, not parameter noise.

(*) Caveat: not if the noise bumps the trajectory out of the basin.

Section

Not the same thing as a projection!
The driven damped pendulum

Time-slice sections of periodic orbits: some thought experiments

- pendulum rotating @ 1 Hz and strobe @ 1 Hz?
- pendulum rotating @ 1 Hz and strobe @ 2 Hz?
- pendulum rotating @ 1 Hz and strobe @ 3 Hz?
- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
- pendulum rotating @ 1 Hz and strobe @ \( \pi \) Hz? (or some other irrational)

What bifurcations look like on a Poincare section

Spatial sections

Cantor set! (remember: not always...)
What about a section of a UPO?

Aside: finding UPOs

- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers

λ₁ and the un/stable manifolds (W⁺ and W⁻)

- If you’re slicing in state space: use the “inside-outside” function
- If you’re slicing in time: use modulo on the timestamp
- See Parker & Chua for more details

Aside: finding those un/stable manifolds

- Linearize the system
- Find the eigenvectors $E_s$ and $E_u$
- Take a step along $E_s$; run time forwards
- Take a step along $E_u$; run time backwards
- See Osinga & Krauskopf paper for more details
These \( \lambda \) & manifolds play a critical role in the control of chaos...

Lyapunov exponents, revisited:
- \( n \)-dim system has \( n \lambda_i \); \( \Sigma \lambda_i < 0 \) for dissipative systems
- \( \lambda_i \) are same for all ICs in one basin
- negative \( \lambda_i \) compress state space along stable manifolds
- positive \( \lambda_i \) stretch it along unstable manifolds
- biggest one (\( \lambda_1 \)) dominates as \( t \to \infty \)
- positive \( \lambda_i \) is a signature of chaos
- calculating them:
  - From equations: eigenvalues of the variational matrix (see variational system notes on CSCI5446 course webpage, which you can access from Liz’s homepage.)
  - From data: various creative algorithms…

Local-linear control of a hyperbolic point

* e.g., via pole placement

Calculating \( \lambda \) (& other invariants) from data
- Associated software: TISEAN
  www.mpipks-dresden.mpg.de/~tisean
Kantz’s algorithm:

1. Choose point K
2. Look at the points around it (ε neighborhood)
3. Measure how far they are from K
4. Average those distances
5. Watch how that average grows with time (Δn)
6. Take the log, normalize over time \( S(Δn) \)
7. Repeat for lots of points K and average the \( S(Δn) \)

If you’re lucky, things look like this.

- The slope of the scaling region—\( \text{iff one exists} \)—is the \( \lambda \).

This is fig 5.3 in Kantz & Schreiber.

If you’re not lucky:

- The slope of the scaling region—\( \text{iff one exists} \)—is the \( \lambda \).

This is fig 5.4 in Kantz & Schreiber.

What do you think those oscillations might be?
Calculating $\lambda$ (& other invariants) from data

Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!

Different colors on that plot from before = different settings for one of those knobs

• Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!
• Use your dynamics knowledge to understand & use those knobs intelligently
• Look at the results plots. For example, do not blindly fit a regression line to something that has no scaling region (this is a good idea in general, of course)

Fractal dimension:

• Capacity
• Box counting
• Correlation ($d_2$ in TISEAN)
• Lots of others:
  • Kth nearest neighbor
  • Similarity
  • Information
  • Lyapunov
  • …
• See Chapter 6 and §11.3 of Kantz & Schreiber

We’ve been assuming that we can measure all the state variables…

But often you can’t.

Rarely do you even know what they are…
How to undo a projection?

Delay-coordinate embedding

"reinflate" that squashed data to get a topologically identical copy of the original thing.

Reconstruction space

Mechanics

TISEAN’s delay command does this
Measure just the $x$ coordinate…

…and then embed:

Takens' theorem

For the right $\tau$ and enough dimensions, the embedded dynamics are diffeomorphic to (and thus have same topology as) the original state-space dynamics.  

* Whitney, Mane, …

Note: the measured quantity must be a smooth, generic function of at least one state variable, and must be uniformly sampled in time.

Choosing $\tau$:

Choosing $m$

$m > 2d$: sufficient to ensure no crossings in reconstruction space (Takens et al.)…

…but that may be overkill, and you rarely know $d$ anyway.

"Embedology" paper: $m > 2d_{\text{box}}$ (box-counting dimension)

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- qualitatively the same shape (topology)
- have same dynamical invariants (e.g., $\lambda$)

Choosing $m$

$m > 2d$: sufficient to ensure no crossings in reconstruction space (Takens et al.)…

…but that may be overkill, and you rarely know $d$ anyway.

"Embedology" paper: $m > 2d_{\text{box}}$ (box-counting dimension)