Power Laws in Financial Time Series

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Abstract

We attempt empirical detection and characterization of power laws in financial time series. Fractional Brownian motion is defined. After testing for multifractality we calculate the multifractal spectrum of the series. The multifractal nature of stock prices leads to volatility clustering (conditional heteroscedasticity) and long memory (slowly decaying autocorrelation). Wavelet Transform Modulus Maxima approach to multifractal spectrum estimation proved certain advantages to the structure function approach as shown by Muzy et al. (1993). We apply wavelet based methodology as wavelets are an appropriate tool for non-stationary time series.

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1 Introduction

Financial markets are typically characterized by instability. Since the late 1980's this has been especially true. This increased instability has led to the emergence of a new sub discipline in finance: risk modeling.

Our goal is to improve understanding of the dynamic systems that lead to the emergence of catastrophic stock market crashes.

Fat-tailed distributions are commonly found in financial contexts such as market returns, trading volumes, wealth and income distributions.

Financial markets seem to display regimes. In some regimes, the market acts like a calm sea; while in others, markets are beset by storms such as bubbles and crisis. We use a variety of analytical techniques to attempt to characterize these regimes in a US stock market index.

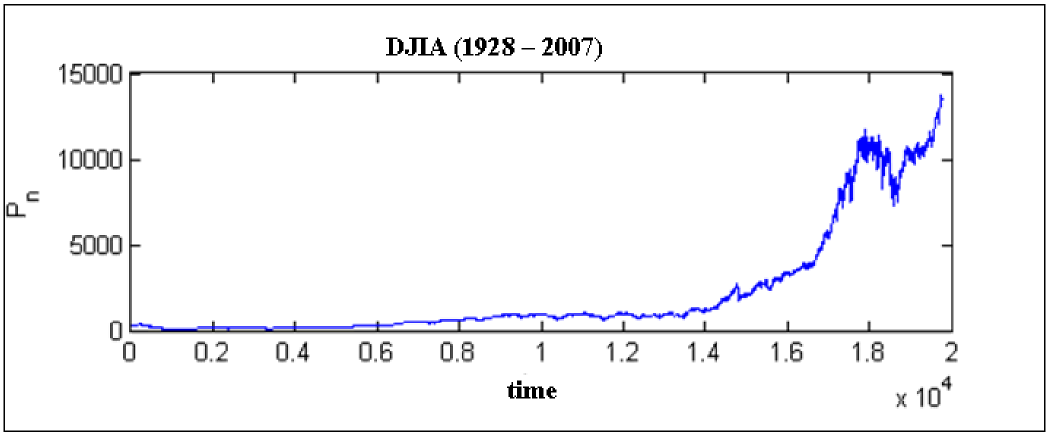

Figure 1: Plot of the original index price series.

2 Power-laws in Financial Time Series

The traditional assumption of independent and normally distributed price changes is now believed to be flawed by many researchers. Instead, empirical evidence indicates that extreme events are much more frequent than Gaussian data (fat-tails) and decay of autocorrelation function are slow (long-memory). Hence, it is proposed that power-laws are more suitable to describe these changes.

We perform an empirical study on power-laws in financial time series. Two kinds of power-laws are estimated, namely power-law distribution of returns, and power-law decay of their autocorrelation functions.

Daily data of Dow-Jones Index from 1928-10-1 to 2007-7-11 are used, which consists of 19778 points. Raw data are illustrated on Fig. 1.

The Logarithmic Returns are defined as

$$r(n) = \ln P(n+1) - \ln P(n) \tag{1}$$

We will focus on the log returns and their absolute values (Fig. 2).

3 Research Questions

1. Log returns have been shown to follow power laws in many markets. We seek to understand how the power law exponents change over time and if they may be used to characterize the crash of 1987.

