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How much spare capacity is necessary for the security of resource networks?

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Abstract

The balance between the supply and demand of some kind of resource is critical for the functionality and security of many complex networks. Local contingencies that break this balance can cause a global collapse. These contingencies are usually dealt with by spare capacity, which is costly especially when the network capacity (the total amount of the resource generated/consumed in the network) grows. This paper studies the relationship between the spare capacity and the collapse probability under separation contingencies when the network capacity grows. Our results are obtained based on the analysis of the existence probability of balanced partitions, which is a measure of network security when network splitting is unavoidable. We find that a network with growing capacity will inevitably collapse after a separation contingency if the spare capacity in each island increases slower than a linear function of the network capacity and there is no suitable global coordinator.

Keywords: Statistical physics of complex networks; Graph theory; Network splitting; Balanced partition; Network capacity

1. Introduction

Complex networks are essential parts of a modern society [1,2]. The resilience of complex network to the malfunctioning of its components and to external disturbances (simulated as the deletion of nodes or edges) has been the subject of a great deal of attention since the work of Albert et al. [3]. The question of resilience has been looked into for a large range of networks, including the Internet [3], metabolic networks [4], food webs [5,6], email networks [7], electrical power grids [8], infrastructure networks [9], and many model networks [3,10–15]. Refs. [13,16] gave good surveys on this issue. Different from previous models, in this paper we study the resource network, in which the balance between the supply and the demand of some kind of resource (e.g., electrical power, oil, and natural gas) is a critical condition for the functionality and security of the network. Local contingencies that break this balance condition may cause performance degradation and even global

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collapse of the entire network. One example is on the North American power infrastructure, which consists of 14 099 nodes (substations) and 19 657 edges (transmission lines) [8], and is considered by many experts as the largest and most complex network of the technological age. In such a large power grid, a transmission line outage due to lightning strike or short-circuit (*local contingencies*) leads to the overload of parallel and nearby lines, which then also trip off. Then the power generated by some generators cannot reach distribution substations and ultimately consumers. *The power generation and consumption in the network is not balanced.* If the control action (such as using the spinning reserve, backup generators, or load shedding) fails to bring the power generation and consumption back to balance in time, the lines continue tripping and the power grid is passively split into several islands. When the balance condition is not satisfied in an island, the splitting continues, and will cause a large-scale blackout (*global collapse*). This is what happened in the July 2, 1996 cascading outage of Western USA power network [17] and the August 14, 2003 blackout of the North American electric power network in the United States and Canada [18]. In the latter case, estimates of total costs in the United States range between $4 billion and $10 billion (US dollars) [19]. In Canada, gross domestic product was down 0.7% in August, there was a net loss of 18.9 million work hours, and manufacturing shipments in Ontario were down $2.3 billion (Canadian dollars) [18].

Another example is on the US petroleum delivery network, which connects the domestic petroleum industries and imports (generation nodes) and the consumers (consumption nodes) together. In the summer of 2005, Hurricanes Katrina and Rita disrupted a substantial portion of production, refining, transportation and marketing sectors of the Gulf Coast oil and natural gas industries. These *local contingencies* broke the balance between demand and supply of the petroleum production, and greatly contributed to the record prices in the US oil market [20] and the decline of US petroleum delivery in 2005 (*global performance degradation*) [21].

Yet another and a most recent example is the pricing dispute between Russia and Ukraine on the natural gas. Russia supplies 25% of western Europe's gas. 80% of the supply comes through Ukraine. Central European nations also rely on Russian gas deliveries via pipelines through Ukraine. Due to a pricing dispute, Russia shut down some delivery systems and halted the gas delivered to Ukraine (*local contingencies*). This caused a shortage in Ukraine and throughout western Europe (*global performance degradation*). Although Moscow and Kiev officials praised a deal to end the pricing dispute on January 4, 2006, many European countries started to realize the vulnerability of their energy systems [22].

These examples show that local contingencies that break the balance between demand and supply can cause global performance degradation and even collapse. Spare capacity (e.g., the spinning reserve and backup generators in the electric power grid, the oil and natural gas reserved for the emergent demand) is usually used to bring the demand and supply back to balance in such emergent situations. However, spare capacity is costly, especially when it takes a big portion of the network capacity. Nowadays, many complex networks are with growing capacity, namely, power generation, oil and gas production are continuously being added. It is crucial to understand the relationship between the spare capacity and the collapse probability. This paper studies how this relationship changes as the network capacity grows.

To study the security aspects of resource networks under contingencies, we introduce the following terminologies. (Some of these terminologies are also defined in a more mathematically rigorous way in Section 2.) (1) A contingency that requires some nodes to be separated from some others will be called a *separation contingency*.[1] (2) The largest difference between the demand and supply that can still be brought back to balance later will be called the *tolerance* of the network.[2] (3) If a network can be partitioned into smaller sub-networks, which are balanced within a tolerance level, then the network is said to *possess the BP*

[1]For example, in a power grid a short circuit in the transmission line may cause the asynchronous among several generators. If the protection mechanism fails to bring these generators back to synchronous in time, then no protection measures can keep the integrity of the power grid. Some controlled system separation should be conducted to separate these asynchronous generators from each other, and keep the customers in each island continuing to be served, so that a blackout can be avoided. The shut down of part of the oil and gas delivery system may also separate some generation nodes from some others.

