# Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Summer School

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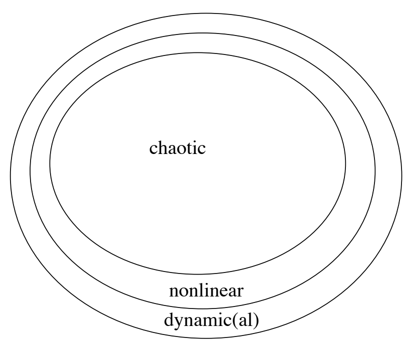
http://ayresriverblog.com

## Chaos

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…



## Chaos

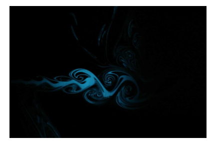
Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and "well-understood"

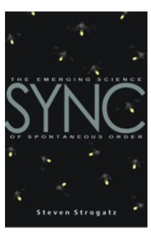
## Where nonlinear dynamics turns up

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers



## Where nonlinear dynamics turns up

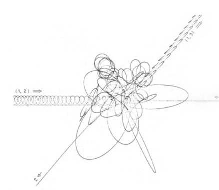
- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies




  - and lots of other electronic, chemical, & biological systems

## Where nonlinear dynamics turns up

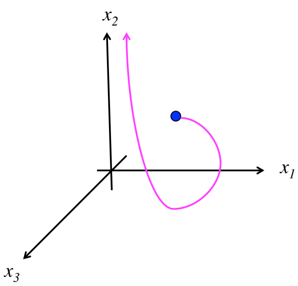
- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….



Hut & Bahcall *Ap J.* ***268***:*319*

- Protein folding
- Population biology
- And many, many other fields (including yours)

## The basic representation in nonlinear dynamics: the state space



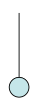

- State variables $x_i$
- Initial condition
- Trajectory

## The damped pendulum

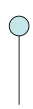
- State variables?
- Initial condition?
- Trajectory?

## Fixed points

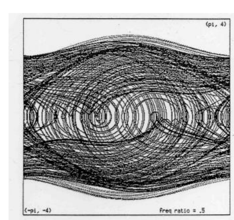
- Stable: perturbations shrink



- Unstable: perturbations grow

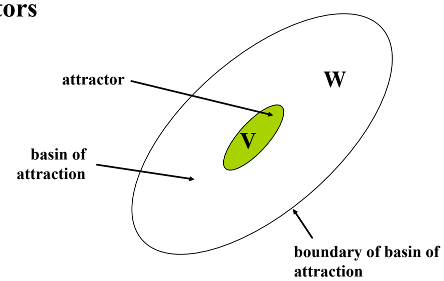


## The *driven* damped pendulum…



…can be chaotic (for some drive parameters)

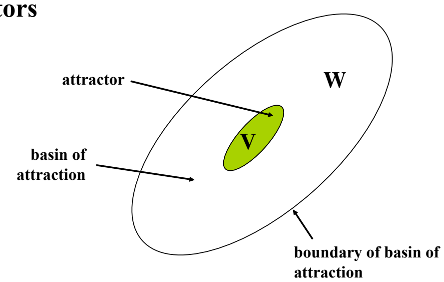
## Attractors




## A useful metaphor:

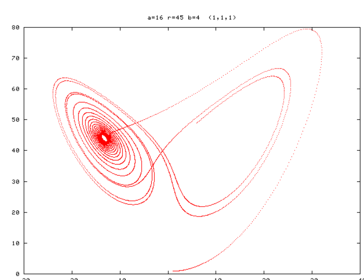
(image from wikipedia)

## Attractors




- Attractors exist only in dissipative systems!
- Dissipation ⟺ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

## The archetype of chaos: Lorenz



## Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

*Massachusetts Institute of Technology*

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

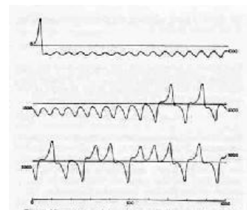


Fig. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

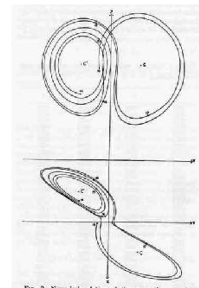
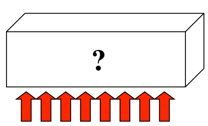


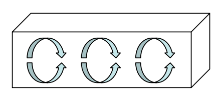
Fig. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

- Equations:

$$x' = a(y - x)$$
$$y' = rx - y - xz$$
$$z' = xy - bz$$

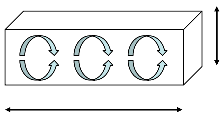


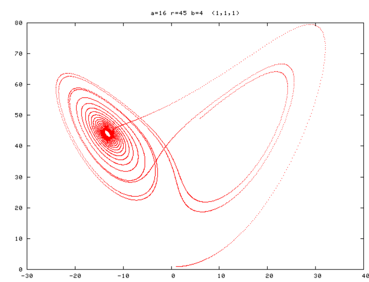

(first three terms of a Fourier expansion of the Navier-Stokes eqns)



- State variables:
  - *x* convective intensity
  - *y* temperature
  - *z* deviation from linearity in the vertical convection profile

- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to $\Delta T$
  - *b* aspect ratio of the fluid sheet

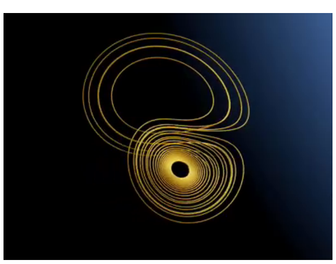




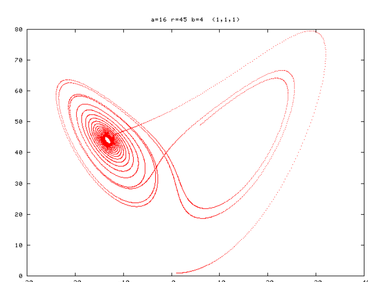

$x' = 16(y - x)$

$y' = 45x - y - xz$

$z' = xy - 4z$





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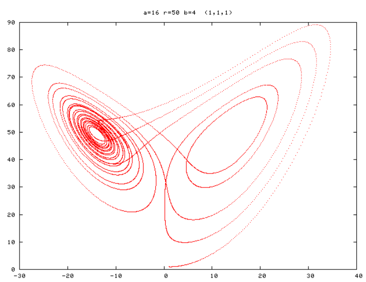
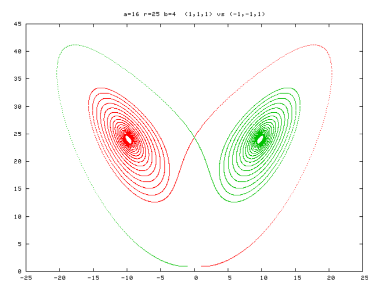



*x' = 16(y - x)*

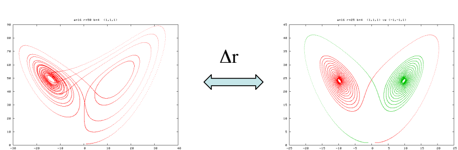
*y' =* **50***x - y - xz*

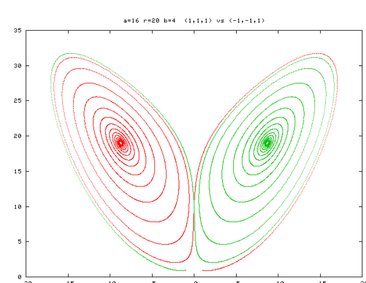
*z' = xy - 4z*

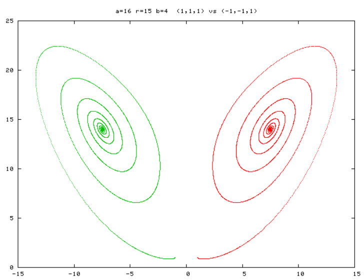



**Recall: bifurcations**

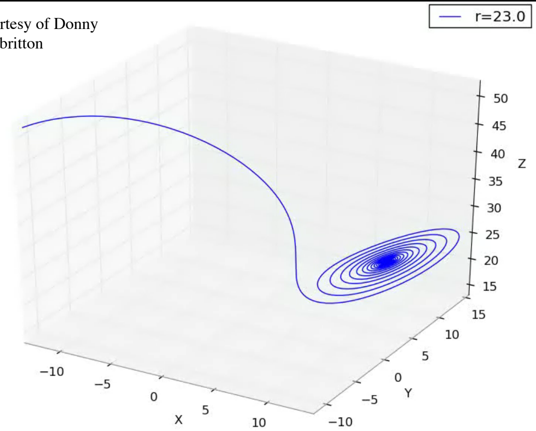
Qualitative changes in the dynamics—i.e., topological changes in the attractor—caused by changes in *parameters:*



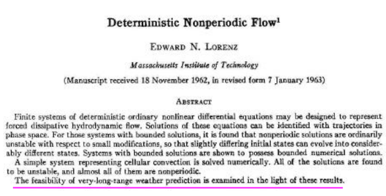






Courtesy of Donny Warbritton




## Before we leave Lorenz…



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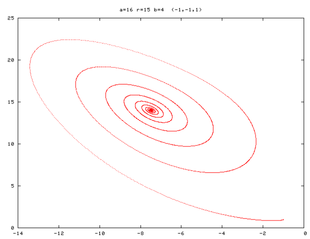
## Attractors

Four types:

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

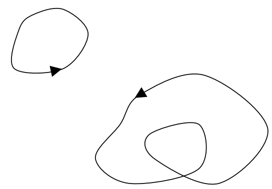
## Attractors

• Fixed point




## Attractors

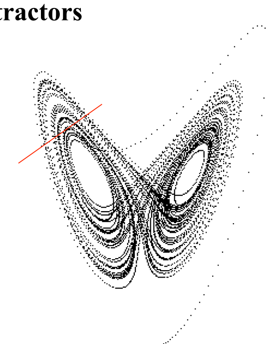
• Limit cycle



## Attractors

• Quasi-periodic orbit…

## “Strange” or chaotic attractors



• *often* fractal

• covered densely by trajectories

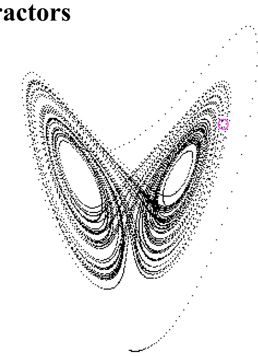
• exponential divergence of neighboring trajectories…

## Fractals and chaos

The connection: *many (most)* chaotic systems have fractal state-space structure.