2. What are the implications of the Autocorrelation Function (ACF) power laws?

3. What are the inferences from the multi-fractal spectrum in financial time series analysis?

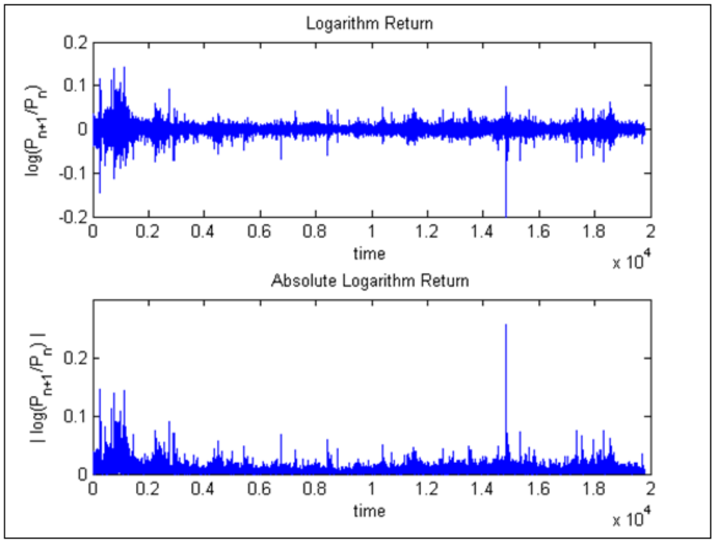

Figure 2: Plots of the continuously compounded return series and their absolute values.

3.1 Power-law distributions of log returns

The existence of power-law distributions in log returns has been verified by many researchers. In this study, we are more interested in how the power-law behavior changes with time, specifically, during the 1987 crash.

The power-law distribution is defined as

$$\Pr(x) \propto x^{-\alpha} \qquad \text{for } x > x_{\min} \tag{2}$$

determined by two parameters, α (exponent) and $x_{\min}$(lower bound).

The algorithms proposed by Clauset et al (2007) are adopted to estimate the power-law parameters as well as their uncertainty.

3.2 Results

The parameter estimates of the power-law distribution for the whole series are reported in Table 1

Data	α (exponent)	x_{min} (lower bound)	D (K-S statistics)
Absolute log returns	3.3536±0.2503	0.0154±0.0056	0.0274
Positive log returns	3.4810±0.1629	0.0154±0.1629	0.0235
Negative log returns	3.3313±0.1984	0.0179±0.0036	0.0342

Table 1.Power-law parameters of the whole series

The K-S statistics is not adequate to tell how well the power-law fits to the empirical data. Clauset et al (2007) also proposed a method to compute the p-values for the power-law hypothesis, which is not conducted in this study due to limited computational power.

3.3 Power-law distributions at different time periods

The parameters are computed using windows of fixed width which slide along the daily time series. Windows with width 2000, 4000 and 6000 days are tested, and the windows slide 20 days each step. We only estimate the parameters of the absolute values of log returns.

The estimations using different window width are shown in Fig. 4.

From the Fig. 4 (b) one can observe that there is a dramatic change of the power law around the 1987 crash. Specifically, decreases sharply from 6.1 to 4.3 when October 12th, 1987 enters the window. See Fig. 5 and Fig. 6 for more details. It's still not clear why there are such big change in the value of and the shape of the CCDF when the 4000-days window moves forward for only 20 days.

3.4 Power-law distributions at different time scales

The log returns at scale s is defined as:

$$r_s(n) = \ln P(n+s) - \ln P(n) \tag{3}$$

The probability distribution of log returns depends on the time scale s, as shown in Fig. 7, which compares the empirical distribution with theoretical Gaussian distribution at different s. For small time scales of daily data, the Complementary Cumulative Distribution Function (CCDF) deviates significantly from the Gaussian and exhibits power-law fat tails. For large time scales of several months and more, it becomes more normally distributed.

The probability distribution of returns depends on the time scale. For small time scales of daily data, the Cumulative Distribution Function (CDF) deviates

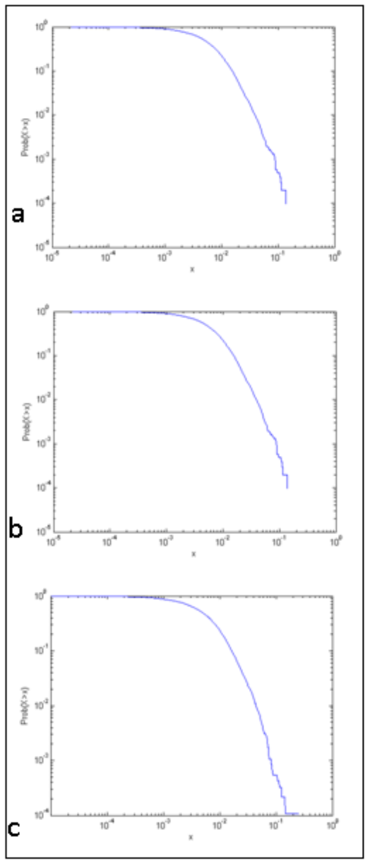

Figure 3: Complementary Cumulative Distribution Function (CCDF) in log-log scale for (a) absolute values of log returns, (b) positive log returns, and (c) negative log returns.