[2]Tolerance is a more general concept than spare capacity, which may have special meaning in engineering systems. When a contingency (e.g., loss of generation or line) happens, if the difference between demand and supply is no greater than the tolerance of the network, then there is some action (using spare capacity or cut off some unimportant demand) that can bring the difference back to zero. Otherwise, no action can keep the balance and the network starts passively splitting (some edges are removed continuously). So the tolerance describes the ability of a network to resist a contingency.

property or *Balanced Partitionable*, or simply *BP*. (4) The term *capacity allocation* will be used to refer to the pattern of the resource generation and consumption at all nodes. (5) The ratio of the number of capacity allocations such that BP holds to the number of ways to allocate the total capacity is called the *BP probability*.[3]

The importance of BP probability is that it provides a measure for the probability of system collapse. The main result of this paper is a necessary condition on the increasing speed of the spare capacity in the capacity growing networks to avoid collapse under separation contingencies. Our main assumptions are as follows.

- First, we arbitrarily specify a separation contingency. The existence of BP is studied with respect to (w.r.t.) this separation contingency when the network capacity increases.
- Second, we consider the BP probability when the load level (the total amount of resource consumption) equals to the network capacity.
- Third, for a load level, we consider all different capacity allocations.

The model studied in this paper is different from previous models in the following sense. (1) The balance between the supply and the demand of the resource is a critical condition for the functionality and security of the network. So far as the authors know, this condition is seldom considered as a critical issue to address in previous study on other networks. (2) The separation contingency considered in our model directly threatens the balance between the supply and the demand, which in turn reduces the security of resource network. But as long as the connectivity of the entire network is sustained, removing nodes and edges (which was the major concern in previous models) does not bring such threat to the security of resource network, unless the removed nodes are with big supply or demand. (3) The BP probability measures the probability that a resource network survives a separation contingency and partitions into sub-networks, in which the local supply and local demand is balanced (within a tolerance). So we use the BP probability to measure the network resilience. In previous models, the network resilience is usually measured by the connectivity of the network after removing some nodes or edges. To summarize, our model captures the critical condition for the security of resource network, that is the balance between the supply and the demand. This allows us to discover some interesting relationships between the tolerance and the network security.

The rest of the paper is organized as follows. In Section 2, we describe the node-weighted graph model of a network which was first introduced by us in Ref. [23] to study the BP problem in the power system context. In Section 3, we first use the numerical examples to motivate and then theoretically develop the main finding of the paper: unless the tolerance increases at least linearly with the growing network capacity, the BP probability of the network decreases to zero. Furthermore, we find that there is a critical value of the linear increasing speed of the tolerance. If the tolerance increases linearly but slower than the critical value, the BP probability still decreases to zero. The critical value is discussed in Section 4. In Section 5, we discuss the implications of this necessary condition. It justifies a common sense in engineering that the tolerance (e.g., spare capacity) should increase proportional to the total resource consumption. We briefly conclude in Section 6.

2. Balanced partition problem based on node-weighted graph

As aforementioned, the BP property of a network depends on many factors. To describe how these factors affect the BP property, a node-weighted (sometimes also called vertex-weighted) undirected graph $G(V, E, W)$ [24] is an appropriate model. Let $V = \{v_1, \ldots, v_n\}$ be a set of nodes; E be a set of edges, each of which $e_{ij}(i \neq j)$ stands for an undirected line between node i and j; $W = \{w_1, \ldots, w_n\}$ be a set of weights of the nodes. To simplify the discussion, we consider only nonzero integer weights. Since the load level is assumed to be equal to the network capacity, the network capacity can be calculated by the sum of all the positive weights. The nodes with positive weights are contained in the set V_g, which will be called generators. The other nodes are with negative weights and are contained in the set V_l, which will be called loads. We assume that the

[3] It is evident that the number of BP is affected by many factors, including the separation contingency, the network topology (i.e., how the nodes are connected to each other), the capacity allocation, the tolerance in each island after the partition, and the transmission capacity limits of the edges.

generation and load are exactly balanced before the contingency, that is, $\sum_{i=1}^{n} w_i = 0$. We specify separation contingencies as contingencies that force the generators (and thus the network) to be separated into several isolated islands. A separation contingency C then can be identified by m (>1) subsets of generators, $(V_{g_1}, V_{g_2}, \ldots, V_{g_m})$, with size $n_{g_1}, n_{g_2}, \ldots, n_{g_m}$, respectively. Note that $n_g = \sum_{u=1}^{m} n_{g_u}$. And we have the following balanced partition problem which was first proposed in the power system context [23].

Balanced partition problem (*BP problem*): Given an undirected, connected, and node-weighted graph $G(V, E, W)$ and a separation contingency $C = (V_{g_1}, V_{g_2}, \ldots, V_{g_m})$ of V such that $\sum_{i=1}^{n} w_i = 0$, is there a cut set $E_c \subset E$ to separate the graph G into m sub-graphs $G_u(V_u, E_u, W_u)(u = 1, \ldots, m)$ such that $V_{g_u} \subset V_u$, and the resource balance conditions are satisfied? i.e.,

$$\left| \sum_{v_i \in V_u} w_i \right| \leqslant d, \quad u = 1, \ldots, m, \tag{1}$$

where $d \geqslant 0$ is a small positive constant called the *tolerance*. Any such cut set E_c is called a BP.

It is usually nontrivial to solve the BP problem. Actually the BP problem in a complete graph (i.e., there is an edge between any two nodes in the graph) has been shown to be NP-complete [23].

We define network BP property as follows.