But **not** "*all*."

## "Strange" or chaotic attractors

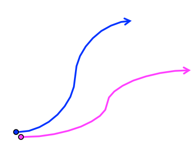


- *often* fractal
- covered densely by trajectories
- exponential divergence of neighboring trajectories…

Courtesy of Mike Neuder

## Lyapunov exponents and chaos

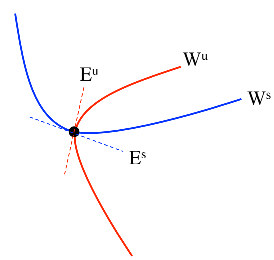
- distance between forward images of two nearby points grows as $e^{\lambda t}$ in the limit, as t → ∞



## Lyapunov exponents: some details

- negative $\lambda_i$ compress state space; positive $\lambda_i$ stretch it
- there are as many $\lambda_i$ as there are state-space dimensions
- long-term average in definition; biggest one ($\lambda_1$) dominates as t → ∞
- $\lambda_i$ are same for all ICs in one basin
- nonlinear analogs of eigenvalues: $n$ $\lambda$ in an $n$-dimensional system
- they parametrize growth/shrinkage along the unstable and stable manifolds $W^u$ and $W^s$
- *positive $\lambda_1$ is a signature of chaos*

## $\lambda_i$ and the un/stable manifolds ($W^u$ and $W^s$)




## Attractors

Four types:

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- quasiperiodic orbits
- chaotic attractors

A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction (plus the basin boundaries) *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc. (which is **not** the case in linear systems!)


(image from wikipedia)

## Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter
- Lyapunov exponent

## Conditions for chaos

**(in continuous-time systems)**

**Necessary:**

- Nonlinear
- At least three state-space dimensions

**Necessary and sufficient:**

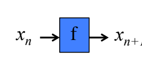
- Cannot be solved in closed form ("nonintegrable," in Hamiltonian parlance)

---

- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: *difference* equation
- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: *differential* equations

## What do those beasts look like and how do we deal with them?

Difference equations:




- e.g., $x_{n+1} = cos(x_n)$
- given state $x$ at time $n$, tells you state at time $n+1$
- solve by iterating

## A canonical difference equation:

The logistic map: $x_{n+1} = R\, x_n\, (1 - x_n)$

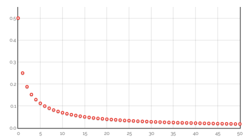
If $R=1$ and $x_0=0.5$, $x_1=1(0.5)(1\text{-}0.5)=0.25$

$x_2=1(0.25)(1\text{-}0.25)=0.1875$

…

Eventually, $x$ settles down at $0$:

Much more on this in Joshua Garland's lecture this afternoon



## What do those beasts look like and how do we deal with them?

Difference equations:

• e.g., $x_{n+1} = R\, x_n\, (1 - x_n)$  $x_n \rightarrow$ [f] $\rightarrow x_{n+1}$

• given state $x$ at time $n$, tells you state at time $n+1$

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Differential equations:

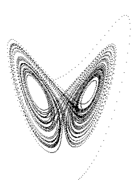
$x \rightarrow$ [f] $\rightarrow x'$

• e.g., $d^2x(t)/dt^2 = -x(t)$

• given state $x$ at time $t$, tells you *the direction in which that state will evolve*

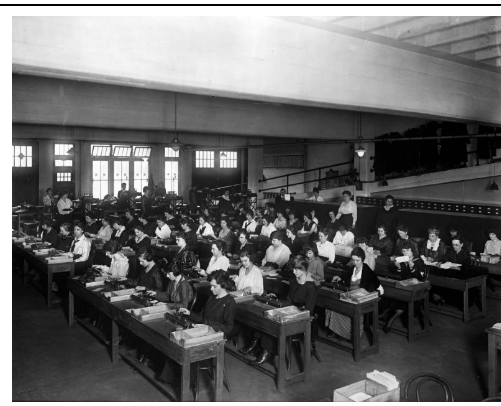
• solve with an ODE solver (see Liz's notes)

## The basic idea behind (one family of) ODE solvers:

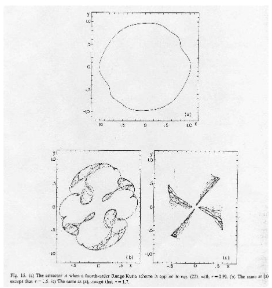


- Follow the slope that the ODE gives you
- Simplest: Euler
- More creative: legion…e.g., `ode45`, `ode34`

All very well if you have a nice modern computer…

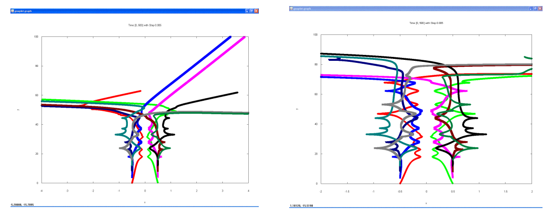


www.computerhistory.org



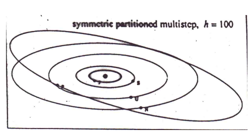
Different timestep

Lorenz, *Physica D* **35:**229

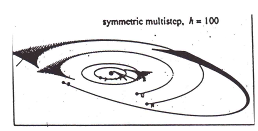


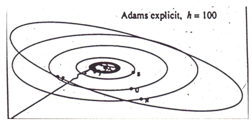
Different arithmetic

N. Ross Ph.D. thesis, Ucolorado, 2008




Different solver algorithm…






**Moral: numerical methods can run amok in "interesting" ways…**

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

**Moral: numerical methods can run amok in "interesting" ways…**

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?
  - *change the timestep* → But beware machine ε…
  - *change the method*
  - *change the arithmetic*

**Another important issue**

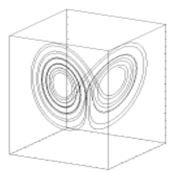
Many solvers, such as Matlab's `ode45`, are *adaptive*: they change the timestep and/or the method itself, on the fly, in order to correctly simulate the dynamics.

(The algorithms for this are interesting; we can talk about them offline.)

That means that the points that are output by tools like `ode45` are *not evenly spaced in time*. That can matter, depending on how you're using that solution…

## So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….



**…??!?**

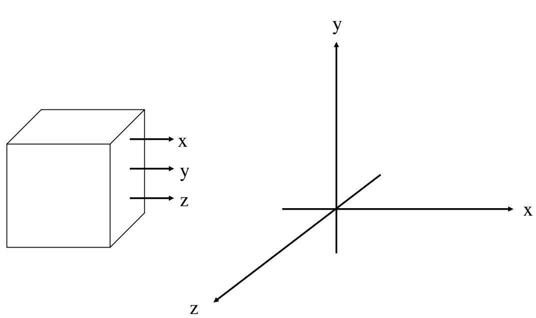
## Shadowing lemma

Every* noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

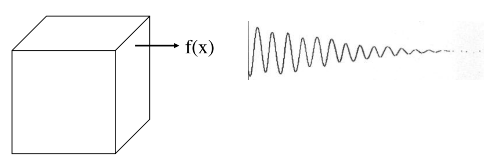
Important: this is for *state* noise, not *parameter* noise.

(*) Caveat: not if the noise bumps the trajectory out of the basin

## That state-space stuff is all very well, but it's a bit utopian.

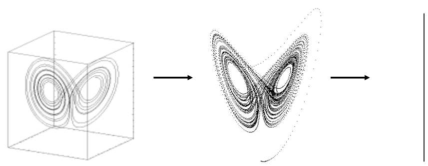



## Reality:




- Rarely do you even *know* what the state variables are.
- Even if you did, you might not be able to *measure* all of them.
- And even if you could, doing so might *change the dynamics*…

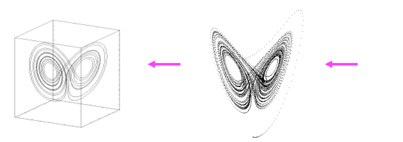
## What we're really asking for when we do time-series analysis on scalar data…



Undoing a projection =gack=

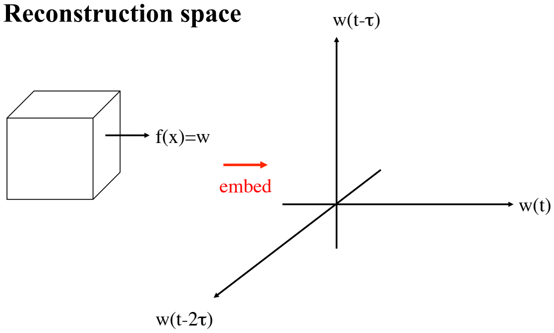
## Delay-coordinate embedding

"reinflate" that squashed data to get a *topologically identical* copy of the original thing.