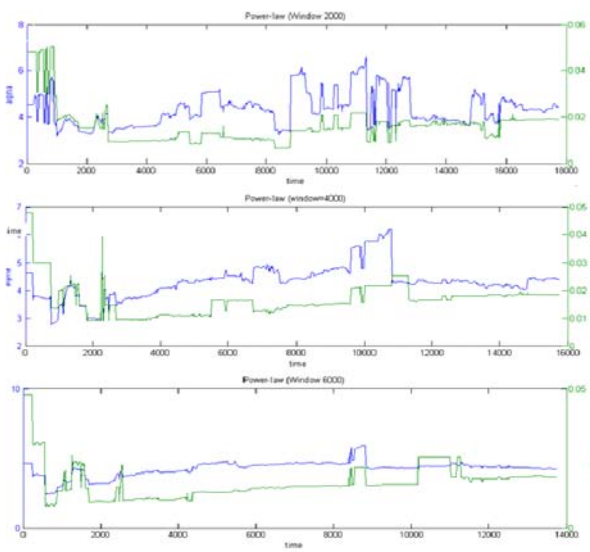

Figure 4: Estimation of power-law parameters with different window width. X-axis is the starting point of the windows. Blue lines: α(exponent). Green lines: $x_{\min}$(lower bound).

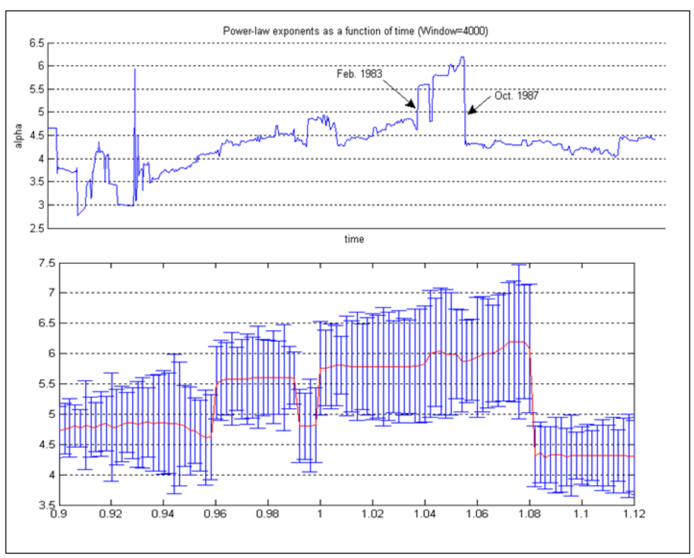

Figure 5: Detailed results of estimation using 4000-day window. (a) Change of power-law exponent across time. (b) Zoom-in of (a) around the 1987 crash with error bars added.

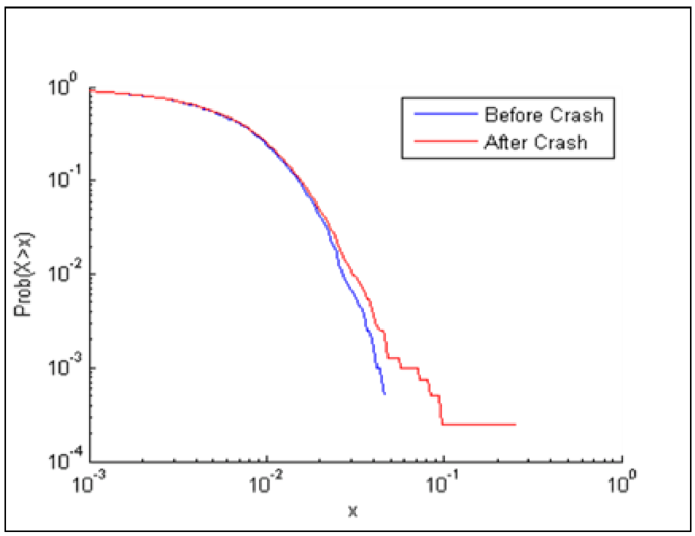

Figure 6: Complementary Cumulative Distribution Function of log returns before and after the crash occurred. Blue lines: from data point 10800 to 14800; red lines: from data point 10820 to 14820. (Data point 14800 is 1987-10-12, Monday)