Definition of network BP property: Given a separation contingency, if the answer to the BP problem is yes, the network is called BP w.r.t. this contingency.

Whether a network has BP property is affected by many factors, including network topology, network capacity, tolerance level, and very importantly the way the generation is dispatched among generators and the load is allocated among load nodes. Networks in equal scale (i.e., the same number of nodes) and with equal capacity may have different BP property after an identical contingency. To see this, let us look at two five-node networks in Fig. 1. Generator nodes are shown as black or grey nodes and load nodes are shown as white nodes. Network (a) has weights $\{1, 1, 1, -1, -2\}$. Network (b) has the same weights except that the weights on node 4 and 5 exchange. The two networks have the equal capacity. Suppose that after a contingency the generator v_1 has to be separated from generators v_2 and v_3. Exact balance is required after the partition, i.e., $d = 0$. Then network (a) has a BP, i.e., to cut edges e_{12}, e_{24}, and e_{45}. But network (b) has no BP. So network (a) is BP w.r.t. this contingency, but network (b) is not. If the tolerance increases to 1, then both networks are BP w.r.t. this contingency.

To study how the capacity and tolerance affects the existence of BP, we fix the network scale and contingency, that is we assume that V_g, V_l, and $V_{g_u}, u = 1, \ldots, m$ in C are known and fixed, but W and the tolerance d are variables. Denote the network capacity, total generation $\sum_{v_i \in V_g} w_i$, by P. Given a network capacity P, let $A(P)$ be the set of all capacity allocations. For example, in the five-node network in Fig. 1, $n_g = |V_g| = 3$, $n_l = |V_l| = 2$, $P = 3$. Note that we assume that all weights are nonzero integers. The set of all capacity allocations is $A(3) = \{\{1, 1, 1, -1, -2\}, \{1, 1, 1, -2, -1\}\}$, corresponding to network (a) and (b) in Fig. 1, respectively. Under separation contingency C, denote the set of capacity allocations that permit a BP by $A_\exists(P, C, d) \subset A(P)$, i.e., capacity allocations such that the answer to the BP problem is yes when the tolerance in each island is d (a nonnegative integer). Consider for example, a separation contingency

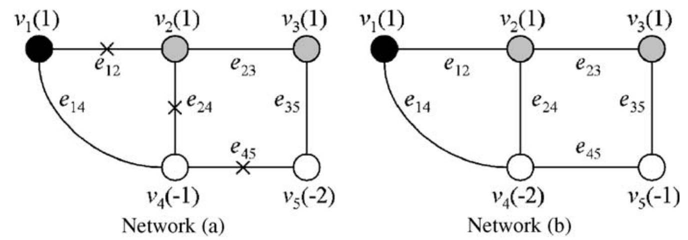

Fig. 1. The effect of capacity allocation on the existence of balanced partition. Network (a) has one balanced partition, but network (b) has no balanced partition.

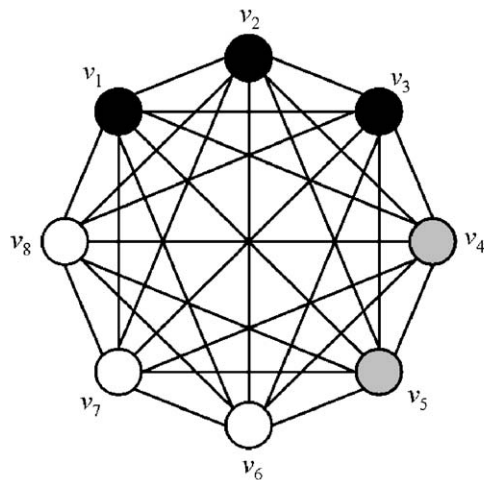

Fig. 2. The eight-node complete graph. Dark nodes and grey nodes are generators. White nodes are loads. A separation contingency requires the dark nodes and grey nodes to be separated from each other.

$C = (\{v_1\}, \{v_2, v_3\})$ with $m = 2$, $n_{g_1} = 1$, $n_{g_2} = 2$ in Fig. 1. We have $A_\exists(3, C, 0) = \{\{1, 1, 1, -1, -2\}\}$, $A_\exists(3, C, 1) = \{\{1, 1, 1, -1, -2\}, \{1, 1, 1, -2, -1\}\}$.The problem we are interested in now can be stated as:

> What is the BP probability of a network with growing capacity, i.e., the asymptotic value of $|A_\exists(P, C, d)|/|A(P)|$ when P goes to infinity?

The term $|A_\exists(P, C, d)|/|A(P)|$ represents the percentage of possible capacity allocations in such a network that can survive a separation contingency C.

3. A necessary condition for the existence of BP

We first use two numerical examples to motivate, and then theoretically discover a necessary condition for the existence of BP. The first example is a small network. Consider an eight-node complete graph with five generators and three loads (Fig. 2). Suppose the separation contingency requires nodes {1,2,3} to separate from nodes {4,5}. For a given network capacity P, we randomly generate 1000 capacity allocations to estimate the BP probability after the contingency. Let the tolerance be a special function of the network capacity P, $T(P) = \lfloor 0.5P^b \rfloor, b = 0.1, 0.2, \ldots, 1.0$, where $\lfloor \cdot \rfloor$ is the floor function since we consider only integer weights. We show the numerical results in Fig. 3. We can see that for $b < 1.0$, the BP probability decreases when P increases. Only when $b = 1.0$, the BP probability stays at 1 for all P.