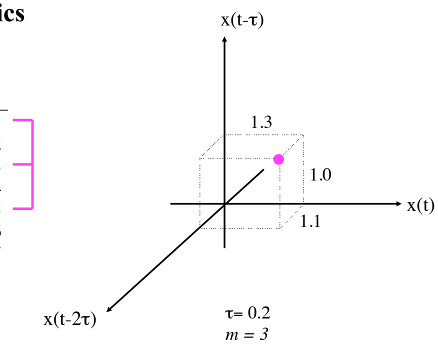
(not quite, but almost…)

## Reconstruction space



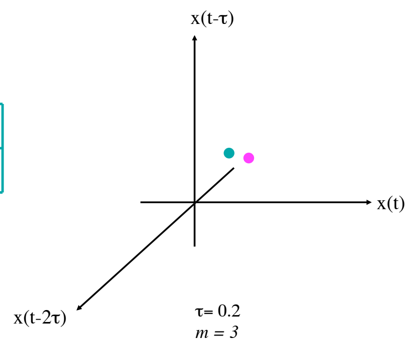

## Mechanics

| x | t |
|---|---|
| 1.3 | 0.1 |
| 1.2 | 0.2 |
| 1.0 | 0.3 |
| 0.8 | 0.4 |
| 1.1 | 0.5 |
| 1.4 | 0.6 |
| 1.6 | 0.7 |

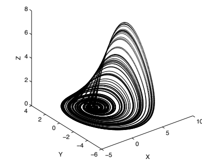



$\tau = 0.2$
$m = 3$

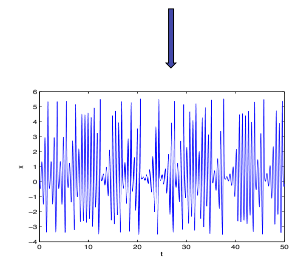
| x | t |
|---|---|
| 1.3 | 0.1 |
| 1.2 | 0.2 |
| 1.0 | 0.3 |
| 0.8 | 0.4 |
| 1.1 | 0.5 |
| 1.4 | 0.6 |
| 1.6 | 0.7 |



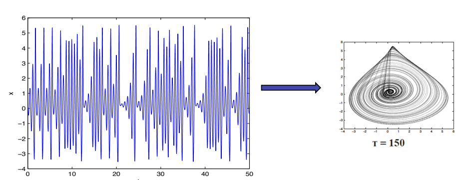

$\tau = 0.2$
$m = 3$



Measure just the *x* coordinate…

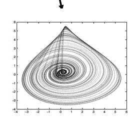


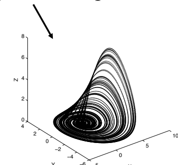
…and then embed:




## Takens* theorem

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (and thus have same topology as) the original state-space dynamics.



* Whitney, Mane, …

Note: the measured quantity must be a smooth, generic function of at least one state variable, and must be uniformly sampled in time.

*Diffeomorphic*: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape (topology)

- i.e., can deform one into the other…

www.shapeways.com/shops/henryseg

*Diffeomorphic*: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape (topology)
- i.e., can deform one into the other
- have same dynamical invariants (e.g., $\lambda$)

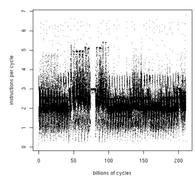
## Calculating $\lambda$ (& other invariants) from data

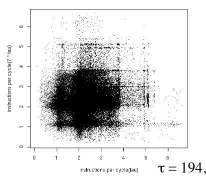
- The bible: H. Kantz & T. Schreiber, *Nonlinear Time Series Analysis*
- Associated software: TISEAN www.mpipks-dresden.mpg.de/~tisean
- A recent review article: EB & H. Kantz, "Nonlinear Time Series Analysis Revisited," *CHAOS* **25**:097610 (2015)

Much more on this in Joshua Garland's lecture tomorrow

## NLTSA* of computer performance dynamics

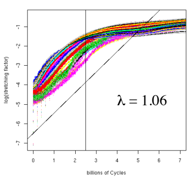
*** nonlinear time-series analysis**



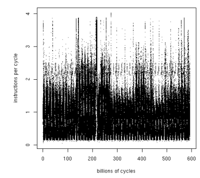


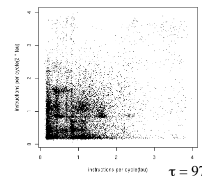
$\tau = 194$, $m = 10$

`bzip2` dynamics on an Intel Core2



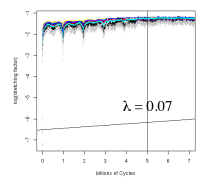

Mytkowicz *et al.*, *Chaos* **19**:033124



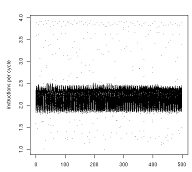


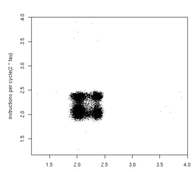
$\tau = 973$, $m = 12$

`bzip2` dynamics on an Intel Pentium 4



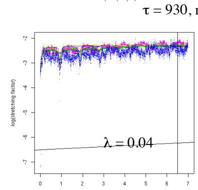

Mytkowicz *et al.*, *Chaos* **19**:033124





$\tau = 930$, $m = 8$

`povray` dynamics on an Intel Core2




Mytkowicz *et al.*, *Chaos* **19**:033124

**Caveat: need enough data…**

## If Δt is not uniform

Theorem (Takens): for $\tau > 0$ and $m > 2d$, reconstructed trajectory is diffeomorphic to the true trajectory

Conditions: evenly sampled in time, smooth generic measurement function

## Interspike interval embedding

idea: lots of systems generate spikes — hearts, nerves, etc.

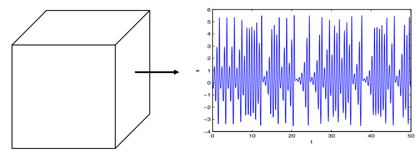
if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

*in which case the embedding theorems still hold.*

(with the Δ*t*s as state variables)

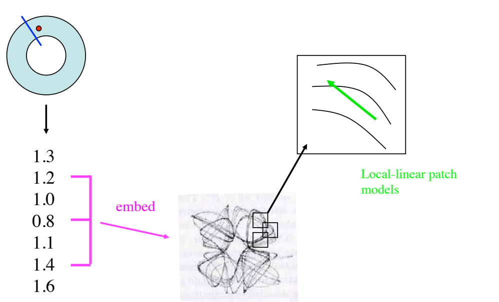
Sauer *Chaos* **5**:127

## Prediction



**…?**

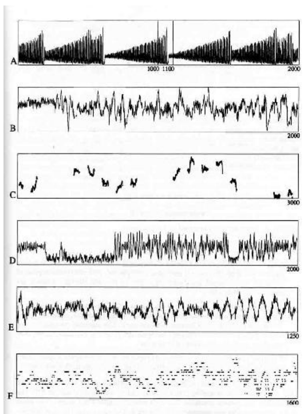
## Predicting the path of a roulette ball…




The Eudaemonic Pie
(or The Newtonian Casino)

## The Santa Fe competition

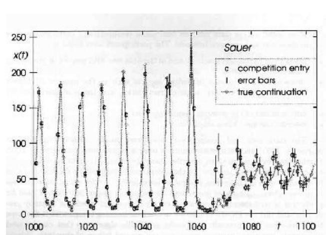
- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past*, Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)



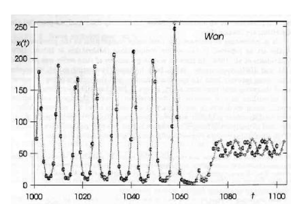

### The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

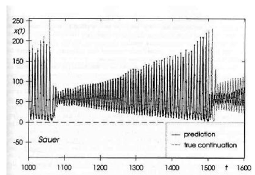
### Embedding + patch models: (Sauer)