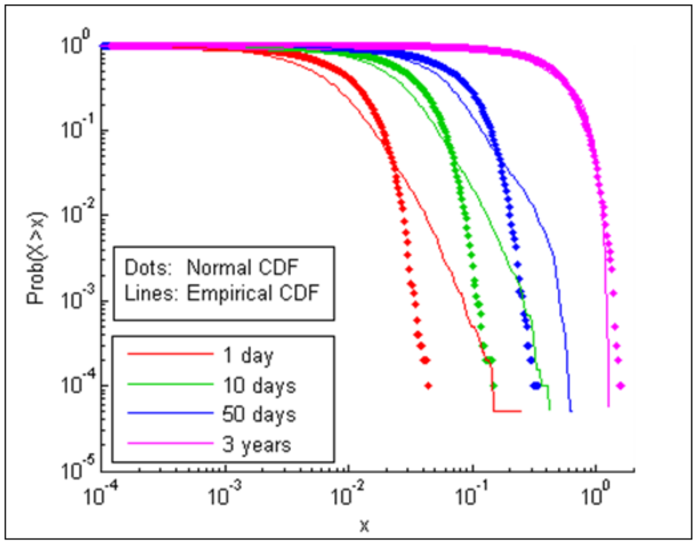

Figure 7: Complementary Cumulative Distribution Function (CCDF) of empirical data (log returns) at different scales, compared with Gaussian distributions with the same means and variances.

significantly from the Gaussian and exhibits power-law fat tails. For large time scales of several months and more, it becomes more normally distributed.

4 Power-law decay of the autocorrelation function

The autocorrelation function (ACF) of log returns and their absolute values are computed as:

$$ACF_r(n) = E[r(t) * r(t+n)] \tag{4}$$

respectively,

$$ACF_{|r|}(n) = E[\mid r(t) \mid * \mid r(t+n) \mid]. \tag{5}$$

Fig. 8 plots the ACF of log returns and their absolute values for lag 0 to 200 days. Fig. 9 gives the ACF in log-log scales of the absolute values of log return for lag 1 to 1000.

One can observe that: the ACF of log returns is effectively zero for lag larger than 1; the ACF of the absolute values of log return (as a straightforward measure of volatility) decreases very slowly and remains positive, even after a lag of 500, before decaying to zero. Such results suggest that the returns of the

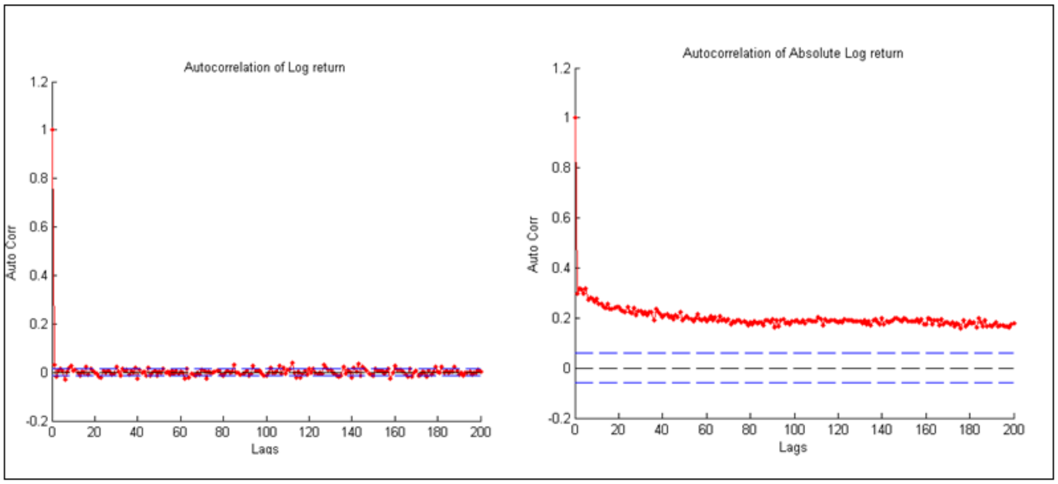

Figure 8: Blue dotted lines are the approximate 95% upper and lower confidence bounds given by Matlab routine autocorr

stock market are uncorrelated while their volatility has long-range dependence. In such long memory process, the decay of autocorrelation function is a power law. Exponent of the ACF α is related to the Hurst exponent H as $\alpha = 2H + 1$. Our results show changing power law exponents of the ACF, which implies changing Hurst exponent over time.