The second example is a large network. Among the 14 099 nodes in the North America power infrastructure, a total of 1633 nodes are power plants; a total of 2179 nodes are distributing substations [8]. So we consider a complete graph with 1633 generators and 2179 loads. Suppose the separation contingency requires 1000 generators to separate from the rest 633 generators. Similar to the first example, we can estimate the BP probability after the contingency by randomly generating 1000 capacity allocations. However, as aforementioned it is computationally infeasible to rigorously check whether there is a BP for each capacity allocation. So, we adopt the method used in practice: we randomly generate 1000 partitions for each capacity allocation. If none of these partitions are BP, we estimate that capacity allocation is not BP. The justification for this estimation method is that in large-scale practical networks we only have time to test a small number of partitions before it is too late to take an active partition operation. Let $T(P) = \lfloor 0.05P^b \rfloor$, $b = 0.1, 0.2, \ldots, 1.0$. We show the numerical results in Fig. 4, which is similar to Fig. 3.[4]

[4]The only difference is that for $b = 0.8$ and 0.9, the BP probability in Fig. 4 is still 1.0. As we will show in the following theoretical analysis, this is because we have not tested large enough P in Fig. 4.

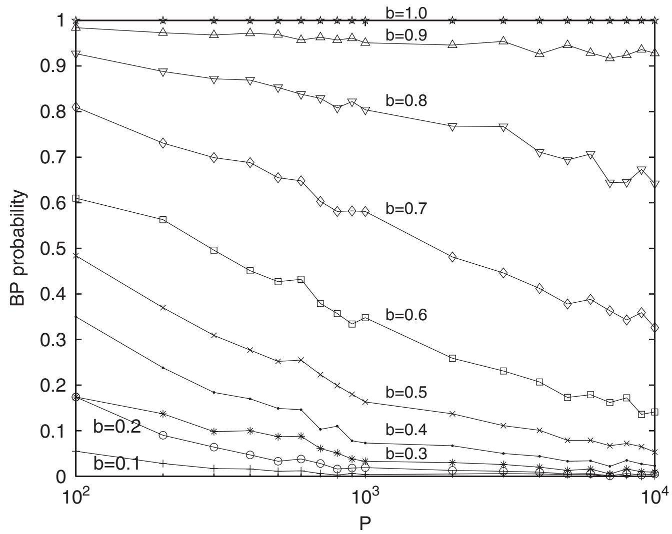

Fig. 3. The effect of the increasing speed of the tolerance $T(P)$ on the BP probability in the small network. $T(P) = \lfloor 0.5P^b \rfloor$. Unless the tolerance increases no slower than linearly, i.e., $b \geqslant 1.0$, the BP probability converges to zero. All points are averaged over 1000 replications.

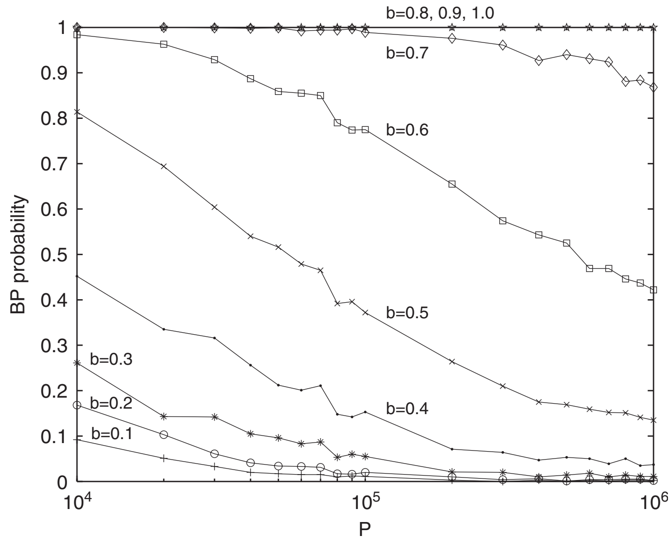

Fig. 4. The effect of the increasing speed of the tolerance $T(P)$ on the BP probability in the large network. $T(P) = \lfloor 0.05P^b \rfloor$. All points are averaged over 1000 replications.

The above two examples motivate us that unless the tolerance increases no less than a linear function of the network capacity, the BP probability decreases to zero. This can be shown theoretically. To be specific, we will show that

Theorem 1 (*A necessary condition for the existence of BP*). *When the tolerance increases slower than a linear function of the growing network capacity, i.e.,* $\lim_{P\to\infty} T(P)/P = 0$, *the network resilience and the BP probability will decrease to zero, i.e.,*

$$\lim_{P\to\infty} \frac{|A_{\exists}(P, C, T(P))|}{|A(P)|} = 0. \tag{2}$$

Note that this necessary condition holds for arbitrary networks.

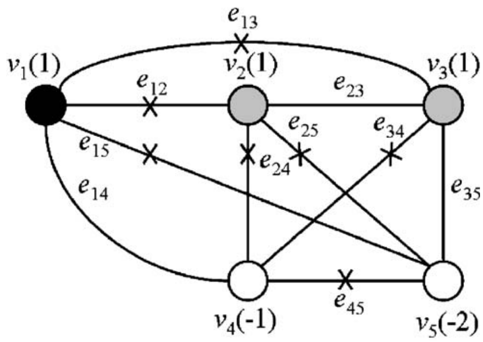

Fig. 5. The complete graph for network (a) in Fig. 1.