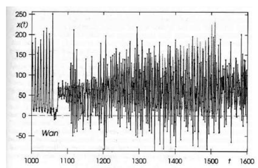

### Neural net: (Wan)



**Further out:**

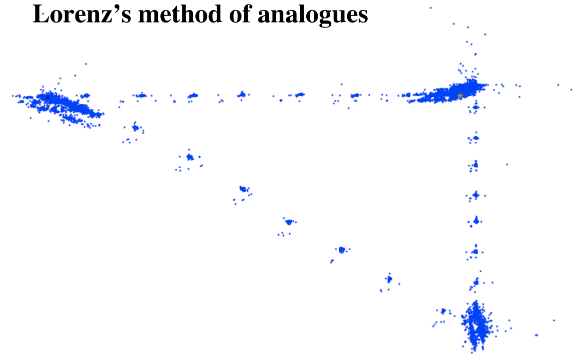


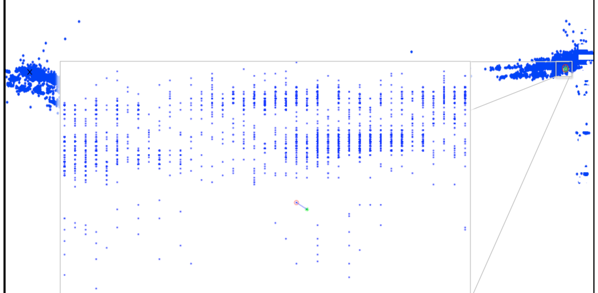

Sauer



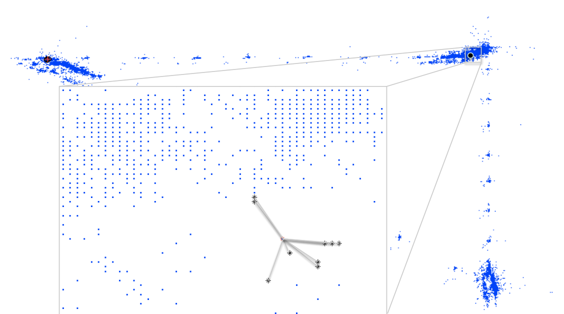

Wan

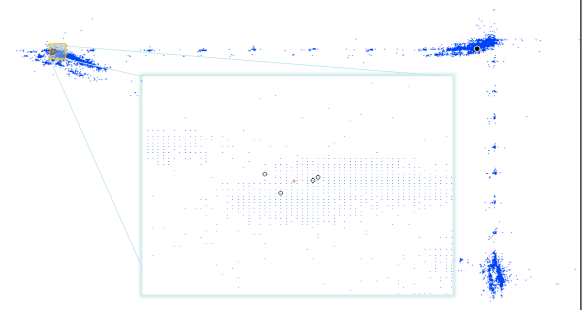
**An even simpler prediction method:**
**Lorenz's method of analogues**

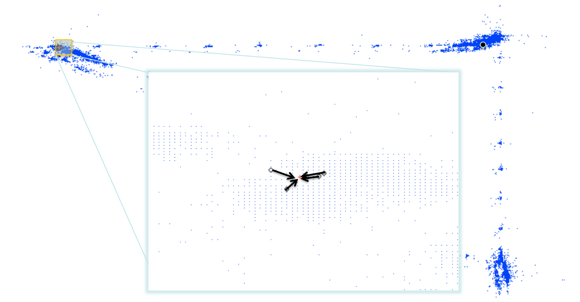




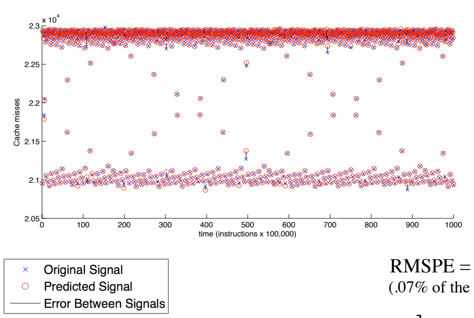
**A *k*-nearest neighbor modification of LMA**







## Using *k*LMA to predict computer dynamics




RMSPE = 11.566
(.07% of the average)

`column_major`
cache misses

Garland/Bradley *Intelligent Data Analysis* 2011

## Noise…

Linear filtering: a very very bad idea if the system is chaotic

**Don't bleach chaotic data**

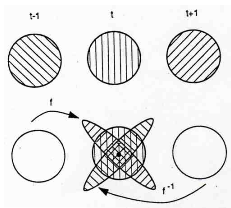
Chaos 3, 771 (1993); https://doi.org/10.1063/1.165936

James Theiler
- Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545;
- Santa Fe Institute, 1660 Old Pecos Trail, Santa Fe, New Mexico 87501

- Frequency is a bad discriminator of signal and noise in chaotic signals
- The math behind linear filters relies on superposition, which doesn't hold in nonlinear systems

Nonlinear alternatives:

- use the stable and unstable manifold structure on a chaotic attractor…

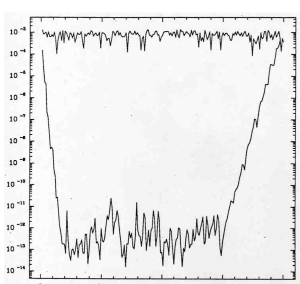



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

## Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
- ➡ noise reduction!
- Works best if manifolds are perpendicular, but requires only transversality

## Results:




Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

## Noise…

Linear filtering: a very very bad idea if the system is chaotic

Nonlinear alternatives:

- use the stable and unstable manifold geometry on a chaotic attractor
- what about using the *topology* of the attractor?

**Topology-based signal separation**

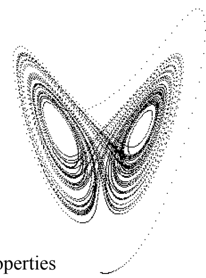
Chaos 14, 305 (2004); https://doi.org/10.1063/1.1705852

V. Robins

- Department of Applied Mathematics, Research School of Physical Sciences and Engineering, The Australian National University,

N. Rooney *and* E. Bradley

## Chaos and control

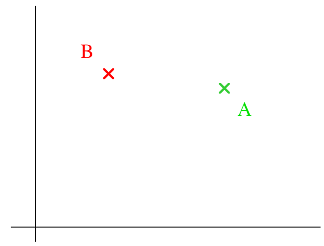


Can we use some of those cool properties and turn them to advantage?

e.g., dense attractor coverage

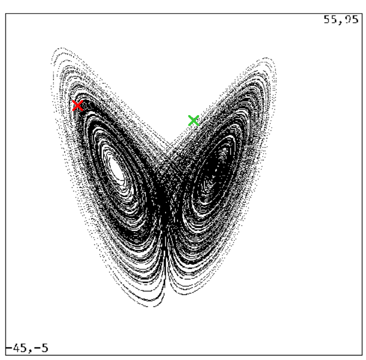
## Control:

getting from A to B, minimizing some cost functional



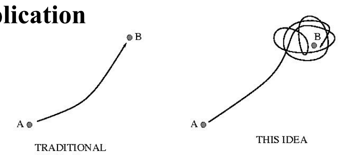

## Lorenz System:

denseness, reachability, and control

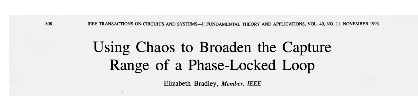



R = 50

## Denseness & reachability in a real engineering application




- can control position/volume/density of attractor — *within limits*
- possibly not reachable any other way
- not for time-critical applications (that “eventually”)

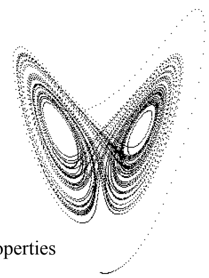


808 IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—I: FUNDAMENTAL THEORY AND APPLICATIONS, VOL. 40, NO. 11, NOVEMBER 1993

Using Chaos to Broaden the Capture Range of a Phase-Locked Loop

Elizabeth Bradley, *Member, IEEE*

## Chaos and control

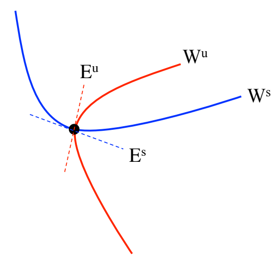


Can we use some of those cool properties and turn them to advantage?

e.g., dense attractor coverage

or the manifold structure?

## Remember those $\lambda_i$ & un/stable manifolds? They play a critical role here…

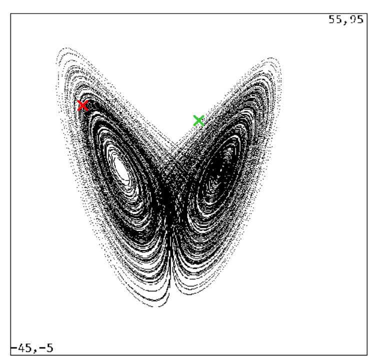



## OGY control

- dense attractor coverage → reachability
- un/stable manifold structure + local-linear control → controllability

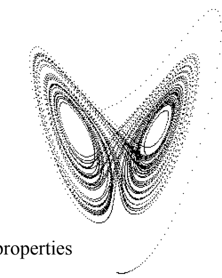
Ott *et al.*, *PRL* **64**:1196

Use local-linear control, designed using the eigen-values and eigen-vectors at that point ✕ to balance a chaotic system on a UPO passing through that point.




*But you're relying on denseness to get you into the controllable region, and that may take a while…*

## Chaos and control



Can we use some of those cool properties and turn them to advantage?

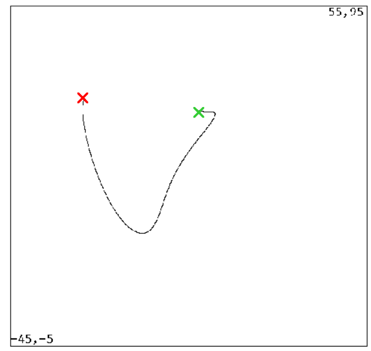
e.g., dense attractor coverage

Or the manifold structure?

Or the sensitive dependence on ICs?