4.1 Hurst exponent - monofractal process

The Hurst exponent analysis, being well suited to distinguish a random series from a non-random one, allows us to give a qualitative description of the behavior of markets. Fractal time series possess cycles and trends, thus exhibiting nonlinear dynamics. Information is not immediately reflected in prices, as the EMH states, but instead manifests itself as a bias in returns.

A random process $x(t)$ is said to be self-similar with Hurst exponent H if for any scale $a > 0$ it obeys the scaling relation

$$x(t) \triangleq a^{-H} x(at) \tag{6}$$

The Hurst statistics provides us with a means to analyze the dependence characteristics of time series and to determine if they are serially, or globally dependent. When the Hurst exponent is $0 < H < 0.5$, the time series is called anti-persistent. This Fractional Brownian Motion increments diffuse more quickly than the Brownian increments as the autocovariance function of the price process is negative. The FBM returns continuously to the initial point. When the Geometric Brownian motion has $H = 0.5$, the time series exhibits

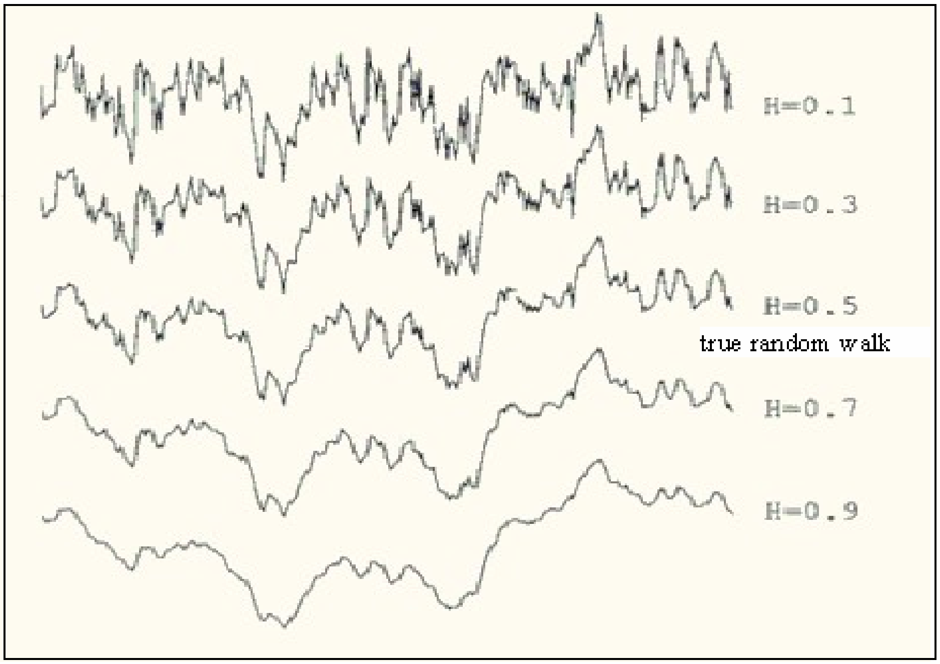

Figure 9: Time series with different Hurst (H) exponents. A smaller Hurst exponent process has a higher fractal dimension (D) and a rougher surface: $D = 2 - H$.

serial, or short term dependence. When $0.5 < H < 1$, the time series has long memory, the price increments are positively correlated and the volatility persistence extends forever: it never dies out and gives rise to empirically observed non-periodic cyclicities. The strength of such persistence is measured by the H-exponent.

Muzy *et al.* (2000) use the non-linearity of the scaling function as evidence of multifractality of price fluctuations of financial time-series. Turiel and Pérez-Vincente (2002), Schmitt *et al.* (2000), and Bacry *et al.* (2001), among others, use the non-linearity of the scaling function to show that the estimate of two moments does not describe the entire distribution, evoking the need of multifractal analyses and models in finance. An important application of multifractal analysis is precisely to characterize all order moments for the validation of a scaling model.