We prove this necessary condition in the rest of this section. For an arbitrary network, we introduce a globally connected network, whose node-weighted graph model is a complete graph, where there are direct edges between any two nodes and all edges have no transmission limit. It can be derived that the BP probability of a complete graph supplies an upper bound for that of a graph with arbitrary topology. To see this, for graph $G(V, E, W)$, we introduce a complete graph $G^c(V, E^c, W)$. The only difference between G and G^c is that there is an edge between any pair of nodes in G^c, but not necessarily in G. If a capacity allocation W in G has a BP E_c, by removing all the edges that are in G^c but not in G (denoted by $E^c \backslash E$), the set of edges $E_c \cup (E^c \backslash E)$ is a BP for capacity allocation W in G^c. For example, for network (a) in Fig. 1 we introduce a globally connected network (c) in Fig. 5. The set of all the edges that are in network (c) but not in network (a) is $E^c \backslash E = \{e_{13}, e_{15}, e_{25}, e_{34}\}$. Network (a) has a BP $E_c = \{e_{12}, e_{24}, e_{45}\}$. Then $E_c \cup (E^c \backslash E) = \{e_{12}, e_{13}, e_{15}, e_{24}, e_{25}, e_{34}, e_{45}\}$ is a BP for network (c).

We will show Eq. (2) holds for complete graphs. Then it is straightforward to show that Eq. (2) holds for arbitrary networks. Our argument is based on counting methods that are familiar in combinatorial mathematics. Recall that $\sum_{u=1}^{m} n_{g_u} = n_g$ and $A(P)$ represents all possible capacity allocations, which consist of the weight allocations among the generators and among the loads. So we have

$$|A(P)| = \binom{P-1}{\sum_{u=1}^{m} n_{g_u} - 1} \binom{P-1}{n_l - 1}, \tag{3}$$

where $\binom{a}{b}$ denotes the number of choices of b elements from a distinguished ones,

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} = \frac{a(a-1)\cdots(a-b+1)}{b!}, \tag{4}$$

where $b!$ is the factorial of b; $\binom{a}{b} = 0$, if $a < b$. Here we have used the fact that the number of ways to assign a units of resource (generation or load) to b nodes is $\binom{a-1}{b-1}$.[5] To simplify the notation, we introduce the relation "$\asymp$" which quantifies the rough idea of "two functions increase at the same speed to infinity". Let k be an integer variable which tends to infinity and let x be a continuous variable tending to some limit. Also let $\phi(k)$ or $\phi(x)$ be a positive function and $f(k)$ or $f(x)$ be any function. Then Ref. [25] defines $f \asymp \phi$ to mean $K_1\phi < f < K_2\phi$ for some positive constants K_1 and K_2. It is trivial to show that

$$\binom{a}{b} \asymp a^b, \quad \binom{a-1}{b-1} \asymp a^{b-1}. \tag{5}$$

Then

$$|A(P)| \asymp P^{\sum_{u=1}^{m} n_{g_u} + n_l - 2}. \tag{6}$$

[5]We regard each unit of resource as a "ball" and put them in a line that should be divided into b groups (each group is corresponding to a node). Balls are indistinguishable. But nodes are distinguishable from each other. So, a grouping is an allocation, vice versa. And a grouping can be regarded as inserting $b-1$ blocks to the $a-1$ interspaces between the a balls. This is how $\binom{a-1}{b-1}$ comes.

Let $C' = (V_{g_1}, \bigcup_{i=2}^{m} V_{g_i})$ be a separation contingency in which only n_{g_1} generators V_{g_1} are forced to be separated from the others, and the rest of the generators need not be further separated from each other. Since the load and generation is exactly balanced before the contingency, we have the observation on the relation between C and C' that

$$A_{\exists}(P, C, T(P)) \subset A_{\exists}(P, C', T(P)),$$

because every capacity allocation such that m islands (with V_{g_u}, $u = 1, \ldots, m$, as generators) are balanced within tolerance $T(P)$ can be regarded as an capacity allocation such that two islands (with V_{g_1} and $\bigcup_{u=2}^{m} V_{g_u}$ as generators) are balanced within tolerance $T(P)$. Thus we have

$$|A_{\exists}(P, C, T(P))| \leqslant |A_{\exists}(P, C', T(P))|.$$

Below is an estimation of $|A_{\exists}(P, C', T(P))|$, which is the number of capacity allocations that lead to BPs separating generators in the set V_{g_1} from the other generators. When there are two islands after the partition, the tolerances in both islands must be equal, i.e., if in one island the generation is d units more than the load, then in the other island, the generation is d units less than the load. Let t_d be the number of capacity allocations that the differences between generation and load are exactly d in both islands. Then we have

$$t_d \leqslant \sum_{P_{g_1}=n_{g_1}}^{P-n_{g_2}} \sum_{n_{l_1}=1}^{n_l-1} \binom{P_{g_1}-1}{n_{g_1}-1} \binom{P-P_{g_1}-1}{\sum_{u=2}^{m} n_{g_u}-1}$$
$$\times \left(\binom{P_{g_1}-d-1}{n_{l_1}-1} \binom{P-P_{g_1}+d-1}{n_l-n_{l_1}-1} + \binom{P_{g_1}+d-1}{n_{l_1}-1} \binom{P-P_{g_1}-d-1}{n_l-n_{l_1}-1} \right) \tag{7}$$
$$\leqslant \sum_{P_{g_1}=n_{g_1}}^{P-n_{g_2}} \sum_{n_{l_1}=1}^{n_l-1} 2 \binom{P-1}{n_{g_1}-1} \binom{P-1}{\sum_{u=2}^{m} n_{g_u}-1} \binom{P-1}{n_{l_1}-1} \binom{P-1}{n_l-n_{l_1}-1} \tag{8}$$
$$\asymp P^{\sum_{u=1}^{m} n_{g_u}+n_l-3}. \tag{9}$$