## Lorenz System:

**SDOIC-based targeting**




Four R switches; 240X faster

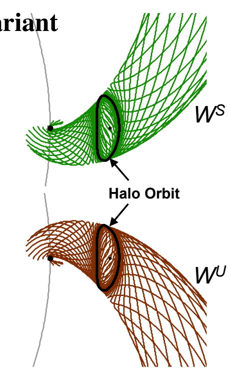
Bradley, *Cybernetics & Systems* **26**:299

**Program in Applied Mathematics**

Erik Bollt

*University of Colorado at Boulder*
*Boulder CO 80309-0526*
*(303) 492-4668*

## Other cool ways to use invariant manifolds




Want to get a spacecraft onto a "halo orbit," which is a UPO of the dynamics.

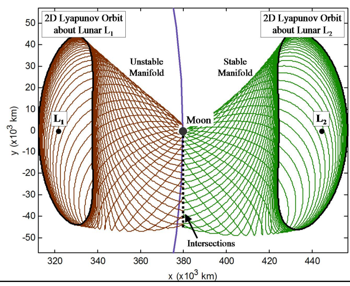
Unstable Periodic Orbits (UPOs) have invariant manifolds:

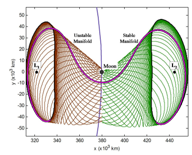
- Stable Invariant Manifold ($W^S$)
  - *The set of all trajectories a particle could use to arrive onto the UPO.*
- Unstable Invariant Manifold ($W^U$)
  - *The set of all trajectories a particle could take after a small perturbation from the UPO.*

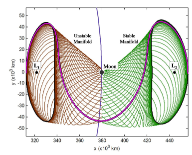
Jeff Parker, PhD thesis, UColorado 2008

## Low-energy (cheap) orbit transfers

- Depart along $W^{U}_{L1}$ & arrive on $W^{S}_{L2}$



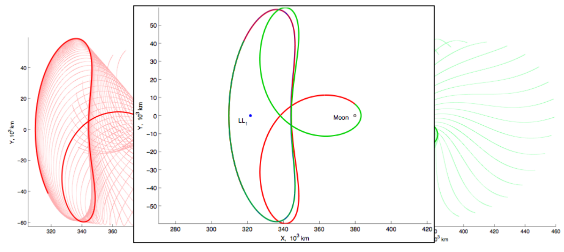





Jeff Parker, PhD thesis, UColorado 2008

## Homoclinic orbits - The best case

- If a trajectory in Stable and Unstable intersect ("homoclinic connection")



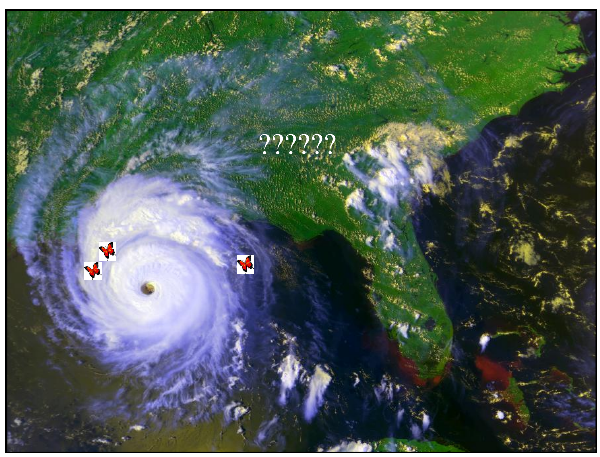

Unstable Manifold of an $LL_1$ Lyapunov Orbit

Stable Manifold of an $LL_1$ Lyapunov Orbit

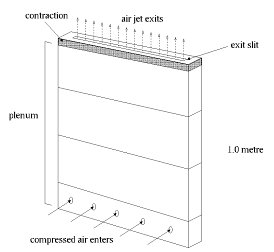
Jeff Parker, PhD thesis, UColorado 2008

**Can we do any of that in spatially extended systems?**

(i.e. harness the butterfly effect, exploit un/stable manifold geometry?)

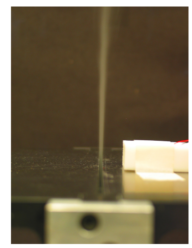


## A 2D jet

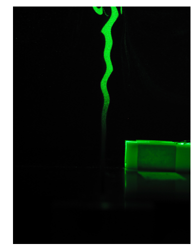



Peacock *et al.*, *Exp. Fluids* **37**:22

## End view



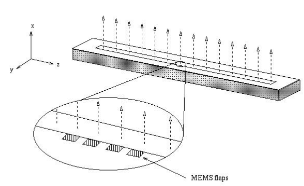
room lighting



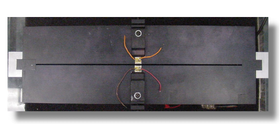
stop-action laser "slice"

aerosolized canola oil

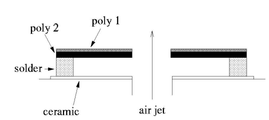
## Forcing the jet flow




Slit: 2.5 X 400 mm



## MEMS actuators



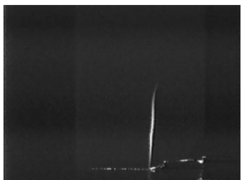

*Video : overhead view at 2Hz, 10V*

Area of individual flap is 1.0 x 0.25mm

Ma *et al.*, *IEEE Trans. Adv. Packaging* **26**:268

## The Butterfly effect in action…

*no forcing*



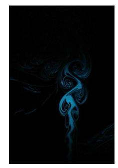
*6Hz forcing*

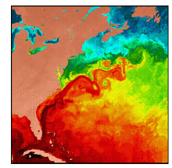
MEMS flap

*Forcing generates coherent structures that enhance entrainment and mixing*

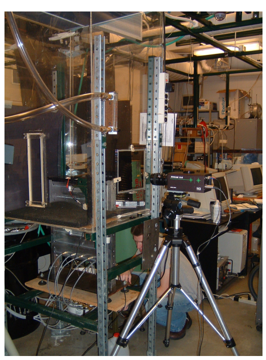
Peacock *et al*., *Exp. Fluids* **37**:22

## Does this have anything to do with reality?





## Measurement & isolation:

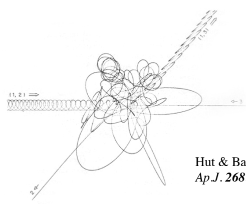


## Another interesting application: chaos in the solar system

- orbits of Pluto, Mars
- Kirkwood gaps
- rotation of Hyperion & other satellites
- …

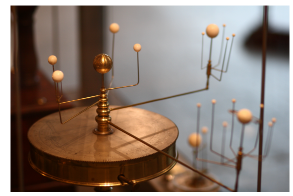
## Solar system stability:

- recall: two-body problem not chaotic
- but three (or more) can be…



Hut & Bahcall
*Ap.J.* ***268****:319*

## Exploring that issue before the digital computer age…

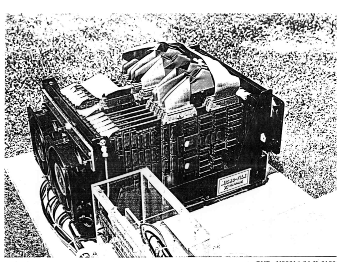


An *orrery*, which is a *mechanical computer* whose gear ratios and circular platters simulate the orbits of the planets

## Exploring that issue, circa 1980:

- write the *n*-body equations for the solar system
- solve them using symplectic ODE solvers on a special-purpose computer

The *digital orrery*
(Wisdom & Sussman)



ONR N00014-86-K-0180

### Numerical Evidence That the Motion of Pluto Is Chaotic

Gerald Jay Sussman and Jack Wisdom

The Digital Orrery has been used to perform an integration of the motion of the outer planets for 845 million years. This integration indicates that the long-term motion of the planet Pluto is chaotic. Nearby trajectories diverge exponentially with an *e*-folding time of only about 20 million years.

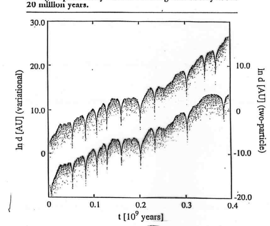


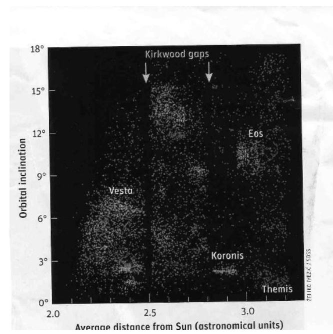

**Figure 3:** The exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithms of the distance measures as a function of time. In the upper trace we see the growth of the variational distance around a reference trajectory. In the lower trace we see how two Plutos diverge with time. The distance saturates near 80AU when the Plutos are on opposite sides of the Sun. The variational method of studying neighboring trajectories does not have the problem of saturation. Note that the two methods are in excellent agreement until the two-trajectory method has nearly saturated.

*Science* **241**:433

## Should we worry?

- No.

## Kirkwood gaps:




From *Sky & Telescope*

## Chaos and the Kirkwood gaps

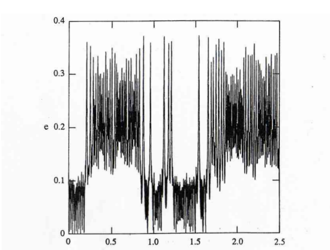



FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. the time is now measured in millions of years. Bursts of high eccentricity behavior are interspersed with intervals of irregular low eccentricity behavior, broken by occasional spikes.