4.2 Multifractals

The index price time series exhibit changes of the Hurst exponent over time as well as deviations from linearity of the scaling exponent $\tau(q)$ suggesting the existence of a multifractal spectrum. In finance, Corazza and Malliaris (2002) draw a similar conclusion for a number of foreign currency markets. Fisher *et al.* (1997), Xu and Gençay (2003), Muller *et al.* (1990) etc. confirm the multifractality in financial data.

Multifractals are observed in turbulence as a result of the self-similar hierarchical structure of energy flow from large to small scale. In finance, Ghashghaie *et al.* (1997), Muller *et al.* (1990) and Schmitt *et al.* (1999) proposed a cascade of information from large to small scale and pointed to similarities with fluid turbulence. Financial fluctuations display intermittence at all scales, unlike turbulence with energy influx at large scales and dissipation at small scales.

Multifractality is related to an underlying multiplicative cascading process. The analysis of the multifractal spectrum should be related to the self-similarity properties of the financial asset returns function. A cascade model describes the distribution of the volatility of returns across scales, not the fluctuation of returns. Note that a multiplicative cascade model predicts strong correlations in volatility, while previously proposed fat tail model distributions assume no correlation.

4.2.1 Methodology

The multifractal spectrum provides information about the statistical distribution of singularities measured by the Hölder exponents. The pointwise Hölder exponent of a function $f(x)$ at an observed point x_0 is the greatest exponent h so that:

$$|f(x) - P_n(x - x_0)| < C\,|x - x_0|^h\,, \tag{7}$$

where C is a constant and P_n is the degree of the polynomial. Statistically self-similar or multiplicative process $X(t)$ is monofractal with Hölder (Hurst) exponent (H) between zero and one, if $\forall\lambda > 0$, $\lambda^{-H}X(\lambda t)$ is the same process as $X(t)$.

Wavelet Transform Modulus Maxima Wavelet transform has proved to be a particularly efficient tool for measuring the local regularity of a function.

The wavelet transform of $f(t) = P(t)$ is defined as

$$W(\tau, a) = \int_{-\infty}^{+\infty} f(t)\psi_{\tau,a}(t)dt, \tag{8}$$

where the analyzing wavelet $\psi_{\tau,a}(t)$ is a function, centered around zero. A common way to build wavelets of order n is to successively differentiate a smoothing function. A popular family of wavelets uses the Gaussian function

$$g_n(t) = g^{(n)}(t) = \frac{d^n}{dt^n}e^{-t^2/2} \tag{9}$$

The wavelet transform is a well adapted tool for studying scaling processes. As wavelet coefficients are stationary and "almost" decorrelated, the q-order moment of a wavelet coefficient at scale a can be estimated by averaging these coefficients at scale a for a given time series.

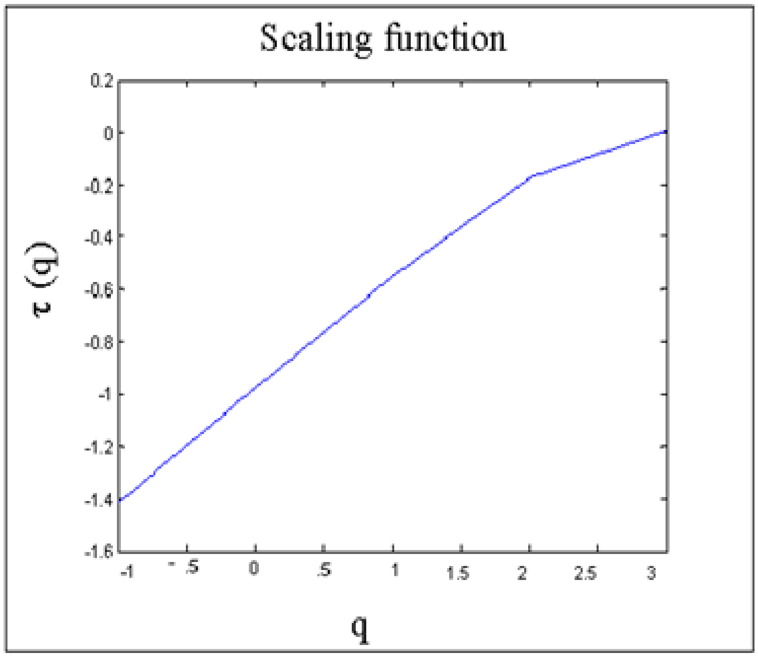

Figure 10: Example of concave scaling function

Multifractal Spectrum Estimation A multifractal formalism based on wavelet transform modulus maxima (WTMM) allows us to determine the whole singularity spectrum $f(\alpha)$ directly from any experimental signal (Muzy *et al.*, 1991). The WTMM approach is the foundation of a unified multifractal description of self-affine distributions, as shown by Muzy *et al.* (1993). We apply WTMM based methodology described in Yalamova (2003).