The inequality (7) is obtained by specifying the weight allocated among the generators, P_{g_1}, and the number of nodes in the first island, n_{l_1}. We can see from above that there are two cases when the differences between generation and load in both islands are exactly d: either the generation in island 1 is d units more than the load (then the generation in island 2 is d units less than the load) or d units less than the load (then the generation in island 2 is d units more than the load). This is how the last summation in Eq. (7) comes. The inequality in Eq. (8) is due to the fact that $\binom{a-i}{b} \leqslant \binom{a}{b}$ for $i = 0, 1, 2, \ldots$. Using Eq. (5), we get Eq. (9) from Eq. (8).

Furthermore, we have

$$|A_{\exists}(P, C', T(P))| = \sum_{d=0}^{T(P)} t_d$$
$$\leqslant \sum_{d=0}^{T(P)} K P^{\sum_{u=1}^{m} n_{g_u}+n_l-3} \tag{10}$$
$$= (T(P)+1) K P^{\sum_{u=1}^{m} n_{g_u}+n_l-3}, \tag{11}$$

where K in Eq. (10) is a positive constant. Combining Eqs. (6) and (11), we have

$$\lim_{P\to\infty} \frac{|A_{\exists}(P, C, T(P))|}{|A(P)|} \leqslant \lim_{P\to\infty} \frac{|A_{\exists}(P, C', T(P))|}{|A(P)|}$$
$$\leqslant \lim_{P\to\infty} \frac{(T(P)+1) K P^{\sum_{u=1}^{m} n_{g_u}+n_l-3}}{|A(P)|}$$
$$\asymp \lim_{P\to\infty} \frac{(T(P)+1) P^{\sum_{u=1}^{m} n_{g_u}+n_l-3}}{P^{\sum_{u=1}^{m} n_{g_u}+n_l-2}}$$
$$= \lim_{P\to\infty} \frac{(T(P)+1)}{P} = 0,$$

where the last equality is due to the assumption that $\lim_{P\to\infty} T(P)/P = 0$. So far we have shown that Eq. (2) holds for all complete graphs. As mentioned above, since the BP probability in a complete graph supplies an upper bound for that in an arbitrary graph, it is straightforward to show that Eq. (2) holds for arbitrary networks. This completes the proof.

4. The critical value of linear increasing speed

In this section, we explore further on the linear increasing speed and will show that there is a critical value of the increasing speed to avoid BP probability being less than 1, which will lead to positive system collapse probability. Before presenting our general analysis, let us use the numerical results on the two networks in Section 3 for motivation. First let us consider the eight-node globally connected network. (The node-weighted graph is shown in Fig. 2.) There are five generators and three loads (i.e., $n_l = 3$). After a separation contingency, three generators are forced to be separated from the other two, i.e., $n_{g_1} = 3$, $n_{g_2} = 2$. What is the asymptotic value of the BP probability when the tolerance increases with different linear speed α, i.e., $T(P) = \lfloor \alpha P \rfloor$? The numerical results are shown in Fig. 6. When $\alpha < 0.43$, the asymptotical value of the BP probability is less than 1; for greater α, this asymptotical value is 1. So the value 0.43 here is a critical value.

Second, we consider the globally connected large-scale network, which is as large as the North American power grid. We also test the asymptotic value of the BP probability when $T(P) = \lfloor \alpha P \rfloor$, and show the results in Fig. 7. The critical value in this network is 0.003.

Now we focus on the theoretical study of the following questions. *Generally is there a critical increasing speed for α to make the asymptotical value of the BP probability be* 1? *If yes*, *what is the critical value*?

Since the BP probability in complete graph supplies an upper bound for that in an arbitrary network, we focus on complete graph in the following discussion. Recall the contingencies $C = (V_{g_1}, \ldots, V_{g_m})$ and $C' = (V_{g_1}, \bigcup_{i=2}^{m} V_{g_i})$. Also please note that $A_\exists(P, C, T(P)) \subset A_\exists(P, C', T(P))$, because the network is originally balanced. So we only need to consider the case that the network is partitioned into two islands, i.e., C'. We have

$$|A_\exists(P, C', \lfloor \alpha P \rfloor)| = \sum_{d=0}^{\lfloor \alpha P \rfloor} |A_\exists(P, C', d)|$$
$$\approx (2\lfloor \alpha P \rfloor + 1)|A_\exists(P, C', 0)|$$

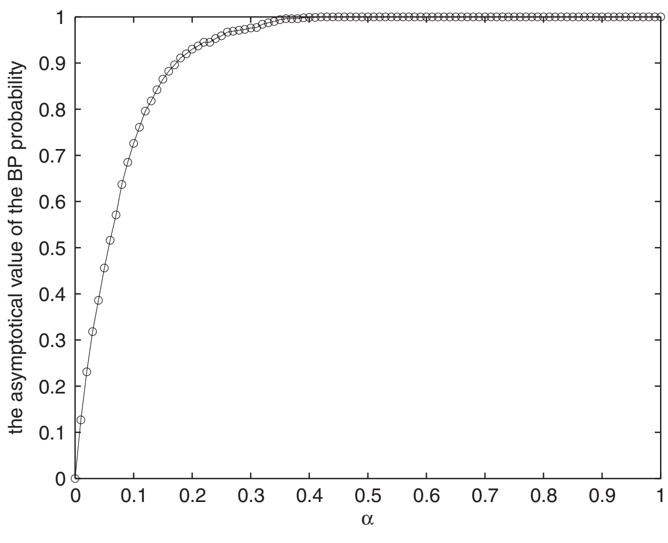

Fig. 6. Different linearly increasing speed of the tolerance affects the BP probability of the network in Fig. 2. $P = 10\,000$. All data are averages over 1000 replications.