Wisdom, *Nuclear Phys. B* **2:**391

## Evidence in favor of the conjecture:

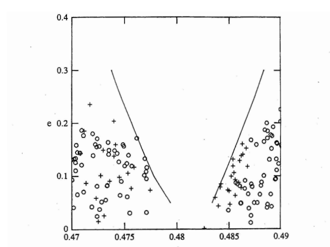



FIGURE 9. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and quasiperiodic region in the gap, but trajectories of both types are planet crossing.

Wisdom, *Nuclear Phys. B* **2:**391

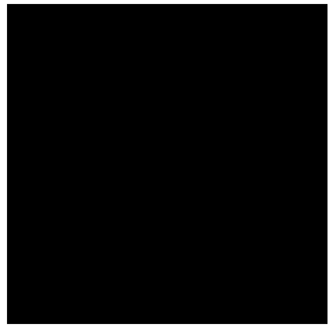
## Chaotic tumbling of satellites:

Voyager and Galileo saw this…


From *Sky & Telescope*

*Ap. J.* **97**:570
*Ap. J.* **98**:1855

## …so did Cassini:



www.nasa.gov/mission_pages/cassini/multimedia/pia06243.html

## Chaotic tumbling of satellites:

This happens for **all** satellites at some point in their history, unless they are perfectly spherical and in perfectly circular orbits (pf: KAM theorem; see Wisdom paper on syllabus)

Some of them are still tumbling chaotically because of their geometry, but most (like the earth and its moon) have settled down into tidal equilibria

## More chaos in the solar system:

- obliquity of Mars (Touma & Wisdom, *Science* **259**:1294)

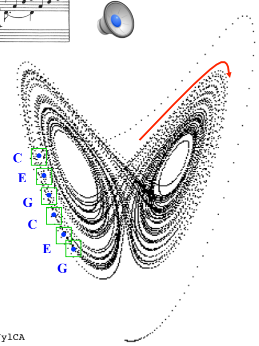
www.solarviews.com

- etc.

## Musical Variations from a Chaotic Mapping

Dabby *Chaos* **6**:95


**Pitch sequence:**
**C, E, G, C, E, G, C, E…**

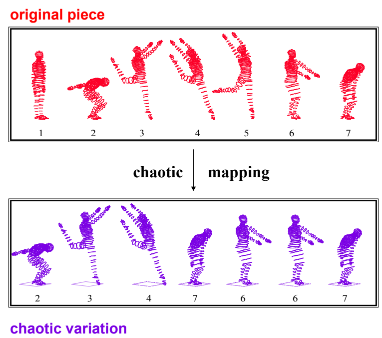



C symbol dynamics

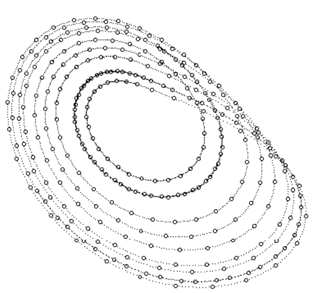
Variation!

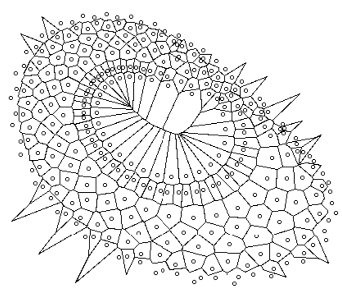
Also fun: `http://www.youtube.com/watch?v=B2XtE9YylCA`

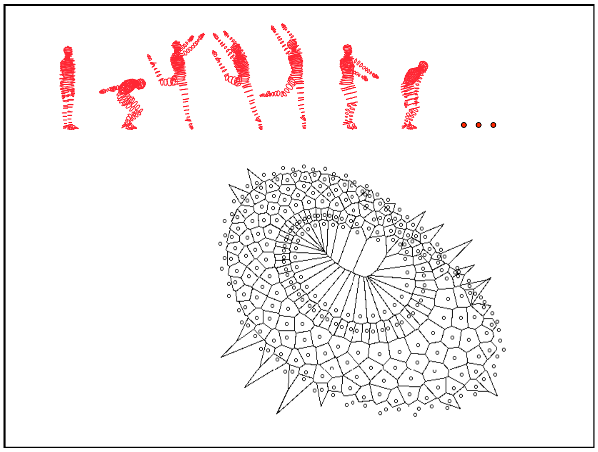
## Chaotic variations on movement sequences