5 Results

The multifractal nature of the index fluctuations is demonstrated on Fig.11 , where the moment ratios strongly depend on the scale. The estimated $\tau(q)$ has a concave shape.

The quantity $f(\alpha)$ is the fractal dimension of the subset of intervals or probability measure on the fractal support. Empirical price data can be compared with multifractal models of volatility distribution (Mandelbrot's cascades), intermittency coefficient, tail index, time reversal invariance. The verification or failure of these models and the overall quest for proper financial price model encourages us to continue our research on multifractality of financial time series.

Another venue in the empirical research related to multifractality goes through

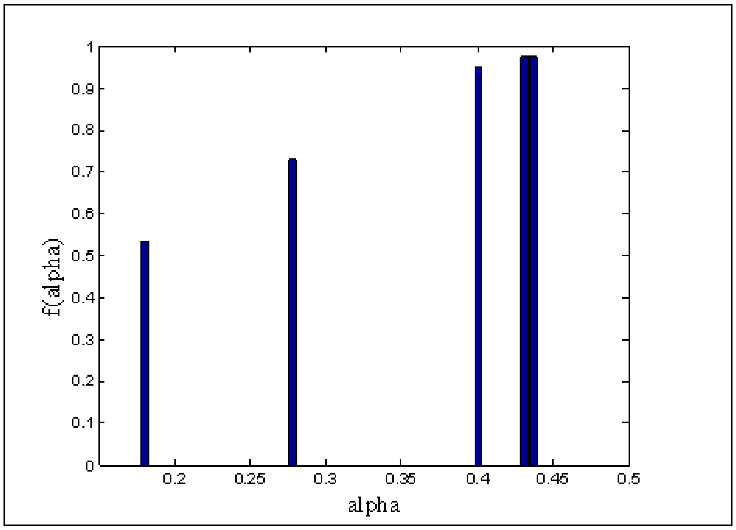

Figure 11: Example of multifractal spectrum for moments of order $-1 < q < 3$. The peak of f(α) yields the box-counting dimension D_0.

D_q generalized dimensions to chaos. With $q = 0$, we find $f'(\alpha) = 0$, therefore the maximum of the multifractal spectrum curve is the box-counting dimension D_0 such as:

$$D_0 = f(\alpha(0)). \tag{10}$$

The information dimension D_1 lies on the $f(\alpha)$-curve where $\alpha = f(\alpha)$ and $f'(\alpha) = 1$:

$$D_1 = f(\alpha(1)). \tag{11}$$

The correlation dimension D_2 is defined:

$$D_2 = 2\alpha(2) = f(\alpha(2)). \tag{12}$$

The research on the generalized dimensions might reveal some changes in the dynamic system related to the boderlines of chaos. It is too early to speculate on the use of the statistical information but McCauley (1993) asserts that the fractal dimensions can be viewed as the ratio of the Boltzman entropy to a corresponding Lyapunov exponent. There might be deeper analogy to explore

6 Conclusions

The existence and significance of the power-law distribution in the returns of the Dow Jones index have been verified and a dramatic changes around the 1987 crash in stock market are documented.

We have also applied a test for multifractality of the time series and calculated the mulrifractal spectrum using wavelet transform modulus maxima method. Changes in the multifractal spectrum around the 1987 crash are discussed in Yalamova (2003). Our goal was to relate information from the multifractal spectrum to the existing multifractal models. On the other side we would like to research further analogy between the multifractal spectrum and chaotic dynamics.

We identify few of the further research questions that stem from this study:

1. Using time series patterns detected around 1987 crash as a starting point, can stock market crashes be predicted?

2. What are the basic behaviors of individual investors leading to fat-tail distributions in financial time series?

3. We need to understand the underlying economic principles for a full understanding of the dynamics surrounding stock market crashes.

4. What is the relationship between multi-fractals and chaos?

Our analysis of the fractal nature of financial time series inspired us to attempt to detect chaos. The initial idea to move into this direction was validated by Halsey *et al.* (1986).

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