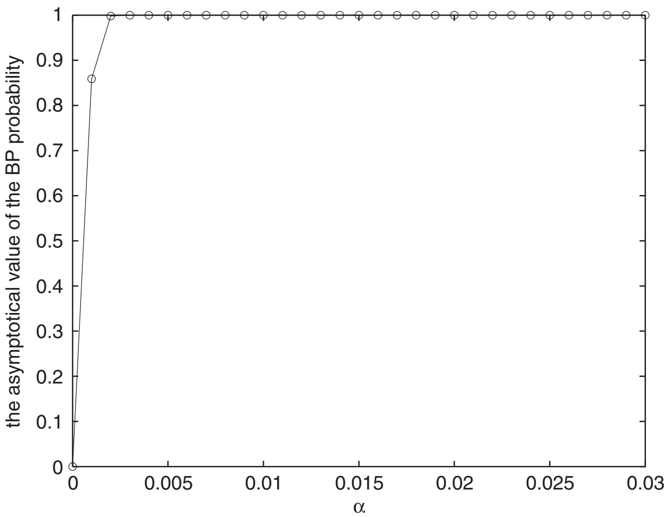

Fig. 7. Different linearly increasing speed of the tolerance affects the BP probability of the globally connected network, which is as large as the North American power grid. $P = 10^{10}$. All data are averages over 1000 replications.

$$\begin{aligned}
&\leqslant (2\lfloor \alpha P \rfloor + 1) \sum_{P_{g_1}=n_{g_1}}^{P-n_{g_2}} \binom{P_{g_1}-1}{n_{g_1}-1}\binom{P-P_{g_1}-1}{n_{g_2}-1} \\
&\quad \times \max_{P_{g_1}=n_{g_1},\dots P-n_{g_2}} \sum_{n_{l_1}=1}^{n_l-1} \binom{n_l}{n_{l_1}}\binom{P_{g_1}-1}{n_{l_1}-1}\binom{P-P_{g_1}-1}{n_l-n_{l_1}-1} \\
&\leqslant (2\lfloor \alpha P \rfloor + 1) \sum_{P_{g_1}=n_{g_1}}^{P-n_{g_2}} \binom{P_{g_1}-1}{n_{g_1}-1}\binom{P-P_{g_1}-1}{n_{g_2}-1} \sum_{n_{l_1}=1}^{n_l-1} \binom{n_l}{n_{l_1}} \frac{P^{n_l-2}}{(n_{l_1}-1)!(n_l-n_{l_1}-1)!}. \qquad (12)
\end{aligned}$$

The second line in Eq. (12) is based on the fact that if the generation in one island is d units greater than the load, then the generation in the other island is d units less than the load. The third line is obtained from the fact that in a BP the load of some (say n_{l_1}) nodes should be equal to the generation in island 1 (P_{g_1}). The last line is based on the fact that $\binom{P_{g_1}}{n_{l_1}-1} \leqslant P^{n_{l_1}-1}$ and $\binom{P-P_{g_1}-1}{n_l-n_{l_1}-1} \leqslant P^{n_l-n_{l_1}-1}$. We also have

$$|A(P)| = \sum_{P_{g_1}=n_{g_1}}^{P-n_{g_2}} \binom{P_{g_1}-1}{n_{g_1}-1}\binom{P-P_{g_1}-1}{n_{g_2}-1}\binom{P-1}{n_l-1}. \qquad (13)$$

The last term in Eq. (13) $\binom{P-1}{n_l-1}$ is the (number of capacity allocations over all the loads. Using Eqs. (12) and (13) we have

$$\begin{aligned}
\lim_{P\to\infty} \frac{|A_\exists(P, C', \lfloor \alpha P \rfloor)|}{|A(P)|} &\leqslant \lim_{P\to\infty} (2\lfloor \alpha P \rfloor + 1) \sum_{n_{l_1}=1}^{n_l-1} \frac{\binom{n_l}{n_{l_1}} P^{n_l-2}}{(n_{l_1}-1)!(n_l-n_{l_1}-1)!\binom{P-1}{n_l-1}} \\
&\leqslant 2\alpha \sum_{n_{l_1}=1}^{n_l-1} \frac{\binom{n_l}{n_{l_1}}(n_l-1)!}{(n_{l_1}-1)!(n_l-n_{l_1}-1)!}. \qquad (14)
\end{aligned}$$

The last line in Eq. (14) is based on the fact that $\binom{P-1}{n_l-1} \asymp P^{n_l-1}$. Only when α is greater than a critical value

$$\alpha_c = \frac{1}{2\sum_{n_{l_1}=1}^{n_l-1} \binom{n_l}{n_{l_1}} (n_l-1)!/(n_{l_1}-1)!(n_l-n_{l_1}-1)!}, \qquad (15)$$