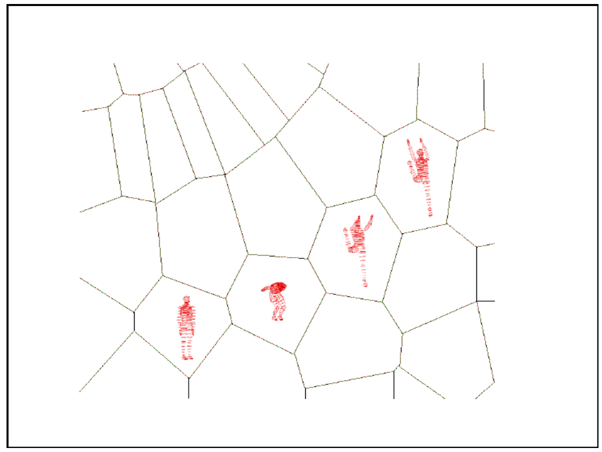


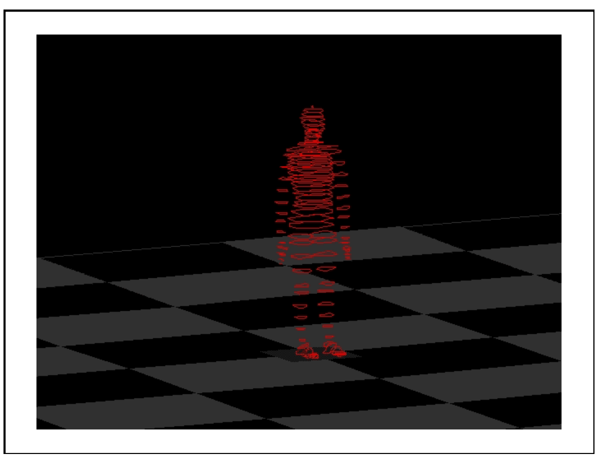

Bradley & Stuart, *Chaos* **8**:800

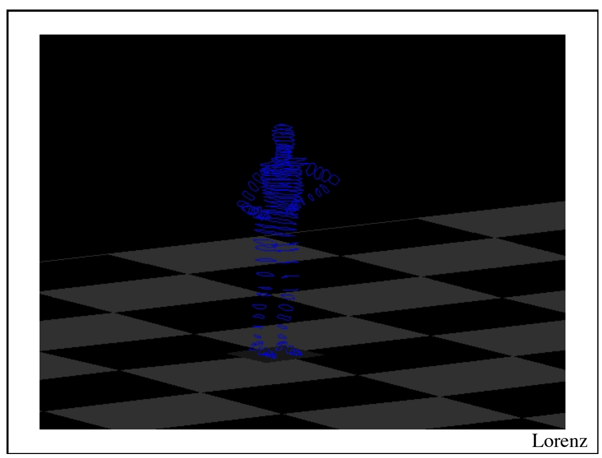


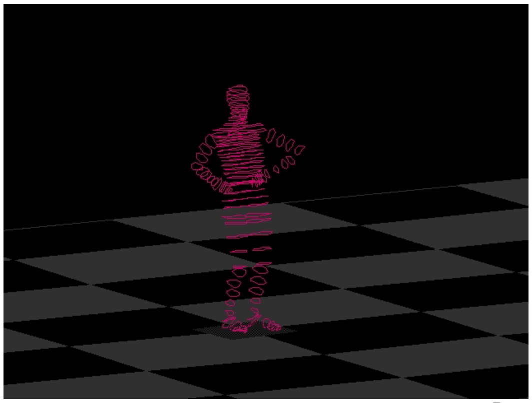




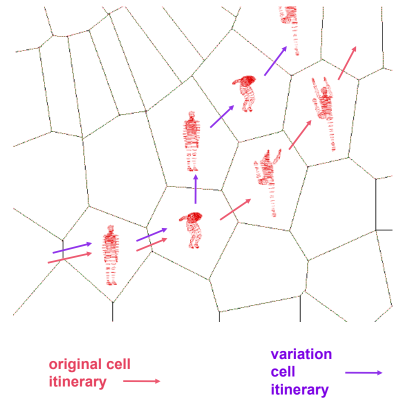


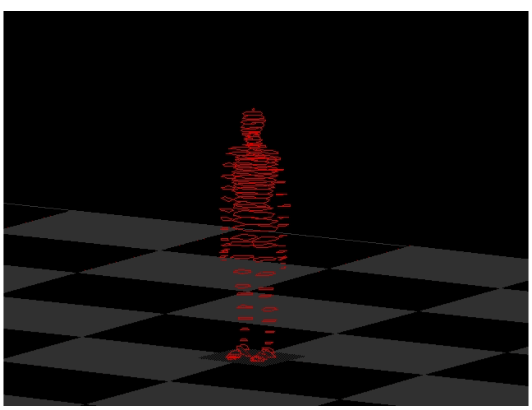



Lorenz

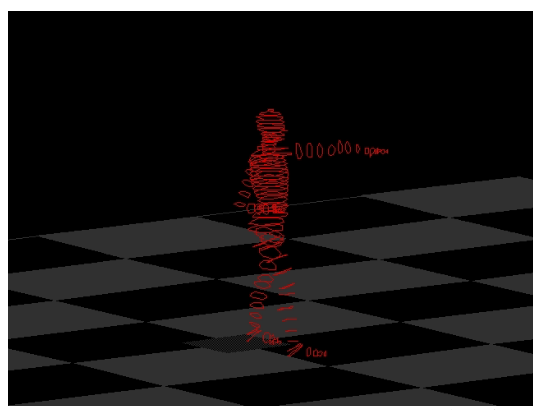


Rossler

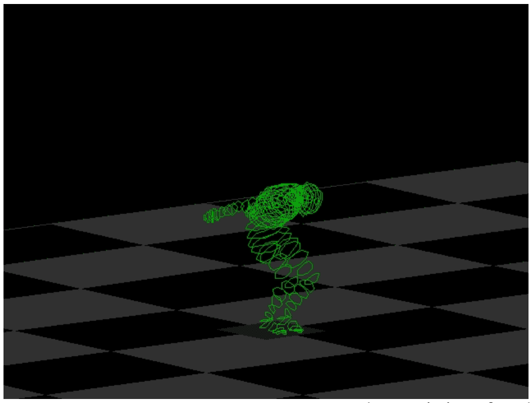


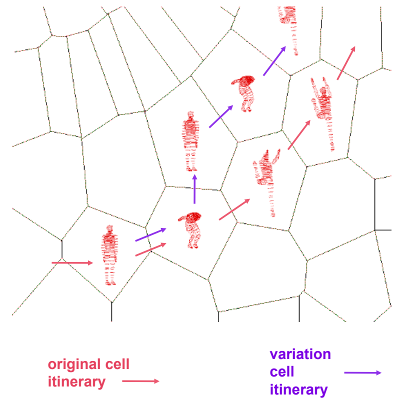
medley—original

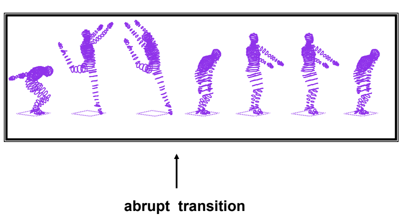


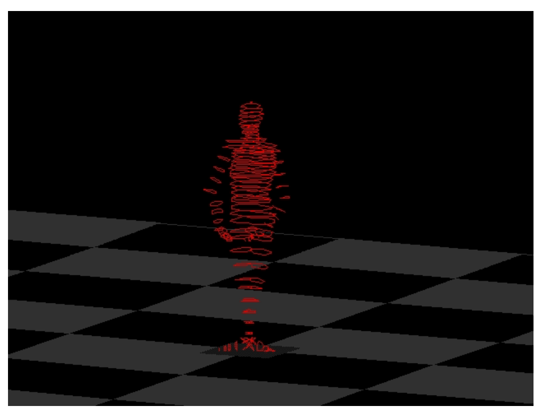
Rossler variation of medley

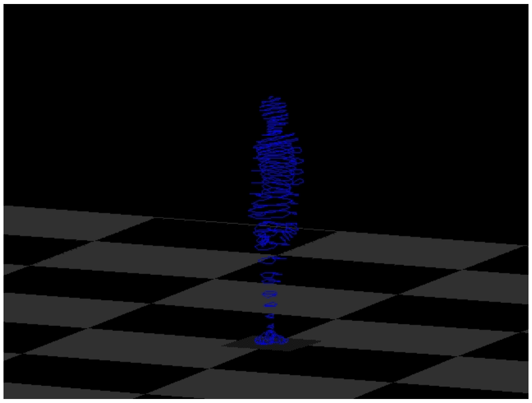


random variation of medley

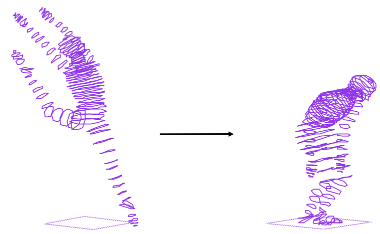


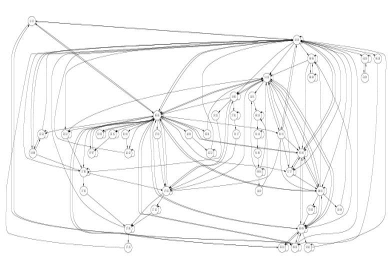


## Interpolation



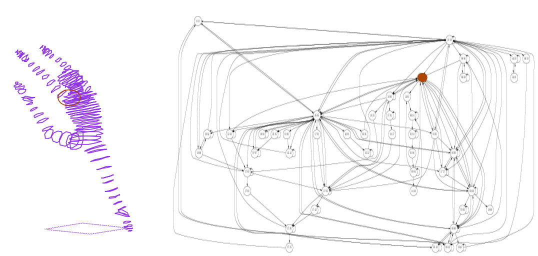
Stuart & Bradley, *ICML* 1998

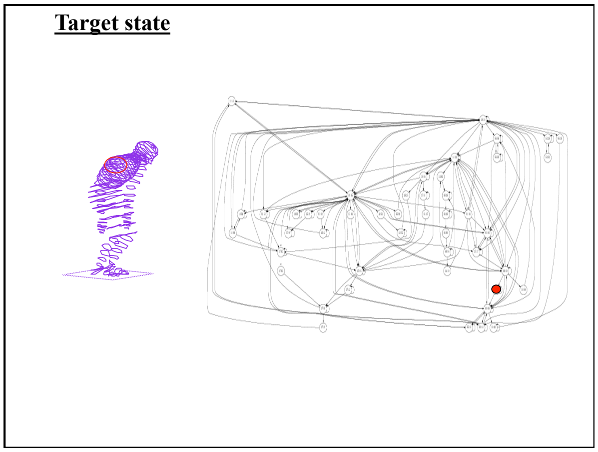
## Corpus-based approach

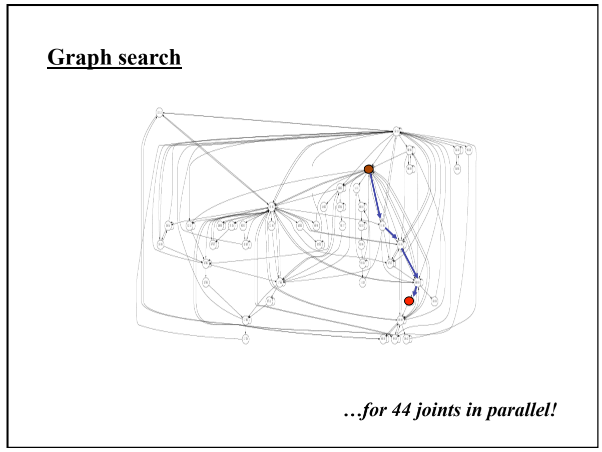


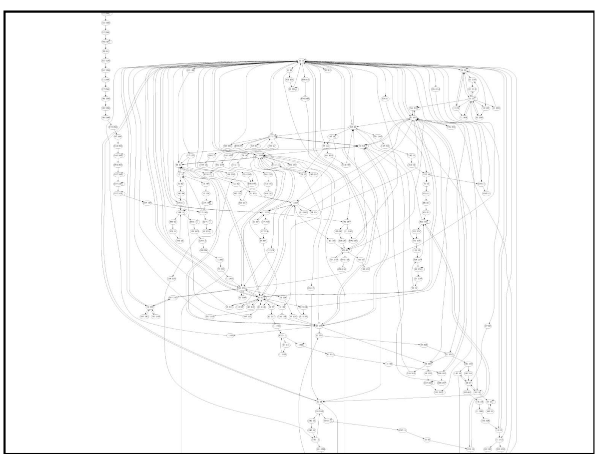
- **graph captures motions of one joint**
- **note: specific to the genre of the corpus!**

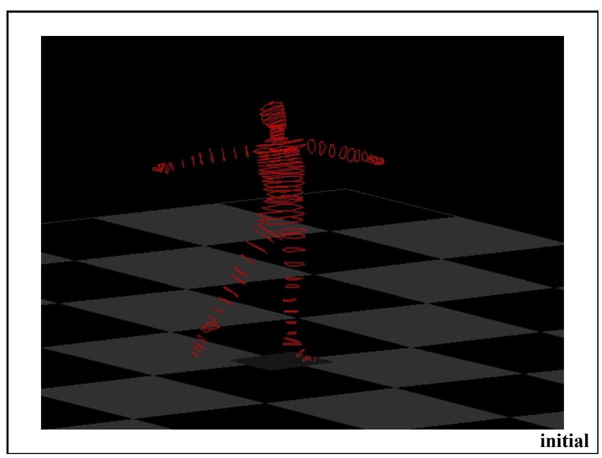
## Initial state

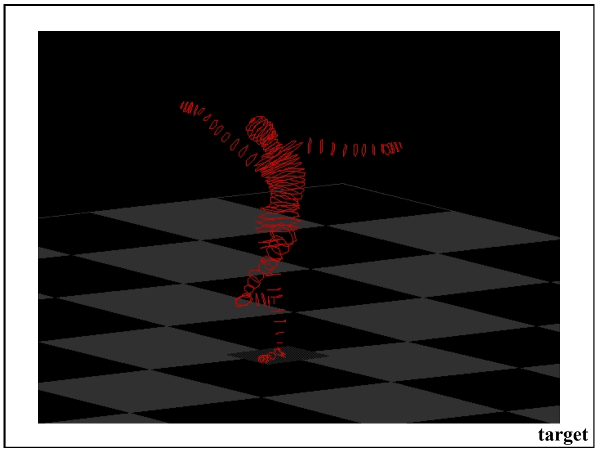


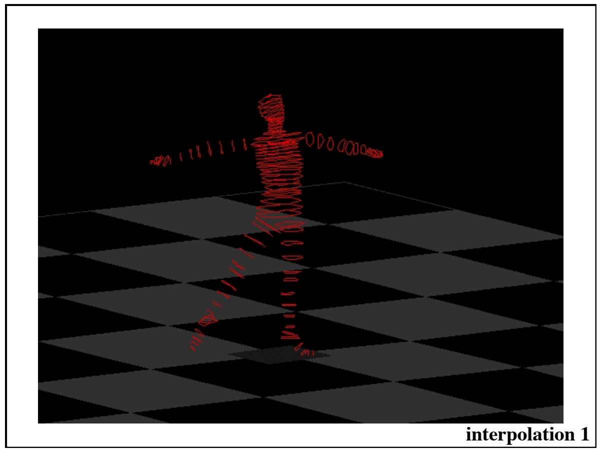

Target state

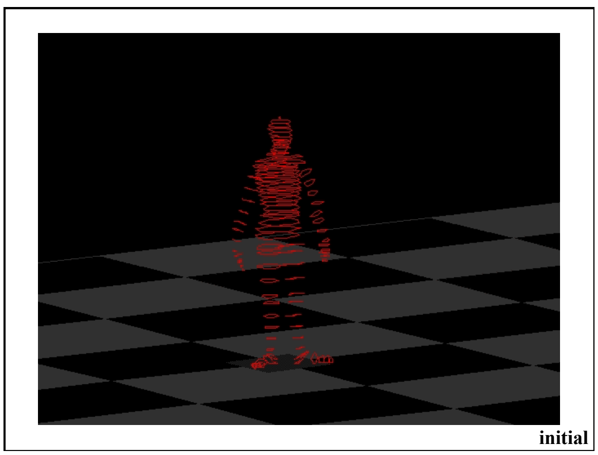

Graph search
…for 44 joints in parallel!

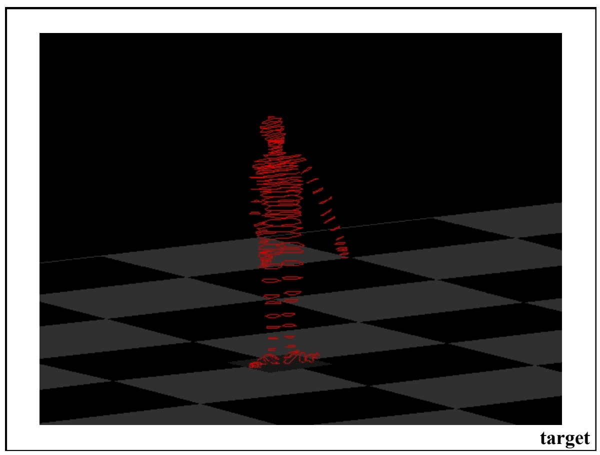


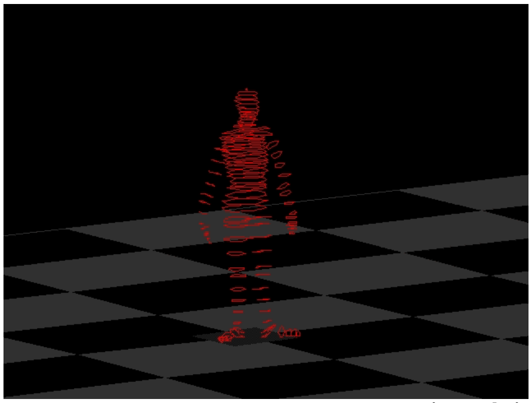

initial


target

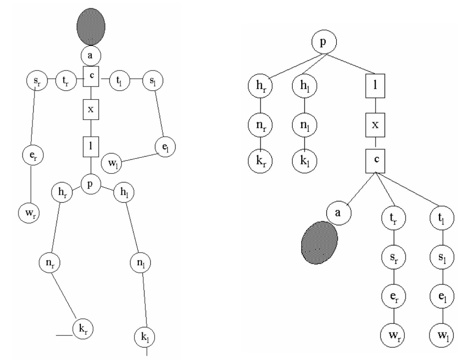

interpolation 1

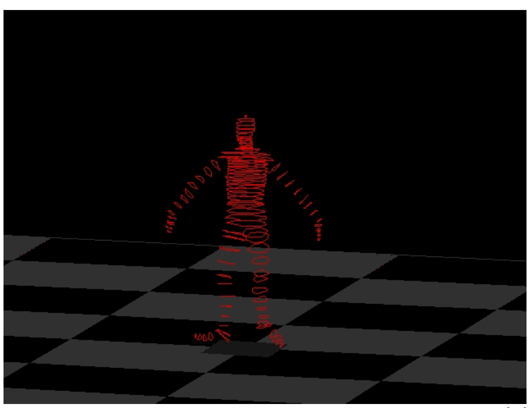

initial


target

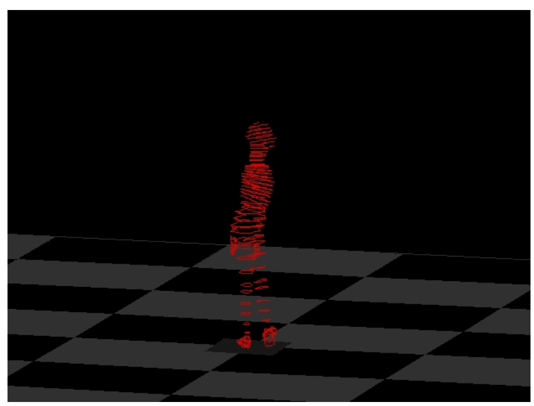


interpolation 2

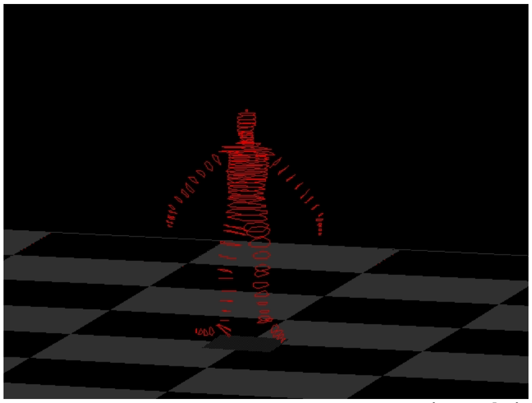




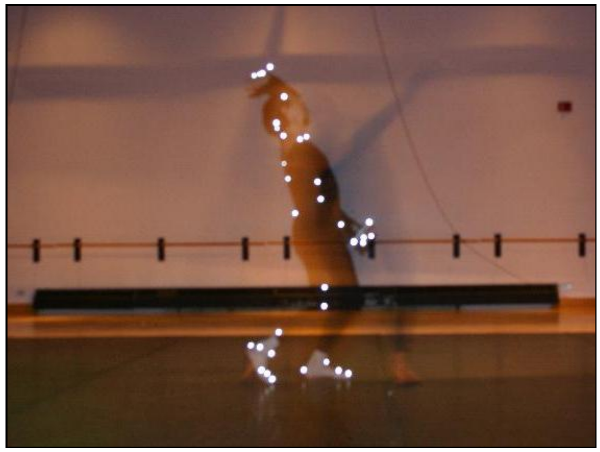
initial

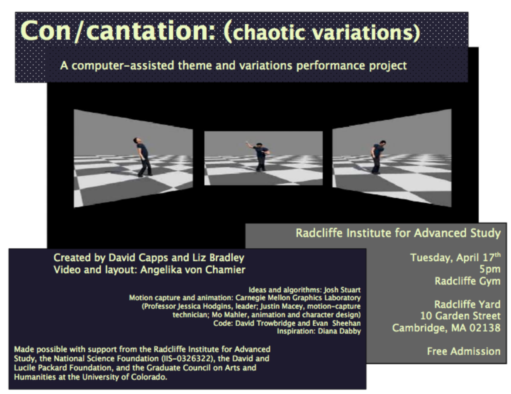


target

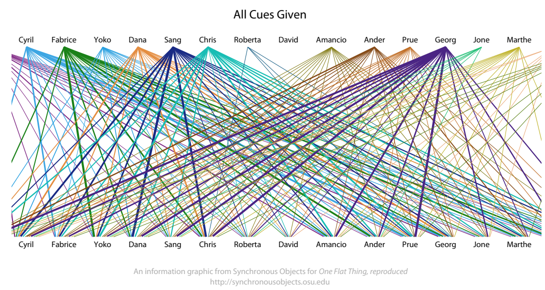


interpolation 3

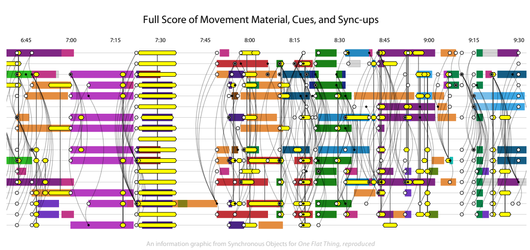




Chaos vs. complexity??




synchronousobjects.osu.edu/content.html

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