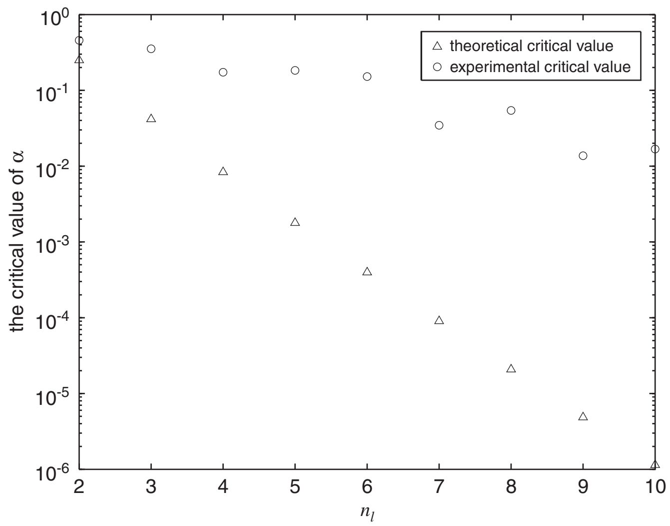

Fig. 8. Critical values of α for different n_l. $n_{g_1} = 3, n_{g_2} = 2, P = 10\,000$. All data are averages over 100 replications.

the asymptotical value of the BP probability may be 1. It is obvious that $\alpha_c > 0$ for finite n_l, which implies that the linearly increasing speed of the tolerance should be large enough, otherwise the BP probability still converges to zero.

We show the numerical results in Fig. 8. The critical values obtained from experiments are always greater than the value obtained from Eq. (15), which supports that Eq. (15) does supply a lower bound of the least linearly increasing speed of the tolerance such that the BP probability converges to 1. We also see that the critical value in Eq. (15) decreases as n_l increases. This implies that increasing the number of loads helps to reduce the increasing speed of the tolerance. This is because when there are more loads the capacity is further separated into smaller units. And there is a larger chance to find some loads that can balance the generations in an island.

In practice usually when the number of loads increases, the number of generators also increases. To test how the critical value changes in this case, we adopt the following experiment. When increasing the total number of nodes in the network, we fix the portion of the loads (e.g., 0.6, which is the portion of the loads in the North American power grid). We also consider the separation contingency that requires to separate a fixed portion of the generators from the rest of the generators. As shown in Fig. 9, except the first point (the smallest case), the critical values variate among a fixed value. This means when the network scale increases the number of generators and loads increase at the same speed, the critical value does not change much.

5. Implications

The necessary condition developed in Section 3 has some interesting implications on network security and resilience.

The impossibilities when improving network resilience: In a network without a global coordinator (who has accurate global information and has the absolute power to decide the capacity allocation), if we do not improve the investment on tolerance facilities at least linearly fast, then it is impossible to keep the network resilience at a high level as the network capacity increases.

Justify the common sense to linearly increasing the tolerance of the network: This is a common sense in engineering. For example, in a power system the spinning reserve is required to be no smaller than 3–5% of the maximal total power consumption. In petroleum industries, the spare capacity is about 5% of the total generation. Eq. (2) justifies this common sense in a quantitative way.

The increasing difficulty for secure generation dispatching in a network with growing capacity: When the network capacity increases but the tolerance is increased slower than linearly, more and more capacity allocations will not be BP under a separation contingency. This will run us into an impossible situation for

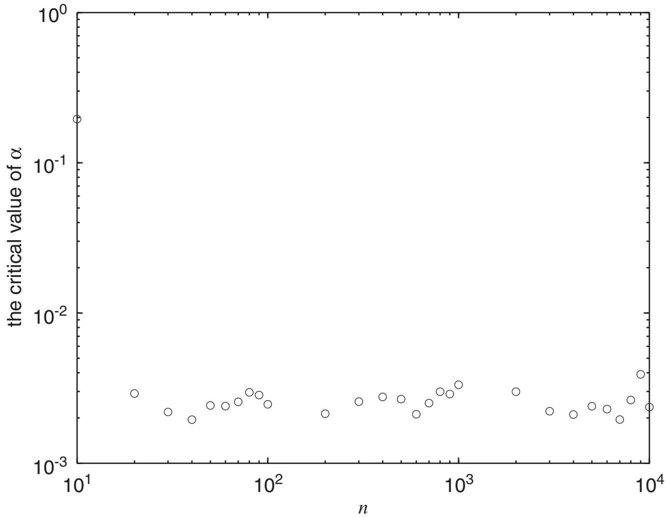

Fig. 9. Critical values of α for different n. $n_l/n = 0.6, n_{g_1}/n_g = 0.6, P = 10^6 n$. All data are averages over 1000 replications.

large-scale networks because it is extremely difficult (if not impossible) to have a general "global coordinator" to ensure such a successful yet restrictive way of precisely controlling the resource generation and consumption of all nodes. Arbitrary and haphazard decisions are not acceptable as they will very likely lead the network to lose BP property.

6. Conclusion

In this paper, a necessary condition for the existence of a balanced partition for a network with growing capacity is obtained: unless the tolerance in each island after the partition increases at least linearly as the capacity grows, the network will eventually collapse with probability 1 when the capacity increases to infinity. It should be noted that even when this condition is satisfied, the network may still collapse. This necessary condition justifies a common sense in engineering that the tolerance (e.g., spare capacity) should increase proportional to the total resource consumption. Some implications on impossibilities and difficulties of the network security and resilience are presented.

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